Electronic Notes in Theoretical Computer Science 65 No. 1 (2002)
URL: http://www.elsevier.nl/locate/entcs/volume65.html

Final Coalgebras And a Solution Theorem for Arbitrary Endofunctors

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Abstract

Every endofunctor F of Set has an initial algebra and a final coalgebra, but they are classes in general. Consequently, the endofunctor F^∞ of the category of classes that F induces generates a completely iterative monad T. And solutions of arbitrary guarded systems of iterative equations w.r.t. F exist, and can be found in naturally defined subsets of the classes TY.

More generally, starting from any category $\mathcal{K}$, we can form a free cocompletion $\mathcal{K}^\infty$ of $\mathcal{K}$ under small-filtered colimits (e.g., Set^∞ is the category of classes), and we give sufficient conditions to obtain analogous results for arbitrary endofunctors of $\mathcal{K}$.

Key words: initial algebra, final coalgebra, completely iterative monad

1 Introduction

In process algebra a system is often described in the form of equations

$$s = (s_1, a_1) \quad \text{or} \quad (s_2, a_2) \quad \text{or} \quad \ldots$$

where s, s_1, s_2, ... are states (from a desired state set S) and a_1, a_2, ... are actions (from a given set Act). Thus, the system is described by a labelled transition system

$$\sigma : S \longrightarrow \mathcal{P}(S \times \mathrm{Act})$$

assigning to every state s the set $\sigma(s)$ of all the possible pairs on the right-hand side. Thus σ represents a system of flat recursive equations, where "flat"

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[4] The support of the Grant Agency of the Czech Republic under the Grant No. 201/02/0148 is gratefully acknowledged.

refers to the fact that $\mathcal{P}$ appears just once, non-iterated, on the right-hand side. A "solution" of that system of equation is a description of the states of the system by the corresponding (extensional) trees, unique up to bisimilarity.

In a number of natural examples, non-flat equations play a rôle. For example the sequence

$$x \equiv 1, 1, 1, \ldots$$

of natural numbers can be presented in the form of the equation

$$x = (1, x).$$

Using the well-established set-theoretical notion for pairs, this means that

$$x = \{\{1\}, \{1, x\}\}$$

This has, for $S = \{x\}$, the form of the (non-flat) iterative equation

$$\sigma : S \longrightarrow \mathcal{P}\mathcal{P}(S + \{1\})$$

It is the aim of this paper to study equations of this kind, and to establish a general result on the existence and uniqueness of solutions.

In our previous work [AAV] and [AAMV] we have studied recursive equations for all "iteratable" endofunctors H of Set, i.e., all endofunctors such that $H(\_)+X$ has a final coalgebra for every set X. This, of course, excluded important functors such as $H = \mathcal{P}$. The same restriction has been considered by Larry Moss [M]. In the present paper we show that the previous result, namely that every guarded system of recursive equations has a unique solution, can be proved for *all* endofunctors H of Set. The trick is that we extend H to an endofunctor of

$$\mathsf{Class}$$

the category of classes and class functions, obtaining an essentially unique functor $H^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$ preserving small-filtered colimits (= large colimits which are λ-filtered for all small cardinals λ). Or, equivalently, to a set-based endofunctor H^∞ in the terminology of Aczel and Mendler [AM]; recall that by their Final Coalgebra Theorem, $H^\infty(\_)+X$ has a final coalgebra, see also [HL]. Then H^∞ is iteratable, and we can thus use the previous results, just moving from sets to classes. But even better: no concrete system of iterative equations actually requires this move from Set to Class! For example, the power-set functor $\mathcal{P}$ is iteratable only when extended to $\mathcal{P}^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$ (the functor assigning to every class the class of all sub*sets*). A final coalgebra of $\mathcal{P}^\infty$ is the coalgebra $B/\!\sim$ where

B is the coalgebra of all extensional trees

and

$\sim$ is the bisimilarity equivalence on B (which we describe in Section 5 below).

Now B is, of course, a proper class and so is $B/\!\sim$ (since a final coalgebra is a fixed point, by Lambek's Lemma, but $\mathcal{P}$ has no fixed points in Set). However, every system of equations (with a set of variables) has a unique solution that lives in a natural small subcoalgebra of B. This is so because every transition system is λ-branching for some cardinal number λ. Thus, it is an iterative equation morphism

$$\sigma : X \longrightarrow \mathcal{P}_\lambda X$$

for the functor $\mathcal{P}_\lambda$ of all subsets of cardinality less than λ. And $\mathcal{P}_\lambda$ is iteratable (in Set) with final coalgebra which is a natural subcoalgebra of that of $\mathcal{P}$. The morale of this is: for every transition system one has a unique solution in $B/{\sim}$, and the solution also lives in a small subcoalgebra (which one can ignore unless one objects to classes too much).

All this has nothing to do with $\mathcal{P}$. We prove that for every endofunctor $H : \mathsf{Set} \longrightarrow \mathsf{Set}$ there is a natural iteratable extension $H^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$. And if $T^\infty Y$ denotes a final coalgebra of $H^\infty(\_) + Y$, then every guarded equation system of H^∞ with parameters in Y has a unique solution in $T^\infty Y$.

Now all this has nothing to do with Set either! For every cocomplete category $\mathcal{K}$ we construct an extension $\mathcal{K}^\infty$ of $\mathcal{K}$ such that every endofunctor H of $\mathcal{K}$ naturally extends to an iteratable endofunctor H^∞ of $\mathcal{K}^\infty$. Thus, guarded equation morphisms have unique solutions in $\mathcal{K}^\infty$.

The above case of non-labelled transition systems was one of the motivations for the introduction of non-well-founded set theory. Our paper could thus be considered as a continuation of the program of Michael Barr [B] of deleting non-well-foundedness from process algebra. There is no question that there is a certain loss of elegance in the process, but we feel that the loss is less heavy than expected. We return to this question in Section 5.

Set-Theoretical Assumptions

We have, essentially, just one, standard, assumption: that a universe of "small" sets has been chosen, so that we can form the category of all small sets. Now assuming that the universe itself is a (non-small) set in some higher universe, we can denote by

$$\aleph_\infty$$

the cardinality of that set. This enables us to identify

small sets with sets of cardinality less than $\aleph_\infty$

and

classes as sets with cardinality at most $\aleph_\infty$.

More precisely, for a set theorist, the universe of small sets can be the $\aleph_\infty$-th member $V(\aleph_\infty)$ of the cumulative hierarchy. However, we will take as

$$\mathsf{Set}$$

the category of all sets of cardinality less than $\aleph_\infty$ (equivalent to $V(\aleph_\infty)$). And we take as

$$\mathsf{Class}$$

the category of all sets of cardinality less than or equal to $\aleph_\infty$.

We call a category $\mathcal{K}$ *locally small* if all objects form a class and every hom-set $\mathcal{K}(A, B)$ is small.

2 Solution Theorem for Iteratable Functors

In the present section we recall results obtained independently by Larry Moss in [M] and our group [AAV], [AAMV]. Throughout this section, $\mathcal{K}$ denotes a category with binary coproducts.

Definition 2.1 A functor $H : \mathcal{K} \longrightarrow \mathcal{K}$ is called *iteratable* provided that for every object X of $\mathcal{K}$ a final coalgebra

$$TX$$

of the functor $H(\_) + X$ exists.

Examples 2.2

(i) Every polynomial endofunctor H_Σ of Set is iteratable. Here Σ is a (possibly infinitary) signature, i.e., a set of operation symbols σ with prescribed arities $\mathrm{ar}(\sigma)$, which are cardinal numbers. And H_Σ assigns to every set X the coproduct

$$\coprod_{\sigma \in \Sigma} X^{\mathrm{ar}(\sigma)}.$$

Here TX is the coalgebra of all (finite or infinite) Σ-labelled trees over X. That is, trees with leaves labelled by nullary operation symbols or variables from X, and inner nodes (of n children) labelled by n-ary operation symbols.

(ii) More generally, every accessible (=bounded) endofunctor of Set is iteratable.

(iii) The power-set functor $\mathcal{P} : \mathsf{Set} \longrightarrow \mathsf{Set}$ is not iteratable.

Notation 2.3 By Lambek's Lemma, the structure arrow $TX \longrightarrow HTX + X$ of the final coalgebra TX is an isomorphism. That is, TX is a coproduct of HTX and X. We denote by

$$\tau_X : HTX \longrightarrow TX \quad (\text{“}TX \text{ is an } H\text{-algebra”})$$

and

$$\eta_X : X \longrightarrow TX \quad (\text{“}TX \text{ contains } X\text{”})$$

the coproduct injections.

Substitution Theorem 2.4 For every morphism $s : X \longrightarrow TY$ in $\mathcal{K}$ there exists a unique extension to a homomorphism $\widehat{s} : TX \longrightarrow TY$ of H-algebras. That is, a unique homomorphism with $s = \widehat{s}\eta_X$.

For a proof see either 2.4 in [M] or 2.11 in [AAV] (somewhat improved by 2.17 in [AAMV]).

Corollary 2.5 *The formation of TX (for all objects X) and $\widehat{s}$ (for all morphisms $s : X \longrightarrow TY$) is a Kleisli triple. The corresponding monad (T, η, μ) has*

$$\mu_X = \widehat{id_{TX}} : TTX \longrightarrow TX \quad \textit{for all objects } X.$$

This monad T is called the *completely iterative monad generated by H.*

Definition 2.6 By an (iterative) *equation morphism* with object X of variables and object Y of parameters is meant a morphism $e : X \longrightarrow T(X + Y)$.

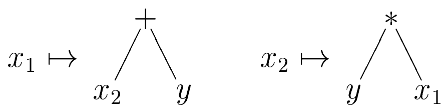
Example 2.7 Let Σ be the signature of two binary operations $+$ and $*$. The iterative system of equations

$$\begin{aligned} x_1 &= x_2 + y \\ x_2 &= \ y * x_1 \end{aligned} \tag{1}$$

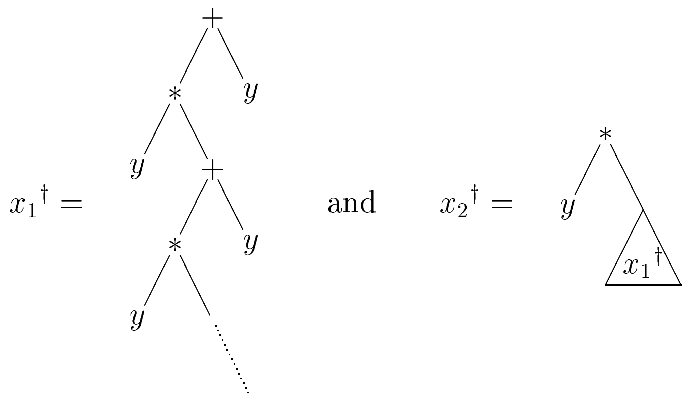
corresponds to the morphism

$$e : \{x_1, x_2\} \longrightarrow T\{x_1, x_2, y\}$$

defined by



This system has a unique solution ${x_1}^\dagger$, ${x_2}^\dagger$, viz,



The solution defines a morphism

$$e^\dagger : X \longrightarrow TY$$

with the following property: if we substitute in the right-hand sides of (1) each x_i by ${x_i}^\dagger$ (and the parameter y by the corresponding tree $\eta_Y(y)$), then $e^\dagger$ is just e with that substitution.

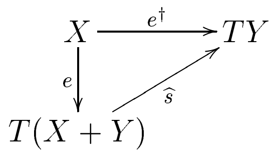
The substitution morphism is

$$s = [e^\dagger, \eta_Y] : X + Y \longrightarrow TY$$

and we extend it, using the Substitution Theorem, to

$$\widehat{s} : T(X + Y) \longrightarrow TY.$$

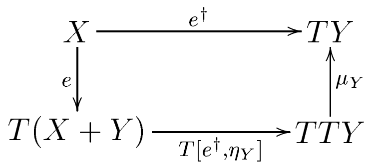
Thus, solutions $e^\dagger$ are morphisms defined by the property that the following triangle



commutes.

Now in every monad we have $\widehat{s} = \mu_Y Ts$, thus, we are led to the following

Definition 2.8 By a *solution* of an equation morphism $e : X \longrightarrow T(X + Y)$ is meant a morphism $e^\dagger : X \longrightarrow TY$ such that the following square



commutes.

Remark 2.9 Some trivial iteration equations, e.g., $x = x$, have many solutions. But "almost" all systems of iterative equations turn out to have a unique solution. The cases we want to exclude are the equations $x = x'$ where the right-hand side is a variable from X. Now given an equation morphism $e : X \longrightarrow T(X + Y)$ recall that $T(X + Y)$ is a coproduct of $HT(X + Y)$ and $X + Y$—thus, it is a coproduct of

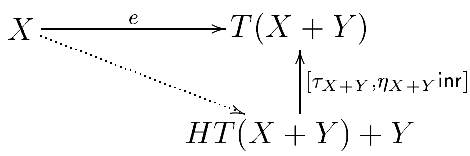
X with injection $X \xrightarrow{\mathsf{inl}} X + Y \xrightarrow{\eta_{X+Y}} T(X + Y)$

and

$HT(X + Y) + Y$ with injection $[\tau_{X+Y}, \eta_{X+Y}\mathsf{inr}] : HT(X + Y) + Y \longrightarrow T(X + Y)$.

It is the first injection that we want to exclude. More precisely, we want e to factorize through the latter one:

Definition 2.10 An equation morphism $e : X \longrightarrow T(X+Y)$ is called *guarded* provided that it factorizes through the coproduct injection $HT(X+Y)+Y \longrightarrow T(X + Y)$:



Solution Theorem 2.11 Every guarded equation morphism has a unique solution.

For the proof see 2.11 in [M] or 3.3 in [AAV] (much improved by 3.4–3.8 in [AAMV]).

Remark 2.12 In particular, every accessible endofunctor of Set (and, more generally, of any locally presentable category) is iteratable, see [AAMV].

3 All Functors Have Initial and Final (Co)Algebras

In the present section we prove that every endofunctor F of Set has an initial F-algebra and a final F-coalgebra, but these can be classes. More precisely, we expand the category Set to the category Class of classes and class functions. Then every functor $F : \mathsf{Set} \longrightarrow \mathsf{Set}$ has a unique extension to a small-accessible functor $F^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$ (see 3.1 and 3.6 below for definitions), and both an initial F^∞-algebra I and a final F^∞-coalgebra T exist. Besides, T is determined by finality w.r.t. all (small) F-algebras in Set.

All this is true for general categories $\mathcal{K}$ satisfying the following assumptions

(1) $\mathcal{K}$ has small colimits (i.e., $\mathcal{K}$ is cocomplete)

(2) $\mathcal{K}$ is (small) cowellpowered

and

(3) $\mathcal{K}$ is locally small (i.e., the objects of $\mathcal{K}$ form a class and the hom-sets $\mathcal{K}(A, B)$ are small sets for all objects A, B of $\mathcal{K}$).

We form a free cocompletion

$$\mathcal{K}^\infty$$

of $\mathcal{K}$ w.r.t. small-filtered colimits (see 3.1). The cocompletion $\mathcal{K}^\infty$ can be described (analogously to the free cocompletion $\mathrm{Ind}(\mathcal{K})$ w.r.t. filtered colimits of Grothendieck [AGV]) as a "suitable" category of all small-filtered diagrams in $\mathcal{K}$. The main example is $\mathsf{Class} = \mathsf{Set}^\infty$, see 3.7.

Then every endofunctor F of $\mathcal{K}$ extends, uniquely up to natural isomorphism, to a small-accessible (see 3.1) endofunctor F^∞ of $\mathcal{K}^\infty$, and F^∞ has an initial algebra and a final coalgebra. There is a substantial difference between the two: for an initial F^∞-algebra, I, we have a formula

$$I = \operatorname*{colim}_{i \in \mathrm{Ord}} F^{(i)} 0$$

naturally expanding the well-known formula

$$I = \operatorname*{colim}_{n \in \omega} F^{(n)} 0 \quad \text{for } F \ \omega\text{-cocontinuous.}$$

That is, we iterate F on an initial object, 0, $\aleph_\infty$-many times (where, recall, $\aleph_\infty$ is the first large ordinal, thus, $\aleph_\infty$, as a well-ordered class, is precisely the same as the class Ord of all small ordinals), we obtain an initial F^∞-algebra.

In contrast, the formula

$$T = \lim_{n\in\omega} F^{(n)}1 \quad \text{for } F\ \omega\text{-continuous}$$

does *not* extend to $T = \lim_{i\in \mathrm{Ord}} F^{(i)}1$. This has two reasons: the transfinite limit does not necessarily exist, and if it does, it need not be a terminal F-coalgebra. However, for $\mathcal{K} = \mathsf{Set}$ we use the ideas of James Worell [W] to show that by forming a limit

$$F^{(\aleph_\infty)}1 = \lim_{i<\aleph_\infty} F^{(i)}1$$

(albeit ouside of Class), the next $\aleph_\infty$ steps

$$F^{(\aleph_\infty+1)}1 = F(F^{(\aleph_\infty)}1), \dots, F^{(\aleph_\infty+i+1)}1 = F(F^{(\aleph_\infty+i)}1), \dots$$

yield a transfinite chain of subsets

$$F^{(\aleph_\infty)}1 \supseteq F^{(\aleph_\infty+1)}1 \supseteq \dots \supseteq F^{(\aleph_\infty+i)}1 \supseteq \dots$$

such that the correct formula for a final F-coalgebra is

$$T = \lim_{i<\aleph_\infty+\aleph_\infty} F^{(i)}1 = \bigcap_{i\in\mathrm{Ord}} F^{(\aleph_\infty+i)}1.$$

3.1 Free Cocompletion Under Small-Filtered Colimits

Recall the concept of a λ-filtered category, for a given infinite cardinal λ: it is a category $\mathcal{D}$ such that every (non-full) subcategory on less than λ morphisms has a cocone in $\mathcal{D}$. Colimits of diagrams with λ-filtered domains are called *λ-filtered colimits*. Basic example: a colimit of a λ-chain. And functors preserving λ-filtered colimits are called *λ-accessible*.

Definition 3.1 A category $\mathcal{D}$ is called *small-filtered* if it has a class of morphisms, and every small subcategory of $\mathcal{D}$ has a cocone in $\mathcal{D}$; that is, $\mathcal{D}$ is λ-filtered for all small cardinals λ.

Colimits of diagrams with small-filtered domains are called *small-filtered colimits*.

A functor preserving small-filtered colimits is called *small-accessible*.

Example 3.2 The well-ordered category Ord of all small ordinals is small-filtered. Thus, a small-accessible functor preserves colimits of transfinite chains.

As a concrete example of a small-accessible functor, consider the usual extension of the power-set functor $\mathcal{P} : \mathsf{Set} \longrightarrow \mathsf{Set}$ to the power-set functor

$$\mathcal{P}^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$$

assigning to every class X the class $\mathcal{P}^\infty X$ of all sub*sets* of X.

Remark 3.3 Peter Aczel and Nax Mendler [AM] call an endofunctor F of Class *set-based* provided that for every element of F, $x \in FX$, there exists a small subset $m : Y \longrightarrow X$ of the class X such that x lies in the image of $Fm : FY \longrightarrow FX$. This is equivalent to F being small-accessible, see the argument in [AP] for "bounded=accessible".

Notation 3.4 Let $\mathcal{K}$ be any category. We denote by

$$E : \mathcal{K} \longrightarrow \mathcal{K}^\infty$$

a free cocompletion of $\mathcal{K}$ under small-filtered colimits.

Explicitly: $\mathcal{K}^\infty$ is a category having small-filtered colimits and E is a full embedding with the following universal property:

> for every functor $F : \mathcal{K} \longrightarrow \mathcal{L}$ where $\mathcal{L}$ has small-filtered colimits there exists a small-accessible extension $F' : \mathcal{K}^\infty \longrightarrow \mathcal{L}$ of F, unique up to a natural isomorphism.

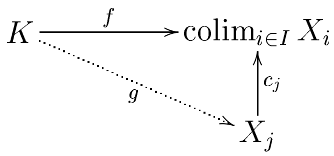
Remark 3.5

(a) Every object K of $\mathcal{K}$ is *small-presentable* in $\mathcal{K}^\infty$. This means that for every morphism

$$f : K \longrightarrow \operatorname*{colim}_{i \in I} X_i$$

from K into a small-filtered colimit in $\mathcal{K}^\infty$ (with a colimit cocone $c_j : X_j \longrightarrow \operatorname{colim}_{i \in I} X_i$) we have that

(i) f factorizes through some c_j:



(ii) the factorization is essentially unique, i.e., given $g' : K \longrightarrow X_j$ with $f = c_j \cdot g'$ then there exists a morphism $x_{jk} : X_j \longrightarrow X_k$ of the given diagram with

$$x_{jk} \cdot g = x_{jk} \cdot g'.$$

Conversely, every small-presentable object K of $\mathcal{K}^\infty$ is a retract of an object of $\mathcal{K}$. Thus, whenever idempotents split in $\mathcal{K}$, then small-presentable objects of $\mathcal{K}^\infty$ are precisely those isomorphic to objects of $\mathcal{K}$.

(b) The universal property of $\mathcal{K}^\infty$ mentioned above can be restated as follows: the functor category $[\mathcal{K}, \mathcal{L}]$ is equivalent to the full subcategory $[\mathcal{K}^\infty, \mathcal{L}]_{\mathrm{sacc}}$ of $[\mathcal{K}^\infty, \mathcal{L}]$ formed by all small-accessible functors under the equivalence functor

$$(\_) \cdot E : [\mathcal{K}^\infty, \mathcal{L}]_{\mathrm{sacc}} \longrightarrow [\mathcal{K}, \mathcal{L}]$$

This explains the following extension of the above notation.

Notation 3.6 Let $\mathcal{K}$ be a locally small category. For every functor $F : \mathcal{K} \longrightarrow \mathcal{K}$ we denote by

$$F^\infty : \mathcal{K}^\infty \longrightarrow \mathcal{K}^\infty$$

the (essentially unique) extension of $F \cdot E : \mathcal{K} \longrightarrow \mathcal{K}^\infty$ to a small accessible endofunctor. For every natural transformation

$$f : F \longrightarrow G \quad \text{in } [\mathcal{K}, \mathcal{K}]$$

we denote by

$$f^\infty : F^\infty \longrightarrow G^\infty \quad \text{in } [\mathcal{K}^\infty, \mathcal{K}^\infty]$$

the unique natural transformation extending $E \cdot f$, i.e., such that

$$f^\infty \cdot E = E \cdot f.$$

Examples 3.7

(i) $\mathsf{Set}^\infty = \mathsf{Class}$. In fact firstly, Class has small-filtered colimits, in fact, all class-indexed colimits. (This is obvious: a coproduct of a class of classes is a class, since $(\aleph_\infty)^2 = \aleph_\infty$, and coequalizers also clearly exist.)

Next, let $F : \mathsf{Set} \longrightarrow \mathcal{L}$ be a functor, where $\mathcal{L}$ has small-filtered colimits. For every class X form the small-filtered diagram $D_X : \mathcal{D}_X \longrightarrow \mathsf{Set}$ of all small subsets A of X and all inclusion functions, and choose a colimit $F'X$ of $F \cdot D_X$ with a colimit cocone

$$c_{A,X} : FA \longrightarrow F'X \quad (A \text{ in } D_X)$$

In case X is small, it is the largest element of D_X and we choose $F'X = FX$ and $c_{A,X} = F(A \longrightarrow X)$.

For every morphism $f : X \longrightarrow Y$ in Class denote by $F'f : F'X \longrightarrow F'Y$ the unique morphism of $\mathcal{L}$ such that for every set $A \subseteq X$ with image $B = f[A]$ in Y and domain-codomain restriction $f_0 : A \longrightarrow B$ the following square

$$\begin{array}{ccc} FA & \xrightarrow{c_{A,X}} & F'X \\ {\scriptstyle Ff_0}\downarrow & & \downarrow{\scriptstyle F'f} \\ FB & \xrightarrow[c_{B,Y}]{} & F'Y \end{array}$$

commutes. It is easy to verify that this defines a functor $F' : \mathsf{Class} \longrightarrow \mathcal{L}$ which preserves small-filtered colimits. Obviously, F' extends F, and is unique up to a natural isomorphism. Thus, Class is a free cocompletion of Set under small-filtered colimits.

(ii) An analogous description can be provided for the cocompletions $\mathcal{K}^\infty$ of other "everyday-life" categories. E.g., if $\mathcal{K} = \mathsf{Pos}$ is the category of small posets and order-preserving maps, then

$$\mathsf{Pos}^\infty$$

is the category of all partially ordered classes and order-preserving maps. The argument is analogous to Class above. Or for $\mathcal{K} = \mathsf{Cpo}$, the category of all small posets with directed joins and continuous (= directed-joins-preserving) maps we have

$$\mathsf{Cpo}^\infty$$

the category of partially ordered classes having joins of directed subsets, and functions preserving such joins.

(iii) Let Ord^+ be the well-ordered category of (a) all small ordinals and (b) a largest object, $\top$. Then $(\mathrm{Ord}^+)^\infty$ is the extension of Ord^+ by a new element, u, satisfying

$$i < u < \top \quad \text{for all } i \in \mathrm{Ord}.$$

Lemma 3.8 *Every locally small, cowellpowered category $\mathcal{K}$ with small colimits is closed under small colimits in $\mathcal{K}^\infty$, and $\mathcal{K}^\infty$ has class-indexed colimits (i.e., colimits with at most $\aleph_\infty$ morphisms in the diagram scheme) and arbitrary multiple pushouts of epimorphisms.*

Proof. The first statement is trivial, since objects of $\mathcal{K}$ are small-presentable in $\mathcal{K}^\infty$ (see Remark 3.5(a) and recall that in small-cocomplete categories idempotents split). The second statement requests just showing that $\mathcal{K}^\infty$ has small colimits: since it has small-filtered colimits, it has, then, class-indexed colimits (given a class-indexed diagram D, consider the small-filtered colimit of the diagram of colimits of all small subdiagrams of D; this is a colimit of D).

The existence of small coproducts in $\mathcal{K}^\infty$ is evident since objects of $\mathcal{K}^\infty$ are small-filtered colimits of objects of $\mathcal{K}$: given a small collection of small-filtered diagrams $D_i : \mathcal{D}_i \longrightarrow \mathcal{K}^\infty$ $(i \in I)$, form the small-filtered diagram

$$\prod \mathcal{D}_i \xrightarrow{\prod D_i} \mathcal{K}^I \longrightarrow \mathcal{K},$$

where the second part is taking coproducts in $\mathcal{K}$. Its colimit is the coproduct of $\operatorname{colim} D_i$ in $\mathcal{K}^\infty$. Analogously with coequalizers: given a parallel pair $f, g : \operatorname{colim} D \longrightarrow \operatorname{colim} D'$ in $\mathcal{K}^\infty$, where D, D' are small-filtered in $\mathcal{K}$, we can find natural transformations $f_i, g_i : Di \longrightarrow D'j_i$ in $\mathcal{K}$ with $f = \operatorname{colim} f_i$ and $g = \operatorname{colim} g_i$. By forming coequalizers $c_i : D'j_i \longrightarrow D''i$ in $\mathcal{K}$ we obtain a small-filtered diagram D'' in $\mathcal{K}$ and a natural transformation $(c_i) : D' \longrightarrow D''$. It is easy to see that $\operatorname{colim} c_i$ is a coequalizer of f and g.

The existence of multiple pushouts of epimorphisms is proved analogously to the proof that locally presentable categories are cowellpowered, see Theorem 2.14 of [GU]. ☐

Remark 3.9 Every F-coalgebra is also an F^∞-coalgebra (since $FA = F^\infty A$ for all $A \in \mathcal{K}$). And every F^∞-coalgebra is a small-filtered colimit of F-coalgebras. This has been proved in [AP_1] (see Theorem IV.2 applied to $\lambda = \aleph_\infty$).

3.2 Initial Algebras and Final Coalgebras

Remark 3.10 Let $\mathcal{K}$ be a locally small, cowellpowered category with small colimits. By Lemma 3.8, for every endofunctor F and every F^∞-coalgebra A there exists a *greatest congruence* on A, i. e., a homomorphism $e : A \longrightarrow A^*$ of F^∞-coalgebras carried by an epimorphism of $\mathcal{K}$ such that every other epimorphic homomorphism $f : A \longrightarrow B$ has a factorization $f^* : B \longrightarrow A^*$ with $f^* f = e$. (Viz., e is a multiple pushout of all f's.)

Theorem 3.11 *Let $\mathcal{K}$ be a locally small, cowellpowered category with small colimits. For every endofunctor F of $\mathcal{K}$ an initial F^∞-algebra, I, exists, in* *fact*

$$I = \operatorname*{colim}_{i \in \mathrm{Ord}} F^{(i)}0,$$

where 0 *is initial in* $\mathcal{K}$, *and* Ord *is the chain of all small ordinals. And a final* F^∞*-coalgebra,* T*, exists, in fact*

$$T = \Big(\coprod_{A \in \mathsf{Coalg}\, F} A \Big)^*$$

is a quotient of the coproduct of all F*-coalgebras modulo the greatest congruence.*

Remark. The statement on the existence of T is a generalization of the Final Coalgebra Theorem of [AM], see also the paper [B] of Barr.

Proof. (1) Following [Ad] define an Ord-chain $F^{(i)}0$ ($i \in \mathrm{Ord}$) with connecting morphisms $w_{ij} : F^{(i)}0 \longrightarrow F^{(j)}0$ ($i, j \in \mathrm{Ord}$, $i \leq j$) in $\mathcal{K}$ by the following transfinite induction over Ord:

$$F^{(0)}0 = 0, \quad F^{(1)}0 = F0, \quad \text{and} \quad w_{01} : 0 \longrightarrow F0 \text{ is uniquely determined.}$$

For the isolated step, given $F^{(i)}0$ and w_{ij} put

$$F^{(i+1)}0 = F(F^{(i)}0) \quad \text{and} \quad w_{i+1,j+1} = Fw_{ij}.$$

For the limit step, assume that j is a small limit ordinal such that the chain $(F^{(i)}0)_{i<j}$ has already been defined. Put

$$F^{(j)}0 = \operatorname*{colim}_{i<j} F^{(i)}0$$

with a colimit cocone

$$w_{ij} : F^{(i)}0 \longrightarrow F^{(j)}0 \qquad (i < j).$$

The requirement that we define a chain makes $w_{j,j+1} : F^{(j)}0 \longrightarrow F(F^{(j)}0)$ uniquely determined:

$$w_{j,j+1} \cdot w_{i+1,j} = w_{i+1,j+1} = Fw_{ij} \quad (\text{for all } i < j).$$

Denote by I a colimit of this (small-filtered) chain in $\mathcal{K}^\infty$. Then F^∞ preserves that colimit, yielding a canonical isomorphism

$$F^\infty I \cong \operatorname*{colim}_{i \in \mathrm{Ord}} F^{(i+1)}0 = \operatorname*{colim}_{i \in \mathrm{Ord}} F^{(i)}0 = I.$$

This is an initial F^∞-algebra, as proved in [Ad].

(2) The collection of all F-coalgebras $A = (X_A, \xi_A : X_A \longrightarrow FX_A)$ is a class because it is a class-indexed union of small sets $\mathcal{K}(X, FX)$. The category $\mathcal{K}^\infty$ has class-indexed coproducts, by Lemma 3.8, thus, the coproduct

$$B = \coprod_{A \in \mathsf{Coalg}\, F} A$$

exists as an F^∞-coalgebra. In fact, the forgetful functor $\mathsf{Coalg}\, F^\infty \longrightarrow \mathcal{K}^\infty$

creates colimits, thus, B is the unique F^∞-coalgebra on the coproduct

$$\coprod_{A \in \mathsf{Coalg}\, F} X_A$$

in $\mathcal{K}^\infty$ forming a coproduct in $\mathsf{Coalg}\, F^\infty$. It follows from Remark 3.9 that B is weakly final; thus, so is B^*. Consequently, B^* is final: suppose that $p, q : C \longrightarrow B^*$ are F^∞-coalgebra homomorphisms. We can form their coequalizer and find that, since B^* has no non-trivial quotients, we have $p = q$. ☐

Remark 3.12 For set functors James Worrell [W] has provided a different, much more natural construction of a final coalgebra T:

$$T = \lim_{i \in \mathrm{Ord}} F^{(\aleph_\infty + i)}1 = F^{(\aleph_\infty + \aleph_\infty)}1.$$

More precisely, given $F : \mathsf{Set} \longrightarrow \mathsf{Set}$, we can form a cochain indexed by Ord (or, which is the same, indexed by the first non-small ordinal $\aleph_\infty$), $F^{(i)}1$ ($i \in \mathrm{Ord}$), by dualizing the chain of the proof of Theorem 3.11:

$$F^{(0)}1 = 1, \quad F^{(1)} = F1 \quad \text{and} \quad w_{10} : F1 \longrightarrow 1 \text{ is unique;}$$

for the isolated steps we put

$$F^{(i+1)}1 = F(F^{(i)}1) \quad \text{and} \quad w_{i+1,j+1} = Fw_{ij}$$

and on limit steps, where j is a limit ordinal, put

$$F^{(j)}1 = \lim_{i<j} F^{(i)}1 \quad \text{with limit cone } w_{ji}\ (i < j).$$

Notice that by forming the class-indexed limit

$$F^{(\aleph_\infty)}1 = \lim_{i \in \mathrm{Ord}} F^{(i)}1 = \lim_{i < \aleph_\infty} F^{(i)}1$$

we can leave not only Set, but also Class: there is no guarantee that $F^{(\aleph_\infty)}1$ is a class! And, whenever it is not a class, then we have not found our final coalgebra yet (since, by Theorem 3.11, T *is* a class). Fortunately, another Ord-indexed cochain repairs the damage.

Let us denote by $\mathsf{Set}^@$ the category of all sets of cardinality at most $2^{\aleph_\infty}$; since $\mathrm{card}(F^{(i)}1) < \aleph_\infty$ for all $i < \aleph_\infty$, it follows that $\mathrm{card}(F^{(\aleph_\infty)}1) \le {\aleph_\infty}^{\aleph_\infty} = 2^{\aleph_\infty}$ and our limit thus lives in $\mathsf{Set}^@$. We have an essentially unique $2^{\aleph_\infty}$-accessible extension

$$F^@ : \mathsf{Set}^@ \longrightarrow \mathsf{Set}^@$$

of F. And this allows us to define an Ord-indexed cochain

$$F^{(\aleph_\infty + i)}1 \quad (i \in \mathrm{Ord})$$

in $\mathsf{Set}^@$ by a transfinite induction which precisely follows the previous one, except that F is now substituted by $F^@$:

$$F^{(\aleph_\infty)}1 \text{ has been defined already,} \quad F^{(\aleph_\infty+1)}1 = F^@(F^{(\aleph_\infty)}1)$$

and

$$w_{\aleph_\infty+1, \aleph_\infty} : F^@(F^{(\aleph_\infty)}1) \longrightarrow F^{(\aleph_\infty)}1$$

is uniquely determined by the commutativity of the following triangles

$$\begin{array}{ccc} F^{@}(F^{(\aleph_\infty)}1) & \xrightarrow{w_{\aleph_\infty+1,\aleph_\infty}} & F^{(\aleph_\infty)}1 \\ & \searrow^{w_{\aleph_\infty+1,i+1}=F^{@}w_{\aleph_\infty i}} & \downarrow w_{\aleph_\infty,i+1} \\ & & F(F^{(i)}1) = F^{(i+1)}1 \end{array}$$

for all $i \in \mathrm{Ord}$. The isolated step is, as above,

$$F^{(\aleph_\infty+i+1)}1 = F^{@}(F^{(\aleph_\infty+i)})$$

$$w_{i+1,j+1} = F^{@}w_{ij}.$$

And limit steps are given by the formation of limits. We denote by

$$F^{(\aleph_\infty+\aleph_\infty)}1 = \lim_{i\in\mathrm{Ord}} F^{\aleph_\infty+i}1$$

a limit of this cochain in $\mathsf{Set}^{@}$ with limit cone $\overline{w_i} : F^{(\aleph_\infty+\aleph_\infty)}1 \longrightarrow F^{(\aleph_\infty+i)}1$. This is an $F^{@}$-coalgebra w.r.t. the unique

$$\tau : F^{@}(F^{(\aleph_\infty+\aleph_\infty)}1) \longrightarrow F^{(\aleph_\infty+\aleph_\infty)}1$$

with

$$\overline{w_{i+1}}\tau = F^{@}\overline{w_i} : F^{@}(F^{(\aleph_\infty+\aleph_\infty)}1) \longrightarrow F^{(\aleph_\infty+i+1)}1$$

for all ordinals $i \in \mathrm{Ord}$.

It has been proved by J. Worrell that this $F^{@}$-coalgebra is final. And, unlike $F^{(\aleph_\infty)}1$, we are now sure that

$$F^{(\aleph_\infty+\aleph_\infty)}1$$

is a class. In fact, the argument that a final $F^{@}$-coalgebra is a class is the same as that presented in Theorem 3.11: all F-coalgebras form a generator of $\mathsf{Coalg}\, F^{@}$, thus, a final $F^{@}$-coalgebra is a quotient of the class-coalgebra $\coprod_{A\in\mathsf{Coalg}\,F} A$.

Remark 3.13

(i) In [W] J. Worrell has shown that the connecting maps starting after $\aleph_\infty$:

$$F^{(\aleph_\infty)}1 \longleftarrow F^{(\aleph_\infty+1)}1 \longleftarrow \cdots \longleftarrow F^{(\aleph_\infty+i)}1 \longleftarrow \cdots$$

are all monomorphisms, i.e., $F^{(\aleph_\infty+i)}1$ is a subobject of $F^{(\aleph_\infty)}1$, and a final F^{∞}-coalgebra is thus an intersection

$$T = \bigcap_{i\in\mathrm{Ord}} F^{(\aleph_\infty+i)}1$$

of these subobjects.

(ii) All the above results hold not only for functors F^{∞}, but for all small-accessible endofunctors of Class.

Definition 3.14 A category $\mathcal{K}$ is called *smooth* provided that it has no non-trivial small-filtered colimits of monomorphisms.

That is, given a small-filtered diagram $D : \mathcal{D} \longrightarrow \mathcal{K}$ of monomorphisms with a colimit $c_d : Dd \longrightarrow K$ (d in $\mathcal{D}$) then some of the colimit morphisms c_d is an isomorphism.

Examples 3.15

(i) Set is smooth. In fact, given a small-filtered diagram D of monomorphisms whose colimit (= union) is a set, then this set is simply Dd for some object d.

(ii) All "everyday-life" categories are smooth, e.g., Pos, Cpo, etc. The argument is similar to that for Set.

(iii) Every locally presentable category is smooth. Given a small-filtered colimit $c_d : Dd \longrightarrow K$ of monomorphisms, then, since K is a λ-presentable object for some λ, the morphism $id_K : K \longrightarrow K$ factorizes through some c_d. Thus, c_d is both a monomorphism and a split epimorphism.

(iv) Categories $\mathcal{K}^\infty$ are typically not smooth, e.g., Class, Pos^∞ or Cpo^∞ are certainly not smooth.

Lemma 3.16 *For every smooth category $\mathcal{K}$ the functor $(\_)^\infty$ from $[\mathcal{K},\mathcal{K}]$ to $[\mathcal{K}^\infty,\mathcal{K}^\infty]$ preserves all existing small-filtered colimits of monomorphisms.*

Proof. Let $(f_i : F_i \longrightarrow F)_{i\in I}$ be a small-filtered colimit of monomorphisms in $[\mathcal{K},\mathcal{K}]$. This means, of course, that for every object K of $\mathcal{K}$ we have a trivial colimit $((f_i)_K : F_iK \longrightarrow FK)_{i\in I}$, since $\mathcal{K}$ is smooth and since colimits in $[\mathcal{K},\mathcal{K}]$ are, whenever they exist, formed pointwise. We are to prove that $(f_i^\infty : F_i^\infty \longrightarrow F^\infty)_{i\in I}$ is a colimit in $[\mathcal{K}^\infty,\mathcal{K}^\infty]$. We know that, since I is a small-filtered category, a colimit $G = \mathrm{colim}_{i\in I} F_i^\infty$ exists in $[\mathcal{K}^\infty,\mathcal{K}^\infty]$ (with colimit cocone $g_i : F_i^\infty \longrightarrow G$). To prove that $G \cong F^\infty$, observe that G preserves small-filtered colimits (since each F_i^∞ does), thus, it is sufficient to show that G extends F. In fact, for every object K of $\mathcal{K}$ we have $i \in I$ such that $(f_i)_K : F_iK \longrightarrow FK$ is an isomorphism. Then $(g_i)_K$ is an isomorphism, making GK essentially equal to $F_i^\infty K = F_iK = FK$. □

Theorem 3.17 *For every smooth category $\mathcal{K}$ the functor*

$$F \mapsto T_F$$

assigning a final coalgebra to every endofunctor of $\mathcal{K}$ preserves existing small-filtered colimits of monomorphisms.

Remark 3.18 What we mean is, of course, the following functor

$$\Phi : [\mathcal{K},\mathcal{K}] \longrightarrow \mathcal{K}^\infty$$

assigning to every F the object T_F of a final F^∞-coalgebra (T_F, τ_F) and to every natural transformation $f : F \longrightarrow G$ the unique homomorphism $\Phi f : T_F \longrightarrow T_G$ of G^∞-coalgebras:

$$\begin{array}{ccccc} T_F & \xrightarrow{\tau_F} & F^\infty T_F & \xrightarrow{(f^\infty)_{T_F}} & G^\infty T_F \\ {\scriptstyle \Phi f}\downarrow & & & & \downarrow{\scriptstyle G\Phi f} \\ T_G & & \xrightarrow[\tau_G]{} & & G^\infty T_G \end{array}$$

Proof. Let

$$(f_i : F_i \longrightarrow F)_{i\in I}$$

be a small-filtered colimit of monomorphisms in $[\mathcal{K},\mathcal{K}]$. We obtain the corresponding diagram of objects T_{F_i} $(i \in I)$, more precisely, we apply Φ to the given diagram. This diagram is small-filtered in $\mathcal{K}^\infty$, thus, it has a colimit

$$(t_i : T_{F_i} \longrightarrow T)_{i \in I}$$

in $\mathcal{K}^\infty$. There is a unique F^∞-coalgebra structure

$$\tau : T \longrightarrow F^\infty T$$

making each t_i a homomorphism of F^∞-coalgebras:

$$\begin{array}{ccccc} T_{F_i} & \xrightarrow{\tau_{F_i}} & F_i^\infty T_{F_i} & \xrightarrow{(f_i^\infty)_{T_{F_i}}} & F^\infty T_{F_i} \\ {\scriptstyle t_i}\downarrow & & & & \downarrow{\scriptstyle F^\infty t_i} \\ T & & \xrightarrow[\tau]{\hspace{6em}} & & F^\infty T \end{array}$$

To prove that (T,τ) is a final F-coalgebra, we only have to consider an F-coalgebra

$$\beta : B \longrightarrow FB$$

see Remark 3.9. In order to prove the existence and uniqueness of a homomorphism $B \longrightarrow T$, we first observe that since F^∞ preserves small-filtered colimits, we have

$$F^\infty T = \operatorname*{colim}_{i \in I} F^\infty T_{F_i}$$

with the colimit cocone $F^\infty t_i$ $(i \in I)$. By Lemma 3.16

$$(f_i^\infty : F_i^\infty \longrightarrow F^\infty)_{i \in I}$$

is a small-filtered colimit in $[\mathcal{K}^\infty, \mathcal{K}^\infty]$.

Consequently, we also have

$$F^\infty T = \operatorname*{colim}_{i \in I} F_i^\infty T_{F_i} \tag{2}$$

with the colimit cocone

$$F_i^\infty T_{F_i} \xrightarrow{f_i^\infty} F^\infty T_{F_i} \xrightarrow{F^\infty t_i} F^\infty T \quad (i \in I).$$

Existence of a homomorphism $B \longrightarrow T$. Since B is small-presentable, see Remark 3.5(a), the morphism

$$\beta : B \longrightarrow \operatorname*{colim}_{i \in I} F_i B$$

factorizes through some $(f_i)_B$:

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ & \searrow^{\beta'} & \uparrow{\scriptstyle (f_i)_B} \\ & & F_i B \end{array}$$

The unique homomorphism $h : B \longrightarrow T_{F_i}$ of F_i^∞-coalgebras defines a homomorphism $\overline{h} = t_i \cdot h : B \longrightarrow T$ of F^∞-coalgebras:

$$\begin{array}{ccccc}
B & \xrightarrow{\beta'} & F_i^\infty B & \xrightarrow{(f_i^\infty)_B} & F^\infty B \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle F_i^\infty h} & & \downarrow{\scriptstyle F^\infty h} \\
T_{F_i} & \xrightarrow{\tau_{F_i}} & F_i^\infty T_{F_i} & \xrightarrow{(f_i^\infty)_{T_{F_i}}} & F^\infty T_{F_i} \\
{\scriptstyle t_i}\downarrow & & & & \downarrow{\scriptstyle F^\infty t_i} \\
T & & \xrightarrow[\tau]{\qquad\qquad} & & F^\infty T
\end{array}$$

Uniqueness of a homomorphism $B \longrightarrow T$. The uniqueness of $\overline{h}$ follows, again, from small presentability, see (ii) in 3.5(a): given a homomorphism $k : B \longrightarrow T_F$ of F^∞-coalgebras, then there is a factorization $k = t_{\overline{i}} \cdot k'$ for some $\overline{i} \in I$, and without loss of generality we can assume $\overline{i} = i$ (since I is small-filtered):

$$\begin{array}{ccccc}
B & \xrightarrow{\beta'} & F_i^\infty B & \xrightarrow{(f_i^\infty)_B} & F^\infty B \\
{\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle F_i^\infty k'} & & \downarrow{\scriptstyle F^\infty k'} \\
T_{F_i} & \xrightarrow{\tau_{F_i}} & F_i^\infty T_{F_i} & \xrightarrow{(f_i^\infty)_{T_{F_i}}} & F^\infty T_{F_i} \\
{\scriptstyle t_i}\downarrow & & & & \downarrow{\scriptstyle F^\infty t_i} \\
T & & \xrightarrow[\tau]{\qquad\qquad} & & F^\infty T
\end{array}$$

If k' is a homomorphism of F_i^∞-coalgebras, then the proof is finished: we have $k' = h$, thus, $k = t_i \cdot h = \overline{h}$. If not, we use the fact that $F^\infty T$ is a small-filtered colimit of $F_i^\infty T_{F_i}$. Now the two morphisms $(\tau_{F_i} \cdot k')$ and $(F_i^\infty k' \cdot \beta')$ are merged by the colimit map $F^\infty t_i \cdot (f_i^\infty)_{T_{F_i}}$ of the colimit (2):

$$\begin{aligned}
F^\infty t_i \cdot (f_i^\infty)_{T_{F_i}} \cdot (F_i^\infty k' \cdot \beta') &= F^\infty t_i \cdot F^\infty k' \cdot (f_i)_B \cdot \beta' \\
&= F^\infty k \cdot \beta \\
&= \tau \cdot k \\
&= \tau \cdot t_i \cdot k' \\
&= F^\infty t_i \cdot (f_i^\infty)_{T_{F_i}} \cdot (\tau_{F_i} \cdot k') \text{definition of } \tau
\end{aligned}$$

Indeed, the first equation uses naturality of f_i, the second one the definitions of k' and β', the third one holds since k is a homomorphism, and the 4th and 5th follow from the definitions of k and τ, respectively. Since B is small-presentable, there is a connecting morphism

$$x_{ij} : F_i \longrightarrow F_j$$

of the original diagram such that the corresponding connecting morphism

$$F_i^\infty T_{F_i} \xrightarrow{(x_{ij}^\infty)_{T_{F_i}}} F_j^\infty T_{F_i} \xrightarrow{F_j^\infty \Phi x_{ij}} F_j^\infty T_{F_j}$$

also merges the pair $\tau_{F_i} \cdot k'$ and $F_i^\infty k' \cdot \beta'$.

It follows that $\Phi x_{ij} \cdot k' : B \longrightarrow T_{F_j}$ is a homomorphism of F_j^∞-coalgebras—

in fact, the following diagram

$$\begin{array}{ccccc}
B & \xrightarrow{\beta'} & F_i^\infty B & \xrightarrow{(x_{ij}^\infty)_B} & F_j^\infty B \\
{\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle F_i^\infty k'} & & \downarrow{\scriptstyle F_j^\infty k'} \\
T_{F_i} & \xrightarrow{\tau_{F_i}} & F_i^\infty T_{F_i} & \xrightarrow{(x_{ij}^\infty)_{T_{F_i}}} & F_j^\infty T_{F_i} \\
{\scriptstyle \Phi x_{ij}}\downarrow & & & & \downarrow{\scriptstyle F_j^\infty \Phi x_{ij}} \\
T_{F_j} & & \xrightarrow[\tau_{F_j}]{\qquad\qquad} & & F_j^\infty T_{F_j}
\end{array}$$

commutes. Consequently, $\Phi x_{ij} \cdot k' = \Phi x_{ij} \cdot h$ (since the right-hand side is also a homomorphism). Therefore

$$k = t_i \cdot k' = t_j \cdot \Phi x_{ij} \cdot k' = t_j \cdot \Phi x_{ij} \cdot h = t_i \cdot h = \overline{h}.$$

□

4 A General Solution Theorem

We apply here the results of Section 3 to show that for *every* endofunctor H of Set we have a solution theorem concerning guarded sets of iterative equations. This is so because the class extension $H^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$ is iteratable, thus, we have the completely iterative monad $T^\sharp$ of H^∞, see 2.5. (If H is iteratable and defines thus a completely iterative monad $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, then $T^\sharp$ is nothing else than the extension $T^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$.) But we can say more: for every infinite cardinal number λ we can form the λ-accessible coreflection, H_λ, of H: to every set X it assigns the union of images of Hi for all inclusions $i : Y \longrightarrow X$ of subsets Y of cardinality less than λ. The functor H_λ is iteratable in Set, see [AAMV], and we denote by T_λ the corresponding completely iterative monad on Set.

We are going to prove that for every set X the class $T^\sharp X$ is a canonical colimit of the sets $T_\lambda X$, where λ is a small cardinal number. Consequently, every iterative system of equations

$$e : X \longrightarrow T^\sharp(X + Y) \quad (X,\, Y \text{ in } \mathsf{Set})$$

for H actually has the form of a morphism

$$\overline{e} : X \longrightarrow T_\lambda(X + Y) \quad \text{for some small cardinal } \lambda$$

followed by the colimit map $T_\lambda(X + Y) \longrightarrow T^\sharp(X + Y)$. We then solve $\overline{e}$ with respect to T_λ and obtain

$$\overline{e}^\dagger : X \longrightarrow T_\lambda Y$$

which, composed with the colimit map $T_\lambda Y \longrightarrow T^\sharp Y$, is the (unique) solution of e.

Notation 4.1 For every endofunctor

$$H : \mathsf{Set} \longrightarrow \mathsf{Set}$$

denote by

$$H_\lambda : \mathsf{Set} \longrightarrow \mathsf{Set} \quad (\lambda \text{ any small cardinal})$$

the subfunctor given by

$$H_\lambda X = \bigcup Hf[HY]$$

where the union ranges over all $f : Y \longrightarrow X$ with $\mathrm{card}(Y) < \lambda$. Let $h_\lambda : H_\lambda \longrightarrow H$ be the inclusion.

We denote by $T_\lambda : \mathsf{Set} \longrightarrow \mathsf{Set}$ the free completely iterative monad of H_λ and by $T^\sharp$ the free completely iterative monad of $H^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$.

Lemma 4.2 $H = \mathrm{colim}\, H_\lambda$ *is a small-filtered colimit of monomorphisms in* $[\mathsf{Set}, \mathsf{Set}]$.

Example 4.3 If $H = \mathcal{P}$ is the power-set functor then for every $\lambda \geq \omega$ we get the functor $\mathcal{P}_\lambda$ of all subsets of cardinalities less than λ.

We now extend the definition of (guarded) equation morphism and solution to arbitrary endofunctors of Set.

Definition 4.4 Let H be an endofunctor of Set.

(i) By an *equation morphism for H* we understand a morphism

$$e : X \longrightarrow T^\sharp(X+Y) \quad \text{for } X, Y \text{ in } \mathsf{Set}$$

It is called *guarded* if it factorizes through $[\tau^\sharp_{X+Y}, \eta^\sharp_{X+Y}\mathsf{inr}]$:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T^\sharp(X+Y) \\ & \searrow & \uparrow {\scriptstyle [\tau^\sharp_{X+Y}, \eta^\sharp_{X+Y}\mathsf{inr}]} \\ & & H^\infty T^\sharp(X+Y) + Y \end{array}$$

(ii) By a *solution* of e we understand a morphism

$$e^\dagger : X \longrightarrow T^\sharp Y$$

such that the following square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & T^\sharp Y \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu^\sharp_Y} \\ T^\sharp(X+Y) & \xrightarrow[T^\sharp[e^\dagger, \eta^\sharp_Y]]{} & T^\sharp T^\sharp Y \end{array}$$

commutes.

Lemma 4.5 *For every accessible functor* $H : \mathsf{Set} \longrightarrow \mathsf{Set}$ *with a free completely iterative monad* T *the functor* $H^\infty : \mathsf{Class} \longrightarrow \mathsf{Class}$ *has a free completely iterative monad with underlying functor* T^∞.

Proof. We prove that $T^\infty X$ is a final coalgebra for $H^\infty(\_) + X$:

(a) If X is a small set, this is trivial:

$$H^\infty(T^\infty X) + X = HTX + X = TX = T^\infty X.$$

Now use Remark 3.9.

(b) If X is a class, express it as a small-filtered union of all of its subsets, and use the fact that H^∞ and T^∞ preserve small-filtered colimits

$$H^\infty(T^\infty X) + X = \operatorname*{colim}_i \Big(H^\infty(T^\infty X_i) + X_i \Big) = \operatorname*{colim}_i T^\infty X_i = T^\infty X$$

and use Remark 3.9 again.

□

Remark 4.6 In [AAMV] we have proved that the formation of free completely iterative monads over accessible endofunctors is (as the name suggests) a universal construction. Therefore, the natural transformation $h^\infty_\lambda : H^\infty_\lambda \longrightarrow H^\infty$ (inclusion) extends to a unique ideal monad morphism $t^\infty_\lambda : T^\infty_\lambda \longrightarrow T^\sharp$. "Ideal" means that

$$t^\infty_\lambda = h^\infty_\lambda * t^\infty_\lambda + id : H^\infty_\lambda T^\infty_\lambda + Id \longrightarrow H^\infty T^\sharp + Id$$

(here, $*$ denotes the horizontal composition of natural transformations).

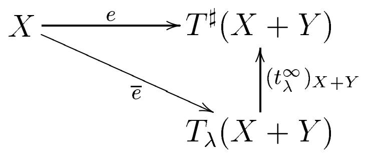
Moreover, the obvious small-filtered diagram formed by all T^∞_λ (λ a small cardinal) has a colimit cocone

$$t^\infty_\lambda : T^\infty_\lambda \longrightarrow T^\sharp$$

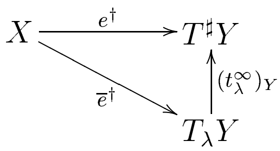
because left adjoints preserve colimits.

General Solution Theorem 4.7 For every endofunctor H of Set, every guarded equation morphism has a unique solution.

Moreover, the solution can be found as follows: we find a factorization



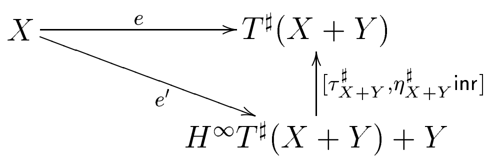
for some small cardinal number λ and some guarded equation morphism $\overline{e}$, and by solving $\overline{e}$ w.r.t. H_λ we solve e w.r.t. H^∞ since the following triangle



commutes.

Remark. The above theorem states that solutions of all guarded equations w.r.t. H are found in the small coalgebras $T_\lambda Y$ for various cardinal numbers λ.

Proof. Suppose that a guarded equation morphism $e : X \longrightarrow T^\sharp(X+Y)$ is given and consider the factorization



Since X is a small set, e' factorizes through some $(h^\infty_\lambda * t^\infty_\lambda)_{X+Y} + id_Y$:

$$\begin{array}{ccc} X & \xrightarrow{e'} & H^\infty T^\sharp(X+Y)+Y \\ & \searrow^{\overline{e}'} & \uparrow{\scriptstyle (h^\infty_\lambda * t^\infty_\lambda)_{X+Y}+id_Y} \\ & & H^\infty_\lambda T^\infty_\lambda(X+Y)+Y \end{array}$$

Observe that the following square

$$\begin{array}{ccc} H^\infty T^\sharp(X+Y)+Y & \xrightarrow{[\tau^\sharp_{X+Y},\, \eta^\sharp_{X+Y}\mathsf{inr}]} & T^\sharp(X+Y) \\ \uparrow{\scriptstyle (h^\infty_\lambda * t^\infty_\lambda)_{X+Y}+id_Y} & & \uparrow{\scriptstyle (t^\infty_\lambda)_{X+Y}} \\ H^\infty_\lambda T^\infty_\lambda(X+Y)+Y & \xrightarrow[{[(\tau^\infty_\lambda)_{X+Y},\, (\eta^\infty_\lambda)_{X+Y}\mathsf{inr}]}]{} & T^\infty_\lambda(X+Y) \end{array}$$

commutes. Thus, by putting

$$\overline{e} = [(\tau^\infty_\lambda)_{X+Y}, (\eta^\infty_\lambda)_{X+Y}\mathsf{inr}] \cdot \overline{e}'$$

we define a guarded equation morphism such that the following triangle

$$\begin{array}{ccc} X & \xrightarrow{e} & T^\sharp(X+Y) \\ & \searrow^{\overline{e}} & \uparrow{\scriptstyle (t^\infty_\lambda)_{X+Y}} \\ & & T^\infty_\lambda(X+Y) = T_\lambda(X+Y) \end{array}$$

commutes. Since t^∞_λ is an ideal monad morphism, it preserves solutions (see 4.11 of [AAMV]), i.e., the following triangle

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & T^\sharp Y \\ & \searrow^{\overline{e}^\dagger} & \uparrow{\scriptstyle (t^\infty_\lambda)_Y} \\ & & T^\infty_\lambda Y = T_\lambda Y \end{array}$$

commutes. $\square$

Remark 4.8 A special case of guarded equation morphisms are the flat ones, i.e., equation morphisms of the form

$$e : X \longrightarrow HX + Y \quad (X,\, Y \text{ in } \mathsf{Set}).$$

We have a natural connecting morphism

$$\rho_{X,Y} : HX + Y \longrightarrow T^\sharp(X+Y)$$

whose left-hand component is

$$HX = H^\infty X \xrightarrow{H^\infty \eta^\sharp_X} H^\infty T^\sharp X \xrightarrow{H^\infty T^\sharp \mathsf{inl}} H^\infty T^\sharp(X+Y) \xrightarrow{\tau^\sharp_{X+Y}} T^\sharp(X+Y)$$

and the right-hand one is

$$Y \xrightarrow{\mathsf{inr}} X + Y \xrightarrow{\eta^\sharp_{X+Y}} T^\sharp(X+Y)$$

Thus, every flat equation morphism $e : X \longrightarrow HX + Y$ yields an equation morphism $\rho_{X,Y}e : X \longrightarrow T^\sharp(X+Y)$ which is easily seen to be guarded. We denote by

$$e^\dagger : X \longrightarrow TY$$

the unique solution of $\rho_{X,Y} e$, for short.

In case of flat equation morphisms we have shown in [AAMV] that

$$\text{solution} = \text{corecursion}.$$

That is, $e^\dagger : X \longrightarrow T^\sharp Y$ is the unique homomorphism from the coalgebra $e : X \longrightarrow HX + Y$ to the final coalgebra $T^\sharp Y$ of $H^\infty(\_) + Y$.

5 Example: Power-Set Functor

We apply the above results to non-labelled transition systems, i.e., to coalgebras of the power-set functor $\mathcal{P} : \mathsf{Set} \longrightarrow \mathsf{Set}$. It has been noticed by several authors [AM], [B], [JPTWW], [RT], [W] that $\mathcal{P}^\infty$ has a very natural *weakly final* coalgebra B (i.e., such that every $\mathcal{P}$-coalgebra A has at least one homomorphism from A to B): the coalgebra of all small extensional trees. Recall that a (rooted, non-ordered) tree is called *extensional* provided that any two distinct nodes with a common parent define non-isomorphic subtrees. Throughout this section trees are always taken up to (graph) isomorphism. Thus, shortly, a tree is extensional if and only if distinct siblings define distinct subtrees. We call a tree *small* if it has only a small set of children (= maximal proper subtrees).

5.1 Coalgebra B

It has as elements all small extensional trees, and the coalgebra structure

$$\beta : B \longrightarrow \mathcal{P}^\infty B$$

is the inverse of tree tupling, i.e., β assigns to every tree t the set of all children of t.

5.2 Final Coalgebra B/∼

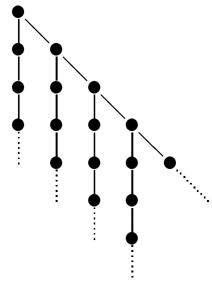
We know from Theorem 3.11 that a final coalgebra exists. Recall here that $\mathcal{P}^\infty$ preserves weak pullbacks. Hence, the greatest congruence coincides with the greatest bisimulation on any $\mathcal{P}^\infty$-coalgebra, see e. g. [R]. Since B is weakly final, it follows that a final coalgebra is a quotient of B modulo the *bisimilarity equivalence* $\sim$ (i.e., the largest bisimulation on B). We are going to describe this equivalence $\sim$. We start by describing one interesting class.

Example 5.1 An extensional tree t is bisimilar to the following tree

Ω

if and only if all paths in t are infinite. Thus, for example, the following tree

Ω'



is bisimilar to Ω. This illustrates that the bisimilarity equivalence is non-trivial. We prove $\Omega \sim \Omega'$ below.

Remark 5.2 For the finite-power-set functor $\mathcal{P}_f$ a nice desription of a final coalgebra has been presented by Michael Barr [B]: let B_f denote the coalgebra of all finitely branching extensional trees. This is a small subcoalgebra of our (large) coalgebra B. We call two trees b, b' in B_f *Barr-equivalent*, notation

$$b \sim_0 b'$$

provided that for every natural number n the tree $b|_n$ obtained by cutting b at level n has the same extensional reflection as the tree $b'|_n$. (An extensional reflection is obtained by identifying pairs of siblings which define identical subtrees until one gets an extensional tree.) For example

$$\Omega \sim_0 \Omega'$$

Barr proved that the quotient coalgebra

$$B_f/\sim_0$$

is a final $\mathcal{P}_f$-coalgebra—that is, $\sim_0$ is the bisimilarity equivalence on B_f.

5.3 The Bisimilarity Equivalence $\sim$

We define, for every small ordinal number i, the following equivalence relation $\sim_i$ on B:

$\sim_0$ is the Barr-equivalence

and in case $i > 0$

$t \sim_i s$ iff for all $j < i$ the following hold:

(1) for each child t' of t there exists a child s' of s such that $t' \sim_j s'$ and

(2) vice versa.

Remark 5.3 We shall show below that the bisimilarity equivalence $\sim$ is the intersection of all $\sim_i$. Notice that this intersection is just the usual construction of a greatest fixed point. Indeed, consider the collection Rel of all binary relations on B. This collection, ordered by set-inclusion, is a class-complete lattice. Define $\Phi : \mathsf{Rel} \longrightarrow \mathsf{Rel}$ as follows:

$$t\ \Phi(R)\ s \quad \text{iff} \quad \text{for every child } t' \text{ of } t \text{ there exists a child } s' \text{ of } s \text{ such that } t'\, R\, s', \text{ and vice versa.}$$

Observe that Φ is a monotone function. Moreover, a binary relation R is a fixed point of Φ if and only if R is a bisimulation on B. Notice that the definition of $\sim_i$ is just an iteration of Φ on the largest equivalence relation $\approx_0$ (i.e., $B \times B$) shifted by ω steps: we have

$$\sim_0 = \Phi^{(\omega)}(\approx_0)$$

where for every relation R the iterations $\Phi^{(i)}(R)$, $i \in \mathrm{Ord}$, are defined inductively as follows: $\Phi^{(0)}(R) = R$, the isolated step is $\Phi^{(i+1)}(R) = \Phi(\Phi^{(i)}(R))$, and for limit ordinals $\Phi^{(i)}(R) = \bigcap_{j<i} \Phi^{(j)}(R)$. Consequently, $\sim_i = \Phi^{(\omega+i)}(\approx_0)$.

That we are indeed constructing the largest fixed point for Φ follows from the following

Lemma 5.4 *Φ preserves intersections of descending* Ord*-chains.*

Proof. Let $(R_i)_{i\in\mathrm{Ord}}$ be a descending chain in Rel and let

$$R = \bigcap_{i\in\mathrm{Ord}} R_i$$

be its intersection. We show that $\Phi(R) = \bigcap_{i\in\mathrm{Ord}} \Phi(R_i)$. In fact, the inclusion from left to right is obvious. To show the inclusion from right to left, suppose that the pair (t, s) is in the right-hand relation. Let t' be any child of t. Then, for any ordinal number $i \in \mathrm{Ord}$ there exists a child s'_i of s with $t\ R_i\ s'_i$. Since s has only a small set of children the set $\{s'_i \mid i \in \mathrm{Ord}\}$ is small, too. Therefore there is a cofinal subset C of Ord such that $\{s'_i \mid i \in C\}$ has only one element, s' say. It follows that $t'\ R_i\ s'$ for all $i \in \mathrm{Ord}$. Hence, $t\ \Phi(R)\ s$, as desired. □

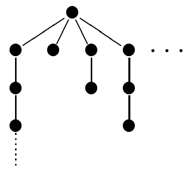
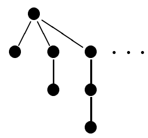
Theorem 5.5 *Two trees $t, s \in B$ are bisimilar iff $t \sim_i s$ holds for all small ordinals i.*

Proof. It follows from Lemma 5.4 that the intersection of all $\sim_i = \Phi^{(i)}(\sim_0)$, $i \in \mathrm{Ord}$ is a fixed point of Φ.

Next form the quotient coalgebra $B/\sim$. Since B is weakly final, so is $B/\sim$. In order to establish that $B/\sim$ is a final $\mathcal{P}^\infty$-coalgebra we must show that for any $\mathcal{P}^\infty$-coalgebra (X, ξ) and any two coalgebra homomorphisms $h, k : (X, \xi) \longrightarrow (B, \beta)$ we have $h(x) \sim k(x)$ for all $x \in X$. We show this by transfinite induction, i. e., we prove that $h(x) \approx_i k(x)$ holds for all $i \in \mathrm{Ord}$.

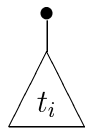
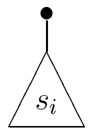
The first step $i = 0$ is obvious and for the induction step suppose that $i > 0$ is any small ordinal number and that for all $x \in X$, $k(x) \approx_j h(x)$ for all $j < i$, where $\approx_j$ denotes $\Phi^{(j)}(\approx_0)$. Consider any child s' of $k(x)$, i. e., $s' = k(x')$ for some $x' \in \xi(x)$ since k is a coalgebra homomorphism. Because h is a coalgebra homomorphism $t' = h(x')$ is a child of $h(x)$ such that $s' \approx_j t'$ for all $j < i$, whence $k(x) \approx_i h(x)$. □

Remark 5.6 Barr showed that $\sim_0$ is the bisimilarity equivalence on the set of finitely branching trees. However, it is not even a bisimulation on B. In order to see this notice that is suffices to find trees that are in $\sim_0$ but not in $\sim_1$. Consider the following trees

$s_0 =$  and $t_0 =$ 

We clearly have $t \sim_0 s$. But $t_0 \not\sim_1 s_0$, since s_0 has a child which is an infinite path while t_0 does not.

Moreover, none of the relations $\sim_i$, $i < \omega$ is a bisimulation. This is easily seen by induction. The base case is the above example, and if t_i and s_i are trees with $t_i \sim_i s_i$ and $t_i \not\sim_{i+1} s_i$, then

$t_{i+1} =$  and $s_{i+1} =$ 

satisfy $t_{i+1} \sim_{i+1} s_{i+1}$ and $t_{i+1} \not\sim_{i+2} s_{i+1}$.

Next we show that $\sim_\omega$ is not a bisimulation. Consider the tree u_ω whose set of children is $\{t_i \mid i < \omega\}$. Further, consider the family v^i_ω of trees that are obtained from u_ω by replacing the ith child by s_i. Now let t_ω be the tree with set of children $\{v^i_\omega \mid i < \omega\}$ and let s_ω be obtained from t_ω by adding one more child u_ω. The following properties are easily established:

$$\begin{aligned} t_i &\not\sim_\omega t_j \text{ for } i < j < \omega, \\ u_\omega &\sim_i v^i_\omega \text{ for all } i < \omega, \\ u_\omega &\not\sim_{i+1} v^i_\omega \text{ for all } i < \omega. \end{aligned}$$

From this it follows clearly that $t_\omega \sim_\omega s_\omega$ but $t_\omega \not\sim_{\omega+1} s_\omega$.

5.4 Free Iterative Monad

We now modify the above to describe a final coalgebra TY of the functor $\mathcal{P}(\_) + Y$, where Y is any small set of parameters.

Let

$$BY$$

denote the coalgebra of all extensional trees with leaves partially labelled in Y (i.e., some leaves are labelled by parameters and some are not). The coalgebra structure

$$\beta : BY \longrightarrow \mathcal{P}^\infty(BY) + Y$$

assigns to every singleton tree labelled by $x \in Y$ the value x (in the second summand) and to any other tree t the (possibly empty) set of all children of t.

Lemma 5.7 *The coalgebra BY of all small extensional trees is a weakly final coalgebra of $\mathcal{P}^\infty(\_) + Y$.*

Proof. Given a coalgebra

$$\xi : A \longrightarrow \mathcal{P}^\infty A + Y$$

(with $A \cap Y = \emptyset$, for simplicity), we define for every $a \in A$ a labelled tree t_a all of whose nodes are labelled in $A + Y$ as follows:

the root of t_a is labelled by a;

given a node of t_a labelled by $x \in A$, then the children of that node correspond to the elements of $\xi(x)$, in case $\xi(x) \subseteq A$, and in case $\xi(x) \in Y$, the node is a leaf.

Let $h : A \longrightarrow BY$ assign to $a \in A$ the tree $h(a) \in BY$ obtained from t_a by deleting all the labels in A. Then h is easily seen to be a homomorphism. □

Definition 5.8 Two trees t, s in BY are called *Barr-similar*, notation

$$t \sim_0 s$$

provided that for every $n \in \omega$ we have $C_n(t) = C_n(s)$ (where C_n denotes the extensional reflection of the cutting at level n, leaving all new leaves unlabelled).

For every small ordinal number $i > 0$ we denote by $\sim_i$ the equivalence on BY with

$t \sim_i s$ iff for every $j < i$ and every child t' of t there is a child s' of s with $t' \sim_j s'$, and vice versa.

Theorem 5.9 *A final coalgebra for $\mathcal{P}^\infty(\_) + Y$ is a quotient*

$$T^\sharp Y = BY/\sim$$

of the coalgebra BY modulo the bisimilarity equivalence given by

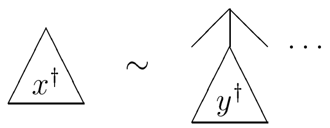
$$t \sim s \quad \textit{iff} \quad t \sim_i s \textit{ for all small ordinals } i.$$

Proof. Analogous to that of Theorem 5.5. □

Corollary 5.10 *Every guarded system of equations $e : X \longrightarrow T^\sharp(X + Y)$ has a unique solution $e^\dagger : X \longrightarrow T^\sharp Y$. In particular, every system of equations*

(3) $$x = A_x \quad x \in X$$

where the right-hand sides are subsets $A_x \subseteq X$, has a solution, i.e., a system $x^\dagger$ $(x \in X)$ of extensional trees such that



$y \in A_x$

holds for all $x \in X$, and these trees are unique up to bisimilarity.

Example 5.11 The equation

$$x = \{x\}$$

has as a solution the tree Ω of Example 5.1. And also the tree Ω'.

Remark 5.12 The possibility of uniquely solving all systems of equations (3) is the basis of non-well-founded set theory. In fact, every system (3) describes a graph on the set X (with edges those pairs (x, y) where $y \in A_x$) and a solution, provided that it is formed by sets rather than extensional trees, is precisely Aczel's concept of decoration of the graph. And Aczel's Antifoundation Axiom states that every graph has a unique decoration.

Now extensional trees are closely related to (well-founded) sets: In well-founded set theory

(a) every set has a graph of the elementhood relation which is extensional and has no infinite paths (i.e., is "well-founded" as a graph)

and

(b) two well-founded, extensional graphs are bisimilar if and only if they are equal.

Thus, non-well-founded set theory extends the concept of set so as to retain (a) and (b) for not necessarily well-founded graphs. Our concept of bisimilarity class of extensional graphs thus exactly corresponds to the concept of non-well-founded set.

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