

Elgot Theories: A New Perspective of Iteration Theories (Extended Abstract)*

Jiří Adámek^{a,2} Stefan Milius^{a,2} Jiří Velebil^{b,1}

^a *Institut für Theoretische Informatik, Technische Universität Braunschweig, Germany*

^b *Faculty of Electrical Engineering, Czech Technical University of Prague, Czech Republic*

Abstract

The concept of iteration theory of Bloom and Ésik summarizes all equational properties that iteration has in usual applications, e.g., in Domain Theory where to every system of recursive equations the least solution is assigned. However, this assignment in Domain Theory is also functorial. Yet, functoriality is not included in the definition of iteration theory. Pity: functorial iteration theories have a particularly simple axiomatization, and most of examples of iteration theories are functorial.

The reason for excluding functoriality was the view that this property cannot be called equational. This is true from the perspective of the category **Sgn** of signatures as the base category: whereas iteration theories are monadic (thus, equationally presentable) over **Sgn**, functorial iteration theories are not. In the present paper we propose to change the perspective and work, in lieu of **Sgn**, in the category of sets in context (the presheaf category of finite sets and functions). We prove that Elgot theories, which is our name for functorial iteration theories, are monadic over sets in context. Shortly: from the new perspective functoriality is equational.

Keywords: iteration theory, Elgot theory, iterative algebra, rational monad

1 Introduction

In Domain Theory one works in a continuous theory and one uses iteration expressed by the fact that for every equation-morphism $e: n \longrightarrow n + k$ there exists the least solution $e^\dagger: n \longrightarrow k$. This dagger operation $e \longmapsto e^\dagger$ enjoys a number of equational properties, e.g., the fact that e^\dagger is a solution of e is the equation $e^\dagger = [e^\dagger, \text{id}_k] \cdot e$. The aim of the concept of iteration theory of Stephen Bloom and Zoltan Ésik was to collect all equational properties of the dagger operation in Domain Theory (and in a substantial number of other applications where iteration

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is used, see the fundamental monograph [12]). The function $e \mapsto e^\dagger$ in Domain Theory is also functorial, that is, for every given k we obtain a functor $(-)^\dagger$ from the category of all equation morphisms $e: n \longrightarrow n + k$ to the slice category of k . This important property of functoriality is studied in various contexts, e.g., Alex Simpson and Gordon Plotkin call it parametrized uniformity in [22], and they say in their introduction that this is “a convenient tool for establishing that the equations of an iteration operator are satisfied”. Larry Moss observed in [21] that functorial iteration theories allow for a particularly simple axiomatization. Functoriality is, however, *not* a part of the definition of iteration theory; this property is called “functorial dagger implication” in the monograph [12]. The name and the non-inclusion into the definition both indicate that Bloom and Ésik do not consider functoriality an equational property. The aim of the present paper is to demonstrate that from a new perspective functoriality *is* equational. Thus *Elgot theories* which is our name for functorial iteration theories, form an important class of equationally specified algebraic theories. They are, as proved by Martin Hyland and by Masahito Hasegawa [18], precisely those theories that are traced cocartesian categories where the trace operation is uniform for all base morphisms.

Recall that for every signature Σ the free continuous theory on Σ is the theory $\mathbb{T}_{\Sigma_\perp}$ of Σ_\perp -trees: one adds to Σ a new nullary symbol \perp , forming a new signature Σ_\perp , and the morphisms from 1 to n in $\mathbb{T}_{\Sigma_\perp}$ are all Σ_\perp -trees (finite and infinite) on n variables. As proved by Bloom and Ésik, the free iterative theory on Σ is the subtheory $\mathbb{R}_{\Sigma_\perp}$ of all *rational* Σ_\perp -trees, that is, trees with finitely many subtrees up to isomorphism. This defines a monad **Rat** on the category **Sgn** of signatures:

$$\mathbf{Rat}(\Sigma) = \text{the signature of rational } \Sigma_\perp\text{-trees.}$$

We have proved recently that the Eilenberg-Moore algebras for this monad **Rat** are precisely the iteration theories, see [6]. It then follows from a general theory of equational presentations due to Max Kelly and John Power [19], recalled briefly in the Appendix below, that iteration theories are equationally presentable over **Sgn**. And the corresponding equations for dagger are precisely those that hold in Domain Theory since they are precisely those that hold in the theories $\mathbb{T}_{\Sigma_\perp}$ or $\mathbb{R}_{\Sigma_\perp}$. In contrast, Elgot theories are not monadic over the category of signatures.

However, free iteration theories exist not only on all signatures, but also on all sets in context, as we proved in [4]. The latter means objects of the functor category **Set** ^{\mathcal{F}} where \mathcal{F} is the category of natural numbers and all functions between them. Thus, a set in context X assigns (like a signature) to every $n \in \mathbb{N}$ a set $X(n)$ which we can consider as the set of all “formulas of type X in n variables”. And (unlike a signature) it assigns to every function $\varphi: n \longrightarrow m$ “changing variable names” a function $X\varphi: X(n) \longrightarrow X(m)$ of the corresponding “renaming of free variables” in formulas. See for example the semantics of λ -calculus presented by M. Fiore et al. [15] where λ -formulas are treated as a set in context. It follows from our results in [4] that for every set in context, $X \in \mathbf{Set}^{\mathcal{F}}$, a *rational theory* \mathbb{R}_X can be constructed analogously to the rational-tree theory for a signature (see also [3] for concrete descriptions of those theories \mathbb{R}_X). Moreover, in [7] we proved that

rational theories of the form \mathbb{R}_{X+1} are Elgot theories. Here we prove that \mathbb{R}_{X+1} is a free Elgot theory on X and that it is a quotient theory of the theory $\mathbb{R}_{\Sigma_{\perp}}$ for some Σ . This gives a monad \mathbf{Rat} on the category $\mathbf{Set}^{\mathcal{F}}$. Our main result is that the Eilenberg-Moore algebras for this monad are precisely the Elgot theories. It then follows from the results of Kelly and Power [19] that Elgot theories are equationally presentable over $\mathbf{Set}^{\mathcal{F}}$. And the corresponding equations for the dagger operation are precisely those that hold in Domain Theory because, once again, we only need to consider the free theories and they are quotients of the theories $\mathbb{R}_{\Sigma_{\perp}}$. The equational presentation of Elgot theories is particularly simple: the solution function $e \mapsto e^{\dagger}$ is requested to be functorial, and satisfy the Parameter Identity and the Bekić Identity, see Definition 2.8.

The first step in the proof of our result is the fact that Elgot’s iterative theories [13] (i. e., theories with unique solutions of all *ideal* equation morphisms) are Elgot theories, see [12], Theorem 4.4.5. Here we work in a more general category theoretic setting; in lieu of theories we consider finitary monads on a locally finitely presentable and hyper-extensive category, see Assumption 3.1. In [2] it was proved that every iterative monad on such a category has unique *strict* solutions of all equation morphisms, and then we proved in [7] that the corresponding dagger operation satisfies all the axioms of Elgot theories.

2 Elgot Theories and Elgot Monads

Assumption 2.1 *Throughout this section \mathcal{K} denotes a locally finitely presentable category, see [16] or [10]. More detailed:*

- (i) \mathcal{K} has colimits, and
- (ii) \mathcal{K} has a small full subcategory \mathcal{F} representing all finitely presentable objects such that every object of \mathcal{K} is a filtered colimit of objects of \mathcal{F} .

An object n is called **finitely presentable** if $\mathcal{K}(n, -)$ preserves filtered colimits. More generally: functors preserving filtered colimits are called **finitary**.

Fact 2.2 A finitary functor $H: \mathcal{K} \rightarrow \mathcal{K}$ is, up to natural isomorphism, fully determined by its restriction $H|_{\mathcal{F}}$ in $\mathcal{K}^{\mathcal{F}}$. In fact, H is the left Kan extension of $H|_{\mathcal{F}}$ along the inclusion $\mathcal{F} \rightarrow \mathcal{K}$. Thus, we have an equivalence of categories

$$\mathcal{K}^{\mathcal{F}} \cong \mathbf{Fin}(\mathcal{K})$$

where $\mathbf{Fin}(\mathcal{K})$ is the category of finitary endofunctors and natural transformations.

Remark 2.3 (Monads and Theories)

- (i) Recall that a *monad* $\mathbb{S} = (S, \eta, \mu)$ consists of an endofunctor $S: \mathcal{K} \rightarrow \mathcal{K}$ and natural transformations $\eta: \text{Id} \rightarrow S$ and $\mu: S \cdot S \rightarrow S$ such that $\mu \cdot \eta S = \text{id}_S = \mu \cdot S \eta$ and $\mu \cdot S \mu = \mu \cdot \mu S$. The monad is called *finitary* if S is a finitary functor.

(ii) The *Kleisli category* $\mathcal{K}_{\mathbb{S}}$ of \mathbb{S} has the same objects as \mathcal{K} and its morphisms

$$f: X \multimap Y$$

are the morphisms $f: X \longrightarrow SY$ of \mathcal{K} . They compose as follows: given $g: Y \multimap Z$ the composite $g \cdot f$ in the Kleisli category is the \mathcal{K} -morphism

$$X \xrightarrow{f} SY \xrightarrow{Sg} SSZ \xrightarrow{\mu_Z} Z.$$

(iii) There is the canonical functor $J: \mathcal{K} \longrightarrow \mathcal{K}_{\mathbb{S}}$ which assigns to $f: X \longrightarrow Y$ the morphism $Jf = \eta_Y \cdot f: X \longrightarrow SY$ in $\mathcal{K}_{\mathbb{S}}$; we will write $f: X \longrightarrow Y$ for $Jf: X \multimap Y$ and call f a *base morphism*.

(iv) The *theory* of \mathbb{S} is denoted by

$$\text{Th}(\mathbb{S});$$

it is the category whose objects are the objects of \mathcal{F} and morphisms are the Kleisli morphisms.

(v) $\text{Th}(\mathbb{S})$ has finite colimits formed on the level of the base category \mathcal{K} . In particular, finite coproducts in \mathcal{K} and in $\text{Th}(\mathbb{S})$ are the same.

Example 2.4 If $\mathcal{K} = \mathbf{Set}$ we can choose \mathcal{F} to be the category of natural numbers and functions between them.

Every finitary monad \mathbb{S} on \mathbf{Set} is equationally presentable: there exists a signature Σ and a set E of equations such that \mathbb{S} is the monad of all free algebras in the variety $\mathbf{Alg}(\Sigma, E)$ presented by E . Then $\text{Th}(\mathbb{S})$ is the category of natural numbers with hom-sets $\text{Th}(\mathbb{S})(1, n)$ formed by terms in n variables of the variety $\mathbf{Alg}(\Sigma, E)$, and $\text{Th}(\mathbb{S})(k, n) = (\text{Th}(\mathbb{S})(1, n))^k$ of k -tuples formed by such terms.

Definition 2.5 Let \mathbb{S} be a finitary monad.

- (i) An *equation morphism* is a morphism $e: n \multimap n + k$ in the theory of \mathbb{S} . We refer to k as the object of *parameters*. (That is, an equation morphism with k as object of parameters is given by a finitely presentable object n and a morphism $e: n \longrightarrow S(n + k)$.)
- (ii) A *solution* of e is a morphism $e^\dagger: n \multimap k$ such that the triangle below commutes in $\text{Th}(\mathbb{S})$:

$$\begin{array}{ccc}
 n & \xrightarrow{e^\dagger} & k \\
 \downarrow e & \searrow [e^\dagger, k] & \\
 n + k & &
 \end{array}
 \tag{1}$$

Example 2.6 For \mathbb{S} as in Example 2.4 the morphism $e: n \longrightarrow S(n + k)$ can be viewed as n recursive equations

$$x_i \approx t_i(x_1, \dots, x_n, y_1, \dots, y_k) \quad i = 1, \dots, n$$

where t_i is a (Σ, E) -term in $n + k$ variables. A solution is then a substitution of terms $x_i^\dagger(y_1, \dots, y_k)$ for variables x_i making each of the formal equations an identity

$$x_i^\dagger = t_i(x_1^\dagger, \dots, x_n^\dagger, y_1, \dots, y_k).$$

Remark 2.7 In the following definition we assume that every equation morphism e is given a solution e^\dagger “canonically”. This means that various “natural” equational properties are requested. It was observed by Larry Moss [21] that, for $\mathcal{K} = \mathbf{Set}$, this definition is equivalent to the definition of functorial iteration theory by Stephen Bloom and Zoltan Ésik [12]—we state this in our setting of finitary monads of \mathcal{K} :

Definition 2.8 An *Elgot monad* is a finitary monad \mathbb{S} together with an operation

$$\frac{e: n \multimap \rightarrow n + k}{e^\dagger: n \multimap \rightarrow k} \quad (\text{for all } n, k \in \mathcal{F})$$

satisfying the following axioms:

Solution: $e^\dagger = [e^\dagger, k] \cdot e$.

Functoriality: Given a “homomorphism of equations”, i. e., a base morphism v with

$$\begin{array}{ccc} n & \xrightarrow{e} & n + k \\ \downarrow v & & \downarrow v+k \\ m & \xrightarrow{f} & m + k \end{array}$$

then $f^\dagger = e^\dagger \cdot v$.

Parameter Identity: Given $u: k \multimap \rightarrow k'$, then $u \cdot e^\dagger = (u \bullet e)^\dagger$ where

$$u \bullet e = n \xrightarrow{e} n + k \xrightarrow{n+u} n + k'. \tag{2}$$

Bekić Identity: Given $e: n \multimap \rightarrow n + m + k$ and $f: m \multimap \rightarrow n + m + k$ form

$$e_R = [e^\dagger, m + k] \cdot f: m \multimap \rightarrow m + k, \quad \text{and} \quad e_L = (n + [e^\dagger_R, k]) \cdot e: n \multimap \rightarrow n + k$$

then $[e, f]^\dagger = [e_L^\dagger, e_R^\dagger]: n + m \multimap \rightarrow k$.

Remark 2.9 An *Elgot theory* is the theory $\text{Th}(\mathbb{S})$ of an Elgot monad \mathbb{S} . Equivalently, $\text{Th}(\mathbb{S})$ is a traced cocartesian category with the trace uniform for base morphisms; see [18].

Example 2.10 We present some examples of Elgot theories (or monads) in \mathbf{Set} .

(i) *Partial-function theory.* We consider the monad \mathbb{S} with $S = \text{Id} + 1$ (whose algebras are pointed sets). Its theory is $\text{Th}(\mathbb{S}) = \text{Pfn}$ the category of natural numbers and partial functions. To every partial function $e: n \multimap \rightarrow n + k$ we assign its iteration $e^\dagger: n \multimap \rightarrow k$ defined in an element x of n iff $e(x), e(e(x)), \dots, e^i(x)$ are defined and $e^i(x)$ lies in k ; then $e^\dagger(x) = e^i(x)$.

(ii) *Multifunction theory.* Here we take the finite-power-set monad \mathbb{P}_f (whose algebras are join semilattices with a least element). Its theory is

$$\text{Th}(\mathbb{P}_f) = \text{Mfn}$$

the category of natural numbers and multifunctions. For every multifunction $a: n \multimap n$ denote by a^* its iteration $a^* = \text{id}_n \cup a \cup (a \cdot a) \cup \dots$. Then the dagger of $e: n \multimap n + k$ is defined as follows: let $a: n \multimap n$ and $b: n \multimap k$ be the multifunctions with $e = a \cup b$, then $e^\dagger = b \cdot a^*$.

(iii) *The free-semigroup theory* $X \mapsto X^+$ is not an Elgot theory. But we can extend it by adding to X^+ an absorbing element \perp (that is, the binary operation of concatenation is extended by $w \cdot \perp = \perp = \perp \cdot w$ for all $w \in X^+$). The resulting monad $SX = X^+ + \{\perp\}$ is iterative, see [8], thus yields an Elgot monad, as we show in Section 3.

(iv) *Infinite-tree theory.* Let Σ be a signature and let $T_\Sigma(n)$ denote the Σ -algebra of all Σ -trees on n variables, that is, (rooted and ordered) trees with leaves labelled in $n + \Sigma_0$ and nodes of $k > 0$ children labeled in Σ_k . This gives rise to a finitary monad \mathbb{T}_Σ . This was first observed by Eric Badouel [11]. Let us add one new nullary operation \perp . We obtain a signature $\Sigma_\perp = \Sigma + \{\perp\}$ for which $\mathbb{T}_{\Sigma_\perp}$ is an Elgot monad.

(v) *Rational-tree theory.* A tree is called *rational* (or regular) if it has up to isomorphism only finitely many subtrees, see [17]. We denote by \mathbb{R}_Σ the submonad of \mathbb{T}_Σ formed by all rational Σ -trees. As proved in [12], the theory of $\mathbb{R}_{\Sigma_\perp}$ is the free iteration theory on the signature Σ . We will see below that this is also the free Elgot theory on Σ .

Definition 2.11 Let (\mathbb{S}, \dagger) and (\mathbb{T}, \ddagger) be Elgot monads. An *Elgot monad morphism* α from (\mathbb{S}, \dagger) to (\mathbb{T}, \ddagger) is a monad morphism $\alpha: \mathbb{S} \longrightarrow \mathbb{T}$ that is solution-preserving, in the sense that for every equation morphism $e: n \longrightarrow S(n + k)$ we have

$$\alpha_k \cdot e^\dagger = (\alpha_{n+k} \cdot e)^\ddagger.$$

The category of Elgot monads and their morphisms is denoted by

$$\mathbf{EM}(\mathcal{K}).$$

We denote its forgetful functor into $\mathcal{K}^{\mathcal{F}}$ by

$$U: \mathbf{EM}(\mathcal{K}) \longrightarrow \mathcal{K}^{\mathcal{F}}.$$

It assigns to every Elgot monad (\mathbb{S}, \dagger) the restriction functor S/\mathcal{F} in $\mathcal{K}^{\mathcal{F}}$.

Remark 2.12

- (i) The aim of our paper is to prove that Elgot theories are monadic over sets in context, that is, if $\mathcal{K} = \mathbf{Set}$ then U is a monadic functor.

- (ii) We will prove a more general result: $\mathbf{EM}(\mathcal{K})$ is monadic over $\mathcal{K}^{\mathcal{F}}$ for every locally finitely presentable category satisfying an additional assumption called hyper-extensivity.

3 Iterative Theories

In this section we prove the main technical result of our paper: free Elgot theories coincide with free iterative theories of Calvin Elgot [13]. This continues the category theoretic extension and generalization of the work of Elgot as presented in [4,5,3,2,7].

Assumption 3.1 *Throughout this section we assume that \mathcal{K} is a locally finitely presentable category which is **hyper-extensive**, that is, every object is a coproduct of connected objects A (where A is called connected if the hom-functor $\mathcal{K}(A, -)$ preserves coproducts). We also assume that a finitary monad $\mathbb{S} = (S, \eta^S, \mu^S)$ is given. monad on*

Example 3.2

- (i) The categories of sets, posets, graphs and unary algebras are hyper-extensive and locally finitely presentable.
- (ii) If \mathcal{K} has both properties, so do all presheaf categories on \mathcal{K} . Thus, $\mathbf{Set}^{\mathcal{F}}$ (equivalently, $\mathbf{Fin}(\mathbf{Set})$) is an example.

Definition 3.3 A finitary monad \mathbb{S} is called *ideal* if there exists a subfunctor $\sigma : S' \rightarrow S$ such that $S = S' + \text{Id}$ with injections σ and η^S , and if μ^S has a restriction $(\mu')^S : S'S \rightarrow S$ with $\sigma \cdot (\mu')^S = \mu^S \cdot \sigma^S$. An ideal monad is called *iterative* if every equation morphism $e : n \rightarrow S(n+k)$ which factorizes through σ_{n+k} (i.e., we have $e = \sigma_{n+k} \cdot e'$ for some $e' : n \rightarrow S'(n+k)$) has a unique solution e^\dagger .

Example 3.4 The monads $SX = X^+ + \{\perp\}$, \mathbb{T}_Σ and \mathbb{R}_Σ from Example 2.10 are iterative.

Remark 3.5 (i) A *strict* endofunctor is an endofunctor H with a chosen morphism $\perp : 0 \rightarrow H1$. Notice that every Elgot monad is strict w.r.t. the solution of $e : 0 \rightarrow 0 + 1$. Also H_{Σ_\perp} is strict, and for every endofunctor H the functor $H + 1$ is strict.

(ii) A *strict natural transformation* between strict functors is a natural transformation preserving \perp (in the obvious sense).

Theorem 3.6 (see [7]) *Every strict iterative monad is an Elgot monad.*

Notation 3.7 *We denote by*

$$\mathbf{IM}_\perp(\mathcal{K})$$

the full subcategory of all strict iterative monads in $\mathbf{EM}(\mathcal{K})$. By abuse of notation, we write $U : \mathbf{IM}_\perp(\mathcal{K}) \rightarrow \mathcal{K}^{\mathcal{F}}$ for the forgetful functor as in Definition 2.11.

Remark 3.8 Observe that a slightly different category and forgetful functor was used in [4]: the category

$$\mathbf{IM}(\mathcal{K})$$

of iterative monads and *ideal monad morphisms*, that is monad morphisms $\alpha: \mathbb{S} \longrightarrow \mathbb{T}$ such that the natural transformation $\alpha: S' + \text{Id} \longrightarrow T' + \text{Id}$ has the form $\alpha = \alpha' + \text{Id}$ for some natural transformation $\alpha': S' \longrightarrow T'$.

We have the forgetful functor

$$U': \mathbf{IM}(\mathcal{K}) \longrightarrow \mathcal{K}^{\mathcal{F}}$$

assigning to every iterative monad $\mathbb{S} = (S' + \text{Id}, \eta^S, \mu^S)$ the restriction of the sub-functor S' to \mathcal{F} : $U'(\mathbb{S}) = S'/\mathcal{F}$.

Theorem 3.9 (see [4]) *The forgetful functor U' has a left adjoint assigning to every finitary endofunctor H of \mathcal{K} the free iterative monad \mathbb{R}_H on H (called the **rational monad** of H).*

Example 3.10 For a given signature Σ the associated polynomial endofunctor of **Set** is given by $H_\Sigma X = \coprod_{i \in \mathbb{N}} X^i \times \Sigma_i$. Its algebras are the classical Σ -algebras in **Set**. The functor H_Σ is finitary, and its rational monad is the monad \mathbb{R}_Σ of Example 2.10(v).

Remark 3.11 Monadic algebras for the rational monad \mathbb{R}_H were characterized in [5] as precisely those H -algebras equipped with an operation of taking solutions of “flat” equation morphisms which satisfies two “natural” axioms. Let us recall this concept that we called Elgot algebras.

Given an algebra $a: HA \longrightarrow A$ for H , *flat equation morphisms* in A are the morphisms $e: n \longrightarrow Hn + A$, $n \in \mathcal{F}$, of \mathcal{K} . For example, if $H = H_\Sigma$ then whereas general equation morphisms $e: n \twoheadrightarrow n + k$ are systems of equations $x_i \approx t_i$ with right-hand sides t_i being general terms, see Example 2.6, the flat equation morphisms

$$e: n \longrightarrow \coprod_{i \in \mathbb{N}} n^i \times \Sigma_i + A$$

have right-hand sides either as elements of A , or as flat terms $\sigma(x_0, \dots, x_{i-1})$ for some $\sigma \in \Sigma_i$ and some variables x_0, \dots, x_{i-1} in n . However, each general system can be “flattened” by introducing new variables.

Definition 3.12 By an *Elgot algebra* for H is meant an algebra $a: HA \longrightarrow A$ together with a function

$$\frac{e: n \longrightarrow Hn + A}{e^\dagger: n \longrightarrow A} \quad (\text{for all } n \in \mathcal{F})$$

such that the following axioms hold:

Solution:

$$\begin{array}{ccc}
 n & \xrightarrow{e^\dagger} & A \\
 e \downarrow & & \downarrow [a, A] \\
 Hn + A & \xrightarrow{He^\dagger + A} & HA + A
 \end{array} \tag{3}$$

Functoriality: Given a “homomorphism of of flat equations”, i. e., a morphism $v : n \longrightarrow m$ with

$$\begin{array}{ccc}
 n & \xrightarrow{e} & Hn + A \\
 \downarrow v & & \downarrow Hv+A \\
 \underline{m} & \xrightarrow{f} & Hm + A
 \end{array} \tag{4}$$

then $f^\dagger \cdot v = e^\dagger$.

Compositionality: Given

$$e : n \longrightarrow Hn + k \quad \text{and} \quad f : k \longrightarrow Hk + A \quad (n, k \in \mathcal{F})$$

form the equation morphisms $f^\dagger \bullet e = (Hn + f^\dagger) \cdot e$ and

$$f \blacksquare e \equiv n + k \xrightarrow{[e, \text{inr}]} Hn + k \xrightarrow{Hn+f} Hn + Hk + A \xrightarrow{\text{can}+A} H(n+k) + A, \tag{5}$$

where $\text{can} = [H \text{ inl}, H \text{ inr}]$ is the canonical morphism. Then we have

$$\begin{array}{ccc}
 n & \xrightarrow{(f^\dagger \bullet e)^\dagger} & A \\
 \downarrow \text{inl} & \nearrow (f \blacksquare e)^\dagger & \\
 n + k & &
 \end{array} \tag{6}$$

Notation 3.13 We denote by

$$\mathbf{Elg}(H)$$

the category of Elgot algebras and their homomorphisms, that is, those morphisms $p : A \longrightarrow B$ that preserve solutions: for every flat equation morphism $e : n \longrightarrow Hn + A$ the corresponding equation morphism $p \bullet e = (Hn + p) \cdot e : n \longrightarrow Hn + B$ fulfils

$$\begin{array}{ccc}
 & & A \\
 & \nearrow e^\dagger & \downarrow p \\
 n & & B \\
 & \searrow (p \bullet e)^\dagger &
 \end{array}$$

Note that every solution-preserving morphism p from (A, a, \dagger) to (B, b, \ddagger) is a homomorphism of H -algebras, i.e., $p \cdot a = b \cdot Hp$. We have the forgetful functor

$$U : \mathbf{Elg}(H) \longrightarrow \mathcal{K}, \quad (A, a, \dagger) \longmapsto A.$$

Theorem 3.14 (See [5].) *The category of Elgot algebras for H is isomorphic to the category of Eilenberg-Moore algebras for the rational monad \mathbb{R}_H , shortly: $U : \mathbf{Elg}(H) \longrightarrow \mathcal{K}$ is monadic with the associated monad \mathbb{R}_H .*

Theorem 3.15 *For every strict finitary endofunctor H the rational monad \mathbb{R}_H is the free Elgot monad on H . That is, for every Elgot monad \mathbb{S} and every strict natural transformation $\lambda: H \longrightarrow S$ there exists a unique Elgot monad morphism $m: \mathbb{R}_H \longrightarrow \mathbb{S}$ extending λ .*

Sketch of proof. (1) The first step in our proof is the verification that for every object A of \mathcal{A} the algebra $\mu_A^S \cdot \lambda_{SA}: H(SA) \longrightarrow SA$ is an Elgot algebra. Its operation $e \longmapsto e^\dagger$ is defined for $e: n \longrightarrow Hn + SA$ as follows: apply the solution operation of the Elgot monad \mathbb{S} to the following equation morphism:

$$n \xrightarrow{e} Hn + SA \xrightarrow{\lambda_n + SA} Sn + SA \xrightarrow{\text{can}} S(n + A).$$

The verification that we indeed have an Elgot algebra is non-trivial, and we must omit the details here.

Since $\eta_A: A \longrightarrow R_H A$ is the free Elgot algebra on A , we obtain the unique Elgot algebra morphism

$$m_A: R_H A \longrightarrow SA \quad \text{with} \quad m_A \cdot \eta_A = \eta_A^S.$$

(2) The next step is to prove that these morphisms m_A form a natural transformation $m: \mathbb{R}_H \longrightarrow \mathbb{S}$ which is a monad morphism and, in fact, a morphism of Elgot monads. The proof is quite involved making use of the axioms of Elgot monads for \mathbb{R}_H and the way the dagger operation of \mathbb{R}_H is defined in several steps, see [7] and [2]. Due to space constraints we have to omit the details.

(3) Finally, one needs to verify that m is the unique extension of λ . □

4 The Monad Rat and its Algebras

Assumption 4.1 *We still assume that \mathcal{K} is a hyper-extensive, locally finitely presentable category. Recall that \mathcal{F} is its small, full subcategory representing all finitely presentable objects.*

Proposition 4.2 *The forgetful functor $U: \mathbf{EM}(\mathcal{K}) \longrightarrow \mathcal{K}^{\mathcal{F}}$ (see Definition 2.11) has a left adjoint*

$$\Phi: \mathcal{K}^{\mathcal{F}} \longrightarrow \mathbf{EM}(\mathcal{K})$$

assigning to every X in $\mathcal{K}^{\mathcal{F}}$ the rational monad \mathbb{R}_{X+1} of $X + 1$.

Proof. Recall that $\mathcal{K}^{\mathcal{F}}$ is equivalent to the category $\mathbf{Fin}(\mathcal{K})$ of finitary endofunctors. Thus, we can work with the forgetful functor in the form $U: \mathbf{EM}(\mathcal{K}) \longrightarrow \mathbf{Fin}(\mathcal{K})$, given by $U(\mathbb{S}) = S$. This is a composite $U = \widehat{U} \cdot W$ of the forgetful functor W into the category $\mathbf{Fin}_\perp(\mathcal{K})$ of all strict finitary endofunctors and strict natural transformations and the functor $\widehat{U}: \mathbf{Fin}_\perp(\mathcal{K}) \longrightarrow \mathbf{Fin}(\mathcal{K})$ forgetting \perp . From Theorem 3.15 and the fact that \widehat{U} has the left-adjoint $X \longmapsto X + 1$ we conclude that U has the left adjoint as stated. □

Corollary 4.3 *The forgetful functor $U : \mathbf{IM}_\perp \longrightarrow \mathcal{K}^{\mathcal{F}}$ of the category of strict iterative monads has a left adjoint.*

In fact, the free Elgot monad \mathbb{R}_{X+1} on the set in context X is a strict iterative monad.

Example 4.4 Here we consider $\mathcal{K} = \mathbf{Set}$.

- (i) The value of Φ at H_Σ , see Example 3.10, is as follows: recall the notation $\Sigma_\perp = \Sigma + \{\perp\}$ from the Introduction and observe that $H_{\Sigma_\perp} = H_\Sigma + 1$. Thus, $\Phi(H_{\Sigma_\perp}) = \mathbb{R}_{\Sigma_\perp}$, the rational Σ_\perp -tree monad.
- (ii) The value of Φ at an arbitrary set in context X (considered as an endofunctor): express X as a quotient of H_Σ for some Σ . For example, the signature $\Sigma_n = X(n)$, for all $n \in \mathbb{N}$, yields, by Yoneda Lemma, an epimorphism (that is, a natural transformation with surjective components) $\varepsilon : H_\Sigma \twoheadrightarrow X$. We extend it to an epimorphism $\bar{\varepsilon} = \varepsilon + 1 : H_{\Sigma_\perp} \twoheadrightarrow X + 1$. Since Φ , being a left adjoint, preserves epimorphisms, we see that $\Phi(X) = \mathbb{R}_{X+1}$ is a quotient of $\mathbb{R}_{\Sigma_\perp}$ via $\Phi(\bar{\varepsilon}) : \mathbb{R}_{\Sigma_\perp} \longrightarrow \mathbb{R}_{X+1}$. In fact, in [3] the monad \mathbb{R}_{X+1} was described concretely: if ε is given by a set E of equations (between flat Σ -terms), then \mathbb{R}_{X+1} is the quotient of $\mathbb{R}_{\Sigma_\perp}$ modulo a potentially infinite application of the equations in E .

Definition 4.5 We denote by \mathbf{Rat} the monad on $\mathcal{K}^{\mathcal{F}}$ given by the adjunction $\Phi \dashv U$ above. Thus, on objects X we have $\mathbf{Rat}(X) = R_{X+1}/\mathcal{F}$, where R_{X+1} is the underlying functor of the rational monad of $X + 1$.

Theorem 4.6 *The forgetful functor U of the category of Elgot monads is monadic, with \mathbf{Rat} as the corresponding monad.*

Proof. We know from Proposition 4.2 that U has a left adjoint and the corresponding monad is \mathbf{Rat} . Thus, we only need to prove that U creates coequalizers of U -split pairs, then monadicity follows from Beck’s Theorem, see [20]. In more detail, suppose we are given a pair of parallel Elgot monad morphisms $\alpha, \beta : (\mathbb{T}, \dagger) \longrightarrow (\mathbb{S}, \dagger)$ and natural transformations

$$T \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\tau} \\ \xleftarrow{\beta} \end{array} S \begin{array}{c} \xrightarrow{\psi} \\ \xleftarrow{\sigma} \end{array} C \quad \text{for } C \text{ in } \mathbf{Fin}(\mathcal{K})$$

such that

$$\psi \cdot \alpha = \psi \cdot \beta, \quad \psi \cdot \sigma = \text{id}_C, \quad \beta \cdot \tau = \text{id}_S, \quad \text{and} \quad \sigma \cdot \psi = \alpha \cdot \tau. \tag{7}$$

We must prove that there exists a unique Elgot monad \mathbb{C} on C such that $\psi : \mathbb{S} \longrightarrow \mathbb{C}$ is an Elgot monad morphism, and moreover, ψ is a coequalizer of α and β in $\mathbf{EM}(\mathcal{K})$.

It is a trivial application of Beck’s Theorem that for the category $\mathbf{FM}(\mathcal{K})$ of finitary monads on \mathcal{K} the forgetful functor $V : \mathbf{FM}(\mathcal{K}) \longrightarrow \mathcal{K}^{\mathcal{F}}$ given by $V(\mathbb{S}) = S/\mathcal{F}$ is monadic. Consequently, V creates the coequalizer above, thus there

exists a unique structure $\mathbb{C} = (C, \eta^C, \mu^C)$ of a finitary monad such that ψ is a monad morphism and a coequalizer of α and β in $\mathbf{FM}(\mathcal{X})$.

Next, we prove that there exists at most one structure $e \mapsto e^*$ of an Elgot monad on \mathbb{C} for which ψ is solution-preserving. In fact, the equation $\psi_k \cdot f^\dagger = (\psi_{n+k} \cdot f)^*$ of Definition 2.11, where $f: n \rightarrow S(n+k)$, implies that e^* must be defined, for every $e: n \rightarrow C(n+k)$, by

$$e^* = \psi_k \cdot (\sigma_{n+k} \cdot e)^\dagger.$$

With this definition ψ preserves solutions: due to (7) we have

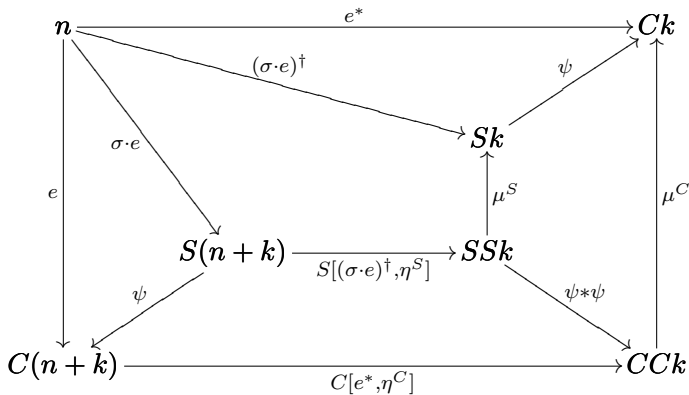
$$(\psi_{n+k} \cdot f)^* = \psi_k \cdot (\sigma_{n+k} \cdot \psi_{n+k} \cdot f) = \psi_k \cdot (\alpha_{n+k} \cdot \tau_{n+k} \cdot f)^\dagger = \psi_k \cdot \alpha_k \cdot (\tau_{n+k} \cdot f)^\ddagger$$

since α is solution preserving. The last morphism is $\psi_k \cdot f^\dagger$ since (7) and the fact that β is solution-preserving yield

$$\psi_k \cdot \alpha_k \cdot (\tau_{n+k} \cdot f)^\ddagger = \psi_k \cdot \beta_k \cdot (\tau_{n+k} \cdot f)^\ddagger = \psi_k \cdot (\beta_{n+k} \cdot \tau_{n+k} \cdot f)^\dagger = \psi_k \cdot f^\dagger$$

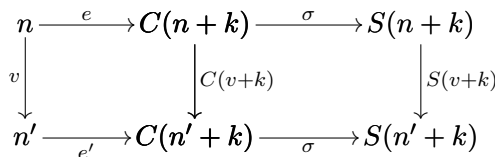
We will verify below that $(-)^*$ satisfies the axioms of Elgot monads. Then it is easy to prove that ψ is the coequalizer of α and β in $\mathbf{EM}(\mathcal{X})$.

(a) Proof of Solution. In the diagram

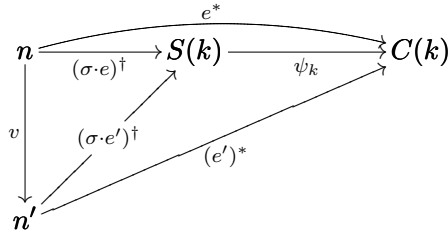


all inner parts commute: this is clear for the right-hand square since $\psi: \mathbb{S} \rightarrow \mathbb{C}$ is a monad morphism, for the middle square due to Solution w.r.t. \mathbb{S} , and the left-hand triangle follows from (7). The lower square is easy to verify.

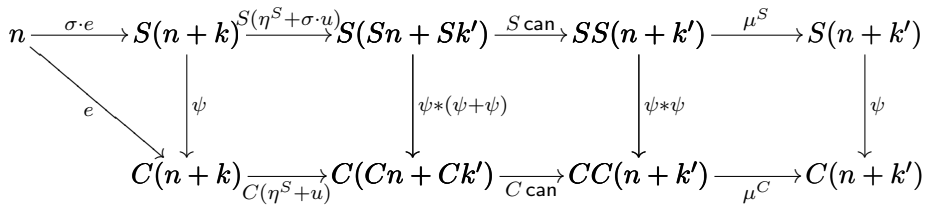
(b) Proof of Functoriality. Every homomorphism of equations v w.r.t. \mathbb{C} yields one w.r.t. \mathbb{S} by the naturality of σ :



The desired triangle follows from Functoriality w.r.t. \mathbb{S} :



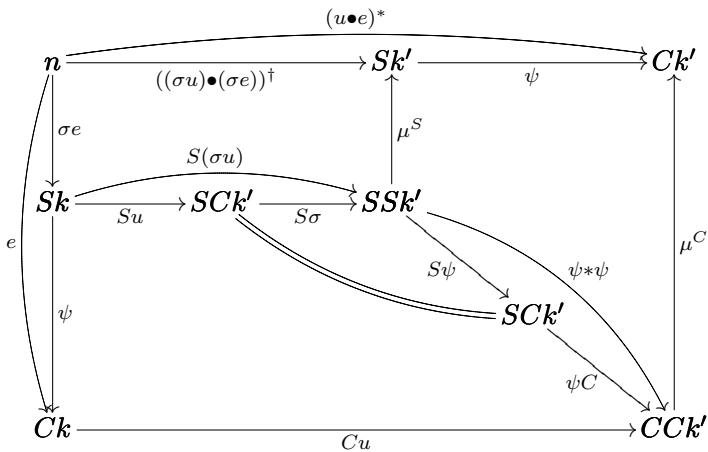
(c) Proof of Parameter Identity. Given $e: n \longrightarrow C(n+k)$ and $u: k \dashrightarrow k'$, we first relate $u \bullet e: n \longrightarrow C(n+k')$ and $(\sigma_{k'} \bullet u) \bullet (\sigma_{n+k} \bullet e): n \longrightarrow S(n+k')$ (recall the definition of \bullet from (2)). In the following diagram we use (2) expressed in the base category \mathcal{K} for the equation morphisms of interest; the commutativity of the diagram



implies

$$(u \bullet e)^* = \psi_{k'} \cdot ((\sigma u) \bullet (\sigma e))^\dagger. \tag{8}$$

To see that the Parameter Identity holds for $(-)^*$ we now verify that the following diagram commutes:



The upper part commutes by (8), the left-hand square by the Parameter Identity for \mathbb{S} , for the inner and left-hand triangles use (7), and all other parts commute since ψ is a monad morphism.

(d) Proof of Bekić Identity. Given $e: n \longrightarrow C(n + m + k)$ and $f: m \longrightarrow C(n + m + k)$ we form the morphisms e_L and e_R for e as in Definition 2.8 applied to \mathbb{C} . And we also form, for $\sigma \cdot e$ and $\sigma \cdot f$, the corresponding morphisms w.r.t. \mathbb{S} and denote them by ε_L and ε_R , respectively. For ε_R we get the diagram (written in \mathcal{X} once more)

$$\begin{array}{ccccccc}
 m & \xrightarrow{f} & C(n+m+k) & \xrightarrow{\sigma} & S(n+m+k) & \xrightarrow{S[(\sigma \cdot e)^\dagger, \eta^S]} & SS(m+k) & \xrightarrow{\mu^S} & S(m+k) \\
 \parallel & & \searrow & & \downarrow \psi & & \downarrow \psi * \psi & & \downarrow \psi \\
 m & \xrightarrow{f} & C(n+m+k) & \xrightarrow{C[e^*, \eta^C]} & CC(m+k) & \xrightarrow{\mu^C} & C(m+k) & & \\
 & & & & & & & & \\
 & & & & & & & & \varepsilon_R
 \end{array}
 \tag{9}$$

which clearly commutes (recall (7)). This implies, since ψ is solution-preserving,

$$e_R^* = \psi_k \cdot \varepsilon_R^\dagger \tag{10}$$

Analogously, for ε_L we have

$$\begin{array}{ccccccc}
 n & & & & & & & & \\
 \downarrow e & & & & \varepsilon_L & & & & \\
 C(n+m+k) & \xrightarrow{\sigma} & S(n+m+k) & \xrightarrow{S(\eta^S + [e_R^\dagger, \eta^S])} & S(Sn + Sk) & \xrightarrow{\mu^S \cdot S \text{ can}} & S(n+k) \\
 & & \downarrow \psi & & \downarrow \psi * (\psi + \psi) & & \downarrow \psi \\
 & & C(n+m+k) & \xrightarrow{C(\eta^C + [e_R^*, \eta^C])} & C(Cn + Ck) & \xrightarrow{\mu^C \cdot C \text{ can}} & C(n+k) \\
 & & & & & & \\
 & & & & & & \varepsilon_L
 \end{array}
 \tag{11}$$

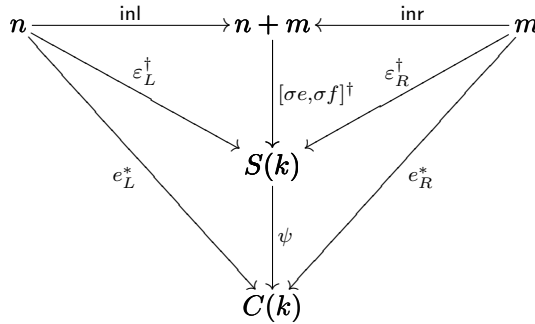
The commutativity of the middle square follows from

$$\begin{array}{ccc}
 S(n+m+k) & \xrightarrow{S(\eta^S + [e_R^\dagger, \eta^S])} & S(Sn + Sk) \\
 \downarrow \psi & & \downarrow \psi(S+S) \\
 C(n+m+k) & \xrightarrow{C(\eta^C + [e_R^*, \eta^C])} & C(Cn + Ck) \\
 & \searrow C(\eta^C + [e_R^*, \eta^C]) & \downarrow C(\psi + \psi) \\
 & & C(Cn + Ck)
 \end{array}$$

The square is the naturality of ψ , the triangle is easy: delete C and consider the components separately using $\psi \cdot \eta^S = \eta^C$ (since ψ is a monad morphism) and (10). From (11) we derive (analogously to (10))

$$e_L^* = \psi_k \cdot \varepsilon_L^\dagger \tag{12}$$

We now see that the Bekić Identity for \mathbb{S} implies that for \mathbb{C} :



The upper triangles follow from $\sigma \cdot [e, f] = [\sigma \cdot e, \sigma \cdot f]$ using Bekić Identity for \mathbb{S} , and the lower ones follow from (10) and (12). □

Remark 4.7 Notice that in the proof of Functoriality the naturality of $\sigma: \mathbb{C} \longrightarrow \mathbb{S}$ is essential, whereas it is not used in the proof of the other axioms. This accounts for the fact that Functoriality is not an axiom for iteration theories, where one works over the category **Sgn** of signatures, see [6]. But for Elgot theories Functoriality is an equational axiom (or rather, an infinite set of axioms) since we are working over the category **Fin**(\mathcal{K}) of finitary endofunctors of \mathcal{K} (or, equivalently, sets in context $\mathcal{K}^{\mathcal{F}}$). We shall further discuss this in the Appendix below.

Corollary 4.8 *Elgot monads are precisely the monadic algebras for the monad Rat on $\mathcal{K}^{\mathcal{F}}$.*

In fact, since U is monadic, we have an isomorphism between the categories of Elgot monads and of algebras for Rat :

$$\mathbf{EM}(\mathcal{K}) \cong (\mathcal{K}^{\mathcal{F}})^{\text{Rat}}.$$

Corollary 4.9 *The axioms of Elgot monads on **Set** precisely summarize all equational properties that the assignment*

$$e^\dagger = \text{least solution of } e$$

has in Domain Theory. More detailed:

- (i) *If an equation over **Set** ^{\mathcal{F}} holds for least solutions in all continuous theories, then that equation follows from the axioms of Elgot monads, and*
- (ii) *Every axiom of Elgot monads holds in all continuous theories.*

In fact, (ii) has been proved by Stephen Bloom and Zoltan Ésik in [12]. To see (i), apply the results of Max Kelly and John Power in the Appendix to the monad Rat . We know that the algebras for Rat form an equational class for some signature Γ on **Set** ^{\mathcal{F}} . Every equation which holds in continuous theories holds in the Σ_\perp -tree theories of Example 2.10(vi). Consequently, it holds in the theories $\mathbb{R}_{\Sigma_\perp}$ of rational Σ_\perp -trees, see Example 2.10(vii), since the definition of e^\dagger is the same

as in $\mathbb{T}_{\Sigma_{\perp}}$. For every free algebra for **Rat** the same equation must hold again since by Example 4.4(ii) these free algebras are quotients of $\mathbb{R}_{\Sigma_{\perp}}$. Consequently, the equation will hold in all algebras for **Rat**.

5 Conclusions

Stephen Bloom and Zoltan Ésik proved that their concept of iteration theory in [12] sums up all equational properties that the formation of the least solutions e^{\dagger} of a recursive equations e possesses in Domain Theory. This, however, assumes that the concept of “equational property” is related to the base category **Sgn** of signatures.

In our paper we take **Set^ℱ**, the category of sets in context, as our base category. It then turns out that the summation of equational properties of the above function $e \mapsto e^{\dagger}$ in Domain Theory is given by Elgot theories—our abbreviation for the concept of iteration theory satisfying the functorial dagger implication from [12]. Elgot theories have a simpler definition than iteration theories, and they precisely correspond to cocartesian traced categories uniform w.r.t. base morphisms, see [18].

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Appendix: The Kelly-Power Equational Presentations

In this appendix we just recall some concepts and results from [19].

Assumption A.1 *Throughout the appendix \mathcal{A} denotes a locally finitely presentable category and $\mathcal{F}(\mathcal{A})$ its full subcategory representing all finitely presentable objects. The copower of M copies of an object $K \in \mathcal{A}$ is denoted by $M \bullet K$.*

Definition A.2 A signature Σ is a collection of objects of \mathcal{A} indexed by $\mathcal{F}(\mathcal{A})$; in symbols: $\Sigma = (\Sigma(p))_{p \in \mathcal{F}(\mathcal{A})}$.

Example A.3

- (i) In case $\mathcal{A} = \mathbf{Set}$ we denote $\mathcal{F}(\mathbf{Set})$ by \mathcal{F} . This is the category of natural numbers and functions. Definition A.2 is the usual concept of a (finitary, one-sorted) signature. Observe that a Σ -algebra can be viewed as a set A together with, for every $p \in \mathbb{N}$, as assignment

$$\frac{p \xrightarrow{a} A}{\Sigma(p) \xrightarrow{\hat{a}} A}$$

which to every p -tuple (a_0, \dots, a_{p-1}) assigns the map $\sigma \longmapsto \sigma_A(a_0, \dots, a_{p-1})$. Or, more compactly, an algebra is a set A together with a morphism

$$\alpha: \coprod_{p \in \mathcal{F}} A^p \times \Sigma(p) \longrightarrow A.$$

- (ii) In the category

$$\mathcal{A} = \mathbf{Set}^{\mathcal{F}}$$

of sets in context the finitely presentable objects are, as proved in [9], precisely the *super-finitary* ones. That is, those sets in context X for which there exists a natural number n such that (a) $X(n)$ and $X(0)$ are finite, and (b) all elements of $X(k)$, $k \in \mathbb{N} \setminus \{0\}$, have the form $Xf(t)$ for some $f: n \longrightarrow k$ and $t \in X(n)$. Then $\mathcal{F}(\mathbf{Set}^{\mathcal{F}})$ denotes a set of representatives of all super-finitary sets in context.

A signature in $\mathbf{Set}^{\mathcal{F}}$ is a collection $\Sigma = (\Sigma_X)_{X \in \mathcal{F}(\mathbf{Set}^{\mathcal{F}})}$ of sets in context.

Definition A.4 By a Σ -algebra is meant an object A of \mathcal{A} together with a morphism

$$\alpha: \coprod_{p \in \mathcal{F}(\mathcal{A})} \mathcal{A}(p, A) \bullet \Sigma(p) \longrightarrow A.$$

This is just an algebra for the endofunctor $H_\Sigma: \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$H_\Sigma X = \coprod_{p \in \mathcal{F}(\mathcal{A})} \mathcal{A}(p, X) \bullet \Sigma(p).$$

Homomorphisms are the usual homomorphisms of algebras for H_Σ .

Remark A.5 The forgetful functor $\Sigma\text{-Alg} \longrightarrow \mathcal{A}$ has a left adjoint which assigns to every object $X \in \mathcal{A}$ the free H_Σ -algebra $\mathbb{F}_\Sigma(X)$ on X . It is easy to see that H_Σ is a finitary functor, in particular, it preserves colimits of ω -chains. Consequently, the standard construction of the free algebra, see [1], applies: $\mathbb{F}_\Sigma(X)$ is the colimit of the chain

$$X \xrightarrow{\text{inl}} X + H_\Sigma X \xrightarrow{\text{id} + H_\Sigma \text{inl}} X + H_\Sigma(X + H_\Sigma X) \longrightarrow \dots$$

Observe that we have a canonical natural transformation $\kappa: H_\Sigma \longrightarrow \mathbb{F}_\Sigma$ given by the right-hand components of the colimit injections $X + H_\Sigma X \longrightarrow \mathbb{F}_\Sigma(X)$.

Definition A.6 By an equation for a signature Σ is meant a parallel pair of morphisms

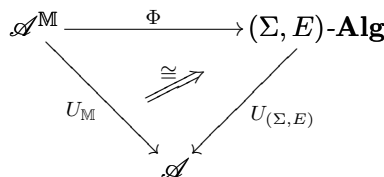
$$u, u': p \longrightarrow \mathbb{F}_\Sigma(r) \quad \text{for } p, r \in \mathcal{F}(\mathcal{A}).$$

A Σ -algebra A satisfies the equation provided that for every homomorphism $h: \mathbb{F}_\Sigma(r) \longrightarrow A$ we have $h \cdot u = h \cdot u'$.

Notation A.7 Given a set E of equations, we denote by $(\Sigma, E)\text{-Alg}$ the full subcategory of $\Sigma\text{-Alg}$ formed by those Σ -algebras that satisfy every equation in E . And we denote the forgetful functor by

$$U_{(\Sigma, E)}: (\Sigma, E)\text{-Alg} \longrightarrow \mathcal{A}.$$

Proposition A.8 (See [19].) The functor $U_{(\Sigma, E)}$ is finitary monadic. That is, there exists a finitary monad \mathbb{M} on \mathcal{A} such that for the forgetful functor $U_{\mathbb{M}}: \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}$ of its Eilenberg-Moore category we have an equivalence functor $\Phi: \mathcal{A}^{\mathbb{M}} \longrightarrow (\Sigma, E)\text{-Alg}$ together with a natural isomorphism $U_{\mathbb{M}} \xrightarrow{\cong} U_{(\Sigma, E)} \cdot \Phi$:



The main result for our purposes is the converse:

Theorem A.9 (See [19].) *Every finitary monad \mathbb{M} on \mathcal{A} has an equational presentation (Σ, E) , that is, a signature Σ , a set E of equations and an equivalence functor $\Phi: \mathcal{A}^{\mathbb{M}} \longrightarrow (\Sigma, E)\text{-Alg}$ with $U_{\mathbb{M}} \cong U_{(\Sigma, E)} \cdot \Phi$.*

Example A.10 The category of all finitary monads in **Set** (or, equivalently, the category of Lawvere theories and theory morphisms) is monadic over $\mathbf{Set}^{\mathcal{F}}$, the category of sets in context—this is an easy application of Beck’s theorem. That is, there exists a signature Σ and a set E of equations describing finitary monads as Σ -algebras satisfying the equations from E . Recall that $\mathbf{Set}^{\mathcal{F}}$ is equivalent to the category $\mathbf{Fin}(\mathbf{Set})$ of finitary set functors. A finitary monad is given by (a) a functor $A \in \mathbf{Fin}(\mathbf{Set})$, (b) a natural transformation $\eta: \text{Id} \longrightarrow A$ and (c) a natural transformation $\mu: AA \longrightarrow A$ satisfying certain axioms. The natural transformation μ can, since A is finitary, be substituted by the collections of assignments

$$\frac{f: m \longrightarrow A}{f': m \cdot m \longrightarrow A}$$

where m is an arbitrary finitely presentable object of $\mathbf{Fin}(\mathbf{Set})$, f an arbitrary natural transformation and $f' = \mu \cdot (f * f)$. This leads us to the following signature Σ^{mon} for a presentation of finitary monads: $\Sigma^{\text{mon}}(m) = m \cdot m$ for all $m \neq 0$ (0 the initial object), and $\Sigma^{\text{mon}}(0) = \text{Id}_{\mathbf{Set}}$. Here a Σ -algebra consists of a finitary functor A , a map

$$\frac{0 \longrightarrow A}{\text{Id} \longrightarrow A}$$

representing a natural transformation $\eta: \text{Id} \longrightarrow A$, and transformation maps

$$\frac{m \longrightarrow A}{m \cdot m \longrightarrow A} \quad (m \neq 0 \text{ finitely presentable})$$

representing μ provided that some equational properties hold. The set E^{mon} of equations we need then guarantees that the above transformation maps represent a natural transformation $\mu: AA \longrightarrow A$ and, together with η , satisfy the monad axioms. In other words, $(\Sigma^{\text{mon}}, E^{\text{mon}})\text{-Alg}$ is the category of Lawvere theories (equivalently, finitary monads on **Set**).

Example A.11 Let us illustrate the equations needed to represent functoriality of iteration theories. We work here with the category $\mathcal{A} = (\Sigma^{\text{mon}}, E^{\text{mon}})\text{-Alg}$ of Lawvere theories of the preceding example as the base category. For every pair n, m of natural numbers we denote by $T_{g:n \rightarrow m}$ the free Lawvere theory on one generator g representing a morphism from n to m . Notice that every theory morphism $u: T_g \longrightarrow X$ is uniquely determined by picking a morphism $u(g) \in X(n, m)$. Clearly, $T_{g:n \rightarrow m}$ is a finitely presentable object of \mathcal{A} .

Let Σ be the signature whose values are $\Sigma(p) = 0$ (the initial algebraic theory) except for $p = T_{e: n \rightarrow n+k}$ where

$$\Sigma(T_{e: n \rightarrow n+k}) = T_{e^\dagger: n \rightarrow k} \quad \text{for all } e: n \longrightarrow n+k.$$

Its polynomial functor assigns to every theory X the theory

$$\begin{aligned} H_{\Sigma}X &= \coprod_{n,k \in \mathbb{N}} \mathcal{A}(T_{e: n \rightarrow n+k}, X) \bullet T_{e^{\dagger}: n \rightarrow k} \\ &= \coprod_{n,k \in \mathbb{N}} X(n, n+k) \bullet T_{e^{\dagger}: n \rightarrow k}. \end{aligned}$$

Its algebras are precisely the *preiteration theories* of Bloom and Ésik [12], i.e., Lawvere theories X together with maps

$$\frac{e \in X(n, n+k)}{e^{\dagger} \in X(n, k)}$$

satisfying no axioms.

For every base morphism (function)

$$v: n \longrightarrow m \quad \text{in Set}$$

we now formulate an equation in the above signature Σ expressing functoriality w.r.t this morphism v : for all morphisms $e: n \longrightarrow n+k$ and $f: m \longrightarrow m+k$ this equation ensures that

$$\begin{array}{ccc} \begin{array}{ccc} n & \xrightarrow{e} & n+k \\ v \downarrow & & \downarrow v+\text{id} \\ m & \xrightarrow{f} & m+k \end{array} & \text{implies} & \begin{array}{ccc} n & & k \\ & \searrow e^{\dagger} & \\ v \downarrow & & \nearrow f^{\dagger} \\ m & & \end{array} \end{array}$$

Our equation $u_v, u'_v: p \longrightarrow \mathbb{F}_{\Sigma}(r)$ works with p free on one generator $g: n \longrightarrow k$,

$$p = T_{g: \rightarrow m}$$

and with r given by the quotient

$$r = T_{e,f} / \approx$$

of the free theory on two generators $e: n \longrightarrow n+k$ and $f: m \longrightarrow m+k$ modulo the smallest congruence \approx with

$$f \cdot v \approx (v + \text{id}) \cdot e$$

Before specifying u_v, u'_v we observe that the congruence classes

$$[e] \in r(n, n+k) \quad \text{and} \quad [f] \in r(m, m+k)$$

yield in

$$H_{\Sigma}(r) = \coprod_{i,j \in \mathbb{N}} r(i, i+j) \bullet T_{h^{\dagger}: i \rightarrow j}$$

two coproduct injections

$$\text{in}_e : T_{h^\dagger:n \rightarrow k} \longrightarrow H_\Sigma(r) \quad \text{and} \quad \text{in}_f : T_{h^\dagger:m \rightarrow k} \longrightarrow H_\Sigma(r),$$

respectively. Hence, in the theory $H_\Sigma(r)$ we have the two parallel morphisms

$$n \xrightarrow{\text{in}_e(h^\dagger)} k \quad \text{and} \quad n \xrightarrow{v} m \xrightarrow{\text{in}_f(h^\dagger)} k$$

(recall that $v : n \longrightarrow m$ is a base morphism in every theory). Using the canonical morphism $\kappa_r : H_\Sigma(r) \longrightarrow \mathbb{F}_\Sigma(r)$ of [A.5](#) we obtain two elements

$$\kappa_r(\text{in}_e(h^\dagger)), \kappa_r(\text{in}_f(h^\dagger) \cdot v) \in \mathbb{F}_\Sigma(r)(n, k)$$

Our equation

$$u_v, u'_v : p = T_g \longrightarrow \mathbb{F}_\Sigma(r)$$

is given by the above two elements. It is easy to see that a preiteration theory satisfies this equation iff the functoriality holds for the given base morphism v . The collection of all these equations indexed by all the base morphisms v yields the axiomatization of functoriality.