What are Iteration Theories?

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Summary. We prove that iteration theories can be introduced as algebras for the monad $\mathbb{R}$at on the category of signatures assigning to every signature Σ the rational-Σ-tree signature. This supports the result that iteration theories axiomatize precisely the equational properties of least fixed points in domain theory: $\mathbb{R}$at is the monad of free rational theories and every free rational theory has a continuous completion.

Key words: Iteration theory, rational theory, monad, Eilenberg-Moore algebra.

> "In the setting of algebraic theories enriched with an external fixed-point operation, the notion of an iteration theory seems to axiomatize the equational properties of all the computationally interesting structures of this kind."
>
> S. L. Bloom and Z. Ésik (1996), see [4]

1 Introduction

In domain theory recursive equations have a clear semantics given by the least solution. The function assigning to every system of recursive equations e the least solution $e^\dagger$ has a number of equational properties. One answer to the question in the title is given by a semantic characterization: iteration theories are those Lawvere theories in which recursive equations have solutions subject to all equational laws that the least-solution-map $e \mapsto e^\dagger$ obeys in domain theory. The same question also has a precise answer given by a list of all the equational axioms involved, see the fundamental monograph [3] of Stephen Bloom and Zoltan Ésik, or Definition 3.1 below. The aim of the present paper is to offer a short and precise syntactic answer:

* The support of the grant 201/06/664 of the Grant Agency of the Czech Republic is gratefully acknowledged.

Iteration theories are precisely the algebras for the rational-tree monad $\mathbb{R}\mathrm{at}$ on the category of signatures.

To be more specific: let **Sgn** be the category of signatures, that is, the slice category $\mathbf{Set}/\mathbb{N}$. Every signature Σ generates a free rational theory in the sense of the ADJ-group: it is the theory $\mathbf{RT}_{\Sigma_\perp}$ of all *rational trees* (which means trees having, up to isomorphism, only finitely many subtrees) on the signature $\Sigma_\perp = \Sigma + \{\perp\}$, for a new nullary symbol $\perp$. This follows from results in [7] and [11], and it yields a free-rational-theory monad

$$\mathbb{R}\mathrm{at}, \qquad \Sigma \mapsto \mathbf{RT}_{\Sigma_\perp}$$

on the category **Sgn** of signatures.

The main result of our paper is the following

Theorem. *The category of iteration theories is isomorphic to the category of Eilenberg-Moore algebras for the rational-tree monad* $\mathbb{R}\mathrm{at}$ *on* **Sgn**.

This theorem gives at first a new semantic characterization of iteration theories as a category of algebras. In addition, it is well-known that a characterization of a category as a category of Eilenberg-Moore algebras for a finitary monad on a locally finitely presentable category means that the category can be presented in the form of identities over the base category, see [8]. Hence, our result bears aspects of a syntactic characterization too.

It is well-known that classical varieties (of one-sorted, finitary algebras) are precisely the finitary, monadic categories over **Set**. We can generalize this to **Sgn** and speak about varieties of theories as categories that are finitary, monadic over **Sgn**. In this sense our main result proves that the iteration theories form a variety which is generated by all rational theories. Is this new? We believe it is—not in its spirit, but in its formal proof. In the monograph [3] the same result is formulated, see Theorem 8.2.5, however, the concept of a variety of theories is not introduced there. In later work, see, e.g., [4], [5], [6], varieties of preiteration theories are defined as classes of theories which satisfy given identities—however, the concept of identity is not defined. We have no doubt that it is possible to introduce the concept of "identity over **Sgn**" and then use it to provide a formal proof, but we decided for a different route using Beck's Theorem to prove the result above. The first (and very crucial) part of such a proof is explicitly contained in the monograph [3]: it is the fact that a free iteration theory on a signature Σ is the rational-tree theory $\mathbf{RT}_{\Sigma_\perp}$ above.

In our previous work we have dealt with iterative theories, e.g., by characterizing the Eilenberg-Moore algebras for the free iterative theories [1]. The present paper uses an analogous method but applied in a second-order setting: whereas Beck's Theorem is used over the given base category in [1], here we apply it to the category of signatures over the base category.

2 Rational Theories

The aim of the present section is to introduce the rational-tree monad $\mathbb{R}\mathrm{at}$ without using the result of [3] that $\mathbb{R}\mathrm{at}$ is the monad of free iteration theories: instead we use the free rational theories of [11].

2.1 Remark. Throughout the paper an (algebraic) *theory* is a category whose objects are natural numbers and every object n is a coproduct $n = 1+1+\cdots+1$ (n copies) with chosen coproduct injections. *Theory morphisms* are the functors which are identity maps on objects and which preserve finite coproducts (on the nose). The resulting category of theories is denoted by

$$\mathbf{Th}.$$

2.2 Examples. (1) Every signature $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ defines the *Σ-tree theory*

$$\mathbf{CT}_\Sigma$$

as follows.

By a *Σ-tree* on a set X of generators is meant a tree[3] labelled so that every leaf is labelled in $X + \Sigma_0$ and every node with $n > 0$ children is labelled in Σ_n. The theory $\mathbf{CT}_\Sigma$ has as morphisms $1 \to k$ all Σ-trees on k generators, thus:

$$\mathbf{CT}_\Sigma(n,k) = \text{all } n\text{-tuples of } \Sigma\text{-trees on } k \text{ generators.}$$

Composition is given by tree substitution.

(2) Analogously, the *finite-Σ-tree theory* $\mathbf{FT}_\Sigma$ is given by

$$\mathbf{FT}_\Sigma(n,k) = \text{all } n\text{-tuples of } \textit{finite } \Sigma\text{-trees on } k \text{ generators.}$$

(3) The theory

$$\mathscr{N}$$

which is the full subcategory of **Set** on natural numbers is initial: for every theory $\mathscr{T}$ we have a unique theory morphism $u_\mathscr{T} : \mathscr{N} \to \mathscr{T}$; we call the morphisms in $u_\mathscr{T}(\mathscr{N})$ *basic*.

2.3 Notation. We denote by

$$U\colon \mathbf{Th} \to \mathbf{Sgn}$$

the forgetful functor assigning to every theory $\mathscr{T}$ the signature $U(\mathscr{T})$ whose n-ary symbols are $\mathscr{T}(1,n)$ for all $n \in \mathbb{N}$.

[3] Trees are understood to be rooted, ordered, labelled possibly infinite trees that one considers up to isomorphism.

2.4 Remark. (i) U has a left adjoint

$$\mathbf{FT}\colon \mathbf{Sgn} \to \mathbf{Th}$$

assigning to every signature Σ the finite-tree theory $\mathbf{FT}_\Sigma$. This gives us a monad on the category of signatures:

$$\mathbb{F}\mathrm{in}\colon \Sigma \mapsto U(\mathbf{FT}_\Sigma).$$

Recall that Jean Bénabou [2] proved that U is *monadic*; that means that **Th** is isomorphic to the category of algebras for the monad $\mathbb{F}\mathrm{in}$.

(ii) We denote by

$$\omega\mathbf{CPO}$$

the category whose objects are posets with joins of ω-chains (a least element is not required; if an object has it we speak about a *strict* CPO). Morphisms are the *continuous* functions; they are monotone functions preserving joins of ω-chains. Morphisms between strict CPO's preserving the least element are called *strict continuous maps.*

2.5 Definition (see [11]). (1) *Theories enriched over* **Pos***, the category of posets and monotone functions, are called* ***ordered theories****. These are the theories with ordered hom-sets such that both composition and cotupling are monotone.*

(2) *An ordered theory is called* ***pointed*** *provided that every hom-set* $\mathscr{T}(n,k)$ *has a least element (notation:* $\bot_{nk}$ *or just* $\bot$*) and composition is left-strict, i.e.,* $\bot_{kr} \cdot f = \bot_{nr}$ *for all* $f \in \mathscr{T}(n,k)$*.*

(3) ***A continuous theory*** *is a pointed theory enriched over the category* $\omega\mathbf{CPO}$*, which means that both composition and cotupling preserve joins of* ω*-chains (but there is no condition on cotupling concerning* $\bot$*).*

2.6 Remark. (1) In an algebraic theory $\mathscr{T}$ *equation morphisms* are morphisms of the form

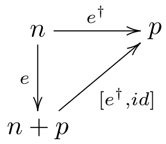
$$e\colon n \to n + p.$$

For example, if $\mathscr{T} = \mathbf{FT}_\Sigma$ then e represents a recursive system of n equations

$$\begin{aligned} x_1 &\approx t_1(x_1, \dots, x_n, z_1, \dots, z_p) \\ &\ \vdots \\ x_n &\approx t_n(x_1, \dots, x_n, z_1, \dots, z_p) \end{aligned} \tag{1}$$

where the right-hand sides are terms in $X + Z$ for $X = \{x_1, \dots, x_n\}$ and $Z = \{z_1, \dots, z_p\}$.

(2) A *solution* of an equation morphism $e\colon n \to n + p$ is a morphism $e^\dagger\colon n \to p$ such that the triangle



commutes. In case of (1) this is a substitution of terms $e^\dagger(x_i)$ for the given variables x_i such that the formal equations of (1) become identities in $\mathbf{FT}_\Sigma$. It is obvious that many systems (1) fail to have a solution in $\mathbf{FT}_\Sigma$ (because the obvious tree expansions are not finite).

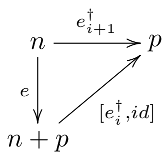
(3) In contrast, in continuous theories all equation morphisms have a solution. In fact, the *least* solution $e^\dagger$ always exists because the endofunction

$$x \mapsto [x, id]\cdot e \qquad \text{of } \mathscr{T}(n,p)$$

is continuous. By Kleene's Fixed-Point Theorem, $e^\dagger$ is the join of the following ω-chain of approximations:

$$e^\dagger = \bigsqcup_{i\in\mathbb{N}} e_i^\dagger : n \to p$$

where $e_0^\dagger = \perp$ and given $e_i^\dagger$ then $e_{i+1}^\dagger$ is the morphism



(2)

(4) Observe that the left-hand coproduct injection e in $\mathscr{T}(n, n+p)$ has the solution $e^\dagger = \perp$ in every continuous theory.

2.7 Example: the continuous theory $\mathbf{CT}_{\Sigma_\perp}$. Given a signature Σ we denote by

$$\Sigma_\perp$$

the extension of Σ by a nullary symbol $\perp \notin \Sigma$ (no ordering assumed a priori!). The theory $\mathbf{CT}_{\Sigma_\perp}$ of $\Sigma_\perp$-trees, see 2.2, carries a very "natural" ordering: given trees t and t' in $\mathbf{CT}_{\Sigma_\perp}(1,k)$ then $t \sqsubseteq t'$ holds iff t can be obtained from t' by cutting away some subtrees and labelling the new leaves by $\perp$. And the ordering of $\mathbf{CT}_{\Sigma_\perp}(n,k)$ is componentwise.

2.8 Notation. We denote by

$$\mathbf{CTh}$$

the category of all continuous theories and strict, continuous theory morphisms. Its forgetful functor $\mathbf{CTh} \to \mathbf{Sgn}$ is the domain restriction of U from Notation 2.3.

2.9 Theorem [11]. *For every signature Σ a free continuous theory on Σ is $\mathbf{CT}_{\Sigma_\perp}$. That is, the forgetful functor $\mathbf{CTh} \to \mathbf{Sgn}$ has a left adjoint given by $\Sigma \mapsto \mathbf{CT}_{\Sigma_\perp}$.*

2.10 Remark. Let $\mathscr{T}$ be a pointed ordered theory. For every equation morphism $e\colon n \to n+p$ we can form the morphisms $e_i^\dagger\colon n \to p$ as in 2.2 and we clearly obtain an ω-chain

$$e_0^\dagger \sqsubseteq e_1^\dagger \sqsubseteq e_2^\dagger \dots \qquad \text{in } \mathscr{T}(n,p).$$

We call these chains admissible and extend this to composites $e_i^\dagger \cdot v$ for morphisms $v\colon m \to n$:

2.11 Definition [11]. *In a pointed ordered theory $\mathscr{T}$ an ω-chain in $\mathscr{T}(m,p)$ is called **admissible** if it has the form $e_i^\dagger \cdot v$ $(i \in \mathbb{N})$ for some morphisms $e\colon n \to n+p$ and $v\colon m \to n$. The theory $\mathscr{T}$ is called **rational** if it has joins of all admissible ω-chains and if cotupling preserves these joins.*

2.12 Examples. (1) Every continuous theory is rational.

(2) The free continuous theory $\mathbf{CT}_{\Sigma_\perp}$ has a nice rational subtheory: the theory

$$\mathbf{RT}_{\Sigma_\perp} \quad \text{of all rational } \Sigma_\perp\text{-trees.}$$

Recall that a $\Sigma_\perp$-tree is called *rational* if it has up to isomorphism only finitely many subtrees, see [7]. It is easy to see that $\mathbf{RT}_{\Sigma_\perp}$ is a pointed subtheory of $\mathbf{CT}_{\Sigma_\perp}$.

2.13 Notation. We denote by

$$\mathbf{RTh}$$

the category of rational theories and order-enriched strict theory morphisms preserving least solutions. That is, given rational theories $\mathscr{T}$ and $\mathscr{R}$, a morphism is a theory morphism $\varphi\colon \mathscr{T} \to \mathscr{R}$ which (i) is monotone and strict on hom-sets and (ii) fulfils $\varphi(e^\dagger) = \varphi(e)^\dagger$ for all $e \in \mathscr{T}(n, n+p)$.

2.14 Proposition [11]. *A free rational theory on a signature Σ is the rational-tree theory $\mathbf{RT}_{\Sigma_\perp}$. More precisely, the forgetful functor*

$$W\colon \mathbf{RTh} \to \mathbf{Sig}$$

(a domain restriction of U in 2.3) has a left adjoint

$$\Sigma \mapsto \mathbf{RT}_{\Sigma_\perp}.$$

2.15 Corollary. *The monad*

$$\mathbb{R}\mathrm{at}$$

of free rational theories on the category **Sgn** *is defined by*

$$\Sigma \mapsto W(\mathbf{RT}_{\Sigma_\perp}).$$

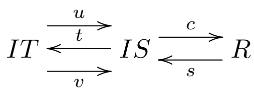
More precisely, to every signature Σ this monad assigns the signature whose n-ary operation symbols are the rational $\Sigma_\perp$-trees on n generators.

*We call $\mathbb{R}\mathrm{at}$ the **rational-tree** monad.*

3 Iteration Theories

In this section we first recall the definition of an iteration theory from [3] and then prove the main result: iteration theories are algebras for the rational-tree monad $\mathbb{R}\mathrm{at}$.

Our proof uses Beck's theorem characterizing categories of $\mathbb{T}$-algebras for a monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathscr{A}$. Recall that a $\mathbb{T}$-algebra is an object A of $\mathscr{A}$ together with a morphism $\alpha\colon TA \to A$ such that $\alpha{\cdot}\eta_A = id_A$ and $\alpha{\cdot}\mu_A = \alpha{\cdot}T\alpha$. The category $\mathscr{A}^{\mathbb{T}}$ of $\mathbb{T}$-algebras and homomorphisms (defined via an obvious commutative square in $\mathscr{A}$) is equipped with the forgetful functor $I\colon \mathscr{A}^{\mathbb{T}} \to \mathscr{A}$ given by $(A, \alpha) \mapsto A$. This functor is a right adjoint, and it *creates coequalizers of I-split pairs*. The latter means that for every parallel pair $u, v\colon T \to S$ in the category $\mathscr{A}^{\mathbb{T}}$ and every diagram



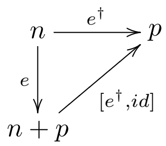
in $\mathscr{A}$ whose mappings satisfy the equations

$$cu = cv \tag{i}$$
$$cs = id \tag{ii}$$
$$ut = id \tag{iii}$$
$$vt = sc \tag{iv}$$

there exists a unique morphism $\bar{c}\colon S \to \overline{R}$ in $\mathscr{A}^{\mathbb{T}}$ with $I\bar{c} = c$, and moreover $\bar{c}$ is a coequalizer of u and v. Beck's theorem states that monadic algebras are characterized by the above two properties of the forgetful functor. More precisely, whenever a functor $I\colon \mathscr{B} \to \mathscr{A}$ is a right adjoint creating I-split coequalizers, then $\mathscr{B}$ is isomorphic to $\mathscr{A}^{\mathbb{T}}$ for the monad $\mathbb{T}$ given by the adjoint situation of I. See [9] for a proof.

3.1 Definition. (See [3], 6.8.1.) *An iteration theory is a theory $\mathscr{T}$ together with a function † assigning to every ("equation") morphism $e\colon n \to n + p$ a morphism $e^\dagger\colon n \to p$ in such a way that the following five axioms hold:*

(1) ***Fixed Point Identity****. This states that $e^\dagger$ is a solution of e, i.e., a fixed point of $[-, id_p]{\cdot}e$:*



(1)

(2) ***Parameter Identity.*** *We use the following notation: given an equation morphism $e\colon n \to n + p$, then every morphism $h\colon p \to q$ yields a new equation morphism*

$$h \bullet e \equiv n \xrightarrow{e} n + p \xrightarrow{id+h} n + q. \tag{2}$$

The parameter identity tells us how the solutions of e *and* $h \bullet e$ *are related: the triangle*

$$\begin{array}{ccc} n & \xrightarrow{e^\dagger} & p \\ & \searrow^{(h\bullet e)^\dagger} & \downarrow h \\ & & q \end{array} \tag{3}$$

commutes.

(3) ***Simplified Composition Identiy.*** *We use the following notation: given morphisms*

$$m \xrightarrow{g} n + p \qquad \text{and} \qquad n \xrightarrow{f} m$$

we obtain an equation morphism

$$f \circ g \equiv m \xrightarrow{g} n + p \xrightarrow{f+id} m + p. \tag{4}$$

The simplified composition identity states that the triangle

$$\begin{array}{ccc} n & \xrightarrow{(g\cdot f)^\dagger} & p \\ f \downarrow & \nearrow_{(f\circ g)^\dagger} & \\ m & & \end{array} \tag{5}$$

commutes.

(4) ***Double Dagger Identity.*** *This is a statement about morphisms of the form*

$$e \colon n \to n + n + p.$$

A solution yields $e^\dagger \colon n \to n+p$ *which we can solve again and get* $(e^\dagger)^\dagger \colon n \to p$. *On the other hand, the codiagonal* $\bigtriangledown \colon n+n \to n$ *yields an equation morphism* $\bigtriangledown \circ e \colon n \to n + p$. *The double-dagger identity states*

$$(\bigtriangledown \circ e)^\dagger = \left(e^\dagger\right)^\dagger \colon n \to p. \tag{6}$$

(5) ***Commutative identity.*** *This is in fact an infinite set of identities: one for every* m*-tuple of basic endomorphisms of* m*:*

$$\varrho_0, \ldots, \varrho_{m-1} \in \mathscr{N}(m, m)$$

and for every decomposition $m = n \cdot k$ *such that the corresponding codiagonal* $\bigtriangledown \colon \coprod_k n \to n$ *in* $\mathscr{N}$ *fulfils*

$$\bigtriangledown \cdot \varrho_j = \bigtriangledown \qquad \text{for } j = 0, \ldots, m-1.$$

The commutative identity concerns an arbitrary morphism

$$f : n \to m + p \quad \textit{in } \mathscr{T}.$$

We can form two equation morphisms: $\bigtriangledown \circ f : n \to n + p$ *(see* (4)*) and*

$$\hat{f} : m \to m + p$$

defined by the individual components $\hat{f} \cdot \mathrm{in}_j : 1 \to m + p$ *for* $j = 0, \ldots, m - 1$ *as follows:*

$$\hat{f} \cdot \mathrm{in}_j \equiv 1 \xrightarrow{\mathrm{in}_j} m \xrightarrow{\bigtriangledown} n \xrightarrow{f} m + p \xrightarrow{\varrho_j + id} m + p. \tag{7}$$

The conclusion is that the triangle

$$\begin{array}{ccc} m & \xrightarrow{\hat{f}^\dagger} & p \\ {\scriptstyle \bigtriangledown}\downarrow & \nearrow_{(\bigtriangledown \circ f)^\dagger} & \\ n & & \end{array} \tag{8}$$

commutes. (Remark: the notation in [3] *for* $\hat{f}$ *is* $\bigtriangledown \cdot f \| (\varrho_0, \ldots, \varrho_{m-1})$ *and instead of* $\bigtriangledown$ *a general surjective base morphism is assumed. The simplification working with* $\bigtriangledown$ *was proved in* [6]*.)*

3.2 Definition. *Let* $(\mathscr{T}, \dagger)$ *and* $(\mathscr{S}, \ddagger)$ *be iteration theories. A theory morphism* $\varphi : \mathscr{T} \to \mathscr{S}$ *is said to* ***preserve solutions*** *if for every morphism* $e \in \mathscr{T}(n, n+p)$ *we have* $\varphi(e)^\ddagger = \varphi(e^\dagger)$*. The category of iteration theories and solution-preserving morphisms is denoted by*

$$\mathbf{ITh}.$$

We denote by $V : \mathbf{ITh} \to \mathbf{Sgn}$ *the canonical forgetful functor (a restriction of* U *in* 2.3*).*

3.3 Example. The rational-tree theory $\mathbf{RT}_{\Sigma_\perp}$ is an iteration theory (for the choice $e^\dagger$ = the least solution of e). In fact, as proved in [3], Theorem 6.5.2, this is a free iteration theory on Σ. In other words:

3.4 Theorem. (See [3].) *The forgetful functor* $V : \mathbf{ITh} \to \mathbf{Sgn}$ *is a right adjoint and the corresponding monad is the rational-tree monad* $\mathbb{R}\mathrm{at}$.

3.5 Theorem. *The forgetful functor* $V : \mathbf{ITh} \to \mathbf{Sgn}$ *is monadic; that means that the category of iteration theories and solution preserving theory morphisms is isomorphic to the category of algebras for the rational-tree monad* $\mathbb{R}\mathrm{at}$ *on* $\mathbf{Sgn}$.

Proof. We are going to use Beck's theorem; due to Theorem 3.4 it is sufficient to verify that $V : \mathbf{ITh} \to \mathbf{Sgn}$ creates coequalizers of V-split pairs. From the result of Bénabou mentioned in Remark 2.4(i) we know that $U : \mathbf{Th} \to \mathbf{Sgn}$ is monadic, thus, it creates coequalizers of U-split pairs. Consequently, our task is as follows: given a parallel pair of solution-preserving morphisms

$$u, v\colon (\mathscr{T}, \ddagger) \to (\mathscr{S}, \dagger) \qquad \text{in } \mathbf{ITh},$$

and given a split coequalizer

$$\mathscr{T} \;\underset{Iv}{\overset{Iu}{\rightrightarrows}}\; \mathscr{S} \;\underset{s}{\overset{c}{\rightleftarrows}}\; \mathscr{R} \quad (t\colon \mathscr{S} \to \mathscr{T}) \qquad \text{in } \mathbf{Sgn},$$

where c is a coequalizer of u and v in $\mathbf{Th}$ and the above equations (i)–(iv) hold, then there exists a unique function $*\colon \mathscr{R}(n, n+p) \to \mathscr{R}(n, p)$ (for all $n, p \in \mathbb{N}$) such that

(a) c is solution preserving:

$$cf^{\dagger} = (cf)^{*} \qquad \text{for all } f \in \mathscr{S}(n, n+p), \tag{9}$$

(b) the axioms of Definition 3.1 hold for $*$, and

(c) c is a coequalizer of u and v in $\mathbf{ITh}$.

In fact, (a) determines $*$ as follows:

$$e^{*} = c(se)^{\dagger} \qquad \text{for all } e \in \mathscr{R}(n, n+p). \tag{10}$$

To see this, put $f = se$, then $cf = e$ due to (ii), thus

$$e^{*} = (cf)^{*} = c(f^{\dagger}) = c(se)^{\dagger}.$$

Conversely, by using (10) we get (9) for every morphism $f \in \mathscr{S}(n, n+p)$

$$\begin{aligned}
(cf)^{*} &= c(scf)^{\dagger} && (10)\\
&= c(vtf)^{\dagger} && \text{(iv)}\\
&= cv(tf)^{\ddagger} && v \text{ in } \mathbf{ITh}\\
&= cu(tf)^{\ddagger} && \text{(i)}\\
&= c(utf)^{\dagger} && u \text{ in } \mathbf{ITh}\\
&= cf^{\dagger} && \text{(iii)}
\end{aligned}$$

We now prove the axioms of iteration theories for $*$ and then we will get immediately (c):

Suppose that $\bar{c}\colon (\mathscr{S}, \dagger) \to (\overline{\mathscr{R}}, @)$ is a morphism of $\mathbf{ITh}$ with $\bar{c}u = \bar{c}v$. We have a unique theory morphism $r\colon \mathscr{R} \to \overline{\mathscr{R}}$ with $\bar{c} = rc$ in $\mathbf{Th}$ and we only need to prove that

$$r(e^{*}) = (re)^{@} \qquad \text{for all } e \in \mathscr{R}(n, n+p). \tag{11}$$

In fact, since $\bar{c}$ is a morphism of **ITh** we have by (10)

$$r(e^*) = rc(se)^\dagger = \bar{c}(se)^\dagger = \big(\bar{c}s(e)\big)^@$$

and it remains to verify $\bar{c}s = r$. This follows from c being an epimorphism since from (iv) and (iii) we get

$$\bar{c}sc = \bar{c}vt = \bar{c}ut = \bar{c} = rc.$$

Consequently, the theorem will be proved by verifying the individual axioms of iteration theories for the function from (10) above. Observe that since c is a theory morphism, it preserves $\circ$, see (4):

$$c(f \circ g) = (cf) \circ (cg). \tag{12}$$

(1) **Fixed Point Identity.** Given $e \in \mathscr{R}(n, n+p)$, we have, since c preserves coproducts:

$$\begin{aligned} e^* &= c(se)^\dagger && (10) \\ &= c\Big(\big[(se)^\dagger, id\big]\cdot se\Big) && (1) \\ &= \big[c(se)^\dagger, id\big]\cdot e && \text{(ii)} \\ &= [e^*, id]\cdot e && (10) \end{aligned}$$

(2) **Parameter Identity.** Given $e \in \mathscr{R}(n, n+p)$ and $h \in \mathscr{R}(p, q)$, then (ii) implies, since c preserves finite coproducts, the equality

$$h \bullet e = (id + h)\cdot e = c(sh \bullet se). \tag{13}$$

Therefore

$$\begin{aligned} (h \bullet e)^* &= c\big[s\cdot c(sh \bullet se)\big]^\dagger && \text{(10) and (13)} \\ &= \big[csc(sh \bullet se)\big]^* && (9) \\ &= \big[c(sh \bullet se)\big]^* && \text{(ii)} \\ &= c(sh \bullet se)^\dagger && (9) \\ &= c\big(sh\cdot(se)^\dagger\big) && (3) \\ &= h\cdot c(se)^\dagger && \text{(ii)} \\ &= h\cdot e^* && (10). \end{aligned}$$

(3) **Simplified Composition Identity.** Given morphisms $g \in \mathscr{R}(m, n + p)$ and $f \in \mathscr{R}(n, m)$, we have

$$
\begin{aligned}
(f \circ g)^* \cdot f &= \Big[c\big((sf) \circ (sg)\big)\Big]^* \cdot f && \text{(ii) and (12)} \\
&= c\big[(sf) \circ (sg)\big]^\dagger \cdot f && (9) \\
&= c\Big(\big[(sf) \circ (sg)\big]^\dagger \cdot sf\Big) && \text{(ii)} \\
&= c(sg \cdot sf)^\dagger && (5) \\
&= \Big[c\big((sg) \cdot (sf)\big)\Big]^* && (9) \\
&= (g \cdot f)^* && \text{(ii).}
\end{aligned}
$$

(4) **Double Dagger Identity.** Given $e \in \mathscr{R}(n, n+n+p)$, since $c(\triangledown) = \triangledown$ (recall that c preserves finite coproducts), we have

$$
\begin{aligned}
e^{**} &= \big(c(se)^\dagger\big)^* && (10) \\
&= c(se)^{\dagger\dagger} && (9) \\
&= c(\triangledown \circ se)^\dagger && (6) \\
&= (c\triangledown \circ cse)^* && \text{(9) and (12)} \\
&= (\triangledown \circ e)^* && \text{(ii) and } c\triangledown = \triangledown.
\end{aligned}
$$

(5) **Commutative Identity.** Given $\varrho_i \in \mathscr{N}(m, m)$ and $f \in \mathscr{R}(n, n+p)$ then first observe

$$c(\widehat{sf}) = \hat{f}. \tag{14}$$

In fact, c preserves coproducts and thus it maps base morphisms (in_j, $\triangledown$, ϱ_j etc.) of $\mathscr{S}$ to the corresponding base morphisms of $\mathscr{R}$. Thus (14) follows from (7). Consequently:

$$
\begin{aligned}
\hat{f}^* &= \big[c(\widehat{sf})\big]^* && (14) \\
&= c(\widehat{sf})^\dagger && (9) \\
&= c\big((\triangledown \circ sf)^\dagger \cdot \triangledown\big) && (8) \\
&= c(\triangledown \circ sf)^\dagger \cdot \triangledown && c\triangledown = \triangledown \\
&= \big[c(\triangledown \circ sf)\big]^* \cdot \triangledown && (9) \\
&= (\triangledown \circ f)^* \cdot \triangledown && \text{(ii) and (12)}
\end{aligned}
$$

This completes the proof. □

4 Conclusions and Future Research

The goal of our paper was to prove that iteration theories of Stephen Bloom and Zoltan Ésik are monadic over the category **Sgn** of signatures. This provides the possibility of using the corresponding monad $\mathbb{R}\mathrm{at}$ (of rational tree signatures) as a means for defining iteration theories. More important is the way our result supports the result that iteration theories precisely sum up the "equational properties" that the dagger function, assigning to every equation morphism e its least solution $e^\dagger$, satisfies in all continuous theories. In fact, since $\mathbb{R}$at is the monad of free rational theories, see [11], and every free rational theory has a solution-preserving completion to a continuous theory, it is obvious that all continuous theories and all rational theories satisfy precisely the same equational laws for $\dagger$. To make such statements precise, one can either define the concept of "equation over **Sgn**", or use instead finitary monads on **Sgn** (in analogy to the classical varieties over **Set**). We decided for the latter.

In the future we intend to study the analogous question where the base category is, in lieu of **Sgn**, the category of all finitary endofunctors of **Set**. We hope that the corresponding monadic algebras will turn out to be precisely the iteration theories that are parametrically uniform in the sense of Simpson and Plotkin [10].

References

1. J. Adámek, S. Milius and J. Velebil, Elgot Algebras, *Logical Methods in Computer Science*, Vol. 2(5:4) (2006), 1–31.
2. J. Bénabou, Structures algébriques dans les catégories, *Cah. Topol. Géom. Différ. Catég.* 10 (1968), 1–126.
3. S. L. Bloom and Z. Ésik, *Iteration Theories*, Springer Verlag, 1993.
4. S. L. Bloom and Z. Ésik, Fixed-point operations on ccc's, Part I, *Theoret. Comput. Sci.* 155 (1996), 1–38.
5. S. L. Bloom and Z. Ésik, The equational logic of fixed points, *Theoret. Comput. Sci.* 179 (1997), 1–60.
6. Z. Ésik, Axiomatizing iteration categories, *Acta Cybernetica* 14 (1999), 65–82.
7. S. Ginali, Regular trees and the free iterative theory, *J. Comput. Syst. Sci.* 18 (1979), 228–242.
8. G. M. Kelly and J. Power. Adjunctions whose units are coequalizers and presentations of finitary enriched monads. *J. Pure Appl. Algebra* 89(1993), 163–179.
9. S. MacLane, *Categories for the Working Mathematician*, second ed., Springer Verlag, 1998.
10. A. Simpson and G. Plotkin, Complete axioms for categorical fixed-point operators, *IEEE Symposium Logic in Computer Science* 1998, 30–41.
11. J. B. Wright, J. W. Thatcher, E. G. Wagner and J. A. Goguen, Rational algebraic theories and fixed-point solutions, *Proc. 17th IEEE Symposium on Foundations of Computing*, Houston, Texas, 1976, 147–158.