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Elgot theories: a new perspective on the equational properties of iteration

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Bloom and Ésik's concept of iteration theory summarises all equational properties that iteration has in common applications, for example, in domain theory, where to every system of recursive equations, the least solution is assigned. This paper shows that in the coalgebraic approach to iteration, the more appropriate concept is that of a functorial iteration theory (called Elgot theory). These theories have a particularly simple axiomatisation, and all well-known examples of iteration theories are functorial. Elgot theories are proved to be monadic over the category of sets in context (or, more generally, the category of finitary endofunctors of a locally finitely presentable category). This demonstrates that functoriality is an equational property from the perspective of sets in context. In contrast, Bloom and Ésik worked in the base category of signatures rather than sets in context, and there iteration theories are monadic but Elgot theories are not. This explains why functoriality was not included in the definition of iteration theories.

1. Introduction

What are the essential equational properties of iteration operators? The monograph Bloom and Ésik (1993) gives the answer in terms of the axioms of iteration theories, and we proved in Adámek *et al.* (2007) that iteration theories are monadic over the category **Sgn** of signatures.

The aim of the present paper is to prove that functorial iteration theories, which we call *Elgot theories*, form a monadic category over the base category

$$\mathbf{Set}^{\mathbb{F}} = \text{sets in context},$$

where $\mathbb{F}$ is the category of finite sets and functions.

This implies that the axioms of functorial iteration theories precisely capture all equational properties that one finds, for example, in domain theory for the operation assigning to every recursive equational system its least solution. Let us explain this in detail.

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In domain theory one works in a continuous theory and uses iteration to get for every equation morphism $e : n \longrightarrow n + k$ the least solution $e^{\dagger} : n \longrightarrow k$. This dagger operation $e \longmapsto e^{\dagger}$ enjoys a number of equational properties, for example, the fact that $e^{\dagger}$ is a solution of e is expressed by the equation $e^{\dagger} = [e^{\dagger}, \mathrm{id}_k]\cdot e$. The aim of Stephen Bloom and Zoltán Ésik's concept of iteration theory is to collect together all equational properties of the dagger operation in domain theory (and in a substantial number of other applications where iteration is used – see the fundamental monograph Bloom and Ésik (1993)). The function $e \longmapsto e^{\dagger}$ in domain theory is also functorial, that is, for every given k, we obtain a functor $\dagger$ from the category of all equation morphisms $e : n \longrightarrow n + k$ to the slice category of k. This important functoriality property has been studied in various contexts: for example, Alex Simpson and Gordon Plotkin call it parametrised uniformity in Simpson and Plotkin (2000), and they state in their introduction that functoriality is 'a convenient tool for establishing that the equations of an iteration operator are satisfied'. Larry Moss observed in Moss (2003) that functoriality allows for a particularly simple axiomatisation of iteration theories – see Definition 2.11 below. Functoriality is, however, not a part of the definition of iteration theory; this property is called *functorial dagger implication* in Bloom and Ésik (1993). The name and the fact that it is not included in the definition both indicate that Bloom and Ésik do not consider functoriality to be an equational property. The aim of the present paper is to demonstrate that when viewed from a new perspective, functoriality *is* equational.

Recall that for every signature Σ, the free continuous theory on Σ is the theory $\mathbb{T}_{\Sigma_{\perp}}$ of $\Sigma_{\perp}$-trees: one adds to Σ a new nullary symbol $\perp$, forming a new signature $\Sigma_{\perp}$, and the morphisms from 1 to n in $\mathbb{T}_{\Sigma_{\perp}}$ are all $\Sigma_{\perp}$-trees (finite and infinite) on n variables. As proved by Bloom and Ésik, the free iteration theory on Σ is the subtheory $\mathbb{R}_{\Sigma_{\perp}}$ of all *rational* $\Sigma_{\perp}$-trees, that is, trees with finitely many subtrees up to isomorphism. This defines a monad Rat on the category **Sgn** of signatures:

$$\mathsf{Rat}(\Sigma) = \text{the signature of rational } \Sigma_{\perp}\text{-trees.}$$

In the present paper we change the perspective and work in $\mathbf{Set}^{\mathbb{F}}$ (or, equivalently, the category of finitary endofunctors of **Set**) rather than in the category of signatures. As shown in Adámek *et al.* (2006a), every finitary endofunctor H on **Set** generates a rational monad $\mathbb{R}_H$; we consider it as an algebraic theory (*cf.* Remark 2.3). One of our main technical results is the following theorem.

Theorem 1.1. $\mathbb{R}_{H+1}$ is the free Elgot theory on H.

(Adding the constant functor 1 to the one-element set corresponds to adding $\perp$ to a signature Σ.)

This gives a monad Rat on the category $\mathbf{Set}^{\mathbb{F}}$. Our main result is the following theorem.

Theorem 1.2. Elgot theories are precisely the monadic algebras for Rat. More precisely, the category of Elgot theories is isomorphic to the category $(\mathbf{Set}^{\mathbb{F}})^{\mathsf{Rat}}$ of Eilenberg–Moore algebras for Rat in $\mathbf{Set}^{\mathbb{F}}$.

It then follows from results given in Kelly and Power (1993), which we will recall in the appendix, that Elgot theories are equationally presentable over $\mathbf{Set}^{\mathbb{F}}$. And the corresponding equations for the dagger operation are precisely those that hold in all continuous theories because, once again, we only need to consider the free theories, and they are quotients of the theories $\mathbb{R}_{\Sigma_\perp}$.

Related work

Iterative and iteration theories. In Elgot (1975), Calvin Elgot introduced iterative theories as those theories in which every ideal equation morphism has a unique solution. Stephen Bloom and Zoltán Ésik started to study equational properties of iteration in the 1980's and they summarised the results for their main concept, iteration theories, in their monograph Bloom and Ésik (1993).

Coalgebras and rational monads. The realisation that iteration can be studied more generally using categories, functors and final coalgebras rather than sets, signatures and trees goes back to Moss (2001). As we have already mentioned, every finitary endofunctor H on **Set** (or, more generally, on every locally finitely presentable category) generates the rational monad $\mathbb{R}_H$, which, considered as a theory, is the free iterative theory on H. In Adámek *et al.* (2006a), we also gave a coalgebraic construction of $\mathbb{R}_H$. The rational monad $\mathbb{R}_H$ assigns to every set X the final locally finite coalgebra for $H(-) + X$ (Bonsangue *et al.* 2009; Milius 2010). As a first step towards the proof of Theorem 1.1, we proved in Adámek *et al.* (2010b) that every strict iterative theory is an Elgot theory – Bloom and Ésik (1993) had already proved this for theories in **Set**. The second step, which is presented below, is the proof that free Elgot theories are given by the free iterative theories.

Elgot algebras. Our current result is a complete higher-order analogue of the result for Elgot algebras proved in Adámek *et al.* (2006b) that every set X in context can be extended (essentially uniquely) to a finitary endofunctor H. We proved that the monad on **Set** corresponding to the free iterative theory $\mathbb{R}_H$ is precisely the monad of free iterative H-algebras. And we characterised the Eilenberg–Moore category of that monad: it is the category of all H-algebras A together with a function † assigning solutions $e^\dagger$ to flat recursive equations e in A in such a way that certain equational properties are fulfilled. These properties are quite analogous to the axioms of Elgot monads (compare Definitions 2.11 and 4.7 below).

Contents of the paper

The following is a brief description of the outline of the rest of the paper:

Section 2. Elgot theories and Elgot monads: In this section, we present the axioms of Elgot theories and give some examples, which are mostly from the monograph Bloom and Ésik (1993).

Section 3. Iterative theories: In this section, we recall the fact, which was established in Adámek *et al.* (2008), that given a strict iterative monad, every equation morphism has a unique strict solution.

Section 4. Elgot algebras: In this section, we recall from Adámek *et al.* (2006b) the concept for every finitary endofunctor H, of Elgot H-algebras as the Eilenberg–Moore algebras for the rational (= free iterative) monad $\mathbb{R}_H$ of H.

Section 5. Free Elgot monads: In this section we present the proof of Theorem 1.1.

Section 6. The Monad Rat and its algebras: In this section, we prove that the natural forgetful functor from the category $\mathsf{EMnd}(\mathscr{K})$ of Elgot monads on $\mathscr{K}$ into the category of finitary endofunctors on $\mathscr{K}$ is monadic. The corresponding monad Rat assigns to every finitary endofunctor H the rational monad $\mathbb{R}_{H+C_1}$. Consequently, the Eilenberg–Moore algebras for Rat are precisely the Elgot monads.

Section 7. Conclusions: In this section, we present our conclusions that the equational properties of iteration operators, as encountered, for example, in domain theory (with respect to the least solutions of recursive equations), are precisely summed up by the axioms of Elgot theories.

Appendix: The appendix gives a short presentation of the results we use from Kelly and Power (1993).

An extended abstract presenting the main result of this paper was published as a conference paper (Adámek *et al.* 2009a). The present paper is a fully revised and extended version containing full details of all proofs.

2. Elgot theories and Elgot monads

Assumption 2.1. Throughout this section, we will use $\mathscr{K}$ to denote a locally finitely presentable category – see Gabriel and Ulmer (1971) or Adámek and Rosický (1994). More precisely, $\mathscr{K}$ has

- (i) colimits; and
- (ii) a small full subcategory $\mathscr{F}(\mathscr{K})$ representing all finitely presentable objects such that every object of $\mathscr{K}$ is a filtered colimit of objects of $\mathscr{F}(\mathscr{K})$.

Recall that an object n is said to be *finitely presentable* if $\mathscr{K}(n, -)$ preserves filtered colimits. More generally, functors preserving filtered colimits are said to be *finitary*. We will just write $\mathscr{F}$ instead of $\mathscr{F}(\mathscr{K})$ whenever there is no risk of confusion.

Fact 2.2. A finitary functor $H : \mathscr{K} \longrightarrow \mathscr{K}$ is, up to natural isomorphism, fully determined by its domain restriction $H \restriction \mathscr{F}$, an object of $\mathscr{K}^{\mathscr{F}}$. Indeed, H is the left Kan extension of $H \restriction \mathscr{F}$. Thus, we have an equivalence of categories

$$\mathscr{K}^{\mathscr{F}} \cong \mathsf{Fin}(\mathscr{K}, \mathscr{K})$$

where $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$ is the category of finitary endofunctors and natural transformations.

Remark 2.3 (Monads and theories).

(1) Recall that a *monad* $\mathbb{S} = (S, \eta, \mu)$ consists of an endofunctor $S : \mathscr{K} \longrightarrow \mathscr{K}$ and natural transformations $\eta : \mathrm{Id} \longrightarrow S$ and $\mu : S{\cdot}S \longrightarrow S$ such that $\mu{\cdot}\eta S = \mathrm{id}_S = \mu{\cdot}S\eta$ and $\mu{\cdot}S\mu = \mu{\cdot}\mu S$. The monad is said to be *finitary* if S is a finitary functor.

(2) The *Kleisli category* $\mathscr{K}_{\mathbb{S}}$ of $\mathbb{S}$ has the same objects as $\mathscr{K}$, and its morphisms

$$f : X \longrightarrow\!\!\!\circ\; Y$$

are the morphisms $f : X \longrightarrow SY$ of $\mathscr{K}$. They compose as follows: given $g : Y \longrightarrow\!\!\!\circ\; Z$, the composite $g{\cdot}f$ in the Kleisli category is the $\mathscr{K}$-morphism

$$X \xrightarrow{f} SY \xrightarrow{Sg} SSZ \xrightarrow{\mu_Z} Z.$$

(3) We use $J : \mathscr{K} \longrightarrow \mathscr{K}_{\mathbb{S}}$ to denote the canonical functor that assigns to $f : X \longrightarrow Y$ the morphism $Jf = \eta_Y{\cdot}f : X \longrightarrow SY$ in $\mathscr{K}_{\mathbb{S}}$. We will write $f : X \longrightarrow Y$ for $Jf : X \longrightarrow\!\!\!\circ\; Y$ and call these morphisms *base morphisms.*

(4) The *theory* of $\mathbb{S}$ is denoted by

$$\mathrm{Th}(\mathbb{S})$$

and is the category whose objects are the objects of $\mathscr{F}$ and whose morphisms are the Kleisli category morphisms. For a finitary monad of **Set** this notion corresponds to the notion of the Lawvere theory of the monad (Lawvere 1963).

(5) $\mathscr{K}_{\mathbb{S}}$ has coproduct formed on the level of the base category $\mathscr{K}$, so finite coproducts in $\mathscr{K}$ and in $\mathrm{Th}(\mathbb{S})$ are the same.

Notation 2.4. We use $\mathsf{inl} : X \longrightarrow X + Y$ and $\mathsf{inr} : Y \longrightarrow X + Y$ to denote injections of coproducts. Occasionally, we also use $\mathsf{inm} : B \longrightarrow A + B + C$ for the middle injection. Given a functor H we use can to denote the canonical morphism

$$[H\,\mathsf{inl}, H\,\mathsf{inr}] : HX + HY \longrightarrow H(X + Y).$$

Example 2.5. If $\mathscr{K} = \mathbf{Set}$, we can choose $\mathscr{F}$ to be the category of natural numbers and functions between them.

Every finitary monad $\mathbb{S}$ on **Set** is equationally presentable, that is, there exists a signature Σ and a set of equations E such that $\mathbb{S}$ is the monad of all free algebras in the variety $\mathbf{Alg}(\Sigma, E)$ presented by E. Then

$$\mathrm{Th}(\mathbb{S})$$

is the category of natural numbers with hom-sets

$$\mathrm{Th}(\mathbb{S})(1, n)$$

formed by terms in n variables of the variety $\mathbf{Alg}(\Sigma, E)$. The general case

$$\mathrm{Th}(\mathbb{S})(k, n) = \big(\mathrm{Th}(\mathbb{S})(1, n)\big)^k$$

is formed by k-tuples of such terms.

Remark 2.6.

(1) A *strict* endofunctor is an endofunctor H together with a chosen global element of $H0$:

$$\bot : 1 \longrightarrow H0$$

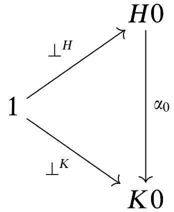
(where 0 is the initial object and 1 the terminal one).

(2) We extend the $\perp$ notation to all composites

$$X \xrightarrow{u} 1 \xrightarrow{\perp} H0 \xrightarrow{Hv} HY \qquad (u, v \text{ unique}).$$

When H is the underlying functor of a monad, this yields a morphism from X to Y in the Kleisli category, which we denote by $\perp_{X,Y} : X \longrightarrow\!\!\!\!\circ\; Y$.

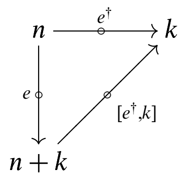
(3) A natural transformation $\alpha : H \longrightarrow K$ between strict functors is said to be *strict* if it preserves $\perp$:



Definition 2.7. Let $\mathbb{S}$ be a finitary monad.

(i) An *equation morphism* is a morphism $e : n \longrightarrow\!\!\!\!\circ\; n + k$ in the theory of $\mathbb{S}$. We refer to n as *the object of variables* and k as the *object of parameters*. That is, an equation morphism with k as object of parameters is given by an object n of variables and a morphism $e : n \longrightarrow S(n + k)$.

(ii) A *solution* of e is a morphism $e^\dagger : n \longrightarrow\!\!\!\!\circ\; k$ such that the triangle



(2.1)

commutes in Th($\mathbb{S}$).

Example 2.8. For $\mathbb{S}$ as in Example 2.5, the morphism $e : n \longrightarrow S(n + k)$ can be viewed as a system of n recursive equations

$$x_i \approx t_i(x_1, \ldots, x_n, y_1, \ldots, y_k) \qquad i = 1, \ldots, n$$

where t_i is a (Σ, E)-term in $n + k$ variables. A solution is then a substitution of terms $x_i^\dagger(y_1, \ldots, y_k)$ for variables x_i making each of the formal equations an identity:

$$x_i^\dagger = t_i\big(x_1^\dagger, \ldots, x_n^\dagger, y_1, \ldots, y_k\big).$$

Remark 2.9. In the following definition we assume that every equation morphism e is given a solution $e^\dagger$ 'canonically'. This means that we are given a function $\dagger$ assigning a solution to every equation morphism. Moreover, various 'natural' equational properties are required:

Functoriality:

For two equation morphisms $e : n \multimap n + k$ and $e' : n' \multimap n' + k$ with the same object k of parameters we introduce the concept of a *homomorphism of equations*: this is a base morphism (see Remark 2.3) $v : n \longrightarrow n'$ such that the square

$$\begin{array}{ccc} n & \xrightarrow{\;e\;} & n+k \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle v+k} \\ n' & \xrightarrow[\;e'\;]{} & n'+k \end{array}$$

commutes. We see that e and e' are essentially the same recursive systems of equations, and the morphism v just renames the variables. We expect that the solution of e' yields the solution of e through the same renaming:

$$\begin{array}{ccc} n & & \\ {\scriptstyle v}\big\downarrow & \searrow^{e^\dagger} & \\ & & k \\ & \nearrow_{(e')^\dagger} & \\ n' & & \end{array}$$

This expresses the fact that $\dagger$ is a functor from the category of equation morphisms and homomorphisms with parameter object k to the slice category $\mathrm{Th}(\mathbb{S})/k$.

Parameter identity:

Suppose $u : k \multimap k'$ is a morphism 'substituting the parameters'. Then every equation morphism $e : n \multimap n + k$ with k as object of parameters defines a new equation morphism with object k' of parameters in the obvious way:

$$u \bullet e \equiv n \xrightarrow{\;e\;} n + k \xrightarrow{\;n+u\;} n + k'. \tag{2.2}$$

Or, on the level of morphisms of $\mathscr{K}$,

$$u \bullet e \equiv n \xrightarrow{\;e\;} S(n+k) \xrightarrow{\;S(\eta^S + u)\;} S(Sn + Sk) \xrightarrow{\;S\,\mathsf{can}\;} SS(n+k) \xrightarrow{\;\mu^S\;} S(n+k),$$

where $\mathsf{can} : Sn + Sk \longrightarrow S(n + k)$ is the canonical morphism (*cf.* Remark 2.10)[1]. We expect that when parameters are again changed through u, the solution of e yields the solution of $u \bullet e$:

$$\begin{array}{ccc} n & \xrightarrow{\;e^\dagger\;} & k \\ & \searrow^{(u\bullet e)^\dagger} & \big\downarrow{\scriptstyle u} \\ & & k' \end{array}$$

[1] From now on we shall frequently omit the subscripts denoting the components of natural transformations in diagrams.

The Bekić Identity:

Consider a pair of equation morphisms

$$e : n \multimap n + m + k$$
$$f : m \multimap n + m + k.$$

We can solve them simultaneously in the sense of forming the equation morphism

$$[e, f] : n + m \multimap n + m + k$$

and solving it:

$$[e, f]^\dagger : n + m \multimap k.$$

We expect to get the same result by performing 'sequential' iteration:

(a) Solve e to get $e^\dagger : n \multimap m + k$.

(b) Form the equation morphism e_R by substituting the variables of n into f using $e^\dagger$,

$$e_R \equiv m \xrightarrow{f} n + m + k \xrightarrow{[e^\dagger, m+k]} m + k, \tag{2.3}$$

and get $e_R^\dagger : m \multimap k$.

(c) Compose $[e^\dagger, \mathsf{inl}] : n + m \multimap m + k$ with $[e_R^\dagger, k]$ to get the desired solution. That is,

$$[e, f]^\dagger \equiv n + m \xrightarrow{[e^\dagger, \mathsf{inl}]} m + k \xrightarrow{[e_R^\dagger, k]} k. \tag{2.4}$$

Remark 2.10. The following definition comes, essentially, from the monograph Bloom and Ésik (1993). Instead of 'Solution', they use the name fixed-point identity, and instead of 'Functoriality' they use the name functorial dagger implication. They call our Bekić Identity (2.4) a pairing identity, and formulate it differently:

$$[e, f]^\dagger \equiv [e_L^\dagger, e_R^\dagger] : n + m \multimap k \tag{2.5}$$

where

$$e_L \equiv n \xrightarrow{e} n + m + k \xrightarrow{n + [e_R^\dagger, k]} n + k. \tag{2.6}$$

However, in the presence of the Parameter Identity, the two conditions (2.4) and (2.5) are equivalent since it is obvious that

$$e_L^\dagger \equiv n \xrightarrow{e^\dagger} m + k \xrightarrow{[e_R^\dagger, k]} k.$$

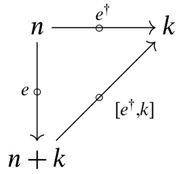
The fact that functorial iteration theories have the simple equational presentation below was first observed by Larry Moss – see the Introduction.

Definition 2.11. An *Elgot monad* is a finitary monad $\mathbb{S}$ together with a function

$$\frac{e : n \multimap n + k}{e^\dagger : n \multimap k} \qquad \text{(for all } n, k \in \mathscr{F}\text{)}$$

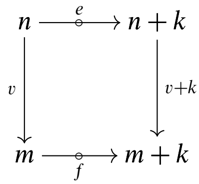
satisfying the following axioms:

Solution:

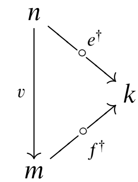


Functoriality:

if

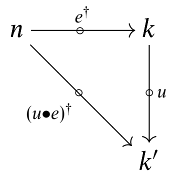


then



for every base morphism $v : n \longrightarrow m$.

Parameter Identity:



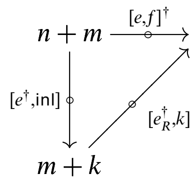
for every morphism $u : k \longrightarrow k'$, where $u \bullet e = (n+u)\cdot e : n \longrightarrow n+k$.

The Bekić Identity:

Given $e : n \longrightarrow n+m+k$ and $f : m \longrightarrow n+m+k$, if we form $e^\dagger : n \longrightarrow m+k$ and put

$$e_R = [e^\dagger, m+k]\cdot f : m \longrightarrow m+k,$$

then



Remark 2.12. Later, we shall work in $\mathscr{K}$ when using the Bekić Identity – it states (employing (2.6) above) that $[e,f]^\dagger = [e_L^\dagger, e_R^\dagger] : n+m \longrightarrow Sk$ where

$$e_R = \mu_{m+k}\cdot S[e^\dagger, \eta_{m+k}]\cdot f$$
$$e_L = \mu_{n+k}\cdot S\,\mathsf{can}\cdot S\big(\eta_n + [e_R^\dagger, \eta_k]\big)\cdot e$$

(*cf.* Remark 2.10).

Remark 2.13. An *Elgot theory* is the theory Th($\mathbb{S}$) of an Elgot monad $\mathbb{S}$ – see Remark 2.3. The operation † of an Elgot monad $\mathbb{S}$ is, by results in Bloom and Ésik (1993), precisely the same as the Conway operator on Th($\mathbb{S}$) in the sense of Simpson and Plotkin (2000), which is uniform for all base morphisms. As proved independently in Hasegawa (1999) and by M. Hyland (unpublished), this is equivalent to giving a trace structure on Th($\mathbb{S}$) in the sense of Joyal *et al.* (1996) that is uniform for base morphisms, where the monoidal operation is the coproduct in Th($\mathbb{S}$) – see also Haghverdi (2000).

Examples 2.14. We will now present some examples of Elgot theories (or monads) in **Set**.

(i) *Partial-function theory*:
Consider the monad $\mathbb{S}$ with

$$S = \mathrm{Id} + 1$$

(with pointed sets as algebras). Its theory

$$\mathrm{Th}(\mathbb{S}) = \mathrm{Pfn}$$

is the category of natural numbers and partial functions. To every partial function

$$e : n \multimap n + k,$$

we assign its iteration

$$e^{\dagger} : n \multimap k$$

defined in an element x of n if and only if $e(x), e(e(x)), \ldots, e^i(x)$ are defined and $e^i(x)$ lies in k. We then have $e^{\dagger}(x) = e^i(x)$.

(ii) *Multifunction theory*:
Take the finite-power-set monad

$$\mathbb{P}_f$$

(whose algebras are join semilattices with a least element). Its theory

$$\mathrm{Th}(\mathbb{P}_f) = \mathrm{Mfn}$$

is the category of natural numbers and multifunctions. For every multifunction $a : n \multimap n$, we use a^* to denote its iteration defined by $a^* = \mathrm{id}_n \cup a \cup (a{\cdot}a) \cup \cdots$ (note that a^* is well defined since n is finite). Then the dagger of $e : n \multimap n + k$ is defined as follows: if $a : n \multimap n$ and $b : n \multimap k$ are the multifunctions with $e = a \cup b$, then

$$e^{\dagger} = b{\cdot}a^*.$$

Observe that (i) is a special case – so the axioms of Elgot theories follow from those for $\mathbb{P}_f$. And this is a special case of the next example (for $C = \{0, 1\}$).

(iii) *Matrix theories* (see Bloom and Ésik (1993, 9.3.10)):
Let

$$(C, +, \cdot, 0, 1)$$

be an ω-complete semiring, that is, a semiring in which $+$ is extended to a summation $\sum_{i \in \mathbb{N}} a_i$ of countable families that is associative and distributes over multiplication. The matrix theory

$$\mathrm{Mat}_C$$

has as morphisms from n to k all $n \times k$ matrices over C. The product of matrices $a \cdot b$ defines composition with swapped order. For every square matrix $a : n \longrightarrow\!\!\circ\, n$, we use a^* to denote its iteration defined by

$$a^* = \sum_{i \in \mathbb{N}} a^i.$$

Then the dagger of $e : n \longrightarrow\!\!\circ\, n + k$ is defined by

$$e^\dagger = b \cdot a^* \qquad \text{for } e = [a \mid b].$$

(iv) *Strict iterative theories*:
Let $\mathbb{S}$ be an iterative theory in **Set** in the sense of Calvin Elgot (Elgot 1975) – we recall this concept in Section 3 below. If the underlying functor is strict, see Remark 2.6, then every equation morphism $e : n \longrightarrow\!\!\circ\, n + k$ has a unique strict solution (defined in Definition 3.14). This makes $\mathbb{S}$ an Elgot monad as proved in Bloom and Ésik (1993, Definition 6.4.1) in the language of theories.

The next three examples are special cases.

(v) *Finite list theory*:
$X \longmapsto X^*$ can be made strict by adding to X^* an absorbing element $\perp$ (that is, the binary operation of concatenation on X^* is extended by $w \cdot \perp = \perp = \perp \cdot w$ for all $w \in X^*$). The resulting monad

$$SX = X^* + \{\perp\}$$

is iterative (Adámek *et al.* 2010a), so it yields an Elgot monad.

(vi) *Infinite-tree theory* (see Bloom and Ésik (1993, 8.2.7)):
Let $\Sigma = (\Sigma_k)_{k<\omega}$ be a signature and let

$$T_\Sigma(n)$$

denote the Σ-algebra of all Σ-*trees on n variables*, that is, (rooted and ordered) trees with leaves labelled in $n + \Sigma_0$ and with nodes with $k > 0$ children labelled in Σ_k. This gives rise to a finitary monad

$$\mathbb{T}_\Sigma,$$

which assigns to a set X the set of all Σ-trees with leaves labelled in $X + \Sigma_0$ so that only finitely many elements of X are used as leaf labels.
In order to get an Elgot monad, we need to add a new nullary symbol, say $\perp$, and put

$$\Sigma_\perp = \Sigma + \{\perp\}.$$

The theory of the monad

$$\mathbb{T}_{\Sigma_\perp}$$

has as hom-sets

$$\mathrm{Th}(\mathbb{T}_\Sigma)(1, n) = T_{\Sigma_\perp}(n)$$

the sets of $\Sigma_\perp$-trees on n variables. There is a canonical ordering $\sqsubseteq$ on this set: if $\Sigma_\perp$-trees $t \in T_{\Sigma_\perp}(n)$ are represented as functions $t : \omega^* \longrightarrow \Sigma_\perp + n$ (in the usual sense except that we make t total by defining $t(w) = \perp$ wherever $w \in \omega^*$ is not a node of the given tree), then

$$t \sqsubseteq t' \qquad \text{iff } t(w) = \perp \text{ or } t(w) = t'(w) \text{ for all } w \in \omega^*.$$

This makes $\mathbb{T}_{\Sigma_\perp}(n)$ a *cpo*, that is, a poset with joins of ω-sequences, having $\perp$ as the least element. Therefore all hom-sets

$$\mathrm{Th}(\mathbb{T}_\Sigma)(k,n) = \mathbb{T}_{\Sigma_\perp}(n) \times \cdots \times \mathbb{T}_{\Sigma_\perp}(n) \qquad (k \text{ times})$$

are cpo's. Moreover, composition and tupling are *continuous*, that is, they preserve the order and all ω-chain joins.

It follows that every equation morphism $e : n \longrightarrow\!\!\!\!\!\circ\;\; n + k$ has the least solution: it can be obtained as the following ω-join

$$e^\dagger = \bigsqcup_{i \in \mathbb{N}} e_i^\dagger : n \longrightarrow\!\!\!\!\!\circ\;\; k$$

where $e_0^\dagger$ is constantly $\perp$ and $e_{i+1}^\dagger = [e_i^\dagger, \mathrm{id}_k] \cdot e$.

Example (i) is the special case when $\Sigma = \varnothing$.

(vii) *Rational-tree theory*:

A tree is said to be *rational* (or regular) if it has, up to isomorphism, only finitely many subtrees (Ginali 1979). We use

$$\mathbb{R}_{\Sigma_\perp}$$

to denote the submonad of $\mathbb{T}_{\Sigma_\perp}$ formed by all rational Σ-trees. This is an Elgot monad since for every equation morphism $e : n \longrightarrow\!\!\!\!\!\circ\;\; n + k$ in the theory of rational trees, the solution $e^\dagger = \bigsqcup e_i^\dagger$ as in (vi) above is formed by rational trees. Therefore, all equational properties of $e \longmapsto e^\dagger$ holding in $\mathbb{T}_{\Sigma_\perp}$ also hold in $\mathbb{R}_{\Sigma_\perp}$.

As proved in Elgot *et al.* (1978), the theory of $\mathbb{R}_{\Sigma_\perp}$ is the free iteration theory on the given signature Σ.

Remark 2.15. The concept of *iteration theory* in Bloom and Ésik (1993) is obtained from Definition 2.11 by replacing *Functoriality* with so-called commutative identities. Examples of iteration theories in the sense of Bloom and Ésik (1993) that are not Elgot theories seem to be quite rare. One example is given in Ésik (1988). Take the signature Σ with one binary operation symbol σ, form the free iteration theory $\mathbb{R}_{\Sigma_\perp}$ and take its quotient $\mathscr{I}$ modulo the equation

$$\sigma(x,x) = \sigma(\sigma(\perp,x), \sigma(x,\perp));$$

more precisely, one takes the quotient modulo the least dagger congruence (*cf.* Bloom and Ésik (1993)) generated by the pair $(a,b) \in \mathbb{R}_{\Sigma_\perp}(1,1)^2$ with

$$a = \nabla \cdot \sigma \qquad \text{and} \qquad b = [[\perp, \mathrm{id}_1] \cdot \sigma, [\mathrm{id}_1, \perp] \cdot \sigma] \cdot \sigma,$$

where $\nabla : 2 \longrightarrow 1$ is the codiagonal, $\sigma : 1 \longrightarrow 2$ and $\perp : 1 \longrightarrow 0$. Then $\mathscr{I}$ is an iteration theory but is not functorial.

Definition 2.16. Let $(\mathbb{S}, \dagger)$ and $(\mathbb{T}, \ddagger)$ be Elgot monads. An *Elgot monad morphism* from $(\mathbb{S}, \dagger)$ to $(\mathbb{T}, \ddagger)$ is a monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ that is *solution-preserving*, that is, for every equation morphism $e : n \longrightarrow S(n+k)$ we have

$$\begin{array}{ccc} n & \xrightarrow{e^\dagger} & S(k) \\ & \searrow^{(\alpha_{n+k} \cdot e)^\ddagger} & \downarrow{\scriptstyle \alpha_k} \\ & & T(k) \end{array}$$

The category of Elgot monads and their morphisms is denoted by

$$\mathsf{EMnd}(\mathscr{K}).$$

We consider it as a concrete category over $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$, or equivalently over $\mathscr{K}^{\mathscr{F}}$, see Fact 2.2. We use

$$U : \mathsf{EMnd}(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K}).$$

to denote its forgetful functor into $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$. This assigns to every Elgot monad $(\mathbb{S}, \dagger)$ the underlying functor S.

Example 2.17. Important Elgot theories in **Set** are those obtained by ω-continuous theories. Suppose a theory $\mathscr{T}$ is ω-continuous, that is, its hom-sets are cpo's and composition is continuous. For every equation morphism $e : n \longrightarrow\!\!\!\!\circ\,\, n + k$ there exists a least solution, *viz.*,

$$e^\dagger = \bigsqcup_{n<\omega} e_n,$$

as above. This yields an Elgot theory – see Bloom and Ésik (1993, Theorem 8.2.15 and Exercise 8.2.17).

Given two ω-continuous theories, every continuous theory morphism between them is an Elgot theory morphism – see in Bloom and Ésik (1993, Proposition 8.2.13).

Remark 2.18.

(i) The aim of our paper is to prove that for $\mathscr{K} = \mathbf{Set}$, Elgot theories are monadic over sets in context. That is, U is a monadic functor.
(ii) We will prove a more general result: $\mathsf{EMnd}(\mathscr{K})$ is monadic over $\mathscr{K}^{\mathscr{F}}$ for every locally finitely presentable category satisfying an additional assumption called hyper-extensivity (see Assumption 3.1).

Remark 2.19. For every Elgot monad $\mathbb{S}$, we can extend the function $e \longmapsto e^\dagger$ to all more general equation morphisms of the form

$$e : n \longrightarrow S(n + A)$$

where $n \in \mathscr{F}$ and A is an arbitrary object of the base category $\mathscr{K}$. Indeed, if we express A as a colimit

$$a_i : k_i \longrightarrow A \qquad (i \in I)$$

of a filtered diagram of objects $k_i \in \mathscr{F}$, then S, being a finitary functor, preserves the following filtered colimit

$$n + A = \mathrm{colim}(n + k_i).$$

Since n is a finitely presentable object, the morphism $e : n \longrightarrow \mathrm{colim}(n + A)$ factorises through one of the colimit morphisms $S(n + a_i)$:

$$\begin{array}{ccc} & & S(n+k_i) \\ & \overset{f}{\nearrow} & \big\downarrow{\scriptstyle S(n+a_i)} \\ n & \xrightarrow[e]{} & S(n+A) \end{array}$$

This gives us an equation morphism $f : n \multimap n + k_i$ with a solution $f^\dagger : n \multimap k_i$. We define

$$e^\dagger \equiv n \xrightarrow{f^\dagger} k_i \xrightarrow{a_i} A.$$

It is easy to prove that $e^\dagger$ is independent of the choice of i and f, and it is a solution of e. In fact, it is easy to see that all the axioms in Definition 2.11 can be extended to the solutions of these more general equation morphisms.

Remark 2.20. Observe that every Elgot monad $\mathbb{S}$ makes its underlying functor S strict by giving a solution

$$\bot = \mathsf{inl}^\dagger : 1 \multimap 0$$

to the equation morphism $\mathsf{inl} : 1 \multimap 1 + 0$. (In Example 2.8 this corresponds to solving the trivial equation $x \approx x$.)

3. Iterative theories

Assumption 3.1. Throughout this section $\mathbb{S} = (S, \eta^S, \mu^S)$ denotes a finitary monad on a locally finitely presentable category $\mathscr{K}$ that is *hyper-extensive*, that is, every object is a coproduct of connected objects A (where A is said to be connected if the hom-functor $\mathscr{K}(A, -)$ preserves coproducts).

Remark 3.2. Hyper-extensive categories were introduced in Adámek *et al.* (2008) as a convenient setting for strict iterative monads (see below). They are closely related to extensive categories, that is, categories in which binary coproducts are disjoint and universal (Carboni *et al.* 1993). Equivalently, for every morphism $f : X \longrightarrow A + B$ into a coproduct, the squares

$$\begin{array}{ccccc} X_0 & \xrightarrow{j_0} & X & \xleftarrow{j_1} & X_1 \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f_1} \\ A & \xrightarrow[\mathsf{inl}]{} & A + B & \xleftarrow[\mathsf{inr}]{} & B \end{array}$$

are pullbacks if and only if the top row of this diagram is a coproduct $X = X_0 + X_1$. Every hyper-extensive category is, of course, extensive, but not conversely. As proved in Adámek *et al.* (2008), a locally finitely presentable, extensive category is hyper-extensive if and only if given pairwise disjoint subobjects $a_i : A_i \longrightarrow B$ ($i \in \mathbb{N}$) each of which is a coproduct injection, then $[a_i] : \coprod_{i \in \mathbb{N}} A_i \longrightarrow B$ is a coproduct injection too.

Examples 3.3.

(1) The categories of sets, posets, graphs and unary algebras are hyper-extensive and locally finitely presentable.
(2) If $\mathscr{K}$ has both properties, so do all presheaf categories on $\mathscr{K}$. Thus the category of sets in context $\mathbf{Set}^{\mathbb{F}}$ is hyper-extensive and locally finitely presentable.
(3) If $\mathscr{K}$ has both properties, so do all slice categories $\mathscr{K}/K$. Thus the category of signatures $\mathbf{Sgn} = \mathbf{Set}/\mathbb{N}$ is an example.

Definition 3.4. A finitary monad $\mathbb{S}$ is said to be *ideal* if there exists a subfunctor $\sigma : S' \rightarrowtail S$ such that $S = S' + \mathrm{Id}$ with injections σ and η^S, and if μ^S has the following restriction $(\mu')^S$:

$$\begin{array}{ccc} S'S & \xrightarrow{(\mu')^S} & S' \\ {\scriptstyle \sigma S}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ SS & \xrightarrow[\mu^S]{} & S \end{array}$$

Examples 3.5.

(1) $SX = X + 1$ is ideal: here S' is the constant functor 1.
(2) $\mathbb{R}_\Sigma$ is ideal: $R'_\Sigma X$ are all trees that are not single variables, and is indeed a subfunctor of R_Σ. Analogously for $\mathbb{T}_\Sigma$.
(3) $SX = X^* + \{\perp\}$ is ideal: $S'X$ is the subfunctor given by all words of lengths other than 1 plus the element $\perp$.
(4) $\mathbb{P}_f$ is not ideal. The complement $P_f X \setminus \eta_X[X]$ is not a subfunctor. For example, if $h : X \longrightarrow Y$ is a constant function, then $P_f h$ maps $P_f X \setminus \{\emptyset\}$ into $\eta_Y[Y]$.

Definition 3.6. Let $\mathbb{S}$ be an ideal monad. An equation morphism $e : n \longrightarrow\!\!\!\!\!\!\circ\;\; n+k$ is said to be *ideal* if it factorises through $S'(n+k)$ in $\mathscr{K}$:

$$\begin{array}{ccc} & & S'(n+k) \\ & \nearrow & \big\downarrow{\scriptstyle \sigma_{n+k}} \\ n & \xrightarrow[e]{} & S(n+k) \end{array}$$

The monad $\mathbb{S}$ is said to be *iterative* if every ideal equation morphism has a unique solution. And a *strict* iterative monad $\mathbb{S}$ is an iterative monad with a strict underlying functor.

Remark 3.7.

(i) The concept of ideal and iterative theory (that is, theory Th($\mathbb{S}$) of an ideal or iterative monad) was introduced in **Set** by Calvin Elgot (Elgot 1975). Our formulation above is based on Aczel *et al.* (2003), where it was proved to be equivalent to Elgot's formulation.

(ii) Adámek *et al.* (2006a) used the more liberal concept of equation morphism, where

$$e : n \longrightarrow S(n + A),$$

with $n \in \mathscr{F}$ and $A \in \mathscr{K}$, is said to be *guarded* if it factorises through $S'(n + A) + A$:

$$\begin{array}{ccc} & & S'(n+A)+A \\ & \nearrow & \Big\downarrow {\scriptstyle [\sigma_{n+A},\, \eta^S_{n+A} \cdot \mathsf{inr}]} \\ n & \xrightarrow[e]{} & S(n+A) \end{array}$$

We proved in Adámek *et al.* (2010a) that (under Assumption 3.1) this does not make any difference: an ideal monad is iterative if and only if every guarded equation morphism has a unique *solution* that is, a unique morphism $e^\dagger : n \longrightarrow SA$ with

$$\begin{array}{ccc} n & \xrightarrow{e^\dagger} & SA \\ {\scriptstyle e}\Big\downarrow & & \Big\uparrow{\scriptstyle \mu^S} \\ S(n+A) & \xrightarrow[S[e^\dagger, \eta^S]]{} & SSA \end{array}$$

Examples 3.8. The monads $SX = X^* + \{\bot\}$, $\mathbb{T}_\Sigma$ and $\mathbb{R}_\Sigma$ are iterative.

Notation 3.9. We use

$$\mathsf{IMnd}(\mathscr{K})$$

to denote the category of iterative monads and *ideal monad morphisms*, that is, monad morphisms $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ such that the natural transformation $\alpha : S' + \mathrm{Id} \longrightarrow T' + \mathrm{Id}$ has the form $\alpha = \alpha' + \mathrm{Id}$ for some natural transformation $\alpha' : S' \longrightarrow T'$.

We have the forgetful functor

$$U' : \mathrm{IM}(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K})$$

assigning to every iterative monad $\mathbb{S} = (S' + \mathrm{Id}, \eta^S, \mu^S)$ the subfunctor S':

$$U'(\mathbb{S}) = S'.$$

We also use

$$\mathsf{IMnd}_\bot(\mathscr{K})$$

to denote the category of strict iterative monads (that is, with strict underlying functors) as a full subcategory of $\mathsf{EMnd}(\mathscr{K})$. Its underlying functor

$$U : \mathrm{IM}_\bot(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K}), \qquad U(\mathbb{S}) = S$$

is the domain restriction of that of $\mathsf{EMnd}(\mathscr{K})$, thus, it differs from U' above.

Theorem 3.10 (Adámek *et al.* 2006a). Every finitary endofunctor H generates a free iterative monad $\mathbb{R}_H$ called the *rational monad* of H. In other words, the forgetful functor $U' : \mathrm{IM}(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K})$ has a left adjoint $H \longmapsto \mathbb{R}_H$.

Notation 3.11. We use $\kappa_H : H \longrightarrow R_H$ to denote the unit of this adjunction.

Example 3.12.

(i) See Elgot *et al.* (1978). For every signature Σ, we have the corresponding polynomial endofunctor of **Set**

$$H_\Sigma X = \coprod_{i \in \mathbb{N}} \Sigma_i \times X^i$$

(whose algebras are the classical Σ-algebras). The free iterative monad is the rational-tree monad $\mathbb{R}_\Sigma$ – see Example 2.14 (vii).

(ii) See Adámek and Milius (2006a). The finite-power-set functor P_f has the rational monad $\mathbb{R}_{P_f}$ with

$$\mathbb{R}_{P_f} X = \text{all rational strongly extensional finitely branching trees on } X,$$

where a tree is said to be strongly extensional if distinct children of any node always yield subtrees that are not bisimilar.

Remark 3.13. We will now turn to solutions of non-ideal equation morphisms. As we observed in Remark 2.20, the case $x \approx x$ (more precisely, $e = \mathsf{inl} : 1 \longrightarrow\!\!\!\!\!\!\circ\;\; 1 + 0$) forces us to work with strict monads.

Larry Moss proved (Moss 2003) that in **Set**, every strict iterative monad $\mathbb{S}$ has the property that, given an equation morphism $e : n \longrightarrow\!\!\!\!\!\!\circ\;\; n + k$, there exists a unique *strict solution* $e^\dagger : n \longrightarrow\!\!\!\!\!\!\circ\;\; k$, which means a solution assigning $\perp$ to all *ungrounded* variables. These are variables $x \in n$ such that the function $e : n \longrightarrow S(n + k)$ when iterated leaves x always in n; more precisely, there is a sequence $x = x_0, x_1, x_2, \ldots$ in n such that for all $i = 0, 1, 2, \ldots$, we have that $e(x_i)$ is the image of x_{i+1} under $\eta^S \cdot \mathsf{inr} : n \longrightarrow S(n + k)$. We proved a categorical generalisation of this in Adámek *et al.* (2008).

Definition 3.14. Let $\mathbb{S}$ be a strict monad and $e : n \longrightarrow S(n + k)$ be an equation morphism.

(1) A subobject $v : V \rightarrowtail n$ is said to be *ungrounded* if e restricts to an endomorphism e' of V:

$$\begin{array}{ccccc} V & & \overset{v}{\rightarrowtail} & & n \\ {\scriptstyle e'}\downarrow & & & & \downarrow{\scriptstyle e} \\ V & \underset{v}{\rightarrowtail} & n & \xrightarrow[\eta^S \cdot \mathsf{inr}]{} & S(n+k) \end{array}$$

(2) A solution $e^\dagger : n \longrightarrow SA$ is said to be *strict* if its restriction to every ungrounded subobject $v : V \longrightarrow n$ is $\perp$ (see Remark 2.6):

$$\begin{array}{ccc} V & & \\ {\scriptstyle v}\downarrow & \searrow^{\perp} & \\ n & \xrightarrow[e^\dagger]{} & Sk \end{array}$$

Theorem 3.15 (Adámek *et al.* 2010b). Every strict iterative monad is an Elgot monad if to each equation morphism e we assign a strict solution $e^\dagger$.

Remark 3.16. This theorem makes sense since strict solutions always exist and are unique. This was proved in Adámek *et al.* (2008), and we will now briefly recall the proof technique since we will need the concepts later.

Notation 3.17. Let $\mathbb{S}$ be an ideal monad. Then:

(1) For every equation morphism $e : n \longrightarrow S(n+k)$, we define the *derived subobjects*

$$i_r : X_r \rightarrowtail X_{r-1} \qquad (r = 1, 2, 3, \dots)$$

of e as follows. Put

$$X_0 = n \qquad \text{and} \qquad i_0 = \eta_n^S \cdot \mathsf{inr} : X_0 \longrightarrow S(n+k).$$

Given i_r, define i_{r+1} by a pullback as follows:

$$\begin{array}{ccccccc} \cdots\ X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X_0 = n \\ \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} \\ X_2 & \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X_0 = n & \xrightarrow[i_0 = \eta_n^S \cdot \mathsf{inr}]{} & S(n+k) \end{array} \tag{3.7}$$

(2) This defines subobjects

$$i_r^* : X_r \rightarrowtail n \qquad r = 1, 2, 3, \dots$$

by composing the first r derived subobjects:

$$i_r^* = i_r \cdot \cdots \cdot i_1.$$

(3) For each $r \geqslant 1$ the morphism i_r is a coproduct injection by extensivity of $\mathscr{K}$. More precisely, we have a subobject $\bar{i}_r : \bar{X}_r \rightarrowtail n$ with $n = X_r + \bar{X}_r$. Moreover, due to extensivity, we have unique morphisms

$$\bar{e}_r : \bar{X}_r \longrightarrow \bar{X}_{r-1} \qquad (r = 2, 3, 4, \dots)$$

forming pullbacks

$$\begin{array}{ccc} \bar{X}_{r+1} & \xrightarrow{\bar{i}_{r+1}} & X_r \\ \downarrow{\scriptstyle \bar{e}_{r+1}} & & \downarrow{\scriptstyle e_r} \\ \bar{X}_r & \xrightarrow[\bar{i}_r]{} & X_{r-1} \end{array} \tag{3.8}$$

We define

$$\bar{i}_r^* \equiv \bar{X}_r \xrightarrow{\bar{i}_r} X_{r-1} \xrightarrow{i_r^*} n.$$

Proposition 3.18 (Adámek *et al.* 2008). Let $\mathbb{S}$ be an ideal monad. Every equation morphism $e : n \multimap n + k$ has the greatest ungrounded subobject equal to the least derived subobject:

there exists $r \geqslant 1$ such that $i_r : X_r \longrightarrow X_{r-1}$ is an isomorphism and then $i_r^* : X_r \longrightarrow n$ is the greatest ungrounded subobject. Then n has a decomposition

$$n = \bar{X}_1 + \cdots + \bar{X}_r + X_r \tag{3.9}$$

with injections $\bar{i}_1^*, \ldots, \bar{i}_r^*$ and i_r, respectively.

Remark 3.19.

(1) The trivial subobject 0 (initial) is always ungrounded. If e has no other ungrounded subobjects, it is said to be *preguarded*. For example, the equation morphism corresponding to the system in Example 3.20 is preguarded.

(2) Let $\mathbb{S}$ be a strict ideal monad and $e : n \longrightarrow S(n+k)$ be an equation morphism with the greatest derived subobject $i_r^* : X_r \longrightarrow n$. Then $\hat{e} : n \longrightarrow S(n+k)$ denotes the equation morphism with the first r components (see (3.9)) equal to those of e, and with $\hat{e} \cdot i_r^* = \bot : X_r \longrightarrow S(n+k)$. This equation morphism $\hat{e}$ is preguarded, and is said to be the *preguarded modification of e*.

Example 3.20. Let $\mathbb{S}$ be the monad of rational binary trees in **Set** (the free iterative monad on one binary operation $*$). Consider the equation morphism e representing

$$\begin{aligned} x_0 &\approx x_1 * x_2 \\ x_1 &\approx x_1 * x_1 \\ x_2 &\approx x_3 \\ x_3 &\approx x_4 \\ x_4 &\approx y_0 * y_1. \end{aligned}$$

Here the derived subobjects of $X = \{x_1, x_2, x_3, x_4\}$ are $X_1 = \{x_2, x_3\}$, $X_2 = \{x_2\}$ and $X_3 = \varnothing$.

Theorem 3.21 (Adámek *et al.* 2008). For every strict iterative monad the following hold:

(1) Every preguarded equation morphism has a unique solution.

(2) Every equation morphism e has the same strict solutions as the preguarded modification $\hat{e}$. Therefore, e has a unique strict solution.

Example 3.22. The unique strict solution of $\mathsf{inl} : X \longrightarrow\!\!\!\circ\; X + Y$ is $\bot_{X,Y}$.

4. Elgot algebras

Assumption 4.1. In this section we use $\mathscr{K}$ to denote a locally finitely presentable category and H to denote a finitary endofunctor.

Recall the rational monad

$$\mathbb{R} = (R, \eta^R, \mu^R)$$

of H from Theorem 3.10. In the present section we recall the description of the monadic algebras for $\mathbb{R}$, called Elgot H-algebras, from Adámek *et al.* (2006b).

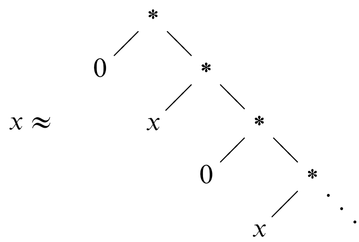
Remark 4.2. Given an algebra $a : HA \longrightarrow A$ for H we are going to study *flat equation morphisms* in A, which are the morphisms

$$e : n \longrightarrow Hn + A \qquad (n \in \mathscr{F})$$

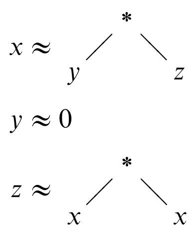
of $\mathscr{K}$. For example, if $H = H_\Sigma$, see Example 3.12, then, whereas general equation morphisms $e : n \multimap\!\!\!\longrightarrow n + k$ are systems of equations $x_i \approx t_i$ with rational right-hand sides, see Example 2.8, the flat equation morphisms

$$e : n \longrightarrow \coprod_{i \in \mathbb{N}} \Sigma_i \times n^i + A$$

have as right-hand sides either elements of A, or 'flat' terms $\sigma(x_0, \ldots, x_{i-1})$ for some $\sigma \in \Sigma_i$ and some variables $x_0, \ldots, x_{i-1}$ in n. However, each general system can be 'flattened' by introducing new variables. For example, given the binary symbol $*$, the equation



can be flattened to



Definition 4.3. Let $a : HA \longrightarrow A$ be an algebra.

(1) We define a *flat equation morphism* to be a morphism

$$e : n \longrightarrow Hn + A \qquad (n \in \mathscr{F}).$$

A *solution* is a morphism $e^\dagger : n \longrightarrow A$ with

$$\begin{array}{ccc} n & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ Hn + A & \xrightarrow[\;He^\dagger + A\;]{} & HA + A \end{array} \tag{4.10}$$

(2) The algebra is said to be *iterative* if every flat equation morphism has a unique solution.

Examples 4.4.

(1) See Adámek *et al.* (2006a). For $H = \mathrm{Id}$ in **Set**, an algebra $a : A \longrightarrow A$ is iterative if and only if a has a unique fixed point

$$a(x_0) = x_0$$

and no cycle of length >1.

(2) See Adámek *et al.* (2006b). Let CMS be the category of complete metric spaces (with distances in $[0,1]$) and non-expanding maps. Let H be a contracting endofunctor of CMS, that is, there exists a constant $k < 1$ such that for all parallel pairs of morphisms $f, g : A \longrightarrow B$, the (supremum) distance of Hf and Hg is at most k times the (supremum) distance of f and g. Then every non-empty algebra for H is iterative. For a specific case, consider the set $C(I)$ of all closed subsets of the unit interval $I = [0,1]$ equipped with the binary operation

$$b(x,y) = \frac{1}{3}x \cup \left(\frac{1}{3}y + \frac{2}{3}\right).$$

This turns $C(I)$ into an iterative algebra for the contracting functor

$$H(X,d) = \left(X \times X, \frac{1}{3}d\right);$$

the unique solution of the equation $x \approx b(x,x)$ is the well-known Cantor set.

(3) See Milius (2005). Let T be a final coalgebra for H. Then its structure morphism $t : T \longrightarrow HT$ is an isomorphism (Lambek 1968), and the algebra (T, t^{-1}) is iterative. Similarly, a final coalgebra for $H(-) + X$ is an iterative algebra for H.

(4) A specific case of (3) is the Σ-algebra $T_\Sigma X$ of all Σ-trees. This algebra is iterative for the polynomial endofunctor H_Σ of **Set**.

(5) See Nelson (1983). The Σ-algebra $R_\Sigma(X)$ of rational trees is also iterative for H_Σ. In fact, this is the free iterative Σ-algebra on X. More generally, we have the following proposition.

Proposition 4.5 (Adámek *et al.* 2006a). Every object of $\mathscr{K}$ generates a free iterative algebra for H, that is, the forgetful functor of the category of iterative algebras and homomorphisms has a left adjoint. The monad of this adjoint situation is the rational monad $\mathbb{R}_H$, see Theorem 3.10.

Remark 4.6.

(1) We use $\varrho_A : HR_H \longrightarrow R_H A$ to denote the algebraic structure of the iterative algebra RH_A on A. The monad unit of $\mathbb{R}_H$ is given by the universal arrows

$$\eta_A : A \longrightarrow R_H A.$$

The monad multiplication is given by the unique homomorphisms

$$\mu_A : R_H R_H A \longrightarrow R_H$$

satisfying $\mu_A \cdot \eta_{R_H A} = \mathrm{id}$. In particular, we have

$$\mu_A \cdot \varrho_{R_H A} = \varrho_A \cdot H\mu_A. \tag{4.11}$$

(2) Iterative Σ-algebras were introduced by Evelyn Nelson (Nelson 1983) – see also Tiuryn (1980) for a related concept.
(3) We are going to use the monadic algebras for the monad $\mathbb{R}_H$. They were described in Adámek *et al.* (2006b) as the algebras for H together with a function $e \longmapsto e^\dagger$ of solutions of all flat equation morphisms subject to axioms that are quite analogous to those of Definition 2.11.

Definition 4.7. We define an *Elgot algebra* for H to be an algebra $a : HA \longrightarrow A$ together with a function

$$\frac{e : n \longrightarrow Hn + A}{e^\dagger : n \longrightarrow A} \qquad (\text{for all } n \in \mathscr{F})$$

satisfying the following axioms:

— *Solution*:
The square in (4.10) commutes.

— *Functoriality*:

$$\text{if} \quad \begin{array}{ccc} n & \xrightarrow{\;e\;} & Hn + A \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle Hv+A} \\ m & \xrightarrow[\;f\;]{} & Hm + A \end{array} \qquad \text{then} \qquad \begin{array}{ccc} n & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle v}\downarrow & \nearrow{\scriptstyle f^\dagger} & \\ m & & \end{array} \tag{4.12}$$

— *Compositionality*:
Given

$$e : n \longrightarrow Hn + k \qquad \text{and} \qquad f : k \longrightarrow Hk + A \qquad (n, k \in \mathscr{F})$$

form $f^\dagger : k \longrightarrow A$ and the equation morphism

$$f \blacksquare e \equiv n + k \xrightarrow{[e,\mathsf{inr}]} Hn + k \xrightarrow{Hn+f} Hn + Hk + A \xrightarrow{\mathsf{can}+A} H(n+k) + A. \tag{4.13}$$

Then

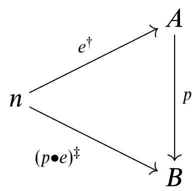
$$\begin{array}{ccc} n & \xrightarrow{(f^\dagger \bullet e)^\dagger} & A \\ {\scriptstyle \mathsf{inl}}\downarrow & \nearrow{\scriptstyle (f \blacksquare e)^\dagger} & \\ n + k & & \end{array} \tag{4.14}$$

Examples 4.8. The following are some examples of Elgot algebras – see Adámek *et al.* (2006b) for details:

(1) Every iterative algebra is an Elgot algebra.
(2) For $H = \mathrm{Id}$ in **Set**, every unary algebra $a : A \longrightarrow A$ with a fixed point x_0 of a defines an Elgot algebra by iterating a and assigning the solution x_0 if the iteration yields no result in finitely many steps.
(3) Every join semilattice is an Elgot algebra for $HX = X \times X$ on **Set**. In contrast, no non-trivial semilattice is iterative.

(4) Continuous algebras on cpos are Elgot algebras. More precisely, let CPO be the category of the partially ordered sets with joins of ω-chains (but not necessarily with a least element) and continuous functions. Let H be a locally continuous endofunctor of CPO, that is, the derived functions from $\mathsf{CPO}(X, Y)$ to $\mathsf{CPO}(HX, HY)$ preserve ω-joins. Then every algebra for H with a least element is an Elgot algebra provided we assign the least solution $e^\dagger$ to every flat equation morphism e.

Notation 4.9. Given Elgot algebras $(A, a, \dagger)$ and $(B, b, \ddagger)$, an *Elgot algebra morphism* is a morphism $h : A \longrightarrow B$ such that for every flat equation morphism $e : n \longrightarrow Hn + A$ the corresponding equation morphism $p \bullet e = (Hn + p)\cdot e : n \longrightarrow Hn + B$ satisfies



As proved in Adámek *et al.* (2006b), this implies that p is a homomorphism of H-algebras, that is, $p\cdot a = b\cdot Hp$. The category of Elgot algebras and their morphisms is denoted by **Elg**(H). We have the forgetful functor defined by

$$V : \mathbf{Elg}(H) \longrightarrow \mathscr{K}, \qquad (A, a, \dagger) \longmapsto A.$$

Theorem 4.10 (Adámek *et al.* 2006b). Elgot algebras form the category of monadic algebras for the rational monad $\mathbb{R}_H$. More precisely, the free iterative algebra on every object A of $\mathscr{K}$ is also a free Elgot algebra, and the forgetful functor $V : \mathbf{Elg}(H) \longrightarrow \mathscr{K}$ is monadic.

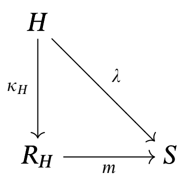
5. Free Elgot monads

The main technical result of the current paper is the fact that for every strict finitary endofunctor H, the free iterative monad $\mathbb{R}_H$, see Theorem 3.10, is also the free Elgot monad on H.

Thus, the aim of this section is to prove (using a series of auxiliary results) the following theorem.

Theorem 5.1. Let H be a strict, finitary endofunctor of a hyper-extensive locally finitely presentable category. Then the rational monad $\mathbb{R}_H$ is the free Elgot monad on H.

To express this in more detail, let $\mathbb{S} = (S, \mu^S, \eta^S, \dagger)$ be an Elgot monad. Then for every strict natural transformation $\lambda : H \longrightarrow S$ there exists a unique Elgot monad morphism $m : \mathbb{R}_H \longrightarrow \mathbb{S}$ with



Before we delve into the details, we will give an overview of the proof of Theorem 5.1. The core idea is to use the fact that the monad $\mathbb{R}_H$ is given objectwise by the free Elgot algebras RA. This is analogous to the well-known proof that free H-algebras form the free monad on H, *cf.* Barr (1970). However, in our case the details are much more involved. Given any Elgot monad $\mathbb{S}$, we will establish in Proposition 5.5 that the H-algebra

$$\gamma_A = (HSA \xrightarrow{\lambda_{SA}} SSA \xrightarrow{\mu^S_A} SA) \tag{5.15}$$

is an Elgot algebra. The corresponding dagger operation is given for every $e : X \longrightarrow HX + SA$ by putting

$$e^* = \bar{e}^\dagger : X \longrightarrow SA \tag{5.16}$$

for the following equation morphism with respect to $\mathbb{S}$

$$\bar{e} \equiv X \xrightarrow{e} HX + SA \xrightarrow{\lambda_X + SA} SX + SA \xrightarrow{\mathsf{can}} S(X + A). \tag{5.17}$$

In order to establish the *Compositionality* of $*$, we will first prove a similar law for (non-flat) equation morphisms $f : X \longrightarrow\!\!\!\!\circ\;\, X + Y$ and $g : Y \longrightarrow\!\!\!\!\circ\;\, Y + A$ – see Lemma 5.3 and Corollary 5.4.

We then use the universal propertry of the free Elgot algebras RA to obtain for every object A a unique Elgot algebra morphism $m_A : RA \longrightarrow SA$ with $m_A{\cdot}\eta_A = \eta^S_A$. In Proposition 5.7, we establish that m is a strict monad morphism such that $m{\cdot}\kappa_H = \lambda$.

Next we verify that m is an Elgot monad morphism, that is, that it preserves the dagger. This step is surprisingly involved – see Lemmas 5.10, 5.14 and 5.15.

Finally, in Lemma 5.16, we prove that m is unique.

Notation 5.2.

(1) We write $\mathbb{R} = (R, \mu, \eta)$ to denote the rational monad of H for short. For every object A, the free Elgot algebra on A is denoted by $\varrho_A : HRA \longrightarrow RA$, and the specified solution operation for flat equation morphisms is denoted by $e \longmapsto e^\ddagger$. Note that these structure morphisms ϱ_A form the components of a natural transformation $\varrho : HR \longrightarrow R$. The universal arrow, see Notation 3.11, was proved in Adámek *et al.* (2006b) to be the composite

$$\kappa_H = H \xrightarrow{H\eta} HR \xrightarrow{\varrho} R.$$

(2) The monad $\mathbb{R}$ is considered to be strict via

$$1 \xrightarrow{\perp} H0 \xrightarrow{(\kappa_H)_0} R0.$$

By Theorem 3.15, we have a function assigning to every guarded equation morphism $e : X \longrightarrow R(X + A)$ with X in $\mathscr{F}$ the unique strict solution, and we denote this function by

$$\frac{X \xrightarrow{e} R(X + A)}{X \xrightarrow{e^\ddagger} RA} \qquad (X \in \mathscr{F}).$$

This corresponds well with the above notation $e^{\ddagger}$ for flat equation morphisms. In fact, every flat equation morphism $e : X \longrightarrow HX + A$ defines a guarded equation morphism

$$\bar{e} \equiv X \xrightarrow{e} HX + A \xrightarrow{\kappa_H + \eta} RX + RA \xrightarrow{\mathsf{can}} R(X + A),$$

and the unique solutions of e and $\bar{e}$ are equal.

(3) We thus reserve $\dagger$ for the structure of the Elgot monad of $\mathbb{S}$. In the sense of Remark 2.19, this is then a function

$$\frac{X \xrightarrow{e} S(X + A)}{X \xrightarrow{e^{\dagger}} SA} \qquad (X \in \mathscr{F}).$$

(4) We extend the notation (4.13) as follows: given

$$f : X \longrightarrow\!\!\!\!\!\circ\;\; X + Y$$
$$g : Y \longrightarrow\!\!\!\!\!\circ\;\; Y + A$$

(see Remark 2.3(2)), we define

$$g \blacksquare f = X + Y \xrightarrow{[f,\mathsf{inr}]} X + Y \xrightarrow{X+g} X + Y + A. \tag{5.18}$$

Recall also from (2.2) the notation

$$g^{\dagger} \bullet f \equiv X \xrightarrow{f} X + Y \xrightarrow{X+g^{\dagger}} X + A.$$

(5) We extend the notation of (2.3) and (2.6) to morphisms $f : X \longrightarrow\!\!\!\!\!\circ\;\; X + Y + A$ and $g : Y \longrightarrow\!\!\!\!\!\circ\;\; X + Y + A$, and define

$$e_R = Y \xrightarrow{g} X + Y + A \xrightarrow{[f^{\dagger}, Y+A]} Y + A$$
$$e_L = X \xrightarrow{f} X + Y + A \xrightarrow{X+[e_R^{\dagger}, A]} X + A.$$

Lemma 5.3. Let $\mathbb{S} = (S, \eta, \mu, \dagger)$ be an Elgot monad and

$$f : X \longrightarrow\!\!\!\!\!\circ\;\; X + Y + A$$
$$g : Y \longrightarrow\!\!\!\!\!\circ\;\; X + Y + A$$

be equation morphisms. Forming the equation morphism

$$g \,\square\, f : X + Y \xrightarrow{[f,\mathsf{inm}]} X + Y + A \xrightarrow{[\mathsf{inl},g,\mathsf{inr}]} X + Y + A,$$

we have

$$(g \,\square\, f)^{\dagger} = [e_L^{\dagger}, e_R^{\dagger}] : X + Y \longrightarrow\!\!\!\!\!\circ\;\; A.$$

Proof. We start be proving an auxiliary property of $\dagger$. Let $e : X \longrightarrow\!\!\!\!\!\circ\;\; X + A$ be an equation morphism, and let $\pi : X' \longrightarrow X$ be an isomorphism in $\mathscr{K}$. Forming the equation morphism

$$e_{\pi} = X' \xrightarrow{\pi} X \xrightarrow{e} X + A \xrightarrow{\pi^{-1}+A} X' + A,$$

we have

$$e^{\dagger}\cdot\pi = (e_{\pi})^{\dagger}. \tag{5.19}$$

Indeed, this follows from the Functoriality of † since π is a homomorphism of equations from e_{π} to e:

$$\begin{array}{ccccccc} X' & \xrightarrow{\pi} & X & \xrightarrow{e} & X+A & \xrightarrow{\pi^{-1}+A} & X'+A \\ \downarrow{\scriptstyle \pi} & & & & & \searrow\!\!= & \downarrow{\scriptstyle \pi+A} \\ X & & & \xrightarrow[e]{\hspace{6em}} & & & X+A \end{array}$$

Now let f and g be as in the statement above and form the equation morphisms

$$h \equiv Y \xrightarrow{g} X+Y+A \xrightarrow{\mathsf{swap}+A} Y+X+A$$
$$k \equiv X \xrightarrow{f} X+Y+A \xrightarrow{\mathsf{swap}+A} Y+X+A,$$

where $\mathsf{swap} = [\mathsf{inr}, \mathsf{inl}] : X+Y \longrightarrow Y+X$ is the canonical isomorphism. Using the denotation

$$h \diamond k \equiv Y+X \xrightarrow{[\mathsf{inl},k]} Y+X+A \xrightarrow{[h,\mathsf{inr}]} Y+X+A,$$

it is then trivial to see that

$$g \,\Box\, f = (h \diamond k)_{\mathsf{swap}},$$

so by (5.19), we obtain

$$(g \,\Box\, f)^{\dagger} = (h \diamond k)^{\dagger}_{\mathsf{swap}} = (h \diamond k)^{\dagger}\cdot \mathsf{swap}. \tag{5.20}$$

Similarly, we have

$$[f,g] = [h,k]_{\mathsf{swap}}$$

so

$$[f,g]^{\dagger} = [h,k]^{\dagger}_{\mathsf{swap}} = [h,k]^{\dagger}\cdot \mathsf{swap}. \tag{5.21}$$

We will now prove the equation

$$(h \diamond k)^{\dagger} = [h,k]^{\dagger}. \tag{5.22}$$

Observe that $h \diamond k = [h,q]$ for the equation morphism

$$q \equiv X \xrightarrow{k} Y+X+A \xrightarrow{[h,\mathsf{inr}]} Y+X+A.$$

Thus, by an application of the Bekić Identity, we have

$$(h \diamond k)^{\dagger} = \left[\varepsilon_L^{\dagger}, \varepsilon_R^{\dagger}\right] : X+Y \longrightarrow A,$$

where

$$\varepsilon_R \equiv X \xrightarrow{k} Y+X+A \xrightarrow{[h,\mathsf{inr}]} Y+X+A \xrightarrow{[h^{\dagger},X+A]} X+A,$$

which means $\varepsilon_R = [h^{\dagger}, X+A]\cdot k$ by *Solution*, and

$$\varepsilon_L \equiv Y \xrightarrow{h} Y+X+A \xrightarrow{Y+[\varepsilon_R^{\dagger},A]} Y+X+A.$$

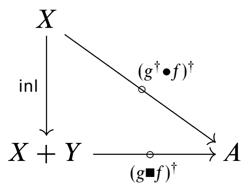
Thus $[\varepsilon_L^\dagger, \varepsilon_R^\dagger] = [h,k]^\dagger$ by another application of the Bekić Identity, and this concludes the proof of (5.22).

Finally, we combine (5.20)–(5.22) to obtain

$$(g \,\square\, f)^\dagger = (h \diamond k)^\dagger \cdot \mathsf{swap} = [h,k]^\dagger \cdot \mathsf{swap} = [e_L^\dagger, e_R^\dagger]$$

by the Bekić Identity once more with e_L and e_R as in the statement of the Lemma. □

Corollary 5.4 (Compositionality). Given $f : X \longrightarrow\!\!\!\!\!\!\circ\;\; X + Y$ and $g : Y \longrightarrow\!\!\!\!\!\!\circ\;\; Y + A$, and forming $g \blacksquare f$ as in (5.18), we have



Indeed, applying Lemma 5.3 to

$$X \xrightarrow{f} X + Y \xrightarrow{\mathsf{inl}} X + Y + A$$

$$Y \xrightarrow{g} Y + A \xrightarrow{\mathsf{inr}} X + Y + A$$

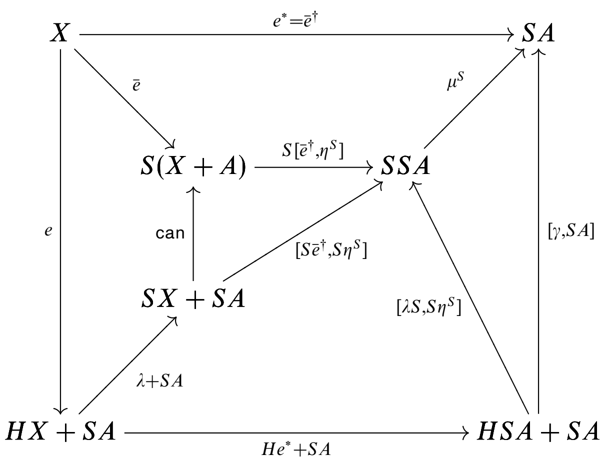
for which $e_R = g$, we get

$$e_L = (X + [g^\dagger, A]) \cdot f = g^\dagger \bullet f.$$

Proposition 5.5. For every object A in $\mathscr{K}$ we can consider SA as an Elgot algebra for H: its algebra structure is $\gamma_A : HSA \longrightarrow SA$ from (5.15). Its dagger operation is given for every $e : X \longrightarrow HX + SA$ by (5.16).

Proof.

(1) The following diagram shows that e^* is a solution:



The right-hand triangle follows from (5.15) and the left-hand part is (5.17). For the lower part, we consider the components separately: the left-hand component with domain HX commutes by naturality of λ and the right-hand one is trivial.

(2) The functoriality of $*$ follows easily from that of $\dagger$. Given two flat equation morphisms $e : X \longrightarrow HX + SA$ and $f : Y \longrightarrow HY + SA$ and $h : X \longrightarrow Y$, and homomorphism of equations, we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{e} & HX + SA & \xrightarrow{\lambda + SA} & SX + SA & \xrightarrow{\mathsf{can}} & S(X+A) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Hh+SA} & & \downarrow{\scriptstyle Sh+SA} & & \downarrow{\scriptstyle S(h+A)} \\ Y & \xrightarrow[f]{} & HY + SA & \xrightarrow[\lambda + SA]{} & SY + SA & \xrightarrow[\mathsf{can}]{} & S(Y+A) \end{array}$$

and, by the Functoriality of $\dagger$, we conclude $\bar{e}^\dagger = \bar{f}^\dagger \cdot h$, that is, $e^* = f^* \cdot h$.

(3) Let us verify that $\blacksquare$ 'commutes' with $\overline{(-)}$ of (5.17):

$$\bar{g} \blacksquare \bar{f} = \overline{g \blacksquare f} \qquad \text{for all } f : X \longrightarrow HX + Y \text{ and } g : Y \longrightarrow HY + SA.$$

Indeed, using (4.13) and (5.17), we see that $\overline{g \blacksquare f}$ is the morphism

$$\begin{array}{l} X + Y \xrightarrow{[f, \mathsf{inr}]} HX + Y \xrightarrow{HX+g} HX + HY + SA \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \lambda+\lambda+SA} \\ \qquad\qquad\qquad\qquad\qquad SX + SY + SA \xrightarrow{\mathsf{can}} S(X + Y + A). \end{array}$$

The right-hand component is therefore $S\,\mathsf{inr} \cdot \bar{g} : Y \longrightarrow S(X + Y + A)$ or, in the Kleisli category, simply

$$\overline{g \blacksquare f} \cdot \mathsf{inr} \equiv Y \xrightarrow{\bar{g}} Y + A \xrightarrow{\mathsf{inr}} X + Y + A.$$

The morphism $\bar{g} \blacksquare \bar{f}$ of (5.18) also has this right-hand component: in the Kleisli category of $\mathbb{S}$ we have

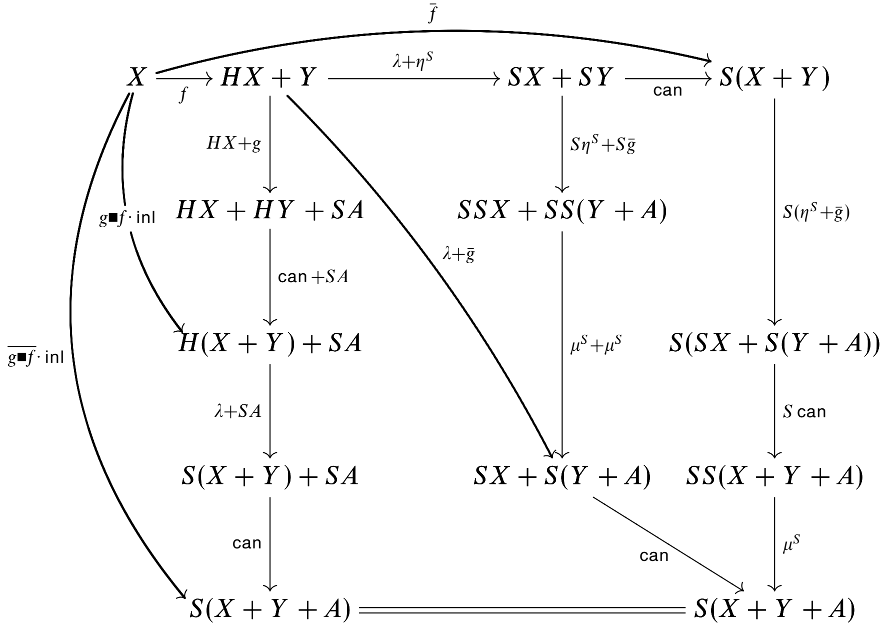
$$(\bar{g} \blacksquare \bar{f}) \cdot \mathsf{inr} = (X + \bar{g}) \cdot \mathsf{inr} = \mathsf{inr} \cdot \bar{g},$$

as desired.

The left-hand components of $\overline{g \blacksquare f}$ and $\bar{g} \blacksquare \bar{f}$ are both equal to

$$\mathsf{can} \cdot (\lambda_X + \bar{g}) \cdot f : X \longrightarrow S(X + Y + A).$$

To see this, we verify that the diagram



commutes. This is clear for both of the left-hand triangles. The upper middle triangle commutes since $\mu^S \cdot S\eta^S = \mathrm{id}$ and from the naturality of η^S. It is easy to see that the lower middle part commutes by verifying the left- and right-hand components separately, and, for the right-hand one, using (5.17). Indeed, it is easy to see that the vertical passage from $HX+HY+HA$ to $S(X+Y+A)$ has the right-hand component

$$HY+SA \xrightarrow{\lambda_Y+SA} SY+SA \xrightarrow{\mathsf{can}} S(Y+A) \xrightarrow{S\,\mathsf{inr}} S(X+Y+A).$$

Finally, for the right-hand part, the left-hand component with domain SX is obvious, and the right-hand component yields $S\,\mathsf{inr}\cdot\mu^S_{Y+A}\cdot S\bar{g}$ in both directions.

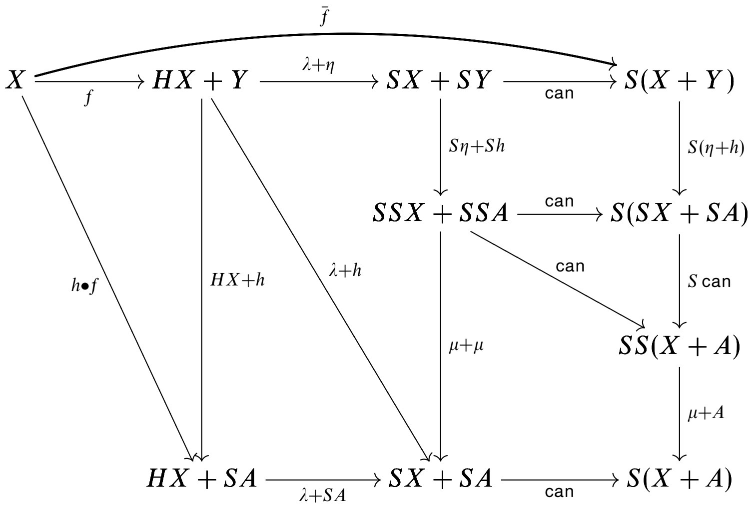
(4) We prove $\overline{h\bullet f} = h\bullet\bar{f}$ for all $f : X \longrightarrow HX+Y$ and $h : X \longrightarrow SA$. Recall that

$$\overline{h\bullet f} \equiv \; X \xrightarrow{f} HX+Y \xrightarrow{HX+h} HX+SA \xrightarrow{\lambda+SA} SX+SA \xrightarrow{\mathsf{can}} S(X+Y)$$

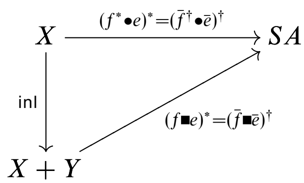
and

$$h\bullet\bar{f} \equiv \; X \xrightarrow{\bar{f}} S(X+Y) \xrightarrow{S(\eta^S_X+h)} S(SX+SA) \xrightarrow{\mathsf{can}} SS(X+A) \xrightarrow{\mu^S_X+A} S(X+A).$$

The proof thus follows from the diagram

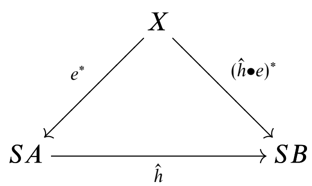


(5) Compositionality of $*$ follows from parts (3) and (4), (5.15) and Corollary 5.4:



□

Observation 5.6. For the Elgot algebras SA of Proposition 5.5, we have that every morphism $h : A \longrightarrow\!\!\!\circ\; B$ establishes an Elgot algebra morphism $\hat{h} = \mu^S \cdot Sh : SA \longrightarrow SB$. Indeed, given $e : X \longrightarrow HX + SA$, the commutativity of the desired triangle



follows from the fact that, analogously to (4) in the proof of Lemma 5.3, we have

$$h \bullet \bar{e} = \overline{\hat{h} \bullet e} : X \longrightarrow\!\!\!\circ\; X + B.$$

Consequently, the Parameter Identity of † (see Definition 2.11) yields

$$\overline{\hat{h} \bullet e}^{\dagger} = (h \bullet \bar{e})^{\dagger} = h \cdot \bar{e}^{\dagger}$$

in the Kleisli category, which is $(\hat{h} \bullet e)^* = \hat{h} \cdot e^*$ in the base category.

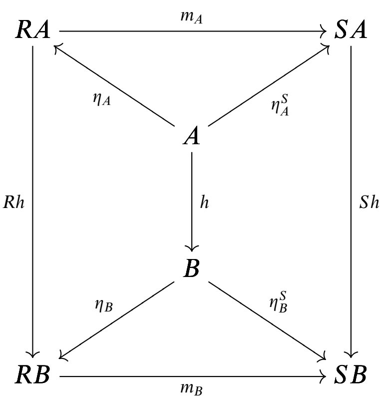
Proposition 5.7. There exists a strict monad morphism $m : \mathbb{R} \longrightarrow \mathbb{S}$ whose components are the (unique) Elgot algebra morphisms m_A from the free Elgot algebra RA to $(SA, \gamma_A, *)$

with $m_A \cdot \eta_A = \eta^S_A$. This monad satisfies

$$\lambda = m \cdot \kappa_H : H \longrightarrow S. \tag{5.23}$$

Proof. For every object A we have a unique morphism $m_A : RA \longrightarrow SA$ of Elgot algebras such that $m_A \cdot \eta_A = \eta^S_A$. We need to prove that m_A is natural in A, preserves the monad multiplication and satisfies (5.23).

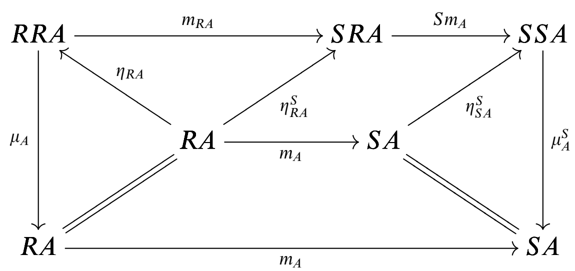
(1) *Naturality*:
For every morphism $h : A \longrightarrow B$ of $\mathscr{K}$, we show that $m_B \cdot Rh = Sh \cdot m_A$. Both sides of this equation are morphisms of Elgot algebras with domain RA (for Sh, apply Observation 5.6 to $\eta^S_B \cdot h$). Hence, it suffices to verify that both homomorphisms are merged when precomposed with the universal arrow $\eta_A : A \longrightarrow RA$:



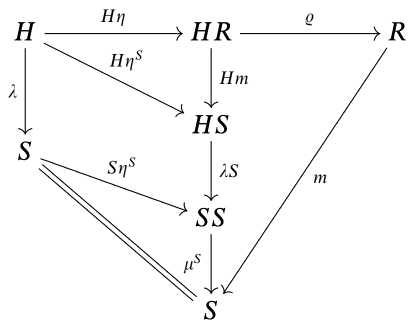
(2) *Preservation of multiplication*:

$$m \cdot \mu = \mu^S \cdot Sm \cdot mR.$$

Again, the components at an object A on both sides are morphisms from the Elgot algebra RRA to SA (for μ^S_A and Sm_A, apply Observation 5.6 to $h = \mathrm{id}_{SA}$ and $h = \eta^S_{SA} \cdot m_A$, respectively). All we need to prove is that the two morphisms are equal when precomposed with $\eta_{RA} : RA \longrightarrow RRA$:



(3) The proof of $\lambda = m{\cdot}\kappa_H$ follows from $\kappa_H = \varrho{\cdot}H\eta : H \longrightarrow R$, see Notation 5.2, and the commutative diagram



The right-hand triangle commutes because m_A is by definition an algebra homomorphism from (RA, ϱ_A) to $(SA, \mu^S_A{\cdot}\lambda_{SA})$, and the other inner parts also clearly commute.

(4) The strictness of m follows from the strictness of λ and part (3): recall that $\bot : 1 \longrightarrow R0$ from Notation 5.2(2) and compute

$$m_0{\cdot}\bot = m_0{\cdot}(\kappa_H)_0{\cdot}\bot = \lambda_0{\cdot}\bot = \bot^S.$$

The fact that $m_A{\cdot}\bot = \bot^S$ holds for an arbitrary object A now follows from the naturality of m. □

Remark 5.8. Observe that in the above proof we applied Observation 5.6 and then the Parameter Identity to show that $\mu^S_A : SSA \longrightarrow SA$ is a morphism of Elgot algebras. For this it is essential that $u : k \multimap k'$ in the definition of the Parameter Identity (see Definition 2.11) is an arbitrary morphism in the Kleisli category – just requiring the Parameter Identity for base morphisms u would be insufficient.

Next we will verify that the monad morphism $m : \mathbb{R} \longrightarrow \mathbb{S}$ is a morphism of Elgot monads, that is, it is dagger preserving. This will be proved step-wise: we first show preservation of the unique solution of guarded equation morphisms in Lemma 5.10; then of preguarded ones in Lemma 5.14; and, finally, in Lemma 5.15, we prove that m preserves the dagger operation.

Remark 5.9. We are going to use the description of the rational monad $\mathbb{R}$ of H from Adámek *et al.* (2006a):

(1) For every object A in the base category $\mathscr{K}$, we form the diagram $D_A : \mathscr{D}_A \longrightarrow \mathscr{K}$ where $\mathscr{D}_A$ is the the category of all flat equation morphisms in A (in other words, $e : X \longrightarrow HX + A$ with $X \in \mathscr{F}$) and their coalgebra homomorphisms with respect to $H(-) + A$, and D_A sends e to X. Then D_A is a small filtered diagram. The underlying functor R of $\mathbb{R}$ is given by

$$RA = \operatorname{colim} D_A.$$

The colimit cocone is denoted by

$$e^\# : X \longrightarrow RA \qquad \text{for } e : X \longrightarrow HX + A \text{ in } \mathscr{D}_A. \tag{5.24}$$

(2) For every $e : X \longrightarrow HX + A$, the morphism $e^{\#}$ is the solution of $\eta_A \bullet e : X \longrightarrow HX + RA$:

$$e^{\#} = (\eta_A \bullet e)^{\ddagger}. \tag{5.25}$$

(3) $\mathbb{R}$ is an ideal monad via a natural transformation $\varrho : HR \longrightarrow R$ such that $R = HR + \mathrm{Id}$ (with coproduct injections ϱ and η). Recall from Example 4.4(2) that $\varrho_A : HRA \longrightarrow RA$ is just the algebra structure of RA.

Lemma 5.10. The monad morphism m preserves solutions of guarded equation morphisms:

$$\begin{array}{ccc} & & HR(X+A)+A \\ & \overset{e_0}{\nearrow} & \big\downarrow {\scriptstyle [\varrho,\eta\cdot \mathsf{inr}]} \\ X & \xrightarrow[e]{} & R(X+A) \end{array} \quad \text{implies} \quad \begin{array}{ccc} & & RA \\ & \overset{e^{\ddagger}}{\nearrow} & \big\downarrow {\scriptstyle m_A} \\ X & & \\ & \underset{(m_{X+A}\cdot e)^{\dagger}}{\searrow} & SA \end{array}$$

Proof.

(1) We first recall from Adámek *et al.* (2006a) how to obtain the unique solution $e^{\ddagger}$ of the guarded equation morphism $e : X \longrightarrow R(X + A)$.
The functor H is finitary, so it preserves the colimit $R(X + A) = \operatorname{colim} D_{X+A}$. Since X is finitely presentable, the morphism

$$e_0 : X \longrightarrow HR(X + A) + A = \operatorname{colim}(H\cdot D_{X+A} + A)$$

factorises through one of the colimit maps $Hh^{\#} + A$ for some $h : W \longrightarrow HW + (X + A)$ in D_{X+A}:

$$\begin{array}{ccc} & & HW+A \\ & \overset{w}{\nearrow} & \big\downarrow {\scriptstyle Hh^{\#}+A} \\ X & \xrightarrow[e_0]{} & HR(X+A)+A \end{array} \tag{5.26}$$

We now form a flat equation morphism

$$\begin{array}{lll} k \equiv W + X & \xrightarrow{[h,\mathsf{inm}]} & HW + X + A \\ & & \big\downarrow {\scriptstyle [HW,w,A]} \\ & & HW + A \xrightarrow{H\,\mathsf{inl}+\eta_A} H(W + A) + RA. \end{array} \tag{5.27}$$

Its solution morphism $k^{\ddagger} : W + X \longrightarrow RA$ yields $e^{\ddagger} : X \longrightarrow RA$ through

$$\begin{array}{ccc} & & W+X \\ & \overset{\mathsf{inr}}{\nearrow} & \big\downarrow {\scriptstyle k^{\ddagger}} \\ X & \xrightarrow[e^{\ddagger}]{} & RA \end{array} \tag{5.28}$$

Thus we need to prove

$$m_A \cdot k^{\ddagger} \cdot \mathsf{inr} = (m_{X+A} \cdot e)^{\dagger}. \tag{5.29}$$

(2) We are going to work with the equation morphisms

$$f \equiv W \xrightarrow{h} HW + X + A \xrightarrow{\lambda + \eta^S} SW + S(X+A) \xrightarrow{\mathsf{can}} S(W + X + A)$$

and

$$g \equiv X \xrightarrow{w} HW + A \xrightarrow{H\,\mathsf{inl} + A} H(W+X) + A \xrightarrow{\lambda + \eta^S} S(W+X) + SA \xrightarrow{\mathsf{can}} S(W + X + A).$$

We first observe that

$$f^{\dagger} = m_{X+A} \cdot h^{\#}. \tag{5.30}$$

Indeed, by (5.16) and (5.25), we have, since m_{X+A} is an Elgot algebra morphism,

$$\begin{aligned} m_{X+A} \cdot h^{\#} &= m_{X+A} \cdot (\eta_{X+A} \bullet h)^{\dagger} \\ &= \big(m_{X+A} \bullet (\eta_{X+A} \bullet h)\big)^{*} \\ &= \big((m_{X+A} \cdot \eta_{X+A}) \bullet h\big)^{*}. \end{aligned}$$

Since (5.17) clearly implies

$$f = \overline{\eta^S_{X+A} \bullet h},$$

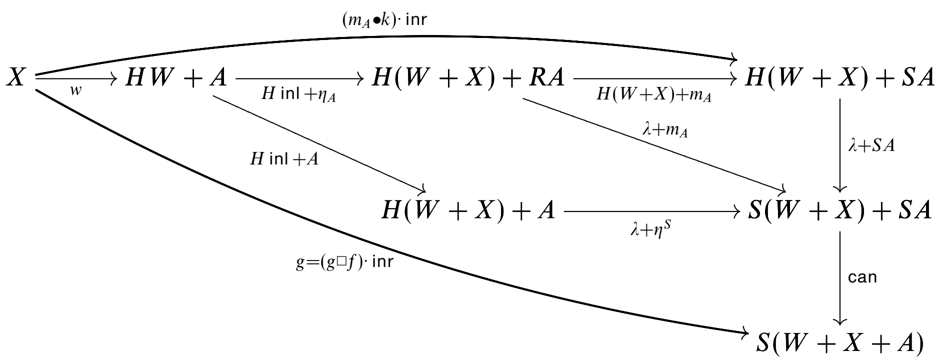
we obtain, by (5.16) and $m \cdot \eta = \eta^S$, the desired equality

$$f^{\dagger} = \left(\overline{\eta^S_{X+A} \bullet h}\right)^{\dagger} = \left(\eta^S_{X+A} \bullet h\right)^{*} = m_{X+A} \cdot h^{\#}. \tag{5.31}$$

(3) We now want to apply Lemma 5.3 to f and g. We first prove

$$g \,\square\, f = \overline{m_A \bullet k} : W + X \longrightarrow S(W + X + A). \tag{5.32}$$

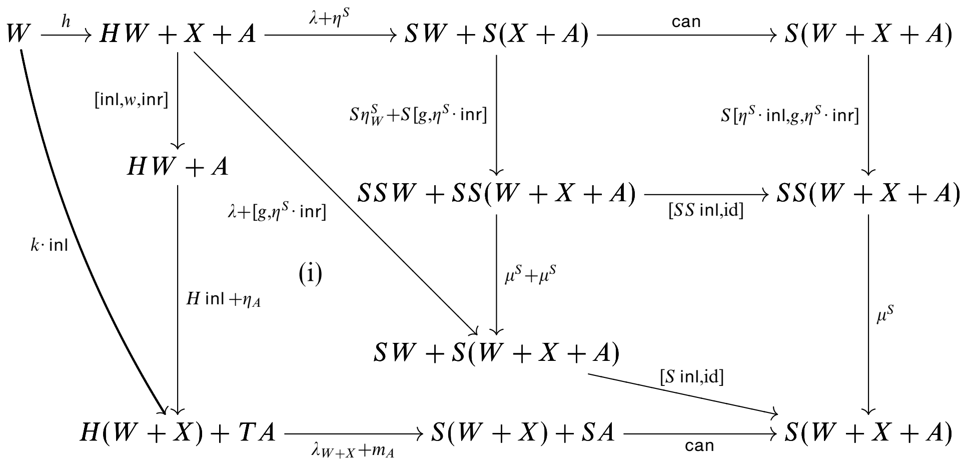
To this end, we consider the coproduct components separately. For the right-hand component, we obtain the commutative diagram



(5.33)

For the middle part, consider the components separately using $\eta^S = m \cdot \eta$ (see Proposition 5.7) – the remaining parts are clear.

The left-hand component of (5.32) is verified by the diagram



The left-hand part is the definition of k, and the three parts to the right of part (i) are clear. For part (i), we consider the three components separately. The left-hand component commutes due to the naturality of λ. The right-hand component commutes due to naturality of η^S and the fact that $m_A \cdot \eta_A = \eta^S_A$. For the middle component with domain X, we compare with the lower part of diagram (5.33) and use $m_A \cdot \eta_A = \eta^S_A$ again. We have now proved (5.32).

The next step of our proof is to apply Lemma 5.3 to the equation morphism $g \,\square\, h$ to obtain the equation

$$\overline{m_A \bullet k}^{\dagger} = [e_L^{\dagger}, e_R^{\dagger}] : W + X \longrightarrow SA, \tag{5.34}$$

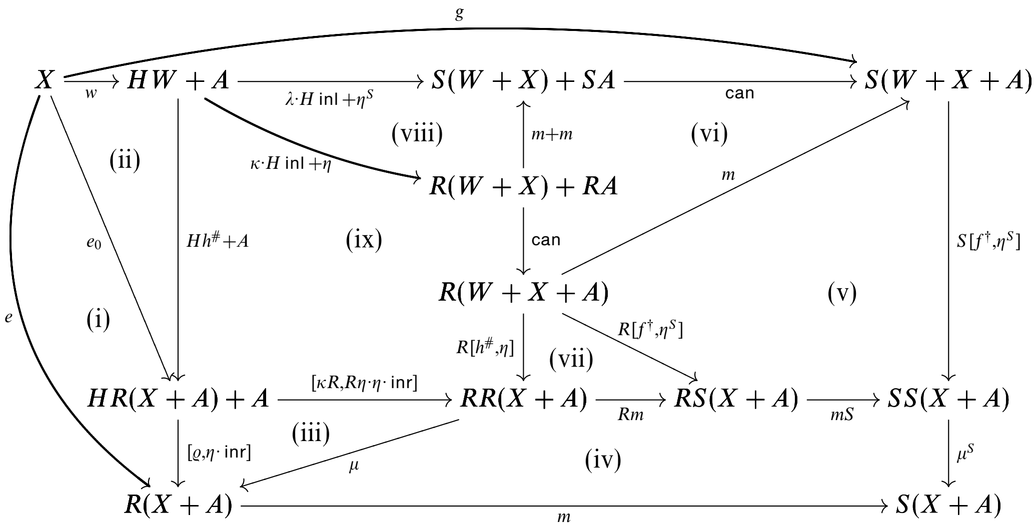
for appropriate equation morphisms $e_L : W \longrightarrow S(W + A)$ and $e_R : X \longrightarrow S(X + A)$. In part 4, we will prove the equation

$$e_R = X \xrightarrow{\;e\;} R(X + A) \xrightarrow{\;m_{X+A}\;} S(X + A). \tag{5.35}$$

This will allow us to conclude our proof through the following computation:

$$\begin{aligned}
(m_{X+A} \cdot e)^{\dagger} &= e_R^{\dagger} && (5.35)\\
&= \overline{m_A \bullet k}^{\dagger} \cdot \mathsf{inr} && (5.34)\\
&= (m_A \bullet k)^{*} \cdot \mathsf{inr} && (5.16)\\
&= m_A \cdot k^{\ddagger} \cdot \mathsf{inr} && m_A \text{ Elgot algebra morphism}\\
&= m_A \cdot e^{\ddagger}. && (5.28)
\end{aligned}$$

(4) The proof of (5.35) follows from the commutative diagram



(i) This part commutes since e is guarded.

(ii) This part commutes by (5.26).

(iii) This part commutes because $\varrho = \mu{\cdot}\kappa R$ (see Notation 5.2) and $\mu{\cdot}R\eta = \mathrm{id}$.

(iv) For this part, we use the fact that m is a monad morphism.

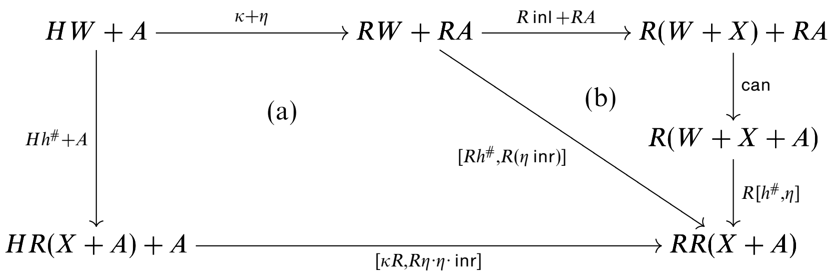
(v) This part is clear.

(vi) This part is clear.

(vii) For this part, we use (5.31) and $m{\cdot}\eta = \eta^S$.

(viii) For this part, we use (5.31) again and $\lambda = m{\cdot}\kappa$ from Proposition 5.7.

(ix) This part follows from the diagram



Part (a) commutes by naturality of κ and η, and part (b) clearly commutes. This completes the proof. □

Notation 5.11. Given a monad morphism $m : \mathbb{R} \longrightarrow \mathbb{S}$, for every equation morphism $e : X \longrightarrow R(X + A)$ with respect to $\mathbb{R}$, we obtain an equation morphism

$$\bar{e} = m_{X+A}{\cdot}e : X \longrightarrow S(X + A)$$

with respect to $\mathbb{S}$. In the following Lemma we refer to the Bekić Identity of Remark 2.12, which was formulated in the base category.

Lemma 5.12. Let $(\mathbb{R}, \ddagger)$ and $(\mathbb{S}, \dagger)$ be Elgot monads and $m : \mathbb{R} \longrightarrow \mathbb{S}$ be a monad morphism. Given equation morphisms

$$f : X \longrightarrow R(X + Y + A)$$
$$g : Y \longrightarrow R(X + Y + A),$$

we use

$$e_R : Y \longrightarrow R(Y + A)$$
$$e_L : X \longrightarrow R(X + A)$$

to denote the morphisms in the Bekić Identity for $\mathbb{R}$ and

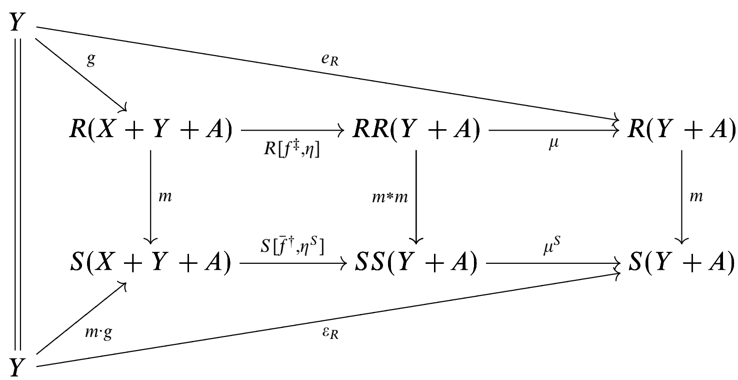
$$\varepsilon_R : Y \longrightarrow S(Y + A)$$
$$\varepsilon_L : X \longrightarrow S(X + A)$$

to denote those for $\mathbb{S}$ (with respect to $\bar{f}$ and $\bar{g}$). Then

(1) Assuming $\bar{f}^\dagger = m_{Y+A} \cdot f^\ddagger$, we conclude $\varepsilon_R = m_{Y+A} \cdot e_R$.
(2) Assuming $\varepsilon_R^\dagger = m_A \cdot e_R^\ddagger$, we conclude $\varepsilon_L = m_{X+A} \cdot e_L$.
(3) Assuming (1), (2) and $\varepsilon_L^\dagger = m_A \cdot e_L^\ddagger = \bar{e}_L^\dagger$, we conclude $[\bar{f}, \bar{g}]^\dagger = m_A \cdot [f, g]^\ddagger$.

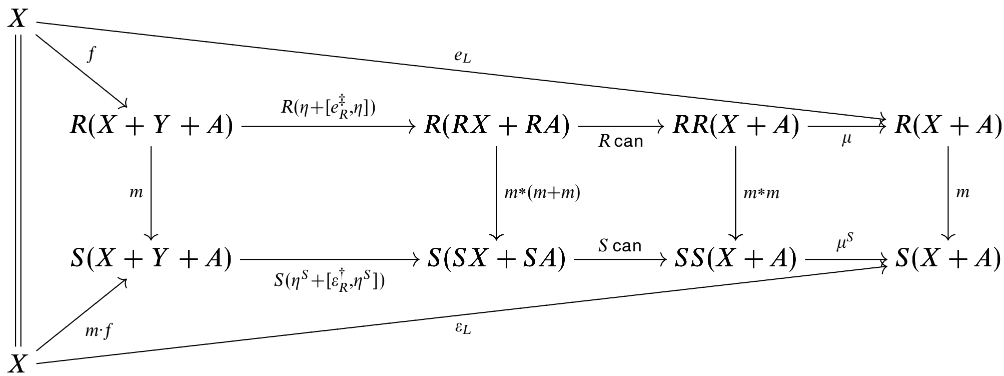
Proof.

(1) The proof follows from the diagram



The upper and lower parts are the definitions of e_R and ε_R, respectively. The right-hand square commutes since m is a monad morphism. For the middle square, we use the naturality of m, the unit law $m \cdot \eta = \eta^S$ and the hypothesis $\bar{f}^\dagger = m \cdot f^\ddagger$. The left-hand part trivially commutes.

(2) Consider the following analogous diagram:



A similar argument to the above shows that this diagram commutes; for the second square from the left, we use the hypothesis $m{\cdot}e_R^\ddagger = \varepsilon_R^\dagger$.

(3) We use the Bekić Identity for both $\mathbb{R}$ and $\mathbb{S}$ to obtain

$$\begin{aligned}
[m{\cdot}f, m{\cdot}g] &= [\varepsilon_L^\dagger, \varepsilon_R^\dagger] && \text{Bekić Identity for } \mathbb{S}\\
&= \big[(m{\cdot}e_L)^\dagger, (m{\cdot}e_R)^\dagger\big] && \text{by (1) and (2)}\\
&= m{\cdot}[e_L^\ddagger, e_R^\ddagger] && \text{by hypothesis}\\
&= m{\cdot}[f,g]^\ddagger && \text{Bekić Identity for } \mathbb{R}.
\end{aligned}$$

□

Remark 5.13. Note that in the previous proof we did not use any equational properties of ‡ or † in items (1) and (2). So the result in these items applies to any monads having a dagger operation (without necessarily satisfying any equations).

Lemma 5.14. The monad morphism m preserves solutions of preguarded equation morphisms (see Remark 3.19).

Proof. Let $e : X \longrightarrow S(X+A)$ be preguarded. This is equivalent to saying that

$$X = \bar{X}_1 + \cdots + \bar{X}_n \quad \text{and} \quad X_n = 0 \qquad (\text{for some } n \text{ in } \mathbb{N})$$

(see Proposition 3.18).

(1) We will use the Bekić Identity repeatedly to obtain a decomposition of

$$e^\ddagger : X \longrightarrow RA$$

as follows: we will find guarded (see Remark 3.7) equation morphisms

$$e_L^i : \bar{X}_i \longrightarrow R(\bar{X}_i + A)$$

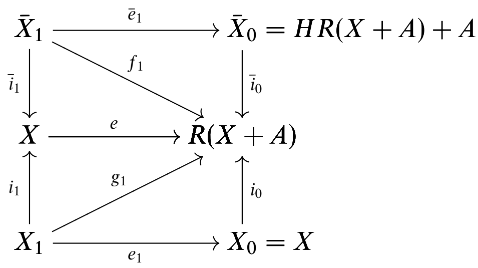
with

$$e^\ddagger = \Big[(e_L^1)^\ddagger, \ldots, (e_L^n)^\ddagger\Big] : \bar{X}_1 + \cdots + \bar{X}_n \longrightarrow R(X+A).$$

(1.1) The first step is to decompose

$$e = [f_1, g_1] : \bar{X}_1 + X_1 \longrightarrow R(\bar{X}_1 + X_1 + A)$$

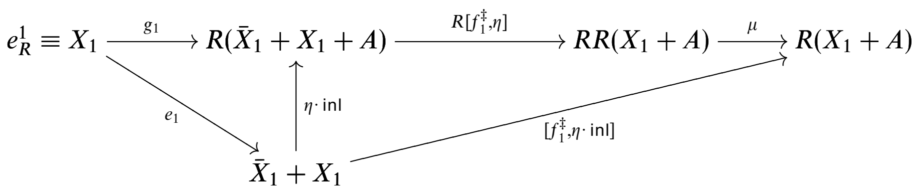
as follows:



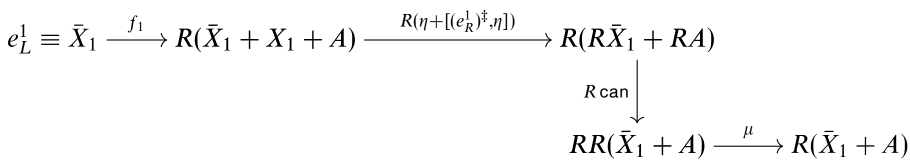
This diagram commutes due to (3.7) and (3.8). Then, by the Bekić Identity, we have

$$e^{\ddagger} = \left[(e_L^1)^{\ddagger}, (e_R^1)^{\ddagger}\right], \tag{5.36}$$

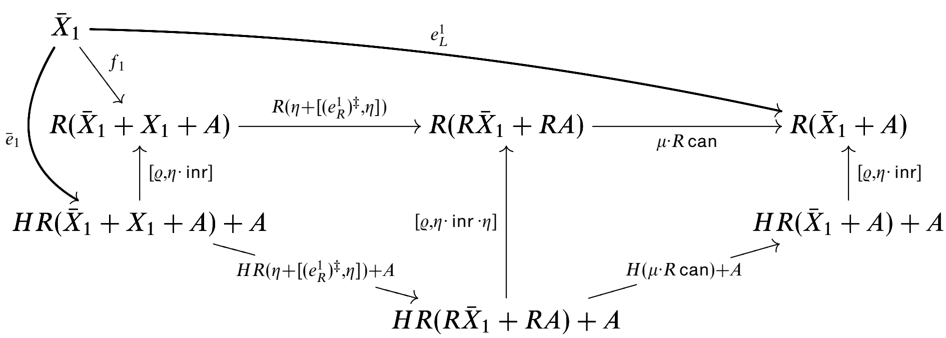
where



and



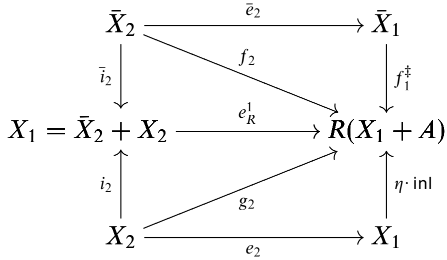
The equation morphism e_L^1 is guarded; using the naturality of ϱ and η, we obtain the commutative diagram



(1.2) Unfortunately, e_R^1 need not be guarded. Thus, we keep applying the Bekić Identity: the following diagram defines equation morphisms f_2, g_2 with

$$e_R^1 = [f_2, g_2] : \bar{X}_2 + X_2 \longrightarrow R(\bar{X}_2 + X_2 + A)$$

as follows:



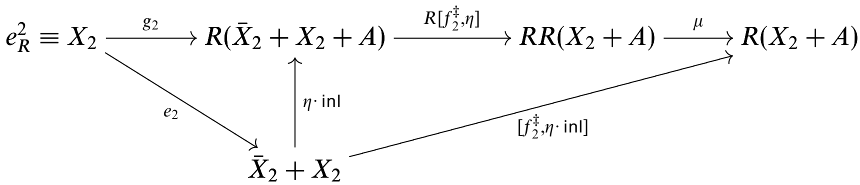
To see that this diagram commutes, recall that by extensivity we have

$$e_1 = \bar{e}_2 + e_2 : X_1 = \bar{X}_2 + X_2 \longrightarrow \bar{X}_1 + X_1,$$

and apply (3.7) and (3.8). Then the Bekić Identity yields

$$(e_R^1)^\ddagger = \left[(e_L^2)^\ddagger, (e_R^2)^\ddagger \right]$$

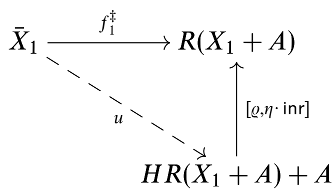
where e_R^2 is given as follows



and e_L^2 is the morphism

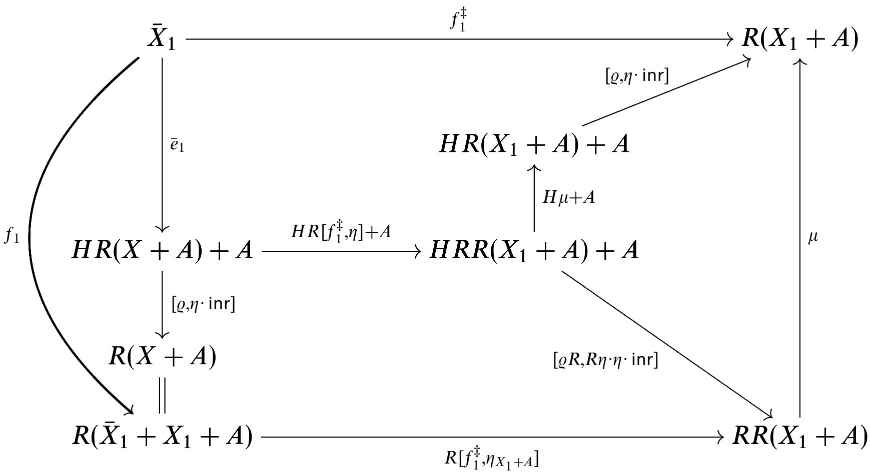
$$e_L^2 \equiv \bar{X}_2 \xrightarrow{f_2} R(\bar{X}_2 + X_2 + A) \xrightarrow{R(\eta + [(e_R^2)^\ddagger, \eta])} R(R\bar{X}_2 + RA) \xrightarrow{R\,\mathsf{can}} RR(\bar{X}_2 + A) \xrightarrow{\mu} R(\bar{X}_2 + A).$$

We now verify that e_L^2 is guarded. To this end, observe first that $f_1^\ddagger$ is 'guarded' in the sense that the triangle

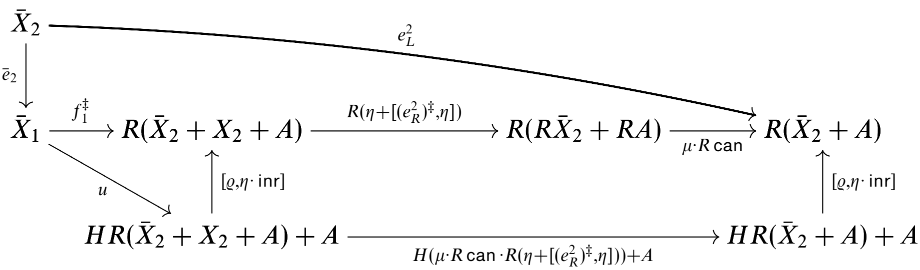


(5.37)

commutes for a suitable morphism u. Indeed, consider the diagram



The outer square commutes since $f_1^\ddagger$ solves f_1. The lower square commutes by the naturality of ϱ and η. To see that the right-hand square commutes, we use the fact that μ is a homomorphism of H-algebras (see Remark 4.6) and satisfies the monad law $\mu\cdot R\eta = \mathrm{id}$. Thus, the upper part commutes as desired. Then we have e_L^2 is guarded due to the diagram



(1.3) As before, the right-hand equation morphism e_R^2 need not be guarded. We define $f_3 = f_2^\ddagger\cdot\bar{e}_3$ and $g_3 = \eta\cdot\mathsf{inl}\cdot e_3$ such that

$$e_R^2 = [f_3, g_3] : \bar{X}_3 + X_3 = X_2 \longrightarrow R(X_2 + A)$$

and apply the Bekić Identity to obtain

$$(e_R^2)^\ddagger = \left[(e_L^3)^\ddagger, (e_R^3)^\ddagger\right]$$

with e_L^3 guarded. We then proceed analogously for $n-1$ steps, where n is the number of summands of X.

(1.n) We thus get, for $i = 1, 2, 3, \ldots, n$, decompositions

$$e_R^{i-1} = [f_i, g_i] : \bar{X}_i + X_i = X_{i-1} \longrightarrow R(X_{i-1} + A)$$

for the equation morphisms

$$f_i \equiv \bar{X}_i \xrightarrow{\bar{e}_i} \bar{X}_{i-1} \xrightarrow{f_{i-1}^{\ddagger}} R(X_{i-1} + A) = R(\bar{X}_i + X_i + A)$$

(guarded) and

$$g_i \equiv X_i \xrightarrow{e_i} X_{i-1} \xrightarrow{\eta \cdot \mathsf{inl}} R(X_{i-1} + A) = R(\bar{X}_i + X_i + A).$$

After n steps, where $X_n \cong 0$, the morphism g_n is guarded by default, and we get

$$\left(e_R^{n-1}\right)^{\ddagger} = \left(e_L^{n}\right)^{\ddagger},$$

which implies

$$\left(e_R^{n-2}\right)^{\ddagger} = \left[\left(e_L^{n-1}\right)^{\ddagger}, \left(e_R^{n-1}\right)^{\ddagger}\right] = \left[\left(e_L^{n-1}\right)^{\ddagger}, \left(e_L^{n}\right)^{\ddagger}\right],$$

and this in turn yields

$$\left(e_R^{n-3}\right)^{\ddagger} = \left[\left(e_L^{n-2}\right)^{\ddagger}, \left(e_L^{n-1}\right)^{\ddagger}, \left(e_L^{n}\right)^{\ddagger}\right],$$

and so on. Thus, we get the desired decomposition in n steps:

$$e^{\ddagger} = \left[\left(e_L^{1}\right)^{\ddagger}, \dots, \left(e_L^{n}\right)^{\ddagger}\right].$$

(2) We are now ready to prove the statement of the lemma:

$$(m_{X+A} \cdot e)^{\dagger} = m_A \cdot e^{\ddagger}.$$

From $e = [f_1, g_1]$, we get

$$m_{X+A} \cdot e = [m_{X+A} \cdot f_1, m_{X+A} \cdot g_1].$$

We want to apply Lemma 5.12 to $f = f_1$ and $g = g_1$, so we check the hypotheses. We have

$$m_{X+1+A} \cdot f_1^{\ddagger} = (m_{\bar{X}_1 + X_1 + A})^{\dagger}$$
$$m_A \cdot (e_L^1)^{\ddagger} = \left(m_{\bar{X}_1 + A} \cdot e_L^1\right)^{\dagger}$$

by Lemma 5.10, since f_1 and e_L^1 are guarded (see item (1.1))
In part 3, we will verify that the equation

$$\left(m_{X_1+A} \cdot e_R^1\right)^{\dagger} = m_A \cdot \left(e_R^1\right)^{\ddagger} \tag{5.38}$$

holds (notice that this does not follow from Lemma 5.10). This will conclude the proof since we then obtain

$$\begin{aligned}(m_{X+A} \cdot e)^{\dagger} &= \left(m_{X+A} \cdot \left[e_L^1, e_R^1\right]\right)^{\dagger} \\ &= m_A \cdot \left[e_L^1, e_R^1\right]^{\ddagger} \\ &= m_A \cdot e^{\ddagger}\end{aligned}$$

by Lemma 5.12.

(3) (The proof of (5.38)). Essentially, we repeat the argument above with e_R^1 instead of e, that is, we apply Lemma 5.12 to $f = f_2$ and $g = g_2$ and obtain (5.38), provided we verify

$$m_{X_2+A} \cdot e_R^2 = m_A \cdot \left(e_R^2\right)^{\ddagger} : X_2 \longrightarrow SA,$$

and so on. The last step then leads to the equation

$$m_{X_n+A} \cdot e_R^n = m_A \cdot \left(e_R^n\right)^{\ddagger} : X_n \longrightarrow SA,$$

which is trivial since $X_n = 0$. □

Lemma 5.15. The monad morphism $m : \mathbb{R} \longrightarrow \mathbb{S}$ of Proposition 5.7 preserves the dagger operation.

Proof. For every equation morphism $e : X \longrightarrow R(X + A)$, we will establish

$$m_A \cdot e^{\ddagger} = \left(m_{X+A} \cdot e\right)^{\dagger} : X \longrightarrow SA. \tag{5.39}$$

Recall from Proposition 3.18 that there exists k such that $i_k^* : X_k \longrightarrow X$ is the greatest ungrounded subobject of e, and then

$$X = X_k + \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_k$$

with coproduct injections i_k^*, and $\bar{i}_1^*, \ldots, \bar{i}_k^*$, respectively (see Notation 3.17). We put

$$Z = X_k$$
$$\bar{Z} = \bar{X}_1 + \cdots + \bar{X}_k$$

for short, so $X = Z + \bar{Z}$ with injections

$$\mathsf{inl} = i_k^*$$
$$\mathsf{inr} = \left[\bar{i}_1^*, \ldots, \bar{i}_k^*\right].$$

We establish some auxiliary results first.

(1) For every equation morphism

$$d \equiv Y \xrightarrow{h} Y \xrightarrow{\mathsf{inl}} Y + A,$$

we prove

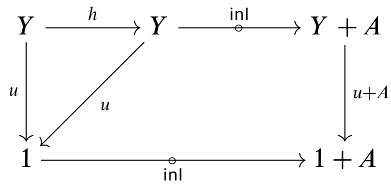
$$d^{\dagger} = \bot : Y \longrightarrow A.$$

(Recall the notation $\bot = \bot_{Y,A}$ from Remark 2.6.)
The statement is true for $Y = 1$: indeed, for $A = 0$, we have $\bot = \mathsf{inl}^{\dagger}$ by definition (*cf.* Remark 2.20), and for arbitrary A, we apply the Parameter Identity to the unique morphism $v : 0 \longrightarrow A$. Then we have

$$1 \xrightarrow{\mathsf{inl}} 1 + 0 \xrightarrow{1+v} 1 + A\,,$$

hence, $\mathsf{inl}^{\dagger} = ((1 + v) \cdot \mathsf{inl})^{\dagger} = v \cdot \mathsf{inl}^{\dagger} = v \cdot \bot = \bot$.

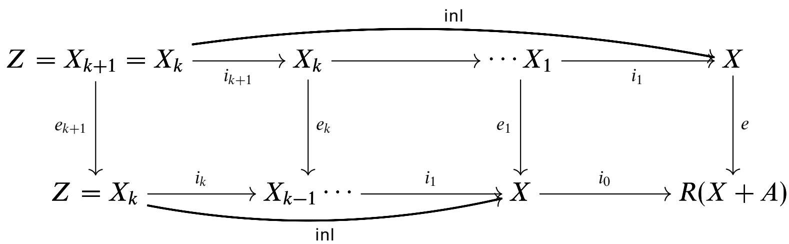
For Y and h arbitrary, we apply *Functoriality* to the unique morphism $u : X \longrightarrow 1$, and since the square



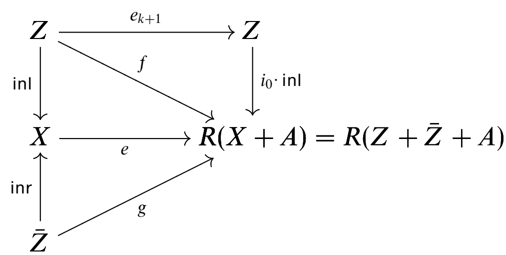
commutes, we conclude

$$(\mathsf{inl} \cdot h)^{\dagger} = \mathsf{inl}^{\dagger} \cdot u = \bot_{1,A} \cdot u = \bot_{Y,A}.$$

(2) Consider the pullbacks defining the derived subobjects of e. By gluing them together, we get the pullback of $i_0 \cdot \mathsf{inl}$ along e in Notation 3.17:



We use the notation f and g as follows:



By the Bekić Identity, we have

$$e^{\ddagger} = [f, g]^{\ddagger} = \left[e_L^{\ddagger}, e_R^{\ddagger}\right] : Z + \bar{Z} \longrightarrow RA, \tag{5.40}$$

where

$$e_R \equiv \bar{Z} \xrightarrow{\;g\;} Z + \bar{Z} + A \xrightarrow{[f^{\ddagger}, \bar{Z}+A]} \bar{Z} + A$$

and

$$e_L \equiv Z \xrightarrow{\;f\;} Z + \bar{Z} + A \xrightarrow{Z + [e_R^{\ddagger}, A]} Z + A.$$

We shall apply Lemma 5.12 to these equation morphisms f and g to conclude the proof and obtain the desired (5.39). We will verify the hypothesis for f and e_L in parts (3) and (4) below. For the remaining hypothesis, for e_R, we need to establish that e_R is preguarded (see part (6)), and for this we shall compute the pullback of g along $\eta \cdot \mathsf{inm} : \bar{Z} \longrightarrow R(Z + \bar{Z} + A)$ in part (5).

(3) We apply part (1) to $h = e_{k+1}$ to show that the unique strict solution of f is simply

$$f^{\ddagger} = \bot : Z \longrightarrow R(\bar{Z} + A). \tag{5.41}$$

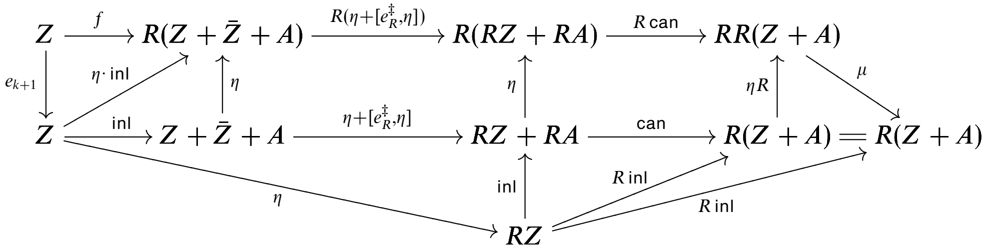
We conclude

$$m_A \cdot f^{\ddagger} = (m_{X+A} \cdot f)^{\dagger}. \tag{5.42}$$

Indeed, the left-hand side is $\bot$ because m is strict, see Proposition 5.7, and the right-hand side of (5.42) is $\bot$ from (1) applied to $h = e_{k+1}$ and using the fact that

$$m_{X+A} \cdot i_0 \cdot \mathsf{inl} = \eta^S_{X+A} \cdot \mathsf{inl}.$$

(4) For e_L, we have a commutative diagram



Consequently, by (1) applied to $h = e_{k+1}$, the unique strict solution is

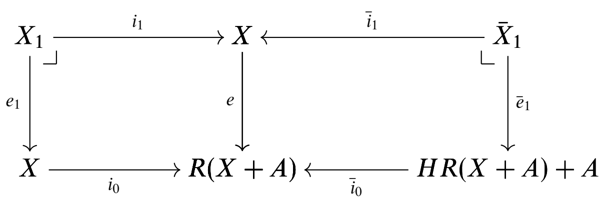
$$e_L^{\ddagger} = \bot.$$

So for the equation morphism $m_{Z+A} \cdot e_L : Z \longrightarrow S(Z + A)$, we get

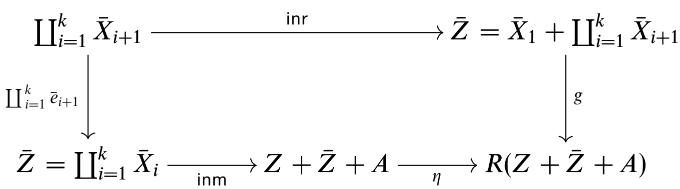
$$\left(m_{Z+A} \cdot e_L\right)^{\dagger} = m_A \cdot e_L^{\ddagger}.$$

Indeed, the left-hand side is $\bot$ by (1) applied to $h = e_{k+1}$, and the right-hand side is $\bot$ since $e_L^{\ddagger} = \bot$ and m is strict.

(5) Recall the notation $i_r, \bar{i}_r$ and $\bar{e}_r$ from Notation 3.17. The case $r = 0$ is given (since $R(X + A) = HR(X + A) + X + A$ and i_0 is the injection of A so $\bar{i}_0 = [\varrho_{X+A}, \eta_{X+A} \cdot \mathsf{inr}]$) by the diagram



We are going to prove that the square



(5.43)

is a pullback. The extensivity of our base category allows us to consider the individual components of g (that is, those of e) separately. The first component is

$$e\cdot\bar{i}_1^* : \bar{X}_1 \longrightarrow R(Z+\bar{Z}+A)$$

whose pullback along $\eta\cdot\,\mathsf{inm}$ is 0. Indeed, $e\cdot\bar{i}_1 = \bar{i}_0\cdot\bar{e}_1$ and we have the following pullbacks

$$\begin{array}{ccccc}
0 & \longrightarrow & & \longrightarrow & \bar{X}_1 \\
\| & & & & \downarrow{\scriptstyle \bar{e}_1} \\
0 & \longrightarrow & A & \xrightarrow{\mathsf{inr}} & HR(X+A)+A \\
\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} & & \downarrow{\scriptstyle [\varrho,\eta\cdot\,\mathsf{inr}]=\bar{i}_0} \\
\bar{Z} & \xrightarrow[\mathsf{inm}]{} & X+A & \xrightarrow[\eta]{} & R(X+A)
\end{array}$$

For $\bar{X}_2$ we get, since $e\cdot\bar{i}_2^* = e\cdot i_1\cdot\bar{i}_2 = i_0\cdot e_1\cdot\bar{i}_2 = i_0\cdot\bar{i}_1\cdot\bar{e}_2$ (see (3.7) and (3.8)), the pullback diagram

$$\begin{array}{ccccc}
\bar{X}_2 & = & = & = & \bar{X}_2 \\
{\scriptstyle \bar{e}_2}\downarrow & & & & \downarrow{\scriptstyle \bar{e}_2} \\
\bar{X}_1 & = & = & = & \bar{X}_1 \\
{\scriptstyle q_1}\downarrow & & & & \downarrow{\scriptstyle \bar{i}_1} \\
\bar{Z} & \xrightarrow{\mathsf{inr}} & Z+\bar{Z}=X & = & X \\
\| & & \downarrow{\scriptstyle \mathsf{inl}} & & \downarrow{\scriptstyle i_0} \\
\bar{Z} & \xrightarrow[\mathsf{inm}]{} & X+A & \xrightarrow[\eta]{} & R(X+A)
\end{array}$$

where $q_1 = \bar{X}_1 \longrightarrow \bar{Z} = \bar{X}_1 + \cdots + \bar{X}_n$ is the coproduct injection. Continuing analogously, for the ith component we have the ith coproduct injection $q_i : \bar{X}_i \longrightarrow \bar{Z}$, and we get the pullback

$$\begin{array}{ccccc}
\bar{X}_{i+1} & = & = & = & \bar{X}_{i+1} \\
{\scriptstyle \bar{e}_{i+1}}\downarrow & & & & \downarrow{\scriptstyle \bar{e}_{i+1}} \\
\bar{X}_i & = & = & = & \bar{X}_i \\
{\scriptstyle q_i}\downarrow & & & & \downarrow{\scriptstyle \bar{i}_n^*} \\
\bar{Z} & \xrightarrow{\mathsf{inr}} & Z+\bar{Z}=X & = & X \\
\| & & \downarrow{\scriptstyle \mathsf{inl}} & & \downarrow{\scriptstyle i_0} \\
\bar{Z} & \xrightarrow[\mathsf{inm}]{} & X+A & \xrightarrow[\eta]{} & R(X+A)
\end{array}$$

yielding the pullback (5.43).

(6) We now prove that the equation morphism

$$e_R \equiv \bar{Z} \xrightarrow{g} R(Z + \bar{Z} + A) \xrightarrow{R[f^\ddagger,\eta]} RR(\bar{Z} + A) \xrightarrow{\mu} R(\bar{Z} + A)$$

is preguarded (that is, its least derived subobject is 0). We verify that the first derived subobject is given by the pullback

$$\begin{array}{ccccccc}
\coprod_{i=1}^{k} \bar{X}_{i+1} & & & \xrightarrow{\quad\mathsf{inr}\quad} & & & \bar{Z} \\
\Big\downarrow{\scriptstyle \coprod_{i=1}^{k} \bar{e}_{i+1}} & & & & & & \Big\downarrow{\scriptstyle g} \\
\bar{Z} & \xrightarrow{\mathsf{inl}} & \bar{Z} + A & \xrightarrow{\mathsf{inr}} & Z + \bar{Z} + A & \xrightarrow{\eta} & R(Z + \bar{Z} + A) \\
\| & & \| & & \Big\downarrow{\scriptstyle [f^\ddagger,\eta]} & & \Big\downarrow{\scriptstyle R[f^\ddagger,\eta]} \\
\bar{Z} & \xrightarrow{\mathsf{inl}} & \bar{Z} + A & \xrightarrow{\eta} & R(\bar{Z} + A) & \xrightarrow{\eta R} & RR(\bar{Z} + A) \\
\| & & \| & & & & \Big\downarrow{\scriptstyle \mu} \\
\bar{Z} & \xrightarrow{\mathsf{inl}} & \bar{Z} + A & & \xrightarrow{\quad\eta\quad} & & R(\bar{Z} + A)
\end{array}$$

The upper square is the pullback (5.43) above, and all the lower inner squares are also pullbacks by extensivity. This is quite obvious for all of them except for the middle square, where it follows from the computation of the two pullbacks of $\eta_{\bar{Z}+A}$ along the two components of $[f^\ddagger, \eta_{\bar{Z}+A}]$: just observe that the pullback of $\eta_{\bar{Z}+A}$ along $f^\ddagger$ is 0 since $f^\ddagger = \perp$ – see (5.41) above. Thus, the first derived subobject of e_R is $\mathsf{inr} : \coprod_{i=1}^{k} \bar{X}_{i+1} \longrightarrow \bar{Z}$.

We are now ready to compute all derived subobjects

$$j_n : \bar{Z}_n \longrightarrow \bar{Z}$$

of e_R. We have

$$j_0 = \eta_{\bar{Z}} \cdot \mathsf{inl} : \bar{Z} \longrightarrow R(\bar{Z} + A)$$

and, as just established,

$$j_1 = \mathsf{inr} : \coprod_{i=1}^{k} \bar{X}_{i+1} \longrightarrow \bar{Z}$$

$$(e_R)_1 = \coprod_{i=1}^{k} \bar{e}_{i+1}.$$

Continuing in the obvious manner, we have

$$\begin{array}{ccccccccc}
\bar{X}_{k+1} & \xrightarrow{\mathsf{inr}} \cdots \xrightarrow{\mathsf{inr}} & \coprod_{i=2}^{k} \bar{X}_{i+1} & \xrightarrow{\mathsf{inr}} & \coprod_{i=1}^{k} \bar{X}_{i+1} & \xrightarrow{\mathsf{inr}} & \bar{Z} \\
\Big\downarrow{\scriptstyle \bar{e}_{k+1}} & & \Big\downarrow{\scriptstyle \coprod_{i=2}^{k} \bar{e}_{i+1}} & & \Big\downarrow{\scriptstyle \coprod_{i=1}^{k} \bar{e}_{i+1}} & & \Big\downarrow{\scriptstyle e_R} \\
\bar{X}_k & \xrightarrow{\mathsf{inr}} \cdots \xrightarrow{\mathsf{inr}} & \coprod_{i=1}^{k} \bar{X}_{i+1} & \xrightarrow{\mathsf{inr}} & \bar{Z} & \xrightarrow{j_0} & R(\bar{Z} + A)
\end{array}$$

However, since X_k is the greatest ungrounded subobject of e, we have $X_k \cong X_{k+1}$ by Proposition 3.18, and this means that $\bar{X}_{k+1} = 0$, so e_R is preguarded. □

The proof of Theorem 5.1 will now be complete if we verify the unicity of the monad morphism $m : \mathbb{R} \longrightarrow \mathbb{S}$ from Proposition 5.7.

Lemma 5.16. Every Elgot monad morphism $m : \mathbb{R} \longrightarrow \mathbb{S}$ with $m \cdot \kappa = \lambda$ is equal to that of Proposition 5.7.

Proof. Since $m \cdot \eta = \eta^S$ (as part of the definition of monad morphism), all we need to prove is that m_A is an Elgot algebra morphism for every object A. This means that

$$m_A \cdot e^{\ddagger} = (m_{X+A} \bullet e)^*$$

for an arbitrary flat equation morphism $e : X \longrightarrow HX + RA$ (see Proposition 5.5). We form the equation morphism

$$\tilde{e} \equiv X \xrightarrow{e} HX + A \xrightarrow{\kappa + RA} RX + RA \xrightarrow{\mathsf{can}} R(X + A)$$

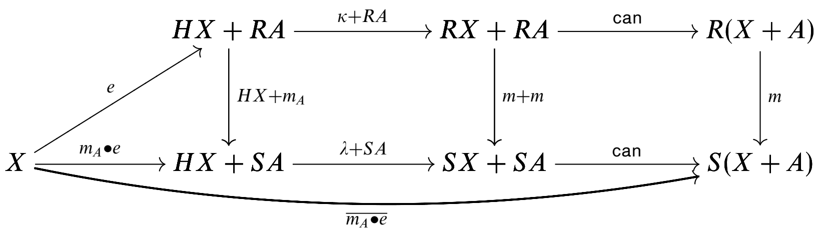
and observe that it is clearly guarded, and the solution $e^{\ddagger}$ of e in RA (as the free Elgot algebra) is equal to the solution $\tilde{e}^{\ddagger}$ of $\tilde{e}$ with respect to the Elgot monad $\mathbb{R}$. Consequently,

$$m_A \cdot e^{\ddagger} = \left(m_{X+A} \cdot \tilde{e}\right)^{\dagger}$$

since m is an Elgot monad morphism. Thus, from Proposition 5.5, all we need to prove is $\overline{m_A \bullet e} = m_{X+A} \cdot \tilde{e}$ since then

$$\left(m_{X+A} \bullet e\right)^* = \left(m_{X+A} \cdot \tilde{e}\right)^{\dagger} = m_A \cdot e^{\ddagger},$$

as desired. The following diagram proves $\overline{m_A \bullet e} = m \cdot \tilde{e} = m_{X+A} \cdot \mathsf{can} \cdot (\kappa + RA) \cdot e$:



The middle square commutes since $m \cdot \kappa = \lambda$, and all other parts are obvious. □

6. The monad Rat and its algebras

Assumption 6.1. We assume that $\mathscr{K}$ is a hyper-extensive, locally finitely presentable category. Recall that $\mathscr{F}$ denotes its small, full subcategory representing all finitely presentable objects. Recall the forgetful functor $U : \mathrm{IM}_{\perp}(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K})$ from Notation 3.9, and the rational monads from Theorem 3.10. We use 1 to denote the constant endfunctor whose value is the terminal object of $\mathscr{K}$.

Proposition 6.2. U has a left adjoint

$$\Phi : \mathsf{Fin}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathrm{IM}_{\perp}(\mathscr{K})$$

assigning to every finitary endofunctor X the free iterative monad $\mathbb{R}_{X+1}$ on $X+1$.

Proof. The forgetful functor U is the composite $U = \hat{U} \cdot U_\perp$ of the forgetful functor

$$U_\perp : \mathrm{IM}_\perp(\mathscr{K}) \longrightarrow \mathsf{Fin}_\perp(\mathscr{K}, \mathscr{K}), \qquad \mathbb{S} \longrightarrow S$$

into the category $\mathsf{Fin}_\perp(\mathscr{K}, \mathscr{K})$ of all strict finitary endofunctors and strict natural transformations, and the functor

$$\hat{U} : \mathsf{Fin}_\perp(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K})$$

forgetting $\perp$. From Theorem 5.1, we conclude that $U_\perp$ has the left adjoint

$$\Phi_\perp : \mathsf{Fin}_\perp(\mathscr{K}, \mathscr{K}) \longrightarrow \mathrm{IM}_\perp(\mathscr{K})$$

given by assigning to X a free iterative monad on X, that is, $\Phi_\perp(X) = \mathbb{R}_X$. And $\hat{U}$ has the obvious left adjoint

$$\hat{\Phi} : \mathsf{Fin}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Fin}_\perp(\mathscr{K}, \mathscr{K}), \qquad X \longmapsto X + 1.$$

Consequently, U has the left adjoint

$$\Phi = \Phi_\perp \cdot \hat{\Phi}, \qquad \Phi X = \mathbb{R}_{X+1}.$$

□

Example 6.3. Let $\mathscr{K} = \mathbf{Set}$.

(i) The value of Φ at H_Σ:
Recall the notation $\Sigma_\perp$ from Example 2.14(vi), and observe that $H_{\Sigma_\perp} = H_\Sigma + 1$. Thus, $\Phi(H_\Sigma) = \mathbb{R}_{\Sigma_\perp}$, the rational tree monad.

(ii) The value of Φ at an arbitrary finitary endofunctor X:
We express X as a regular quotient of H_Σ for some Σ. For example, the signature

$$\Sigma_n = X(n) \qquad \text{for all } n \in \mathbb{N}$$

yields, by the Yoneda Lemma, a natural transformation

$$\varepsilon : H_\Sigma = \coprod_{n \in \mathbb{N}} \coprod_{X(n)} \mathbf{Set}(n, -) \longrightarrow X.$$

Its components ε_n are clearly surjective for all $n \in \mathbb{N}$, and since X is finitary, all components of ε are surjective, that is, ε is a regular epimorphism. We can extend it to a regular epimorphism

$$\bar{\varepsilon} = \varepsilon + 1 : H_{\Sigma_\perp} \longrightarrow X + 1.$$

Since Φ, being a left adjoint, preserves regular epimorphisms, we see that $\Phi(X) = \mathbb{R}_{X+1}$ is a regular quotient of $\mathbb{R}_{\Sigma_\perp}$ through

$$\Phi(\varepsilon) : \mathbb{R}_{\Sigma_\perp} \longrightarrow \mathbb{R}_{X+1}.$$

In fact, Adámek and Milius (2006a) described the monad $\mathbb{R}_{X+1}$ concretely: if ε is given by a set E of equations (between flat Σ-terms), then $\Phi(\varepsilon)$ is the quotient of $\mathbb{R}_{\Sigma_\perp}$ modulo the congruence of potentially infinite applications of the equations in E.

Definition 6.4. We use Rat to denote the monad on $\mathsf{Fin}(\mathscr{K},\mathscr{K})$ given by the adjunction $\Phi \dashv U$ above. Thus, on objects X we have

$$\mathsf{Rat}(X) = R_{X+1},$$

where R_{X+1} is the underlying functor of the free iterative monad on $X+1$.

Remark 6.5. Recall that a functor $U : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *monadic* if there exists a monad $\mathbb{M}$ on $\mathscr{B}$ such that:

(a) $\mathscr{A}$ is isomorphic to the category $\mathscr{B}^{\mathbb{M}}$ of Eilenberg–Moore algebras for the monad $\mathbb{M}$, say, with an isomorphism $I : \mathscr{A} \longrightarrow \mathscr{B}^{\mathbb{M}}$.
(b) The forgetful functor $U^{\mathbb{M}} : \mathscr{B}^{\mathbb{M}} \longrightarrow \mathscr{B}$ is related to U by $U = U^{\mathbb{M}} \cdot I$.

By Beck's Theorem, see Mac Lane (1998), U is monadic if and only if it has a left adjoint and creates U-split coequalisers. This means that, if we are given morphisms $a, b : T \longrightarrow S$ in $\mathscr{A}$ and morphisms ψ, σ and τ in $\mathscr{B}$

$$UT \underset{Ub}{\overset{Ua}{\rightrightarrows}} US \underset{\sigma}{\overset{\psi}{\rightleftarrows}} C \qquad (\tau : US \longrightarrow UT)$$

such that

$$\psi \cdot Ua = \psi \cdot Ub,$$
$$\psi \cdot \sigma = \mathrm{id}_{UC},$$
$$Ub \cdot \tau = \mathrm{id}_{US},$$

and

$$Ua \cdot \tau = \sigma \cdot \psi,$$

then there exists precisely one morphism $c : S \longrightarrow S'$ in $\mathscr{A}$ such that $\psi = Uc$. Moreover, c is a coequaliser of a and b.

Example 6.6. A trivial application of Beck's Theorem shows that for the category $\mathsf{FMnd}(\mathscr{K})$ of finitary monads on $\mathscr{K}$ the forgetful functor

$$V : \mathsf{FMnd}(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K},\mathscr{K})$$

given by $V(\mathbb{S}) = S$ is monadic.

Theorem 6.7. The forgetful functor of the category of Elgot monads

$$U : \mathsf{EMnd}(\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K},\mathscr{K}), \qquad U(S, \eta, \mu, \dagger) = S$$

is monadic, with Rat as the corresponding monad.

Proof. We know from Theorem 5.1 that U has a left adjoint and the corresponding monad is Rat. Thus, we only need to prove that U creates U-split coequalisers, and then monadicity follows from Beck's Theorem. In more detail, given a pair of parallel Elgot monad morphisms

$$\alpha, \beta : (\mathbb{T}, \ddagger) \longrightarrow (\mathbb{S}, \dagger)$$

and given, for C in $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$, natural transformations

$$T \; \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \; S \;\; (\tau : S \to T), \qquad S \; \underset{\sigma}{\overset{\psi}{\rightleftarrows}} \; C$$

such that

$$\psi \cdot \alpha = \psi \cdot \beta, \tag{6.44}$$

$$\psi \cdot \sigma = \mathrm{id}_C, \tag{6.45}$$

$$\beta \cdot \tau = \mathrm{id}_S, \tag{6.46}$$

and

$$\sigma \cdot \psi = \alpha \cdot \tau, \tag{6.47}$$

we must prove that there exists a unique Elgot monad $\mathbb{C}$ on C such that $\psi : \mathbb{S} \longrightarrow \mathbb{C}$ is an Elgot monad morphism. Moreover, ψ is the coequaliser of α and β in $\mathsf{EMnd}(\mathscr{K})$.

From Example 6.6, we deduce that there exists a unique structure

$$\mathbb{C} = (C, \eta^C, \mu^C)$$

of a finitary monad such that ψ is a monad morphism and

$$\psi \text{ is the coequaliser of } \alpha \text{ and } \beta \text{ in } \mathsf{FMnd}(\mathscr{K}). \tag{6.48}$$

There exists at most one structure $e \longmapsto e^*$ of an Elgot monad on $\mathbb{C}$ for which ψ is an Elgot monad morphism. Indeed, the equation of Definition 2.16

$$\psi_k \cdot f^\dagger = (\psi_{n+k} \cdot f)^* \qquad \text{for } f : n \longrightarrow S(n+k)$$

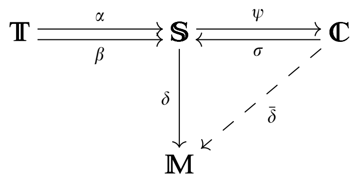
implies that e^* must be defined, for every $e : n \longrightarrow C(n+k)$, by

$$\begin{array}{ccc} n & \xrightarrow{\quad e^* \quad} & Ck \\ & {\scriptstyle (\sigma_{n+k} \cdot e)^\dagger} \searrow \quad \nearrow {\scriptstyle \psi_k} & \\ & Sk & \end{array} \tag{6.49}$$

To show this, we simply use (6.45) with $f = \sigma_{n+k} \cdot e$. We now prove that with this definition ψ preserves the dagger (of arbitrary equation morphisms f). To this end, we compute

$$\begin{aligned}
(\psi_{n+k} \cdot f)^* &= \psi_k \cdot (\sigma_{k+k} \cdot \psi_{n+k} \cdot f)^\dagger && \text{by (6.49)} \\
&= \psi_k \cdot \big(\alpha_{n+k} \cdot \tau_{n+k} \cdot f\big)^\dagger && \text{by (6.47)} \\
&= \psi_k \cdot \alpha_k \cdot \big(\tau_{n+k} \cdot f\big)^\ddagger && \alpha \text{ dagger preserving} \\
&= \psi_k \cdot \beta_k \cdot \big(\tau_{n+k} \cdot f\big)^\ddagger && \text{by (6.44)} \\
&= \psi_k \cdot \big(\beta_{n+k} \cdot \tau_{n+k} \cdot f\big)^\dagger && \beta \text{ dagger preserving} \\
&= \psi_k \cdot f^\dagger && \text{by (6.46).}
\end{aligned}$$

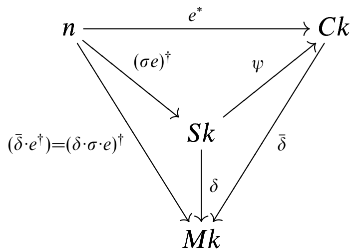
We will verify below that $*$ satisfies the axioms of Elgot monads. Then it is easy to see that ψ is the coequaliser of α and β in $\mathsf{EMnd}(\mathscr{K})$. Indeed, given an Elgot monad $\mathbb{M}$ and an Elgot monad morphism $\delta : \mathbb{S} \longrightarrow \mathbb{M}$



with $\delta{\cdot}\alpha = \delta{\cdot}\beta$, we already know from (6.48) that there exists a unique monad morphism $\bar{\delta} : \mathbb{C} \longrightarrow \mathbb{M}$ with $\delta = \bar{\delta}{\cdot}\psi$. To see that $\bar{\delta}$ is an Elgot monad morphism, observe that $\bar{\delta} = \delta{\cdot}\sigma$, which follows from (6.45):

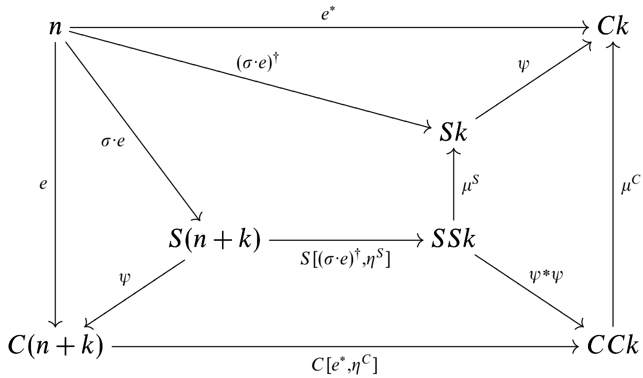
$$\delta{\cdot}\sigma = \bar{\delta}{\cdot}\psi{\cdot}\sigma = \bar{\delta}.$$

Since δ is dagger-preserving, we can conclude that $\bar{\delta}$ is too: given $e : n \longrightarrow C(n+k)$, we have

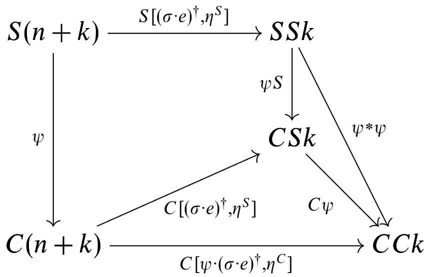


We will now verify the axioms of Elgot monads for $*$.

— *Solution*:
In the diagram



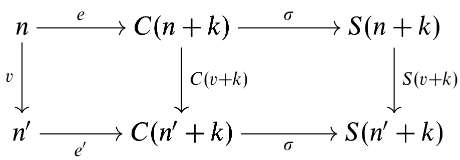
all inner parts commute: this is clear for the right-hand square since $\psi : \mathbb{S} \longrightarrow \mathbb{C}$ is a monad morphism; the middle square commutes from *Solution* with respect to $\mathbb{S}$; and the left-hand triangle commutes from (6.45). The lower square also commutes:



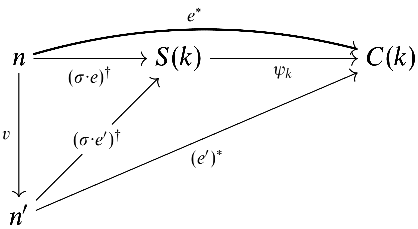
The left-hand square commutes because of the naturality of ψ, and for the lower part, delete C and consider the components separately.

— *Functoriality*:

Every homomorphism v of equations with respect to $\mathbb{C}$ yields one with respect to $\mathbb{S}$ by the naturality of σ:



The desired triangle follows from *Functoriality* with respect to $\mathbb{S}$:



— *Parameter Identity*:

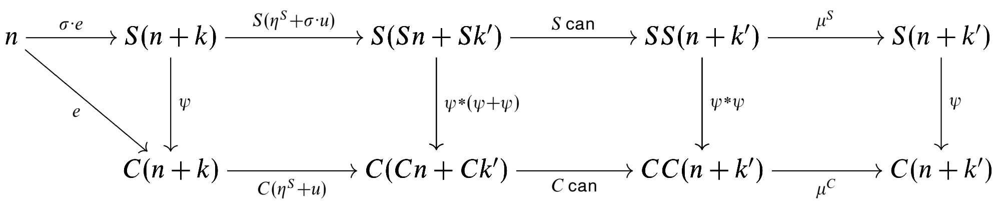
Given $e : n \longrightarrow C(n+k)$ and $u : k \longrightarrow Ck'$, we first relate

$$u \bullet e : n \longrightarrow C(n+k')$$

and

$$(\sigma_{k'}\cdot u) \bullet (\sigma_{n+k}\cdot e) : n \longrightarrow S(n+k').$$

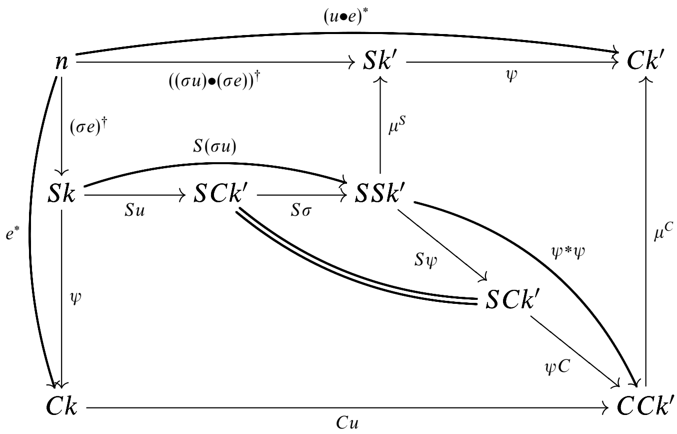
To this end, we consider the diagram



which commutes: the left-hand triangle commutes by (6.45); the right-hand square commutes since ψ is a monad morphism; the middle square commutes by the naturality of ψ; and for the left-hand square, we use this naturality, (6.45) and $\psi{\cdot}\eta^S = \eta^C$. The commutativity of the outside of the above diagram establishes that $\psi_{n+k'}{\cdot}((\sigma u)\bullet(\sigma e)) = u \bullet e$, and since ψ preserves the dagger, we obtain the equation

$$(u \bullet e)^* = \psi_{k'}{\cdot}((\sigma u) \bullet (\sigma e))^\dagger. \tag{6.50}$$

To see that the Parameter Identity holds for $*$, we now verify that the following diagram commutes:



The upper triangle commutes by (6.50). The upper left-hand square commutes by the Parameter Identity for $\mathbb{S}$. The left-hand triangle commutes since ψ is dagger-preserving, so we can apply (6.45). For the inner triangle, we use (6.45), and all other parts commute since ψ is a monad morphism.

— *Bekić Identity*:
Given

$$e : n \longrightarrow C(n+m+k)$$
$$f : m \longrightarrow C(n+m+k)$$

we form for $\sigma{\cdot}e$ and $\sigma{\cdot}f$ the morphisms e_L and e_R as in Remark 2.12. We also form, for e and f, the corresponding morphisms with respect to $\mathbb{C}$ and denote them by ε_L and ε_R, respectively. We clearly have, using (6.45) and (6.49),

$$(\psi{\cdot}\sigma{\cdot}e)^* = e^* = \psi{\cdot}(\sigma{\cdot}e)^\dagger.$$

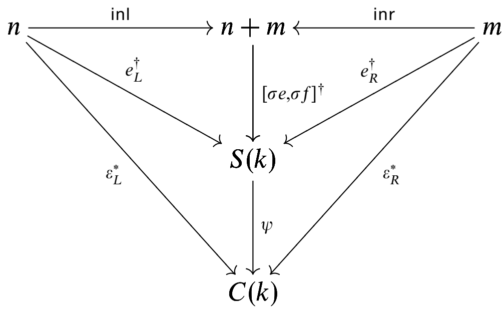
Thus, by Lemma 5.12 (1), we obtain $\psi_{m+k}{\cdot}e_R = \varepsilon_R$, and, since ψ is dagger preserving,

$$\varepsilon_R^* = \psi_k{\cdot}e_R^\dagger. \tag{6.51}$$

Then, by an application of Lemma 5.12 (2), we obtain $\psi_{n+k}{\cdot}e_L = \varepsilon_L$, and dagger preservation of ψ yields

$$\varepsilon_L^* = \psi_k{\cdot}e_L^\dagger. \tag{6.52}$$

We now see that the Bekić Identity for $\mathbb{S}$ implies that for $\mathbb{C}$, we have



The upper triangles follow from $\sigma{\cdot}[e, f] = [\sigma{\cdot}e, \sigma{\cdot}f]$ and the lower ones from (6.51) and (6.52). □

Remark 6.8. Notice that in the proof of *Functoriality* the naturality of $\sigma : \mathbb{C} \longrightarrow \mathbb{S}$ is essential, whereas it is not used in the proof of the other axioms. This accounts for the fact that *Functoriality* is not an axiom for iteration theories, where one works over the category **Sgn** of signatures, *cf.* Adámek *et al.* (2007). But for Elgot theories, *Functoriality* is an equational axiom (or rather, an infinite set of axioms) since we are working over the category $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$ of finitary endofunctors of $\mathscr{K}$ (or, equivalently, over the category $\mathscr{K}^{\mathscr{F}}$). We shall discuss this further in Appendix A below.

Corollary 6.9. Elgot monads are precisely the monadic algebras for the monad Rat on $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$.

Indeed, since U is monadic, we have an isomorphism between the categories of Elgot monads and of Rat-algebras:

$$\mathsf{EMnd}(\mathscr{K}) \cong \left(\mathscr{K}^{\mathscr{F}}\right)^{\mathsf{Rat}}.$$

As we mentioned in Example 2.17, continuous theories are Elgot theories. We derive from Corollary 6.9 that all equational laws that hold for $e \mapsto e^\dagger$ in Elgot theories can be derived from those laws that hold in all continuous theories. This is analogous to Bloom and Ésik (1993, Theorem 8.2.15), except that there the concept of equation relates to the category **Sgn** of signatures as the base category (so functoriality is not an equational property).

In the next corollary we use the phrase 'equational property' to mean the concept of equations from Kelly and Power (1993) related to the category $\mathbf{Set}^{\mathbb{F}}$ of sets in context as the base category. We recall these concepts briefly in the Appendix.

Corollary 6.10. The axioms of Elgot theories on **Set** precisely summarise all equational properties that the assignment

$$e^\dagger = \text{least solution of } e$$

has for continuous theories.

Proof. We use Max Kelly and John Power's results on the monad Rat on $\mathsf{Fin}(\mathbf{Set}, \mathbf{Set})$ (or, equivalently, on $\mathbf{Set}^{\mathbb{F}}$). We conclude that the algebras of Rat form an equational class for some signature Γ on $\mathbf{Set}^{\mathbb{F}}$. Every equation that holds in all continuous theories holds in the theories $\mathbb{T}_{\Sigma_\perp}$ of Example 2.14 (vi). Consequently, it holds in the theories $\mathbb{R}_{\Sigma_\perp}$ of rational $\Sigma_\perp$-trees, see Example 2.14 (vii), since for an equation morphism e that uses only rational trees the definition of $e^\dagger$ is the same as in $\mathbb{T}_{\Sigma_\perp}$. For every free algebra for Rat the same equation must again hold since by Example 6.3 (ii) these free algebras are quotients of $\mathbb{R}_{\Sigma_\perp}$. Consequently, that equation will hold in all algebras for Rat. □

7. Conclusions

This paper is the culmination of a line of research in which we deal with a category theoretic approach to the semantics of iteration. In a nutshell, the results presented here concern the question of what the essential equational properties of iteration operators are.

Calvin Elgot *et al.* proved in Elgot *et al.* (1978) that every signature Σ generates a free strict iterative theory: it is the theory $\mathbb{R}_{\Sigma_\perp}$ of rational $\Sigma_\perp$-trees. This was generalised in Adámek and Milius (2006a) to sets in context, that is, objects X of $\mathbf{Set}^{\mathbb{F}}$ (where $\mathbb{F}$ is the category of natural numbers and all functions): X generates a free strict iterative theory $\mathbb{R}_{X+1}$, which is a quotient theory of $\mathbb{R}_{\Sigma_\perp}$ for some signature Σ. We thus obtain a monad Rat on the category $\mathbf{Set}^{\mathbb{F}}$ of sets in context: this is the monad

$$\mathsf{Rat} : X \longmapsto \mathbb{R}_{X+1}$$

of free strict iterative theories. The aim of the current paper has been to describe the Eilenberg–Moore algebras for this monad. It is easy to see that these algebras have the form of pairs $(\mathscr{T}, \dagger)$ where $\mathscr{T}$ is a (Lawvere) theory and $e \longmapsto e^\dagger$ is a function of solutions of recursive equations e in $\mathscr{T}$. The substantial question is: what properties of $\dagger$ characterise the category $(\mathbf{Set}^{\mathbb{F}})^{\mathsf{Rat}}$ of Eilenberg–Moore algebras for Rat? Whatever these properties are, they are satisfied by the function

$$e^\dagger = \text{least solution of } e$$

in continuous theories. Indeed, the equational properties of $\dagger$ are *precisely* those that hold in all free algebras for Rat, which are the above mentioned theories $\mathbb{R}_{X+1}$. Since each of them is a quotient of some $\mathbb{R}_{\Sigma_\perp}$, it is sufficient to consider these theories (for all signatures). Observe that $\mathbb{R}_{\Sigma_\perp}$ is a subtheory of the free continuous theory $\mathbb{T}_{\Sigma_\perp}$ on Σ (which is a theory of all $\Sigma_\perp$-trees) and that least solutions for equation morphisms e in $\mathbb{R}_{\Sigma_\perp}$ are formed in the same way in $\mathbb{R}_{\Sigma_\perp}$ and $\mathbb{T}_{\Sigma_\perp}$. Consequently, the characterisation of the algebras for the monad Rat is a summary of the equational properties of the least-solution operation in continuous theories. From the work of Stephen Bloom and Zoltán Ésik, we then deduce that there are a number of other applications of iteration satisfying the same equational properties (Bloom and Ésik 1993).

We have proved that the Eilenberg–Moore algebras for the monad Rat are precisely the

Elgot theories.

This is our abbreviation of the concept 'iteration theory satisfying the functorial dagger implication' of Bloom and Ésik (1993). That is, we proved that the category of monadic algebras for Rat is isomorphic to the category EMnd(**Set**) of Elgot theories and morphisms.

Since $\mathbf{Set}^{\mathbb{F}}$ is a locally presentable category and Rat is a finitary monad, we deduce that Elgot theories have an equational presentation over $\mathbf{Set}^{\mathbb{F}}$ in the sense of Max Kelly and John Power (Kelly and Power 1993) (where convenient, we have used, in place of $\mathbf{Set}^{\mathbb{F}}$, the equivalent category Fin(**Set**, **Set**) of finitary set functors).

Previously, we proved an analogous result where the role of the base category was played by the category **Sgn** of signatures (which is the slice category $\mathbf{Set}/\mathbb{N}$) rather than $\mathbf{Set}^{\mathbb{F}}$: we have a monad Rat on **Sgn** given by $\Sigma \longmapsto \mathbb{R}_{\Sigma_\perp}$, that is, once again the monad of free strict iterative theories. The Eilenberg–Moore algebras for this monad are precisely the iteration theories (Adámek *et al.* 2007). Thus, the question of what all the equational properties of † in continuous theories are has two different answers depending on the perspective: from the perspective of **Sgn**, they are precisely the axioms of iteration theories (Bloom and Ésik 1993); from the perspective of $\mathbf{Set}^{\mathbb{F}}$, the equational properties are the (simpler) axioms of Elgot theories.

Appendix A. The Kelly-Power equational presentations

Assumption A.1. Throughout this appendix we use $\mathscr{A}$ to denote a locally finitely presentable category and $\mathscr{F}(\mathscr{A})$ to denote its full subcategory representing all finitely presentable objects. In this appendix, we recall some concepts and results from Kelly and Power (1993), and illustrate them for the category $\mathscr{A} = \mathbf{Set}^{\mathbb{F}}$ of sets in context. The coproduct of M copies of an object $A \in \mathscr{A}$ is denoted by $M \bullet A$.

Definition A.2. A *signature* Σ is a collection of objects of $\mathscr{A}$ indexed by $\mathscr{F}(\mathscr{A})$:

$$\Sigma = (\Sigma(n))_{n \in \mathscr{F}(\mathscr{A})}.$$

Examples A.3.

(1) When $\mathscr{A} = \mathbf{Set}$ we have $\mathscr{F}(\mathbf{Set})$ the full subcategory of natural numbers, which we denote by $\mathscr{F}$. Definition A.2 is the usual concept of a (finitary, one-sorted) signature. Observe that a Σ-algebra can be viewed as a set A together with, for every $n \in \mathbb{N}$, an assignment

$$\frac{n \xrightarrow{a} A}{\Sigma(n) \xrightarrow{\hat{a}} A},$$

which to every n-tuple (x_i) assigns the map $\sigma \longmapsto \sigma(x_i)_{i<n}$ for all $\sigma \in \Sigma_n$. Or, more compactly, an algebra is a set A together with a morphism

$$\alpha : \coprod_{n \in \mathscr{F}} A^n \times \Sigma(n) \longrightarrow A.$$

(2) Let $\mathscr{A} = \mathbf{Set}^S$ be the category of S-sorted sets. A finitely presentable object is an S-sorted set $\{X_s\}_{s\in S}$ such that $\coprod_{s\in S} X_s$ is finite. We can represent them by words

$$\mathscr{F}(\mathbf{Set}^S) = S^*$$

in the sense that the word $s_1 \dots s_k$ is the set whose sort s consists of

$$\{i = 1, \dots, k; s_i = s\}.$$

A signature Σ is then a collection of S-sorted sets $\Sigma(w)$ for $w \in S^*$: for $w = s_1 \dots s_n$ the sort t of $\Sigma(w)$ is the set of all operation symbols $s_1 \times \dots \times s_n \longrightarrow t$.
Again, a Σ-algebra is given by an S-sorted set A and an assignment, for every $w \in S^*$, of S-sorted maps

$$\frac{w \xrightarrow{a} A}{\Sigma(w) \xrightarrow{\hat{a}} A}.$$

(3) In the category

$$\mathscr{A} = \mathbf{Set}^{\mathbb{F}}$$

of sets in context, the finitely presentable objects are, as proved in Adámek *et al.* (2009b), precisely the *super-finitary* ones. That is, those sets in context X for which there exists a natural number n such that:

(a) $X(n)$ and $X(0)$ are finite.

(b) All elements of $X(k)$, $k \in \mathbb{N} - \{0\}$, have the form $Xf(t)$ for some $f : n \longrightarrow k$ and $t \in X(n)$.

Consequently,

$$\mathscr{F}(\mathbf{Set}^{\mathbb{F}})$$

denotes a set of representatives of all super-finitary sets in context.
A signature in $\mathbf{Set}^{\mathbb{F}}$ is a collection $\Sigma = (\Sigma_X)_{X \in \mathscr{F}(\mathbf{Set}^{\mathbb{F}})}$ of sets in context.

Definition A.4. We write Σ-*algebra* to mean an object A of $\mathscr{A}$ together with a morphism

$$\alpha : \coprod_{n \in \mathscr{F}} \mathscr{A}(n, A) \bullet \Sigma(n) \longrightarrow A.$$

This is just an algebra for the endofunctor $H_\Sigma : \mathscr{A} \longrightarrow \mathscr{A}$ defined by

$$H_\Sigma X = \coprod_{n \in \mathscr{F}} \mathscr{A}(n, X) \bullet \Sigma(n).$$

Remark A.5. In other words, a Σ-algebra consists of an object A and an assignment, for every $n \in \mathscr{F}(A)$, a function $\widehat{(-)} : \mathscr{A}(n, A) \longrightarrow \mathscr{A}(\Sigma(n), A)$:

$$\frac{n \xrightarrow{a} A}{\Sigma(n) \xrightarrow{\hat{a}} A}.$$

Example A.6.

(i) In $\mathbf{Set}^{\mathbb{F}}$, the pointed sets in context (that is pairs (A, a) where $A \in \mathbf{Set}^{\mathbb{F}}$ and a is a morphism from I, the inclusion functor $\mathscr{F} \hookrightarrow \mathbf{Set}$, to A) are Σ-algebras. We put

$$\Sigma(0) = I$$
$$\Sigma(n) = 0 \qquad \text{for all } n \neq 0,$$

where 0 denotes the constant presheaf on the empty set.

(ii) Binary algebras in $\mathbf{Set}^{\mathbb{F}}$ are pairs (A, α) where $A \in \mathbf{Set}^{\mathbb{F}}$ and $\alpha : A \times A \longrightarrow A$ is a natural transformation. These algebras can be represented as algebras for a signature as follows. Let Q_k be the (super-finitary) set in context given by $Q_k(n) = n^k$. Then, by the Yoneda Lemma, $\mathbf{Set}^{\mathbb{F}}(Q_k, A) \cong A^k$. Thus, we put

$$\Sigma(Q_2) = Q_1$$
$$\Sigma(n) = 0 \qquad \text{for } n \neq Q_2.$$

Definition A.7. A *homomorphism* from a Σ-algebra (A, α) to a Σ-algebra (B, β) is a morphism $f : A \longrightarrow B$ satisfying

$$f \cdot \hat{a} = \widehat{f \cdot a} \qquad \text{for all } a : n \longrightarrow A.$$

Equivalently, the square

$$\begin{array}{ccc} H_\Sigma A & \xrightarrow{\alpha} & A \\ {\scriptstyle H_\Sigma f}\downarrow & & \downarrow{\scriptstyle f} \\ H_\Sigma B & \xrightarrow[\beta]{} & B \end{array}$$

commutes. We use

$$\Sigma\text{-}\mathbf{Alg}$$

to denote the category of Σ-algebras and homomorphisms.

Remark A.8. The forgetful functor $\Sigma\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}$ has a left adjoint, which assigns to every object $X \in \mathscr{A}$ the *term algebra* $\mathbb{F}_\Sigma(X)$, which means the free H_Σ-algebra on X. Following Adámek (1974), we can describe it as a colimit

$$\mathbb{F}_\Sigma(X) = \operatorname*{colim}_{k \in \omega} W_k$$

of the following chain:

$$W_0 = X \qquad \text{(every variable is a term)}$$

and

$$W_{k+1} = X + H_\Sigma W_k$$

(corresponding to the rule that an n-tuple of terms and a symbol in Σ_n yield a new term). The connecting maps are

$$w_0 = \mathsf{inl} : X \longrightarrow X + H_\Sigma X$$

and

$$w_{k+1} = X + H_\Sigma w_k.$$

We use

$$\kappa : H_\Sigma \longrightarrow \mathbb{F}_\Sigma$$

to denote the universal map.

Definition A.9. We write an '*equation* for a signature Σ' to mean a parallel pair of morphisms

$$u, u' : p \longrightarrow \mathbb{F}_\Sigma(r) \qquad \text{for } p, r \in \mathscr{F}(\mathscr{A}).$$

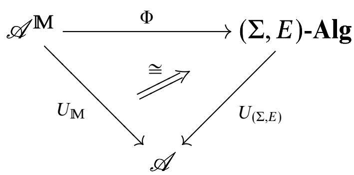
A Σ-algebra A *satisfies* the equation provided we have $h{\cdot}u = h{\cdot}u'$ for every homomorphism $h : \mathbb{F}_\Sigma(r) \longrightarrow A$.

Notation A.10. Given a set E of equations, we use (Σ, E)-**Alg** to denote the full subcategory of Σ-**Alg** formed by those Σ-algebras that satisfy every equation in E. And we denote the forgetful functor by

$$U_{(\Sigma,E)} : (\Sigma, E)\textbf{-Alg} \longrightarrow \mathscr{A}.$$

The following proposition follows from the results in Kelly and Power (1993).

Proposition A.11. The functor $U_{(\Sigma,E)}$ is finitary monadic. In more detail, there exists a finitary monad $\mathbb{M}$ on $\mathscr{A}$ such that for the forgetful functor $U_\mathbb{M} : \mathscr{A}^\mathbb{M} \longrightarrow \mathscr{A}$ of its Eilenberg–Moore category, we have an equivalence functor $\Phi : \mathscr{A}^\mathbb{M} \longrightarrow (\Sigma, E)$-**Alg** together with a natural isomorphism $U_\mathbb{M} \xrightarrow{\cong} U_{(\Sigma,E)}{\cdot}\Phi$



However, for our purposes, the main result is the converse.

Theorem A.12 (Kelly and Power 1993). Every finitary monad $\mathbb{M}$ on $\mathscr{A}$ has an equational presentation (Σ, E), that is, a signature Σ, a set E of equations and an equivalence functor $\Phi : \mathscr{A}^\mathbb{M} \longrightarrow (\Sigma, E)$-**Alg** with $U_\mathbb{M} \cong U_{(\Sigma,E)}{\cdot}\Phi$.

Example A.13. The category of all finitary monads in **Set** (or, equivalently, the category of Lawvere theories and theory morphisms) is monadic on $\mathbf{Set}^\mathbb{F}$, the category of sets in context, as mentioned in Example 6.6. Thus, there exists a signature Σ and a set $\mathscr{E}$ of equations describing finitary monads as Σ-algebras satisfying the given equations. Recall that $\mathbf{Set}^\mathbb{F}$ is equivalent to the category $\mathsf{Fin}(\mathbf{Set}, \mathbf{Set})$ of finitary set functors. A finitary monad is given by:

(a) a functor $A \in \mathsf{Fin}(\mathbf{Set}, \mathbf{Set})$;
(b) a natural transformation $\eta : \mathrm{Id} \longrightarrow A$; and
(c) a natural transformation $\mu : A{\cdot}A \longrightarrow A$ satisfying certain axioms.

Note that, since A is finitary, any natural transformation $\mu : A{\cdot}A \longrightarrow A$ can be equivalently presented by the m-indexed collection of maps $\widehat{(-)} : \mathbf{Set}^{\mathbb{F}}(m, A) \longrightarrow \mathbf{Set}^{\mathbb{F}}(m{\cdot}m, A)$ given by

$$(f : m \longrightarrow A) \qquad \longmapsto \qquad (\hat{f} = \mu{\cdot}(f * f) : m{\cdot}m \longrightarrow A),$$

where m ranges through all finitely presentable objects of $\mathsf{Fin}(\mathbf{Set}, \mathbf{Set})$ and f is an arbitrary natural transformation (*cf.* Remark A.5). Indeed, expressing A as a filtered colimit of finitely presentable objects m_i, with the colimit cocone $f_i : m_i \longrightarrow A$ for $i \in I$, and using the fact that A is finitary and that, since I is filtered, the embedding $\Delta : I \longrightarrow I \times I$ is cofinal, we can deduce that $A{\cdot}A$ is a colimit of $m_i{\cdot}m_i$ with colimit cocone $f_i * f_i = (f_i m_i){\cdot}(Af_i) : m_i{\cdot}m_i \longrightarrow A{\cdot}A$. Thus, all $f * f$ as above form a collectively epimorphic cocone, and giving $\mu : A{\cdot}A \longrightarrow A$ is equivalent to giving a collection of maps $\widehat{(-)}$ as above.

This leads us to the following signature Σ^{mon}:

$$\Sigma^{\mathsf{mon}}(m) = m{\cdot}m \qquad \text{for all } m \neq 0 \text{ (0 the initial object)}$$

and

$$\Sigma^{\mathsf{mon}}(0) = \mathrm{Id}_{\mathbf{Set}}\,.$$

Here a Σ-algebra consists of a finitary set functor A, a map

$$\frac{0 \longrightarrow A}{\mathrm{Id} \longrightarrow A}$$

representing a natural transformation $\eta : \mathrm{Id} \longrightarrow A$, and maps

$$\frac{m \longrightarrow A}{m{\cdot}m \longrightarrow A} \qquad (m \neq 0 \text{ finitely presentable})$$

representing μ (provided some equational properties hold). The set $\mathscr{E}^{\mathsf{mon}}$ of equations we need then guarantees that the above transformation maps represent a natural transformation from $A{\cdot}A$ to A and satisfy the monad axioms.

Example A.14. We will now illustrate the equations needed to represent the *Functoriality* of iteration theories, see Definition 2.11. We work here with the category $\mathscr{A} = (\Sigma^{\mathsf{mon}}, \mathscr{E}^{\mathsf{mon}})$**-Alg** of algebraic theories of the preceding example as the base category. For every pair n, m of natural numbers, we can form the free Lawvere theory on one generator (say g) representing a morphism from n to m, and use the notation

$$T_{g:n \longrightarrow m}.$$

This is clearly a finitely presentable object of $\mathscr{A}$.

Let Σ be the signature in $\mathscr{A}$ having value 0 (the initial algebraic theory) except for

$$\Sigma(T_{e:n \longrightarrow n+k}) = T_{e^\dagger:n \longrightarrow k} \qquad \text{for all } e : n \longrightarrow n+k.$$

Its polynomial functor assigns to every theory X the theory

$$\begin{aligned} H_\Sigma X &= \coprod_{n,k \in \mathbb{N}} \mathscr{A}\big(T_{e:n \longrightarrow n+k}, X\big) \bullet T_{e^\dagger:n \longrightarrow k} \\ &= \coprod_{n,k \in \mathbb{N}} X(n, n+k) \bullet T_{e^\dagger:n \longrightarrow k}. \end{aligned}$$

Its algebras are precisely the *preiteration theories* of Bloom and Ésik (1993), that is, theories X together with maps

$$\frac{e \in X(n, n+k)}{e^{\dagger} \in X(n,k)}$$

satisfying no axioms.

For every base morphism (function)

$$v : n \longrightarrow m \qquad \text{in } \mathbf{Set}$$

we now formulate an equation in the above signature Σ expressing functionality restricted to v, that is:

$$\begin{array}{ccc} n & \xrightarrow{e} & n+k \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v+\mathrm{id}} \\ m & \xrightarrow[f]{} & m+k \end{array} \qquad \text{implies} \qquad \begin{array}{ccc} n & & \\ {\scriptstyle v}\downarrow & \searrow^{e^{\dagger}} & k \\ m & \nearrow_{f^{\dagger}} & \end{array}$$

Our equation $u_v, u'_v : p \longrightarrow \mathbb{F}_{\Sigma}(r)$ works with the free theory p on one generator $g : n \longrightarrow k$,

$$p = T_g,$$

and with r given by the quotient

$$r = T_{e,f}/\approx$$

of the free theory on two generators $e : n \longrightarrow n+k$ and $f : m \longrightarrow m+k$ modulo the smallest congruence $\approx$ with

$$f \cdot v \approx (v + \mathrm{id}) \cdot e.$$

Before specifying u_v, u'_v, we observe that the congruence classes

$$[e] \in r(n, n+k)$$
$$[f] \in r(m, m+k)$$

yield in the theory

$$H_{\Sigma}(r) = \coprod_{i,j \in \mathbb{N}} r(i, i+j) \bullet T_{h^{\dagger} : i \longrightarrow j}$$

two coproduct injections

$$\mathsf{in}_e : T_{h^{\dagger} : n \longrightarrow k} \longrightarrow H_{\Sigma}(r)$$
$$\mathsf{in}_f : T_{h^{\dagger} : m \longrightarrow k} \longrightarrow H_{\Sigma}(r),$$

respectively. Hence, in that theory, we have the two parallel morphisms

$$n \xrightarrow{\mathsf{in}_e(h^{\dagger})} k \qquad \text{and} \qquad n \xrightarrow{v} m \xrightarrow{\mathsf{in}_f(h^{\dagger})} k$$

(recall that $v : n \longrightarrow m$ is a base morphism in every theory). Using the canonical morphism $\kappa_r : H_{\Sigma}(r) \longrightarrow \mathbb{F}_{\Sigma}(r)$ of Remark A.8, we obtain two parallel morphisms in $\mathbb{F}_{\Sigma}(r)$:

$$\kappa_r\big(\mathsf{in}_e(h^{\dagger})\big), \kappa_r\big(\mathsf{in}_f(h^{\dagger}) \cdot v\big) : n \longrightarrow k.$$

Our equation

$$u_v, u'_v : p = T_g \longrightarrow \mathbb{F}_\Sigma(r)$$

is given by the above two elements of $\mathbb{F}_\Sigma(r)(n,k)$ (as responses to the generator g). It is easy to see that a preiteration theory satisfies this equation if and only if *Functoriality* holds for the given base morphism v. The collection of all these equations indexed by all the base morphisms v yields the axiomatisation of *Functoriality*.

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