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How Iterative are Iterative Algebras?

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Abstract

Iterative algebras are defined by the property that every guarded system of recursive equations has a unique solution. We prove that they have a much stronger property: every system of recursive equations has a unique strict solution. And we characterize those systems that have a unique solution in every iterative algebra.

Keywords: iterative algebra, guarded equation, strict solution, extensive category
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1 Introduction

The aim of the present paper is to show that iterative algebras, i.e. algebras with unique solutions of all guarded systems of recursive equations, have solutions of unguarded systems as well. In fact, we introduce a natural concept of a"strict" solution (which is one that assigns to every ungrounded variable the result $\perp$) and prove that iterative algebras have unique strict solutions of all systems of recursive equations.

The motivation for our paper is two-fold. Firstly, in the paper of Evelyn Nelson [15] which introduced iterative algebras as a means to study the iterative theories of Calvin Elgot [10] (see also a very similar concept of Jerzy Tiuryn [16]) a complete

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characterization of all uniquely solvable systems is provided. We show in our paper a categorical generalization of this: we introduce the concept of a a preguarded system of equations, and prove that these are precisely the systems with a unique solution in every iterative algebra. Secondly, our paper is the first step in a "reconciliation" of iterative algebras and iteration algebras of Stephen Bloom and Zoltán Ésik [8]. The latter are algebras where all systems of recursive equations have solutions, and a choice of solutions subject to axioms is performed; the motivation stems from continuous algebras on CPO's, where recursive equations always have the least solution. The "reconciliation" mentioned above has two steps: one, the subject of the present paper, is to show that every iterative algebra has a "canonical" solution of every system of recursive equations. The other step, which we attend to in the paper [2] under preparation, is to show that these canonical solutions satisfy the axioms of iteration algebras. Observe that for *ungrounded variables* which are those where the given system of equations contains a cycle of length 1:

$$x \approx x$$

or 2

$$\begin{aligned} x &\approx y \\ y &\approx x \end{aligned}$$

or 3, etc., the least solution always assigns the value $\perp$. And, on the other hand, ungrounded variables obviously force us, when considering unique solutions in iterative algebras, to restrict ourselves to systems that are (in a specified sense) guarded because one cannot require that for example $x \approx x$ has a unique solution! Based on ideas of [8] we work with algebras having a global constant $\perp$, and then we define a *strict solution* of a system of recursive equations as a solution assigning $\perp$ to every ungrounded variable. Our main result is:

iterative algebras have unique strict solutions

(of arbitrary recursive systems). This holds for H-algebras where H is a finitary endofunctor of a suitable category (such as $\mathbf{Set}$ or $\mathbf{Set}^I$ or $\mathbf{Pos}$). Recall that free H-algebras form a monad F so that every algebra A can be described as a monadic algebra $\widehat{\alpha}\colon FA \longrightarrow A$. Recursive systems of equations can be represented by morphisms

$$e\colon X \longrightarrow F(X + A) \tag{1}$$

where X is a finitely presentable object (of variables). An equation morphism e is called *guarded* if it is disjoint from the injection of variables

$$i_0 \equiv X \xrightarrow{\text{inl}} X + A \xrightarrow{\eta_{X+A}} F(X + A).$$

A *solution* is a morphism $e^\dagger\colon X \longrightarrow A$ such that the

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle \widetilde{\alpha}} \\ F(X+A) & \xrightarrow[F[e^\dagger, A]]{} & FA \end{array}$$

commutes. By definition, an algebra A is iterative if and only if for every guarded equation morphism there exists a unique solution. In order to formulate, in the present generality, the idea of ungrounded variables, we compute the "first derived" subobject $i_1 : X_1 \rightarrowtail X$ as a pullback of the above embedding $i_0 : X \longrightarrow F(X+A)$ along e. In the category of sets $X_1 \subseteq X$ represents the variables that e maps to X. And e_1 is the restriction of e. Then we form the "second derived" subobject X_2 (representing variables that e maps to X_1) as a pullback of X_1 along e_1, etc:

$$\begin{array}{cccccccc} & X_3 & \xrightarrow{\quad i_3 \quad} & X_2 & \xrightarrow{\quad i_2 \quad} & X_1 & \xrightarrow{\quad i_1 \quad} & X \\ \cdots & {\scriptstyle e_3}\big\downarrow & & {\scriptstyle e_2}\big\downarrow & & {\scriptstyle e_1}\big\downarrow & & \big\downarrow{\scriptstyle e} \\ & X_2 & \xrightarrow[\quad i_2 \quad]{} & X_1 & \xrightarrow[\quad i_1 \quad]{} & X_0 = X & \xrightarrow[i_0]{} & F(X+A) \end{array}$$

Each i_n is easily seen to be a coproduct injection, and thus $X_n = X_{n+1} + \overline{X}_{n+1}$ where $\bar{i}_{n+1}\colon \overline{X}_{n+1} \rightarrowtail X_n$ is the complementary coproduct injection of $i_{n+1}\colon X_{n+1} \rightarrowtail X_n$. In the category of sets $\overline{X}_1 \subseteq X$ are the variables that e maps outside of X, then $\overline{X}_2$ are the variables that need two steps to be mapped outside of X, etc.

Definition. *An equation morphism with* $X = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots$ *is called pre-guarded.*

In order to prove our theorem above, we demonstrate that in an iterative algebra

(i) every pre-guarded equation morphism has a unique solution, and

(ii) every equation morphism $e\colon X \longrightarrow F(X+A)$ can be modified to a pre-guarded equation morphism $f\colon X \longrightarrow F(X+A)$ such that solutions of f are precisely the strict solutions of e.

We work at the beginning with cia's (completely iterative algebras), where the restriction that the object X of variables be finitely presentable is lifted. This makes the theory of pre-guardedness and strictness simpler. Iterative algebras are then treated in the last section.

Related Work. For endofunctors of **Set** the unique existence of strict solutions has been proved by Larry Moss [14] and Stephen Bloom *et al.* [6], [7]. Our purely categorical proof is independent.

2 Extensive Categories, cia's and Iterative Algebras

The aim of this section is to shortly recall the three concepts in the title as a preparation for the theory presented further. Given an endofunctor H of a category $\mathscr{A}$

with finite coproducts, an H*-algebra* consists of an object A of $\mathscr{A}$ and a morphism $\alpha\colon HA \longrightarrow A$. A *flat equation morphism* in A is a morphism of the form

$$e\colon X \longrightarrow HX + A \tag{2}$$

and a *solution* of e is a morphism $e^\dagger\colon X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array}$$

commutes. The algebra A is called *completely iterative* (or, shortly *cia*), see [13], if every flat equation morphism has a unique solution. Example: let TZ be the terminal coalgebra of $H(-) + Z$. Then the coalgebra structure is invertible, whence TZ is a coproduct of HTZ and Z

$$TZ = HTZ + Z \tag{3}$$

with injections

$$\begin{array}{ll} \tau_Z\colon HTZ \longrightarrow TZ & \text{(“}TZ\text{ is an }H\text{-algebra”)} \\ \eta_Z\colon Z \longrightarrow TZ & \text{(“embedding of variables”).} \end{array}$$

In fact, TZ is a free cia on Z with η_Z as the universal arrow. We denote by T the monad of free cias for H. Its unit is η and the multiplication μ is given by the unique homomorphism $\mu_Z : TTZ \longrightarrow TZ$ extending identity on TZ.

2.1 Definition [1]. *An endofunctor H is called* ***iteratable*** *if TZ, a terminal coalgebra of $H(-) + Z$, exists for every Z.*

2.2 Example. Let Σ be a signature, i.e., a sequence of sets $(\Sigma_n)_{n\in\mathbb{N}}$. Σ-algebras in **Set** are H-algebras for the polynomial functor

$$H_\Sigma Z = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

H_Σ is iteratable, and $T_\Sigma Z$ can be described as the algebra of all Σ-trees on Z, i.e., trees with leaves labelled in $Z + \Sigma_0$ and nodes with $n > 0$ successors labelled in Σ_n. Recall that a free H_Σ-algebra on a set Z is the algebra $F_\Sigma Z$ of all finite Σ-trees on Z. Thus, equations in the sense of the introduction, see (1), are a special case of the following concept:

2.3 Definition. *Let H be an iteratable endofunctor. An* ***equation morphism*** *in a cia A is a morphism of the form*

$$e\colon X \longrightarrow T(X + A).$$

*It is called **guarded** if it factors through the right-hand injection of* $T(X + A) = X + [A + HT(X + A)]$,

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & T(X+A) \\ & \searrow & \uparrow \text{inr} \\ & & A + HT(X+A) \end{array}$$

2.4 Notation. If A is a cia, we denote by $\widetilde{\alpha}\colon TA \longrightarrow A$ the unique homomorphism with

$$\widetilde{\alpha}\cdot\eta_A = \mathrm{id}\,.$$

The proof of the following theorem is a straightforward adaptation of Theorem 3.9 in [13].

2.5 Theorem. *In a cia every guarded equation morphism* $e\colon X \longrightarrow T(X+A)$ *has a unique solution, i.e., there exists a unique* $e^\dagger\colon X \longrightarrow A$ *such that the square*

$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widetilde{\alpha}} \\ T(X+A) & \xrightarrow[T[e^\dagger, A]]{} & TA \end{array} \tag{4}$$

commutes.

2.6 Remark. Recall that a category $\mathscr{A}$ is called *locally finitely presentable*, see [12] or [4], if it has colimits and a set $\mathscr{A}_{fp}$ of *finitely presentable objects* (i.e., objects A such that $\hom(A, -)$ preserves filtered colimits) such that every object is a filtered colimit of objects in $\mathscr{A}_{fp}$. Examples: **Set**, $\mathbf{Set}^I$, **Pos**, **Vec** are finitely presentable categories. A functor $H\colon \mathscr{A} \longrightarrow \mathscr{A}$ is called *finitary* if it preserves filtered colimits. Every finitary functor has free algebras, and as proved by Michael Barr in [5], this yields a monad F of free H-algebras. Analogously as for the cia's we have $FZ = HFZ + Z$, where the coproduct injections are the H-algebra structure and the universal arrow.

2.7 Definition. *Let H be a finitary endofunctor. A **finitary equation morphism** is a morphism of the form*

$$e\colon X \longrightarrow F(X+A),$$

*where X is finitely presentable. It is called **guarded** if it factors through the right-hand coproduct injection of* $F(X + A) = X + [A + HF(X + A)]$.

2.8 Definition. *An H-algebra is called **iterative** if every finitary flat equation morphism, i.e., (2) with X finitely presentable, has a unique solution.*

2.9 Remark. In every iterative algebra A every finitary, guarded equation morphism $e\colon X \longrightarrow F(X + A)$ has a unique solution $e^\dagger = \widehat{\alpha}\cdot F[e^\dagger, A]\cdot e$ (where $\widehat{\alpha}\colon FA \longrightarrow A$ is the unique homomorphism extending id_A). See [3].

2.10 Example [15]. For $H = H_\Sigma$ the subalgebra $R_\Sigma Z \subseteq T_\Sigma Z$ of the Σ-tree algebra formed by all *rational trees*, i.e., trees which have up to isomorphism only finitely many subtrees, is iterative. This is a free iterative Σ-algebra on Z.

2.11 Notation. We denote by R the monad of free iterative H-algebras. It exists for every finitary functor H, and we have $RZ = HRZ + Z$, similarly as for free algebras and free cias. See [3]. This allows us to define, in analogy to Definition 2.3, *rational equation morphisms* as morphisms $e\colon X \longrightarrow R(X + A)$, X finitely presentable, and call them *guarded* provided that they factor through the right-hand coproduct injection of $R(X + A) = X + [A + HR(X + A)]$. Every iterative algebra has a unique solution $e^\dagger$ of every rational, guarded equation morphism $e\colon X \longrightarrow R(X + A)$, i.e., a unique morphism $e^\dagger = \widetilde{\alpha}\cdot R[e^\dagger, A]\cdot e$ where $\widetilde{\alpha}\colon RA \longrightarrow A$ is the unique homomorphism extending id_A.

2.12 Definition [9]. *A category is called **extensive** if it has finite coproducts which are*

(a) *disjoint, i.e., coproduct injections are monomorphisms and the intersection of coproduct injections of $A + B$ is always* 0 *(initial object), and*

(b) *universal, i.e., for every morphism $f\colon C \longrightarrow A_1 + A_2$ pullbacks of the coproduct injections along f exist and turn C into the corresponding coproduct:*

$$\begin{array}{ccccc} A_1' & \longrightarrow & C = A_1' + A_2' & \longleftarrow & A_2' \\ \downarrow & & \downarrow f & & \downarrow \\ A_1 & \xrightarrow[\mathrm{inl}]{} & A_1 + A_2 & \xleftarrow[\mathrm{inr}]{} & A_2 \end{array}$$

2.13 Notation. We denote, for every coproduct injection $i\colon A \longrightarrow C$, by $\bar{i}\colon \overline{A} \longrightarrow C$ the complementary coproduct injection, i.e., $C = A + \overline{A}$ with injections i and $\bar{i}$.

2.14 Definition. *A category is called ω-**extensive** if it has countable coproducts which are* (a) *disjoint and* (b) *universal, i.e., for every morphism $f\colon C \longrightarrow \coprod_{n\in\mathbb{N}} A_n$ pullbacks of coproduct injections along f exist and turn C into the corresponding coproduct.*

2.15 Examples. (1) **Set** is ω-extensive. The category of finite sets is an example of an extensive category that is not ω-extensive.

(2) Posets, graphs, and unary algebras form ω-extensive categories.

(3) Free completions under countable coproducts are always ω-extensive.

(4) If $\mathscr{K}$ is ω-extensive then so is each functor category $[\mathscr{A}, \mathscr{K}]$, $\mathscr{A}$ small.

3 Pre-Guarded Equation Morphisms

3.1 Assumption. Throughout this section H denotes an iteratable endofunctor of an ω-extensive category, see Definitions 2.1 and 2.14. Coproduct injections of binary coproducts are called inl and inr.

3.2 Definition. *Given an equation morphism $e\colon X \longrightarrow T(X + A)$ the **derived***

subobjects $X_n \rightarrowtail X$, $n = 1, 2, 3, \dots$ *are defined by the following pullbacks*

$$
\begin{array}{ccccccccc}
 & X_3 & \xrightarrow{\ i_3\ } & X_2 & \xrightarrow{\ i_2\ } & X_1 & \xrightarrow{\ i_1\ } & X & \\
\cdots & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} & \\
 & X_2 & \xrightarrow[\ i_2\ ]{} & X_1 & \xrightarrow[\ i_1\ ]{} & X & \xrightarrow[\ i_0=\mathrm{inl}\ ]{} & T(X+A) &
\end{array}
$$

where i_0 is the left-hand coproduct injection of $T(X + A) = X + [A + HT(X + A)]$, see (3) above.

3.3 Remark. Since i_0 is a coproduct injection, so is i_1, and e_1 is a domain-codomain restriction of e. Analogously, since i_1 is a coproduct injection, so is i_2, and e_2 is a domain-codomain restriction of e_1, etc. We denote by

$$\bar{i}_n \colon \overline{X}_n \longrightarrow X_{n-1} \qquad (n = 1, 2, 3, \dots)$$

the complementary coproduct injection, thus, $X_{n-1} = X_n + \overline{X}_n$ for $n = 1, 2, 3, \dots$ We consider $\overline{X}_n$ as a subobject of X via

$$\overline{X}_n \xrightarrow{\ \bar{i}_n\ } X_{n-1} \xrightarrow{\ i_{n-1}\ } X_{n-2} \longrightarrow \cdots \xrightarrow{\ i_1\ } X. \tag{5}$$

3.4 Definition. *An equation morphism $e \colon X \longrightarrow T(X+A)$ is called* ***pre-guarded*** *provided that X is a coproduct of the above subobjects $\overline{X}_n$; shortly*

$$X = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots .$$

3.5 Example. If $\mathscr{A} = \mathbf{Set}$ and $H = H_\Sigma$, then e represents, for $X = \{x_1, x_2, x_3, \dots\}$, equations

$$x_i \approx t_i(x_1, x_2, x_3, \dots, a_1, a_2, a_3, \dots)$$

where the right-hand sides t_i are (possibly infinite) Σ-trees on $X + A$. The variables of $X_1 = e^{-1}(X_0)$ are precisely those x_i where t_i is a single variable in X. That is, those x_i where the corresponding equation has the form $x_i \approx x_{i'}$. We conclude that X_1 are precisely the unguarded variables. To put it positively, $\overline{X}_1$ consists of all the guarded variables. Here we have $e_1 \colon X_1 \longrightarrow X$, $x_i \longmapsto x_{i'}$, and thus x_i lies in $X_2 = e_1^{-1}(X_1)$ if and only if $x_{i'}$ is unguarded. Consequently, for every $x_i \in X_2$ we have equations $x_i \approx x_{i'}$ and $x_{i'} \approx x_{i''}$. In other words, $\overline{X}_2$ consists of all variables reaching a guarded variable in one step (of applying e). Analogously, $x_i \in X_3$ if and only if we have equations $x_i \approx x_{i'}$, $x_{i'} \approx x_{i''}$ and $x_{i''} \approx x_{i'''}$ or, equivalently, $\overline{X}_3$ consists of all variables reaching a guarded variable in two steps, etc. To say

$$X = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots$$

means that every variable reaches a guarded variable in finitely many steps.

3.6 Remark. As demonstrated in Example 3.5, the intuition behind the subobjects $\overline{X}_1, \overline{X}_2, \overline{X}_3, \dots$ is such that $\overline{X}_1$ consists of all guarded variables. If e is a guarded equation morphism, then $X = \overline{X}_1$. If e is pre-guarded, we always have a passage $\overline{X}_n \longrightarrow \overline{X}_1$, for all $n \geq 1$, which to every variable assigns the guarded variable eventually reached by applying e finitely many times. To formulate this categorically, we need the following

3.7 Notation We form a pullback of $e_n \colon X_n \longrightarrow X_{n-1}$ along the complement $\bar{i}_n$ of i_n, see Remark 3.3; for $i = i_n$ this gives us pullbacks

$$\begin{array}{ccccc} \overline{X}_{n+1} & \xrightarrow{\bar{i}_{n+1}} & X_n & \xleftarrow{i_{n+1}} & X_{n+1} \\ {\scriptstyle \bar{e}_{n+1}}\downarrow & & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e_{n+1}} \\ \overline{X}_n & \xrightarrow[\bar{i}_n]{} & X_{n-1} & \xleftarrow[i_n]{} & X_n \end{array} \qquad (n \geq 1)$$

The canonical passage from $\overline{X}_n$ to $\overline{X}_1$ is the composite $\bar{e}_2 \cdot \dots \cdot \bar{e}_n$. This defines a morphism

$$u = [\mathrm{id}, \bar{e}_2, \bar{e}_2 \cdot \bar{e}_3, \dots] \colon \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots \longrightarrow \overline{X}_1. \tag{6}$$

3.8 Construction. Let A be a cia. For every pre-guarded equation morphism $e \colon X \longrightarrow T(X + A)$, $X = \coprod_{n \geq 1} \overline{X}_n$, we define, using (6), a guarded equation morphism as follows

$$f \equiv \overline{X}_1 \xrightarrow{\bar{i}_1} X \xrightarrow{e} T(X + A) \xrightarrow{T(u+A)} T(\overline{X}_1 + A). \tag{7}$$

Solutions of e and f are closely related:

3.9 Theorem. *The equation morphism f is guarded and fulfils*

(a) *if $e^\dagger$ a solution of e, then $e^\dagger \cdot \bar{i}_1 \colon \overline{X}_1 \longrightarrow A$ is a solution of f, and*

(b) *if $f^\dagger$ a solution of f, then $f^\dagger \cdot u \colon X \longrightarrow A$ is a solution of e.*

Proof. (1) We verify that f is guarded. Put

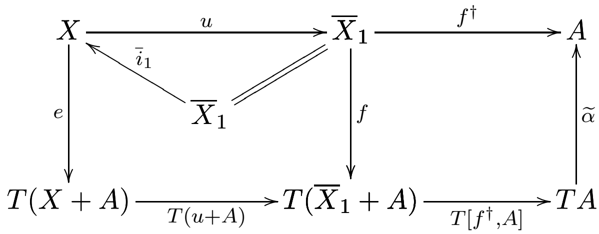
$$j_0 = \mathrm{inl} \colon \overline{X}_1 \longrightarrow T(\overline{X}_1 + A) = \overline{X}_1 + A + HT(\overline{X}_1 + A)$$

and compute a pullback of f along j_0:

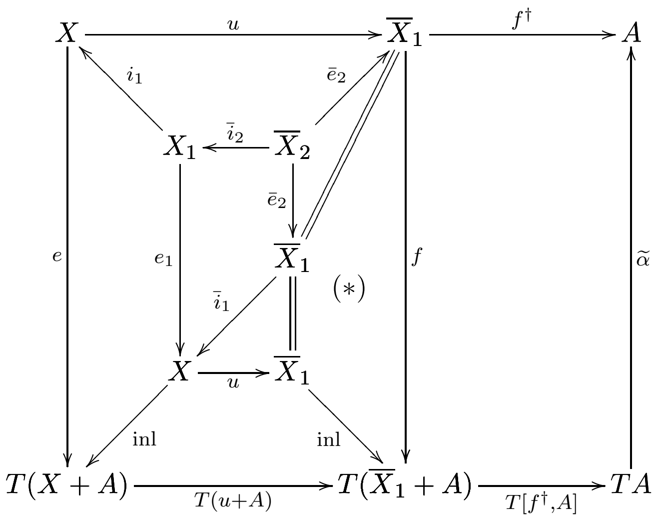
$$\begin{array}{ccc} 0 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow{\scriptstyle \bar{i}_1} \\ X_1 & \xrightarrow{i_1} & X \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle e} \\ X & \xrightarrow{i_0 = \mathrm{inl}} & T(X + A) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle T(u+A) = u + [A + HT(u+A)]} \\ \overline{X}_1 & \xrightarrow[j_0 = \mathrm{inl}]{} & T(\overline{X}_1 + A) \end{array}$$

(2) Proof of (b). Given a solution $f^\dagger \colon \overline{X}_1 \longrightarrow A$ of f, we prove that $f^\dagger \cdot u \colon X \longrightarrow A$ is a solution of e, i.e., $f^\dagger \cdot u = \widetilde{\alpha} \cdot T[f^\dagger \cdot u, A] \cdot e \colon X \longrightarrow A$. This equation will

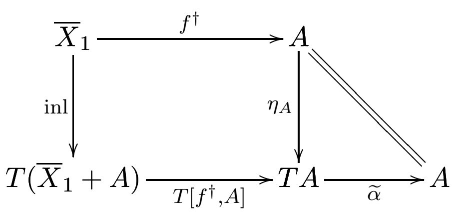
be proved by considering the individual components of $X = \coprod \overline{X}_n$, see (5). For $n = 1$ we use the definition (7) of f and obtain the commutative diagram



For $n = 2$, the coproduct injection is $i_1 \cdot \bar{i}_2 \colon \overline{X}_2 \longrightarrow X$; thus we consider the diagram

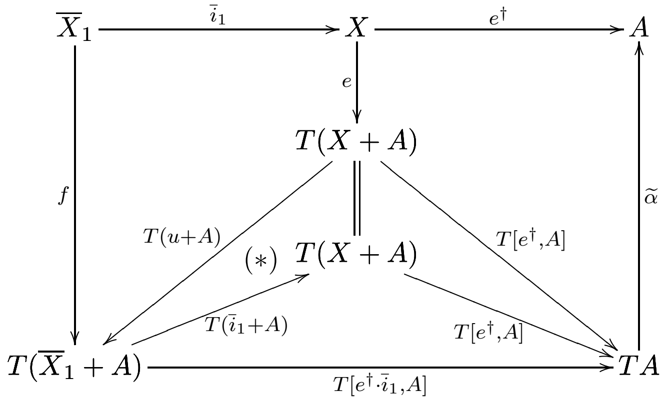


All the inner parts except the one denoted by (∗) clearly commute. The part (∗) commutes when composed with the passage to A, $\widetilde{\alpha} \cdot T[f^\dagger, A] \colon T(\overline{X}_1 + A) \longrightarrow A$, i.e., this morphism merges the parallel pair f, $\mathrm{inl} \colon \overline{X}_1 \longrightarrow T(\overline{X}_1 + A)$. In fact, by the commutativity of the right-hand square in the above diagram it suffices to observe that $f^\dagger = \widetilde{\alpha} \cdot T[f^\dagger, A] \cdot \mathrm{inl}$:



The cases $n = 3, 4, \ldots$ are analogous to the case $n = 2$.

(3) Proof of (a). Let $e^\dagger\colon X \longrightarrow A$ be a solution of e. We are to prove that the outward square of the following diagram



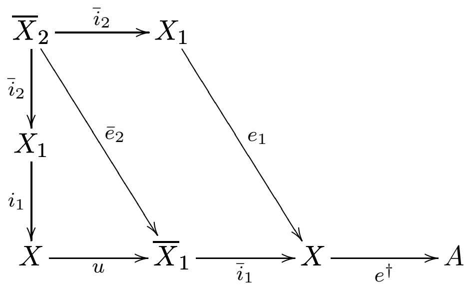
commutes. All the inner parts except that denoted by $(*)$ commute. For $(*)$ it is sufficient to prove that $T[e^\dagger, A]$ merges id and $T(\bar{i}_1 + A)\cdot T(u + A)$. Therefore, the proof of (a) will be finished by proving

$$e^\dagger = e^\dagger\cdot\bar{i}_1\cdot u\colon X \longrightarrow A. \tag{8}$$

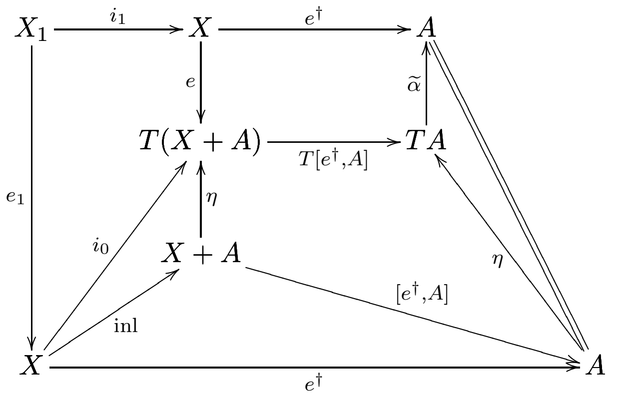
We consider the individual components $\overline{X}_n$ of $X = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots$, see (5):

For $n = 1$ use $u\cdot\bar{i}_1 = \mathrm{id}$ to obtain $e^\dagger\cdot\bar{i}_1 = (e^\dagger\cdot\bar{i}_1\cdot u)\cdot\bar{i}_1$.

For $n = 2$ we are to prove the equation $e^\dagger\cdot i_1\cdot\bar{i}_2 = (e^\dagger\cdot\bar{i}_1\cdot u)\cdot i_1\cdot\bar{i}_2$. Consider the diagram



from which the right-hand side of the desired equation is expressed as $e^\dagger \cdot e_1 \cdot \bar{i}_2$. It remains to verify $e^\dagger \cdot i_1 = e^\dagger \cdot e_1$ which follows from the next diagram



Cases $n = 3, 4, \ldots$ are analogous. □

3.10 Corollary. *In every cia all pre-guarded equation morphisms have unique solutions.*

In fact, the morphism u is an epimorphism, due to $u \cdot \bar{i}_1 = \mathrm{id}$, thus the unique existence of $e^\dagger$ follows from the unique existence of $f^\dagger$ via (a) and (b) above.

3.11 Remark. How about the converse: if $e\colon X \longrightarrow T(X+A)$ has unique solutions in all cia's, is e pre-guarded? The answer is affirmative whenever T satisfies mild side conditions: see Proposition 4.11 below.

4 Strict Solutions

4.1 Assumption. Throughout this section $\mathscr{A}$ denotes a category which

(a) is ω-extensive

(b) has a terminal object, 1, and

(c) has the property that given pairwise disjoint subobjects $A_n \rightarrowtail B$ $(n \in \mathbb{N})$ each of which is a coproduct injection, then the induced morphism $\coprod_{n\in\mathbb{N}} A_n \rightarrowtail B$ as also a coproduct injection.

Moreover, H denotes an iteratable functor for which a morphism

$$\bot\colon 1 \longrightarrow H0$$

has been chosen.

4.2 Notation. For every equation morphism an intersection of the derived subobjects $X_n \rightarrowtail X$ (see Definition 3.2) is denoted by

$$i_\infty\colon X_\infty \longrightarrow X.$$

4.3 Remark. For every equation morphism $e\colon X \longrightarrow T(X+A)$ we see that

(a) an intersection X_∞ of all derived subobjects exists, and

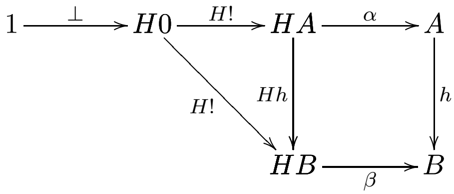
(b) $X = X_\infty + \coprod_{n\geq 1} \overline{X}_n$ (with i_∞ and (5) as injections).

In fact, using Assumption 4.1(c), where $A_n = \overline{X}_{n+1}$, we see that for $y\colon Y = \coprod_{n\geq 1} \overline{X}_n \longrightarrow X$ with components (5) there is a complement $\bar{y}\colon \overline{Y} \longrightarrow X$. It is easy to verify that this is the desired intersection.

4.4 Notation. $\perp$ is a global constant of H, i.e., every H-algebra $HA \xrightarrow{\alpha} A$ obtains the corresponding global element

$$\perp_A \equiv 1 \xrightarrow{\perp} H0 \xrightarrow{H!} HA \xrightarrow{\alpha} A.$$

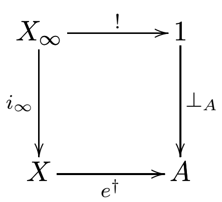
All homomorphisms $h\colon A \longrightarrow B$ preserve this global constant: $h\cdot\perp_A = \perp_B$. In fact, consider the commutative diagram below:



In particular for any object Y we have a global element of TY which we denote by $\perp$ for short:

$$\perp \equiv 1 \longrightarrow H0 \xrightarrow{H!} HTY \xrightarrow{\tau_Y} TY$$

4.5 Definition. *Let A be a cia and $e\colon X \longrightarrow T(X+A)$ an equation morphism with a solution $e^\dagger\colon X \longrightarrow A$. We call $e^\dagger$ **strict** if its restriction to X_∞ is $\perp_A$:*



4.6 Construction. Let A be a cia. For every equation morphism

$$e\colon X \longrightarrow T(X+A)$$

we define a pre-guarded equation morphism

$$f\colon X \longrightarrow T(X+A)$$

by changing the left-hand component of $e\colon X_\infty + \coprod \overline{X}_n \longrightarrow T(X+A)$ to $\perp$:

$$f\cdot \mathrm{inl} \equiv X_\infty \xrightarrow{!} 1 \xrightarrow{\perp} T(X+A)$$
$$f\cdot \mathrm{inr} = e\cdot \mathrm{inr}\colon \coprod \overline{X}_n \longrightarrow T(X+A)$$

where inl and inr are the coproduct injections of $X = X_\infty + \coprod \overline{X}_n$.

4.7 Theorem. *The equation morphism f is pre-guarded and fulfils*

(a) *every strict solution of e is a solution of f, and*

(b) *every solution of f is a strict solution of e.*

Proof. (1) f is pre-guarded. Let $Z_0 = \coprod \overline{X}_n$ and denote by $j_0 = \mathrm{inr}\colon Z_0 \longrightarrow X$ the coproduct injection. Let $j_k\colon Z_k \longrightarrow Z_{k-1}$, $k \geq 1$, denote the derived subobjects of f. We will prove that

$$Z_k = \overline{X}_{k+1} + \overline{X}_{k+2} + \cdots\,, \quad \text{and} \quad j_k = \mathrm{inr}\colon Z_k \longrightarrow \overline{X}_k + Z_k,$$

and that the corresponding morphism opposite f_{k-1} is

$$f_k = \bar{e}_{k+1} + \bar{e}_{k+2} + \cdots : Z_k \longrightarrow Z_{k-1} \qquad (k \geq 1).$$

This proves obviously that f is pre-guarded since $\bigcap_{k\in\mathbb{N}} Z_k = 0$.

Case $k = 1$: To find a pullback of $f = [\bot!, e\cdot j_0]$ along $i_0\colon X \longrightarrow T(X + A)$, we just compute a pullback of $e\cdot j_0$ along i_0: in fact the component $\bot!$ contributes nothing to the pullback because it factors through $\bar{i}_0$, the complement of i_0, and $\mathscr{A}$ is extensive. Here is the pullback of $e\cdot j_0$ along i_0:

$$\begin{array}{ccc}
\overline{X}_2 + \overline{X}_3 + \cdots = Z_1 & \xrightarrow{\mathrm{inr}} & \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots \\
\Big\downarrow{\scriptstyle \mathrm{inr}} & & \Big\downarrow{\scriptstyle j_0=\mathrm{inr}} \\
X_\infty + \overline{X}_2 + \overline{X}_3 + \cdots = X_1 & \xrightarrow{i_1=\mathrm{inr}} & X = X_\infty + \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots \\
\Big\downarrow{\scriptstyle e_1} & & \Big\downarrow{\scriptstyle e} \\
X & \xrightarrow[i_0]{} & T(X+A)
\end{array}$$

Consequently, we have $Z_1 = \overline{X}_2 + \overline{X}_3 + \cdots$ with $j_1 = \mathrm{inr}\colon Z_1 \longrightarrow X = X_\infty + \overline{X}_1 + Z_1$, and the corresponding morphism $f_1\colon Z_1 \longrightarrow X$ is

$$f_1 \equiv Z_1 \xrightarrow{\mathrm{inr}} X_\infty + Z_1 \xrightarrow{e_1} X.$$

Case $k = 2$: We compute a pullback of $f_1 = e_1 \cdot \mathrm{inr}$ along j_1:

$$\begin{array}{ccc} ? & \longrightarrow & Z_1 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{inr}} \\ \coprod_{n\geq 2} P_n & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ \coprod_{n\geq 2} \overline{X}_n = Z_1 & \xrightarrow[j_1]{} & X \end{array}$$

by computing first a pullback P_n of e_1 along the n-th component $\overline{X}_n \longrightarrow X$, $n \geq 2$, of j_1, see (5)

$$\begin{array}{ccccccccc} P_n = \overline{X}_{n+1} & \xrightarrow{\bar{i}_{n+1}} & X_n & \xrightarrow{i_n} \cdots \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 \\ \downarrow{\scriptstyle \bar{e}_{n+1}} & & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} \\ \overline{X}_n & \xrightarrow[\bar{i}_n]{} & X_{n-1} & \xrightarrow[i_{n-1}]{} \cdots \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X \end{array}$$

The connecting maps are $\bar{e}_n \colon P_n \longrightarrow \overline{X}_n$ and $i_2 \cdot \cdots \cdot i_n \cdot \bar{i}_{n+1} \colon P_n \longrightarrow X_1$. Thus, due to extensivity, a pullback of e_1 along j_1 is $\coprod_{n\geq 2} \overline{X}_{n+1} = Z_2$ with the connecting maps $\coprod_{n\geq 2} \bar{e}_{n+1} \colon Z_2 \longrightarrow Z_1$ and $\mathrm{inr} \colon Z_2 \longrightarrow X_1 = X_\infty + \overline{X}_2 + Z_2$. The pullback of $f_1 = e_1 \cdot \mathrm{inr}$ along j_1 is thus

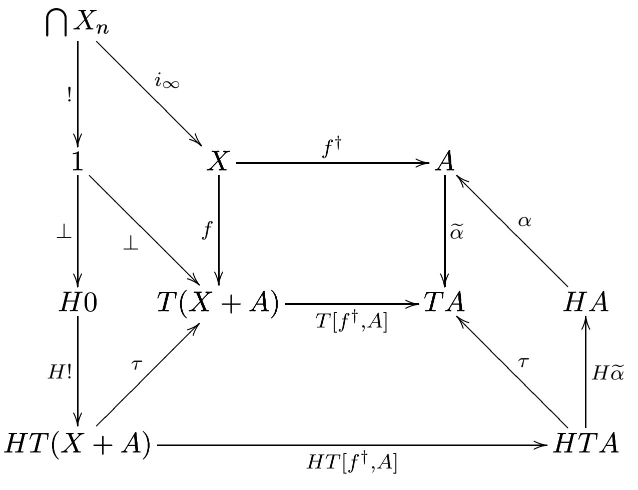
$$\begin{array}{ccc} \overline{X}_3 + \overline{X}_4 + \overline{X}_5 + \cdots = Z_2 & \xrightarrow{\mathrm{inr}} & Z_1 = \overline{X}_2 + \overline{X}_3 + \overline{X}_4 + \cdots \\ \| & & \downarrow{\scriptstyle \mathrm{inr}} \\ \overline{X}_3 + \overline{X}_4 + \overline{X}_5 + \cdots = Z_2 & \xrightarrow{\mathrm{inr}} & X_1 = X_\infty + \overline{X}_2 + \overline{X}_3 + \overline{X}_4 + \cdots \\ \downarrow{\scriptstyle \coprod_{n\geq 2} \bar{e}_{n+1}} & & \downarrow{\scriptstyle e_1} \\ Z_1 & \xrightarrow[j_1]{} & X \end{array}$$

We obtain $Z_2 = \overline{X}_3 + \overline{X}_4 + \overline{X}_5 + \cdots$, $j_2 = \mathrm{inr}$, and $f_2 = \coprod_{n\geq 2} \bar{e}_{n+1}$.

Case $k \geq 3$: Here we use the obvious pullbacks

$$\begin{array}{ccccc} \cdots\ Z_4 & \longrightarrow & Z_3 & \xrightarrow{\mathrm{inr}} & Z_2 \\ \downarrow{\scriptstyle \coprod_{n\geq 4} \bar{e}_{n+1}} & & \downarrow{\scriptstyle \coprod_{n\geq 3} \bar{e}_{n+1}} & & \downarrow{\scriptstyle \coprod_{n\geq 2} \bar{e}_{n+1}} \\ \cdots\ Z_3 & \longrightarrow & Z_2 & \xrightarrow[\mathrm{inr}]{} & Z_1 \end{array}$$

(2) Proof of (b). If $f^\dagger$ is a solution of f, then $f^\dagger$ is strict:

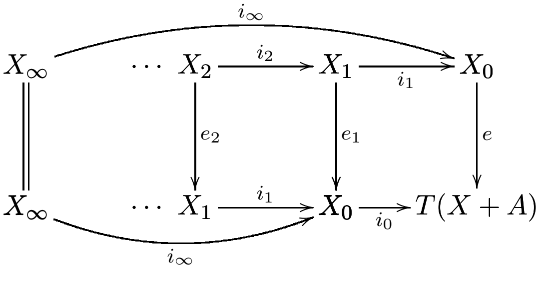


We see that the passage from $H0$ to HA is $H!$ (because $\widetilde{\alpha}\cdot T[f^\dagger, A]\cdot ! = !\colon 0 \longrightarrow A$), thus $f^\dagger\cdot i_\infty = \alpha\cdot H!\cdot\bot\cdot ! = \bot_A\cdot !$ as required.

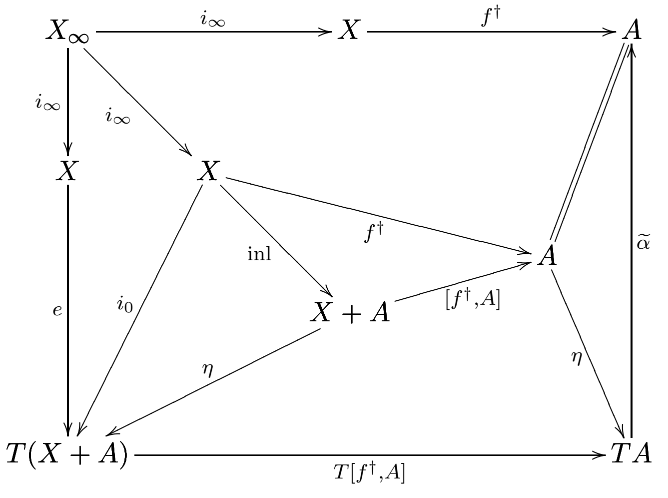
And $f^\dagger$ is a solution of e, i.e., the equation

$$\widetilde{\alpha}\cdot T[f^\dagger, A]\cdot e = f^\dagger\colon X_\infty + \coprod \overline{X}_n \longrightarrow A \tag{9}$$

holds (see (4) in the introduction): for the right-hand component $j_0\colon \coprod \overline{X}_n \longrightarrow X$ this follows from $e\cdot j_0 = f\cdot j_0$. For the left-hand one form a limit of the pullbacks defining i_n and e_n:



to conclude $e \cdot i_\infty = i_0 \cdot i_\infty$. Thus, the diagram



commutes, proving the left-hand component of (9).

(3) Proof of (a). If $e^\dagger$ is a strict solution of e, then we are to prove that the equation $\widetilde{\alpha} \cdot T[e^\dagger, A] \cdot f = e^\dagger$ holds (cf. (4)): for the right-hand component with domain $\coprod \overline{X}_n$ this follows from the fact that $f \cdot j_0 = e \cdot j_0$. For the left-hand component use the fact that both $e^\dagger$ and f yield $\perp$ (in A and $T(X + A)$, respectively) and that $\widetilde{\alpha} \cdot T[e^\dagger, A]$ preserves $\perp$, being a homomorphism (see Notation 4.4). ☐

4.8 Corollary. *In every cia every equation morphism has a unique strict solution.*

4.9 Remark. We will now turn our attention to the question of whether an equation having a unique solution in every cia must be pre-guarded. In the case of $\mathscr{A}$ = **Set**, the answer is affirmative whenever $H1$ has at least two elements. In general categories we need the following

4.10 Definition. *We say that the free cia monad T is* ***nontrivial*** *if it preserves monomorphisms and has at least two global constants,*

$$\operatorname{card} \mathscr{A}(1, T0) \geq 2.$$

4.11 Proposition. *Suppose that morphisms from non-initial objects to* 1 *are epimorphisms. If the free cia monad is nontrivial, then every equation morphism $e \colon X \longrightarrow T(X + A)$ with a unique solution in TA is pre-guarded.*

Remark. We consider e as an equation in TA via $X \xrightarrow{e} T(X + A) \xrightarrow{T(X+\eta)} T(X + TA)$.

Proof. Suppose that e is not pre-guarded. For every global element $b \colon 1 \longrightarrow T0$ we can find a solution $e_b^\dagger \colon X \longrightarrow TA$ such that

$$e_b^\dagger \cdot i_\infty \equiv X_\infty \longrightarrow 1 \xrightarrow{b} T0 \xrightarrow{T!} TA.$$

The proof is precisely the proof of Theorem 4.7 where $\alpha\colon HA \longrightarrow A$ is the replaced by $\tau_A\colon HTA \longrightarrow TA$ (with $\widetilde{\tau}_A = \mu_A$) and $\perp$ is replaced by b. We will prove that e has more than one solution by showing that $e_b^\dagger$ determines b; for that we just observe that $T!\colon T0 \longrightarrow TA$ is a monomorphism. In fact, $!\colon 0 \longrightarrow A$ is a monomorphism since in every extensive category initial objects are strict, and T preserves monomorphisms. □

4.12 Example. Suppose that our base category is $\mathscr{A} = \mathbf{Set}$.

(1) Whenever $H1$ has more than one element then H has a nontrivial free cia monad. In fact, T preserves monomorphisms: see Proposition 6.1 in [3]. And to prove $\operatorname{card} T0 \geq 2$, we decompose $H = H' + H''$ with $H'1 \neq \emptyset$ and $H''1 \neq \emptyset$. This can be done by chosing any $a \in H1$ and defining $H'X$ and $H''X$ as the inverse images of $\{a\}$ and $H1 - \{a\}$, respectively, under $H! = HX \longrightarrow H1$. Consider coalgebras

$$A \equiv 1 \xrightarrow{\text{const}\, a} H'1 \hookrightarrow H1 \qquad \text{and} \qquad B \equiv 1 \xrightarrow{\text{const}\, b} H''1 \hookrightarrow H1$$

($a \in H'1$, $b \in H''1$). It is clear that the unique homomorphism $A \longrightarrow T0$ is disjoint with the unique homomorphism $B \longrightarrow T0$. Therefore, $\operatorname{card} T0 \geq 2$.

(2) Conversely, whenever for every equation morphism e the implication

$$e \text{ has unique solution } \Longrightarrow e \text{ is pre-guarded}$$

holds, then $H1$ must have more than one element. In fact, $\operatorname{card} H1 = 1$ implies that $T0$, a terminal H-coalgebra, has a unique element. Then the equation $x \approx x$ has a unique solution in $T0$.

5 Iterative Algebras

5.1 Assumption. In this section $\mathscr{A}$ is a locally finitely presentable, ω-extensive category such that every finitely presentable object is a finite coproduct of indecomposable, finitely presentable objects. And H is a finitary endofunctor for which a morphism

$$\perp\colon 1 \longrightarrow H0$$

has been chosen.

5.2 Definition. *For a rational equation morphism $e\colon X \longrightarrow R(X+A)$, (see 2.11), we define **derived subobjects** $X_n \rightarrowtail X$ precisely as in Definition 3.2, just replacing T by R everywhere.*

5.3 Remark. We thus have pullbacks

$$\begin{array}{ccccccccc} & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X \\ \cdots & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} \\ & X_2 & \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X & \xrightarrow[i_0=\mathrm{inr}]{} & R(X+A) \end{array}$$

We also use the remaining notation $\bar{i}_n \colon \overline{X}_n \longrightarrow X_{n-1}$ and $\bar{e}_n \colon \overline{X}_n \longrightarrow X_{n-1}$ as in Section 3.

5.4 Lemma. *Every rational equation morphism e has a least derived subobject, i.e., there exists n with $X_n = X_{n-1}$ (more precisely: such that i_n is an isomorphism).*

Proof. Let $e \colon X \longrightarrow R(X + A)$ be a rational equation morphism. By assumption, X is a coproduct of k indecomposable objects, $X = Y_1 + \cdots + Y_k$. For every coproduct injection $z \colon Z \longrightarrow X$ we obtain the corresponding morphisms $z_i \colon Z_i \longrightarrow Y_i$ with $Z = Z_1 + \cdots + Z_k$ and $z = z_1 + \cdots + z_k$. Since each z_i is a coproduct injection of Y_i, either $Z_i = 0$ or $Z_i = Y_i$. Consequently, there are (in case $Y_i \not\cong 0$ for every i) precisely 2^k subjects of X which are coproduct injections. Since the subobjects $\overline{X}_n \rightarrowtail X$, $n \in \mathbb{N}$, are pairwise disjoint, it follows that there exists an $m \in \mathbb{N}$ such that $\overline{X}_m \cong 0$. Thus $X_m \cong X_{m+1} + \overline{X}_{m+1} \cong X_{m+1}$. $\square$

5.5 Definition. *A rational equation morphism e is called* ***pre-guarded*** *provided that it has a trivial derived subobject, i.e., $X_n \cong 0$ for some n.*

Remark. This is equivalent to $X_\infty = 0$ (due to Lemma 5.4). Thus, e is pre-guarded iff $X = \coprod \overline{X}_n$, compare Definition 3.4.

5.6 Theorem. *In every iterative algebra all pre-guarded rational equation morphisms have unique solutions.*

Proof. This is completely analogous to the proof in Section 3, see Theorem 3.9 and Corollary 3.10. Given the pre-guarded rational equation morphism $e \colon X \longrightarrow R(X + A)$, we have $X_n = 0$, i.e., $X = \overline{X}_1 + \cdots + \overline{X}_n$ and we define a guarded equation morphism

$$f \equiv \overline{X}_1 \xrightarrow{\bar{i}_1} X \xrightarrow{e} R(X + A) \xrightarrow{R(u+A)} R(\overline{X}_1 + A)$$

where $u \colon X \longrightarrow \overline{X}_1$ has components $\mathrm{id}_{\overline{X}_1}, \bar{e}_1, \bar{e}_1 \cdot \bar{e}_2, \ldots, \bar{e}_1 \cdot \bar{e}_2 \cdot \cdots \cdot \bar{e}_n$. Observe that since u is a split epimorphism and X is finitely presentable, so is $\overline{X}_1$. Thus, f is a rational equation morphism. Since f is guarded, it has a unique solution $f^\dagger \colon X \longrightarrow A$, see Remark 4.6. The rest is as in Section 3. $\square$

5.7 Definition. *Let $e \colon X \longrightarrow R(X + A)$ be a rational equation morphism in an iterative algebra A. A solution $e^\dagger \colon X \longrightarrow A$ of e is called* ***strict*** *if its restriction to some derived subobject is $\bot_A$, i.e., there exists n for which the square*

$$\begin{array}{ccc} X_n & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle \bot_A} \\ X & \xrightarrow[e^\dagger]{} & A \end{array}$$

commutes.

5.8 Theorem. *In every iterative algebra every finitary equation morphism has a unique strict solution.*

Proof. This is completely analogous to Section 4, see Theorem 4.7 and Corollary 4.8: choose n such that $X_n = X_{n+1}$, see Lemma 5.4, then the role of X_∞ in Section 4 is now played by X_n. □

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