Some Remarks on Finitary and Iterative Monads

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Abstract. For every locally finitely presentable category $\mathcal{A}$ we introduce finitary Kleisli triples on $\mathcal{A}$ and show that they bijectively correspond to finitary monads on $\mathcal{A}$. We illustrate this on free monads and free iterative monads.

Keywords: monad, monoid, Kleisli triple

Mathematics Subject Classification (2000): 18C15, 18C35

1. Introduction

There are many presentations of finitary varieties of algebras: as models of algebraic theories, or as equational classes of Σ-algebras, or as finitary monads in Set, etc. We are going to add another one, using finitary Kleisli triples. Recall that a *Kleisli triple* on a category $\mathcal{A}$ is a triple of functions $(T, \eta, (\_)^*)$ where to every object X of $\mathcal{A}$ an object TX and a morphism $\eta_X : X \longrightarrow TX$ are assigned, and to every morphism $s : X \longrightarrow TY$ a morphism $s^* : TX \longrightarrow TY$ of $\mathcal{A}$ is assigned so that the following axioms

(Unit) $\eta_X{}^* = id_{TX}$,
(Ext) $s^* \cdot \eta_X = s$,

and

(Comp) $r^* \cdot s^* = (r^* \cdot s)^*$,

hold for all objects X and all morphisms $s : X \longrightarrow TY$ and $r : Y \longrightarrow TZ$.

Kleisli triples represent all monads on $\mathcal{A}$ in the following sense:

1. every monad (T, η, μ) generates a Kleisli triple given by the object-part of T, η and the following function $(\_)^*$:

$$X \xrightarrow{s} TY \quad \mapsto \quad TX \xrightarrow{Ts} TTY \xrightarrow{\mu_Y} TY$$

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and

2. every Kleisli triple is generated by a unique monad.

See Exercise 1.3(12) in [13].

The purpose of the present paper is to show that finitary monads (i.e., monads such that T preserves filtered colimits) are, analogously, presented by finitary Kleisli triples:

DEFINITION 1.1. A *finitary Kleisli triple* on a category $\mathcal{A}$ is a triple $(T, \eta, (\_)^*)$ of functions: T assigns to every finitely presentable object X of $\mathcal{A}$ an object TX, η assigns to X a morphism $\eta_X : X \longrightarrow TX$ and $(\_)^*$ assigns to every morphism $s : X \longrightarrow TY$ with X and Y finitely presentable a morphism $s^* : TX \longrightarrow TY$ so that the above axioms (Unit), (Ext) and (Comp) hold when restricted to X, Y and Z finitely presentable.

EXAMPLE 1.2. The theory of monoids can be described by the finitary Kleisli triple with $TX = \mathtt{List}\, X$, the set of all strings of elements of X (for every finite set X), $\eta : x \mapsto (x)$, and s^* sending a list of lists to the corresponding (flattened) list.

We will prove that finitary monads on a locally finitely presentable category $\mathcal{A}$ correspond bijectively to finitary Kleisli triples. More precisely, the above correspondence of 1. and 2. yields an equivalence between the category of finitary monads on $\mathcal{A}$ (and monad morphisms) and the category of all finitary Kleisli triples (and their morphisms introduced below). We then illustrate the concept of a finitary Kleisli triple on two fundamental examples.

2. Locally Finitely Presentable Categories

ASSUMPTION 2.1. Throughout the paper $\mathcal{A}$ denotes a locally finitely presentable (LFP) category. Recall from [11] or [6] that a category $\mathcal{A}$ is LFP provided that it has

(a) colimits

and

(b) a set of finitely presentable objects whose closure under filtered colimits is all of $\mathcal{A}$.

NOTATION 2.2. $\mathcal{A}_{fp}$ denotes a set of representatives of all finitely presentable objects of $\mathcal{A}$ (considered as a full subcategory of $\mathcal{A}$) and

$$J : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$$

denotes the inclusion functor.

For every object X of $\mathcal{A}$ let $D_X : J/X \longrightarrow \mathcal{A}$ be the canonical diagram of all morphisms from objects of $\mathcal{A}_{fp}$ into X with $D_X(JA \longrightarrow X) = JA$. Then X is a canonical colimit of D_X.

REMARK 2.3. $\mathcal{A}$ is a free cocompletion of $\mathcal{A}_{fp}$ under filtered colimits, see [6]. More precisely, if $\mathcal{B}$ is a category with filtered colimits, denote by $[\mathcal{A}_{fp}, \mathcal{B}]$ the functor category and by $\mathrm{Fin}[\mathcal{A}, \mathcal{B}]$ the full subcategory of $[\mathcal{A}, \mathcal{B}]$ of all finitary functors. Then the functor

$$(\_) \cdot J : \mathrm{Fin}[\mathcal{A}, \mathcal{B}] \longrightarrow [\mathcal{A}_{fp}, \mathcal{B}] \tag{2.1}$$

is an equivalence whose pseudoinverse is the functor

$$\mathrm{Lan}_J(\_) : [\mathcal{A}_{fp}, \mathcal{B}] \longrightarrow \mathrm{Fin}[\mathcal{A}, \mathcal{B}]$$

of left Kan extension along J.

The equivalence (2.1) (for the case $\mathcal{B} = \mathcal{A}$) allows us to transport the strict monoidal structure of $\mathrm{Fin}[\mathcal{A}, \mathcal{A}]$ given by composition and $Id_{\mathcal{A}}$ to (now a nonstrict) monoidal structure on the category $[\mathcal{A}_{fp}, \mathcal{A}]$. This has been done in [12] and, for the sake of reference, we describe the monoidal structure of $[\mathcal{A}_{fp}, \mathcal{A}]$ here.

For any pair $R, S : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$ of functors we are to define a functor

$$R \otimes S : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$$

in such a way that $\mathrm{Lan}_J(R \otimes S)$ is isomorphic to the composite $\mathrm{Lan}_J R \cdot \mathrm{Lan}_J S$, i.e., such that $\mathrm{Lan}_J(\_)$ becomes a monoidal functor. To do this, we have to recall the way how left Kan extensions can be computed.

NOTATION 2.4. Given a set S and an object A, let us denote by $S \bullet A$ the sum $\coprod_S A$. In this notation, the usual coend formula for a left Kan extension $\mathrm{Lan}_J F : \mathcal{A} \longrightarrow \mathcal{A}$ of $F : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$ along $J : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$ takes the following form

$$(\mathrm{Lan}_J F)X = \int^Y \mathcal{A}(JY, X) \bullet FY \tag{2.2}$$

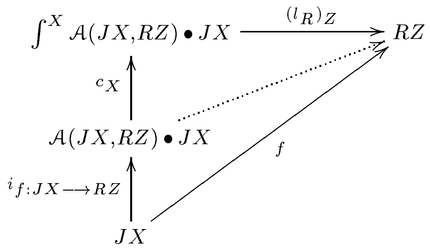
when evaluated at an object X in $\mathcal{A}$.

DEFINITION 2.5. The following defines a monoidal structure on $[\mathcal{A}_{fp}, \mathcal{A}]$:

1. The unit is the inclusion functor $J : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$.
2. For R, S in $[\mathcal{A}_{fp}, \mathcal{A}]$, define $R \otimes S$ on objects by

$$\Big(R \otimes S\Big)X = \int^{Y} \mathcal{A}(JY, SX) \bullet RY \quad \text{for every } X \text{ in } \mathcal{A}_{fp}. \tag{2.3}$$

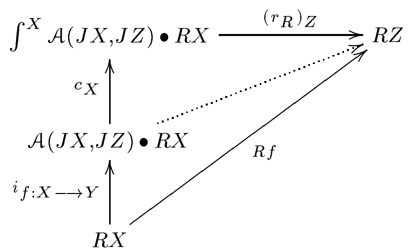
3. The left unit isomorphism $l_R : J \otimes R \longrightarrow R$ is defined as the unique morphism making the following diagrams



(2.4)

commutative (Z in $\mathcal{A}_{fp}$), where c_X denotes the colimit morphism, i_f the coproduct injection and the dotted arrow is induced by the cocone of all $f : JX \longrightarrow RZ$.

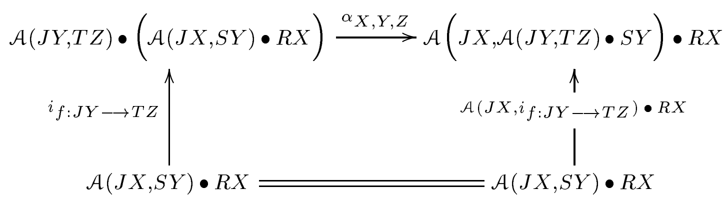
4. The right unit isomorphism $r_R : R \otimes J \longrightarrow R$ is defined as the unique morphism making the following diagrams



(2.5)

commutative (Z in $\mathcal{A}_{fp}$), where c_X denotes the colimit morphism, i_f the coproduct injection and the dotted arrow is induced by the cocone of all $f : X \longrightarrow Z$ (use the fact that J is full and faithful).

5. To define the associativity isomorphism $a_{R,S,T} : (R \otimes S) \otimes T \longrightarrow R \otimes (S \otimes T)$, define first, for every triple X, Y, Z of finitely presentable objects, the morphism $\alpha_{X,Y,Z}$ by commutativity of the following diagram



(2.6)

where i_f denote the coproduct injections.

Then $a_{R,S,T}$ is defined by "integrating" α, i.e., as the unique morphism making the following diagrams

$$\begin{array}{ccc}
\int^Y \mathcal{A}(JY,TZ)\bullet\left(\int^X \mathcal{A}(JX,SY)\bullet RX\right) & \xrightarrow{(a_{R,S,T})_Z} & \int^X \mathcal{A}\left(JX,\int^Y \mathcal{A}(JY,TZ)\bullet SY\right)\bullet RX \\
{\scriptstyle c_Y}\uparrow & & \uparrow{\scriptstyle c_X} \\
\mathcal{A}(JY,TZ)\bullet\left(\int^X \mathcal{A}(JX,SY)\bullet RX\right) & & \mathcal{A}\left(JX,\int^Y \mathcal{A}(JY,TZ)\bullet SY\right)\bullet RX \\
{\scriptstyle \mathcal{A}(JY,TZ)\bullet c_X}\uparrow & & \uparrow{\scriptstyle \mathcal{A}(JX,c_Y)\bullet RX} \\
\mathcal{A}(JY,TZ)\bullet\left(\mathcal{A}(JX,SY)\bullet RX\right) & \xrightarrow[\alpha_{X,Y,Z}]{} & \mathcal{A}\left(JX,\mathcal{A}(JY,TZ)\bullet SY\right)\bullet RX
\end{array} \tag{2.7}$$

commutative (Z in $\mathcal{A}_{fp}$). Here, again, c_X and c_Y denote the various colimit morphisms.

In what follows, when speaking of the category $[\mathcal{A}_{fp},\mathcal{A}]$, we will always consider it equipped with the monoidal structure described above.

REMARK 2.6. Recall the concept of a monoid (T,η,μ) in the monoidal category $[\mathcal{A}_{fp},\mathcal{A}]$: here $T:\mathcal{A}_{fp}\longrightarrow\mathcal{A}$ is a functor, and $\eta: J\longrightarrow T$, $\mu: T\otimes T\longrightarrow T$ are natural transformations such that the following diagrams

$$\begin{array}{ccccc}
J\otimes T & \xrightarrow{\eta\otimes T} & T\otimes T & \xleftarrow{T\otimes\eta} & T\otimes J \\
& {\scriptstyle l_T}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle r_T} & \\
& & T & &
\end{array} \tag{2.8}$$

and

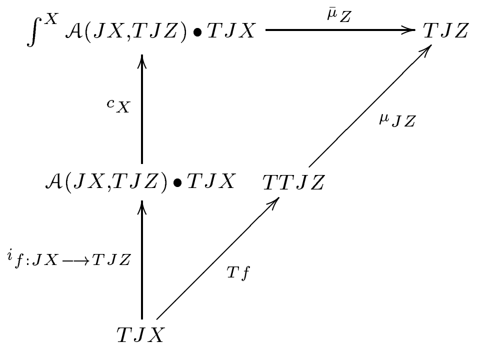
$$\begin{array}{ccccc}
(T\otimes T)\otimes T & \xrightarrow{a_{T,T,T}} & T\otimes(T\otimes T) & \xrightarrow{T\otimes\mu} & T\otimes T \\
{\scriptstyle\mu\otimes T}\downarrow & & & & \downarrow{\scriptstyle\mu} \\
T\otimes T & & \xrightarrow[\mu]{} & & T
\end{array} \tag{2.9}$$

commute.

The equivalence of Remark 2.3 assigns to every finitary monad (T,η,μ) on $\mathcal{A}$ the corresponding monoid $(TJ,\eta J,\bar{\mu})$, where

$$\bar{\mu}: TJ\otimes TJ\longrightarrow TJ$$

is defined by commutativity of the following diagrams:



(2.10)

where Z is in $\mathcal{A}_{fp}$.

And to every monad morphism

$$\rho : (T, \eta^T, \mu^T) \longrightarrow (S, \eta^S, \mu^S)$$

it assigns the monoid homomorphism

$$\rho J : (TJ, \eta^T J, \bar{\mu}^T) \longrightarrow (SJ, \eta^S J, \bar{\mu}^S)$$

This yields an equivalence

$$\mathrm{Mon}_f(\mathcal{A}) \simeq \mathrm{Monoid}[\mathcal{A}_{fp}, \mathcal{A}]$$

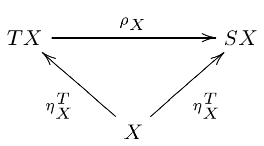
between the categories of finitary monads on $\mathcal{A}$ and monoids in $[\mathcal{A}_{fp}, \mathcal{A}]$ as described in [12].

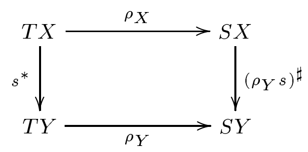
3. Finitary Kleisli Triples

REMARK 3.1. Recall that $\mathcal{A}$ denotes an LFP category. We have defined finitary Kleisli triples in the introduction, and we are going to prove that they naturally represent finitary monads on $\mathcal{A}$. Let us now introduce *morphisms* of finitary Kleisli triples that will correspond to the usual monad morphisms. By a morphism from a finitary Kleisli triple $(T, \eta^T, (\_)^*)$ to another one, $(S, \eta^S, (\_)^\sharp)$ is meant a a function ρ assigning to every finitely presentable object X a morphism

$$\rho_X : TX \longrightarrow SX$$

such that the following diagrams





commute (for all finitely presentable objects X, Y and all morphisms $s : X \longrightarrow TY$, respectively).

Whereas the categories $\mathrm{Mon}_f(\mathcal{A})$ and $\mathrm{Monoid}[\mathcal{A}_{fp}, \mathcal{A}]$ are equivalent, we prove now a stronger result for the category

$$\mathrm{Tri}_f(\mathcal{A})$$

of all finitary Kleisli triples in $\mathcal{A}$: it is actually *isomorphic* to the category of monoids in $[\mathcal{A}_{fp}, \mathcal{A}]$.

As a consequence we will show that every finitary Kleisli triple is generated by an (essentially unique) finitary monad.

THEOREM 3.2. *The categories* $\mathrm{Tri}_f(\mathcal{A})$ *and* $\mathrm{Monoid}[\mathcal{A}_{fp}, \mathcal{A}]$ *are isomorphic.*

Proof. We define a functor

$$E : \mathrm{Tri}_f(\mathcal{A}) \longrightarrow \mathrm{Monoid}[\mathcal{A}_{fp}, \mathcal{A}]$$

which we prove to be an isomorphism of categories.

(I) To define E on objects, suppose that $(T, \eta, (\_)^*)$ is a finitary Kleisli triple.

The assignment $X \mapsto TX$ can be extended to a functor $T : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$; put $Tf = (\eta_Y \cdot f)^*$ for every $f : X \longrightarrow Y$. In fact, the equality

$$T(id_X) = id_{TX}$$

follows immediately from (UNIT) and preservation of composition is derived for $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ as follows:

$$(\eta_Z g)^* \cdot (\eta_Y f)^* = ((\eta_Z g)^* \eta_Y f)^* = (\eta_Z g f)^*$$

where the first equality is by (COMP) and the second by (EXT).

The collection $\eta_X : JX \longrightarrow TX$ becomes a natural transformation: the equality

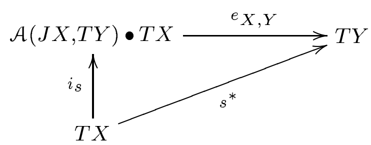
$$Tf \cdot \eta_X = \eta_Y \cdot f$$

for every $f : X \longrightarrow Y$ follows from (EXT).

Observe further that to give the map $s \mapsto s^*$ is equivalent to giving, for every X, Y in $\mathcal{A}_{fp}$, an action

$$e_{X,Y} : \mathcal{A}(JX, TY) \bullet TX \longrightarrow TY$$

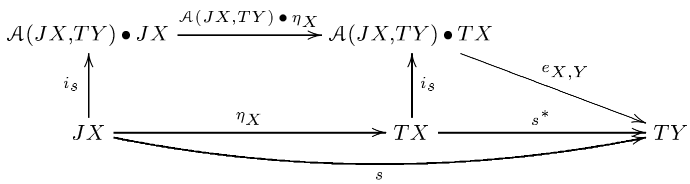
The correspondence is as follows:



(3.11)

where by i_s we denote the coproduct injection of $s : JX \longrightarrow TY$.

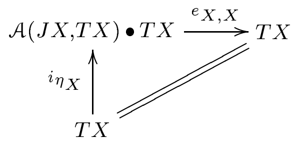
The axiom (Ext) is equivalent to the fact that the following diagrams



(3.12)

commute for every $s : JX \longrightarrow TY$.

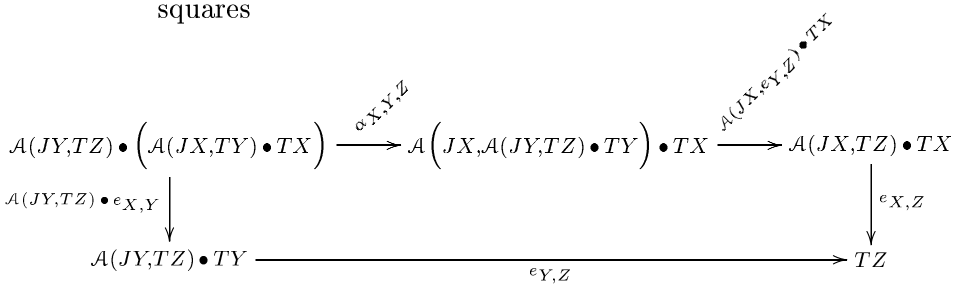
The axiom (Unit) is equivalent to the fact that the triangles



(3.13)

commute for every X in $\mathcal{A}_{fp}$.

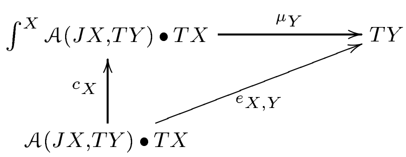
Finally, the axiom (Comp) is equivalent to commutativity of the squares



(3.14)

where $\alpha_{X,Y,Z}$ denotes the morphism of (2.6).

Define $\mu_Y : (T \otimes T)Y \longrightarrow TY$ by "integrating" (3.11), i.e., by commutativity of the triangles

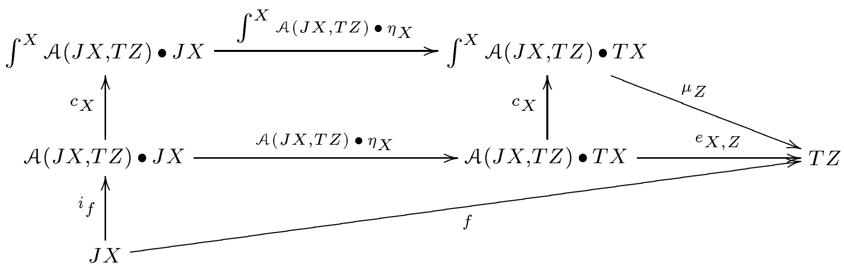


(3.15)

for every X, where c_X denotes the colimit morphism. It is clear that this definition is natural in Y, thus we have a natural transformation $\mu : T \otimes T \longrightarrow T$.

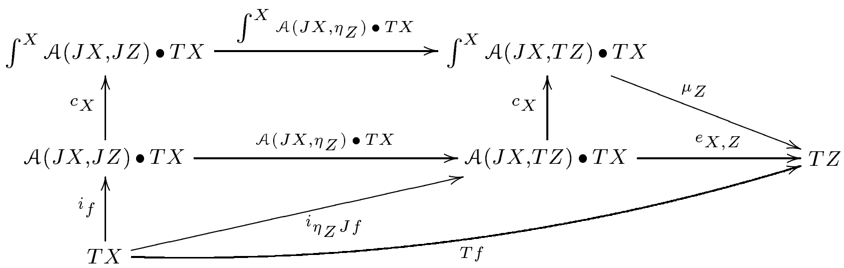
Let us prove that the above triple (T, η, μ) is a monoid in $[\mathcal{A}_{fp}, \mathcal{A}]$.

(a) The equality $\mu_Z \cdot (\eta \otimes T)_Z = (l_T)_Z$ follows from the following diagram (for any morphism $f : JX \longrightarrow TZ$):



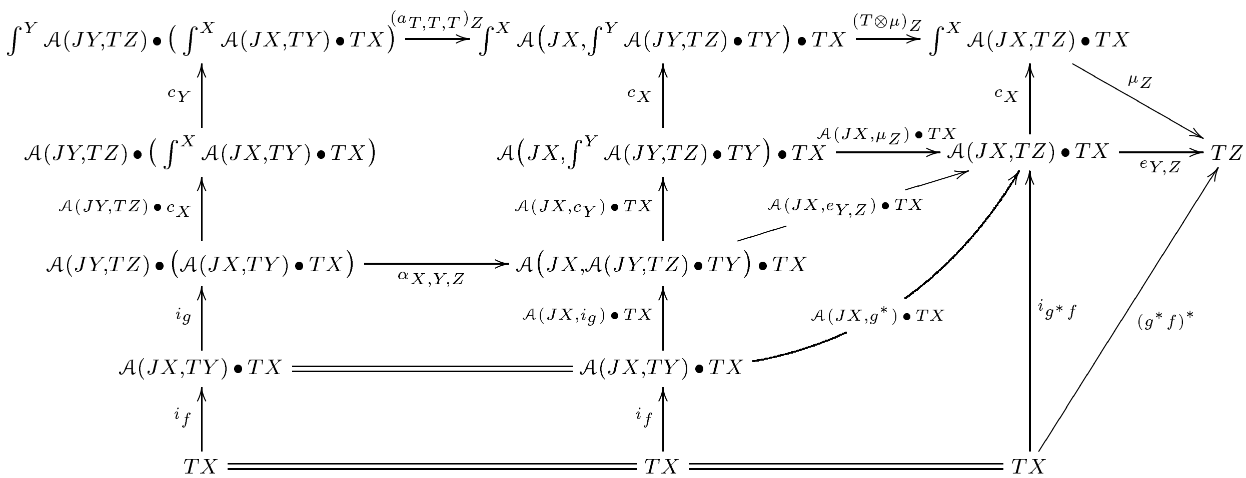
which commutes due to (2.4) and (3.12).

(b) The equality $\mu_Z \cdot (T \otimes \eta)_Z = (r_T)_Z$ follows from the following commutative diagram (for any morphism $f : X \longrightarrow Z$):

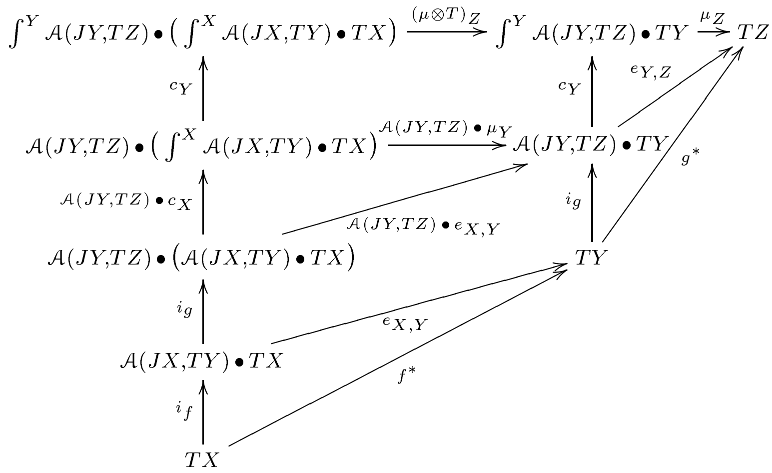


which commutes due to (2.5) and (3.11) because $Tf = (\eta_Z Jf)^*$.

(c) The equality $\mu_Z \cdot (T \otimes \mu)_Z \cdot (a_{T,T,T})_Z = \mu_Z \cdot (\mu \otimes T)_Z$ is deduced as follows: by precomposing the left-hand side with all the colimit morphisms we obtain the following commutative diagram:



Similarly, when precomposing the right-hand side of the desired equality with all the colimit morphisms, we obtain the following commutative diagram:



Since $g^* f^* = (g^* f)^*$ for all $f : JX \longrightarrow TY$ and $g : JY \longrightarrow TZ$, the above two diagrams prove that $\mu_Z \cdot (T \otimes \mu)_Z \cdot (a_{T,T,T})_Z = \mu_Z \cdot (\mu \otimes T)_Z$ holds.

(II) To define E on morphisms

$$\rho : (T, \eta^T, (\_)^*) \longrightarrow (S, \eta^S, (\_)^\sharp)$$

of finitary Kleisli triples, denote by

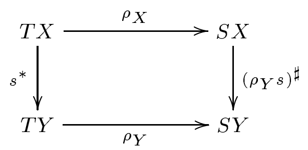
$$E\rho : E(T, \eta^T, (\_)^*) \longrightarrow E(S, \eta^S, (\_)^\sharp)$$

the monoid homomorphism with the same components $\rho_X : TX \longrightarrow SX$. This is indeed a monoid homomorphism:

(a) To verify that ρ is a natural transformation recall that, for $f : X \longrightarrow Y$ in $\mathcal{A}_{fp}$, we have defined

$$Tf = (\eta^T_Y f)^* \quad \text{and} \quad Sf = (\eta^S_Y f)^\sharp$$

Then, using the equality $\rho_Y \cdot \eta^T_Y = \eta^S_Y$ and commutativity of the square



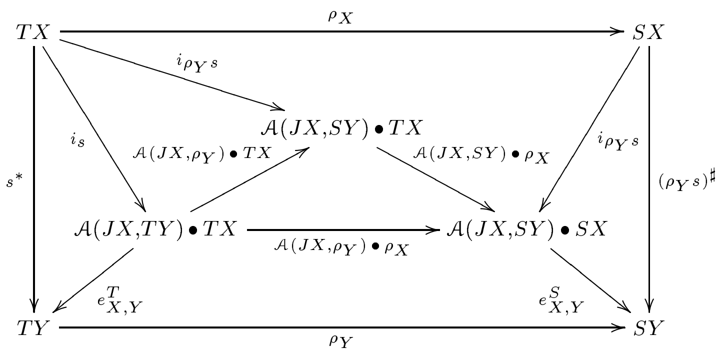
(3.16)

for $s = \eta^T_Y f : X \longrightarrow TY$, we conclude naturality of ρ.

(b) To verify $\rho_Y \mu_Y^T = \mu_Y^S(\rho_Y \otimes \rho_Y)$, we show the commutativity of the following squares

$$\begin{array}{ccc} \int^X \mathcal{A}(JX,TY) \bullet TX & \xrightarrow{\int^X \mathcal{A}(JX,\rho_Y) \bullet \rho_X} & \int^X \mathcal{A}(JX,SY) \bullet SX \\ {\scriptstyle \mu_Y^T}\downarrow & & \downarrow{\scriptstyle \mu_Y^S} \\ TY & \xrightarrow[\rho_Y]{} & SY \end{array} \tag{3.17}$$

Firstly, notice that the commutativity of (3.17) is equivalent to the commutativity of the lower square in the following diagrams for every $s : JX \longrightarrow TY$



(3.18)

But this diagram clearly commutes since its outer square is just (3.16) and the coproduct injections i_s form an epimorphic family.

(III) To prove that E is invertible, consider first the object part: given a monoid (T, η, μ) in $[\mathcal{A}_{fp}, \mathcal{A}]$, we define a finitary Kleisli triple $(T, \eta, (\_)^*)$ as follows: use (3.15) to define $e_{X,Y}$ and (3.11) to define s^*. Then the axioms of Kleisli triples are verified as follows:

(Ext): $s^*\eta_X = s$ is equivalent to (3.12), which commutes because the diagram in I(a) above commutes.

(Unit): ${\eta_X}^* = id_X$ is equivalent to (3.13), which commutes because the diagram in I(b) above commutes.

and

(Comp): $r^*s^* = (r^*s)^*$ is equivalent to (3.14), which commutes because the two diagrams in I(c) above commute.

And it is obvious that the given monoid is equal to

$$E(T, \eta, (\_)^*).$$

For the morphism part, let $\rho : E(T, \eta^T, (\_)^*) \longrightarrow E(S, \eta^S, (\_)^\sharp)$ be a monoid homomorphism. Then the function $X \mapsto \rho_X$ fulfills $\rho_X \eta^T_X = \eta^S_X$, and we only have to verify that diagram (3.16) commutes for every $s : X \longrightarrow TY$. That is, the outward square of (3.18) commutes. In fact, since ρ is a monoid homomorphism we have

$$\mu^S_Y \cdot (\rho \otimes \rho)_Y = \rho_Y \cdot \mu^T_Y,$$

thus, (3.17) commutes. This implies that the lower part of (3.18) commutes, which yields the commutativity of the whole diagram (3.18) since the other inner parts trivially commute. □

COROLLARY 3.3. *The category of all finitary monads is equivalent to the category of all finitary Kleisli triples: the functor*

$$E : \mathrm{Mon}_f(\mathcal{A}) \longrightarrow \mathrm{Tri}_f(\mathcal{A})$$

assigning to every finitary monad the corresponding triple is an equivalence.

COROLLARY 3.4. *Every finitary Kleisli triple on an LFP category is generated by a finitary monad.*

Proof. By Corollary 3.3 every finitary Kleisli triple $(T, \eta, (\_)^*)$ is *isomorphic* to one generated by a finitary monad $\widetilde{\mathbb{T}} = (\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$. Let i be the corresponding isomorphism in $\mathrm{Tri}_f(\mathcal{A})$ with components

$$i_X : TX \longrightarrow \widetilde{T}X \quad (X \text{ in } \mathcal{A}_{fp})$$

It follows immediately that for the functor $T : \mathcal{A}_{fp} \longrightarrow \mathcal{A}$, given on morphism $f : X \longrightarrow Y$ by $Tf = (\eta_Y f)^*$, we have a natural isomorphism i from T to $\widetilde{T}$ restricted to $\mathcal{A}_{fp}$. Denote by $T^0 : \mathcal{A} \longrightarrow \mathcal{A}$ a finitary extension of T, then there is a unique natural isomorphism

$$i^0 : T^0 \longrightarrow \widetilde{T}$$

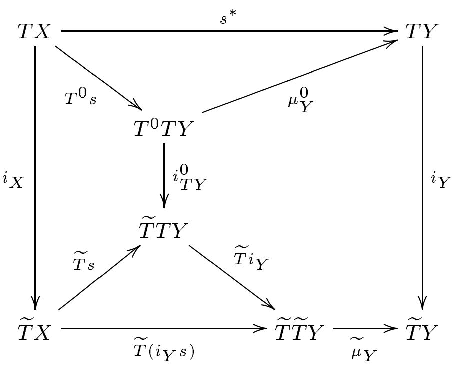
with the above components $i^0_X = i_X$ for finitely presentable objects X. We obtain a unique monad

$$\mathbb{T}^0 = (T^0, \eta^0, \mu^0)$$

for which $i^0 : \mathbb{T}^0 \longrightarrow \widetilde{\mathbb{T}}$ is a monad isomorphism. For finitely presentable X we have $\eta^0_X = \eta_X$ (since $\eta_X = i_X^{-1} \widetilde{\eta}_X$) and thus, to prove that $\mathbb{T}^0$ generates the given finitary Kleisli triple, we only need to prove that

$$s^* = \mu^0_Y \cdot T^0 s$$

for all $s : X \longrightarrow TY$ with X and Y finitely presentable. Since i is a morphism of Kleisli triples, the outer square in the following diagram



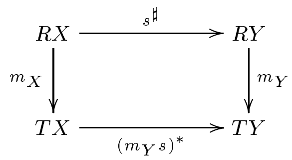
commutes. The left-hand square commutes by naturality of i^0 and the right-hand part by $i^0 : \mathbb{T}^0 \longrightarrow \widetilde{\mathbb{T}}$ being a monad morphism. Since i_Y is an isomorphism, it follows that the upper triangle commutes. □

REMARK 3.5. For every monad $\mathbb{T}$ we have, of course, a corresponding finitary Kleisli triple, and we show below that this describes the finitary coreflection of $\mathbb{T}$. Let us observe here that a *finitary submonad* of $\mathbb{T}$, i.e., a monomorphism of monads $m : \mathbb{R} \longrightarrow \mathbb{T}$ with $\mathbb{R}$ finitary, is completely described by the following data:

(i) a subobject $m_X : RX \longrightarrow TX$ of TX for every finitely presentable object X through which η_X factors,

and

(ii) a morphism $s^\sharp : RX \longrightarrow RY$, for every $s : X \longrightarrow RY$, which is a domain-codomain restriction of $(m_Y s)^* : TX \longrightarrow TY$



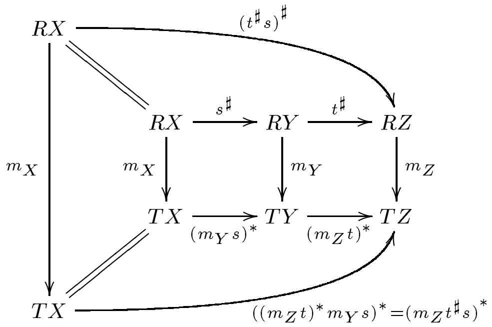
In fact, given such data, we obtain a unique triple

$$(R, \eta', (\_)^\sharp)$$

where $\eta'_X : X \longrightarrow RX$ is given by the factorization $\eta_X = m_X \eta'_X$ in (i) above. The verification of the three axioms of Kleisli triples is simple, based on the fact that every m_X is a monomorphism. For example, the axiom (Comp)

$$t^\sharp s^\sharp = (t^\sharp s)^\sharp$$

follows from the commutativity of the following diagram:



More detailed: to prove that the upper triangle commutes, we compose it with the monomorphism m_Z, and use the fact that all other inner parts as well as the perimeter commute.

COROLLARY 3.6. *The category* $\mathrm{Mon}_f(\mathcal{A})$ *of finitary monads on* $\mathcal{A}$ *is coreflective in the category* $\mathrm{Mon}(\mathcal{A})$ *of all monads on* $\mathcal{A}$*. A coreflection of a monad* $\mathbb{T}$ *is the finitary monad generating the same finitary Kleisli triple as* $\mathbb{T}$*.*

Proof. In fact, by Corollary 3.4 we have a finitary monad $\mathbb{T}^0 = (T^0, \eta^0, \mu^0)$ generating the finitary Kleisli triple of $\mathbb{T} = (T, \eta, \mu)$. Since $T^0 = \mathrm{Lan}_J(TJ)$, there is a canonical natural transformation

$$\psi : T^0 \longrightarrow T$$

namely, the T-component of the counit of left Kan extension.

The monad structure of T^0 is given by Remark 2.6: the unit

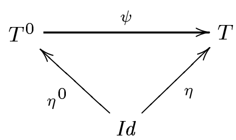
$$\eta^0 : Id \longrightarrow T^0$$

is defined as $\mathrm{Lan}_J(\eta J) : \mathrm{Lan}_J J \longrightarrow \mathrm{Lan}_J(TJ)$ (use the fact that $\mathrm{Lan}_J J \cong Id$) and the multiplication

$$\mu^0 : T^0 T^0 \longrightarrow T^0$$

is given by $\mathrm{Lan}_J \bar{\mu}$, where $\bar{\mu} : T^0 J \otimes T^0 J \longrightarrow T^0 J$ is defined as in diagram (2.10).

We now show that $\psi : T^0 \longrightarrow T$ is a monad morphism. The triangle



commutes by definition of the counit of left Kan extension.

The commutativity of the square

$$\begin{array}{ccc} T^0T^0 & \xrightarrow{\psi*\psi} & TT \\ {\scriptstyle \mu^0}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^0 & \xrightarrow[\psi]{} & T \end{array}$$

follows from commutativity of the square

$$\begin{array}{ccc} T^0J\otimes T^0J & \xrightarrow{id\otimes id} & TJ\otimes TJ \\ {\scriptstyle \bar\mu}\downarrow & & \downarrow{\scriptstyle \bar\mu} \\ T^0J & \xrightarrow[id]{} & TJ \end{array}$$

(use the fact that $T^0J = TJ$ and the definition of the counit of left Kan extension again) but the latter square commutes trivially.

To prove universality of ψ, consider a finitary monad (S, η^S, μ^S) and a monad morphism $\alpha : S \longrightarrow T$. Since $S = \mathrm{Lan}_J(SJ)$, it suffices to show that the unique natural transformation

$$\mathrm{Lan}_J(\alpha J) : S \longrightarrow T^0$$

which makes the triangle

$$\begin{array}{ccc} T^0 & \xrightarrow{\psi} & T \\ {\scriptstyle \mathrm{Lan}_J(\alpha J)}\uparrow & \nearrow{\scriptstyle \alpha} & \\ S & & \end{array}$$

commutative, is a monad morphism. This follows from the fact that the diagrams

$$\begin{array}{ccc} SJ & \xrightarrow{\alpha J} & TJ \\ {\scriptstyle \eta^S J}\nwarrow & & \nearrow{\scriptstyle \eta J} \\ & J & \end{array} \quad \text{and} \quad \begin{array}{ccc} SJ\otimes SJ & \xrightarrow{\alpha J\otimes \alpha J} & TJ\otimes TJ \\ {\scriptstyle \bar\mu^S}\downarrow & & \downarrow{\scriptstyle \bar\mu} \\ SJ & \xrightarrow[\alpha J]{} & TJ \end{array}$$

commute. □

4. Free Kleisli Triples

One consequence of Theorem 3.2 is that every finitary endofunctor H of an LFP category $\mathcal{A}$ generates a free finitary Kleisli triple, viz., the Kleisli triple associated with the free-algebra monad of H. That is, we have a concrete description of a left adjoint of the forgetful functor

$$U : \mathrm{Tri}_f(\mathcal{A}) \longrightarrow \mathrm{Fin}[\mathcal{A}, \mathcal{A}]$$

assigning to every finitary Kleisli triple the corresponding underlying finitary endofunctor of $\mathcal{A}$. This follows from the result of M. Barr that free monads are precisely the free-algebra monads, see [7].

In what follows $\mathsf{Alg}\, H$ denotes the category of all H-algebras, i.e, pairs (A, a) where $a : HA \longrightarrow A$ is a morphism, and their homomorphisms. The natural forgetful functor is denoted by $V : \mathsf{Alg}\, H \longrightarrow \mathcal{A}$. Recall from [2] that since H is finitary, V has a left adjoint, i.e., every object A of $\mathcal{A}$ generates a free H-algebra, FA, which can be described as a colimit of the following ω-chain in $\mathcal{A}$:

$$A \xrightarrow{\mathsf{inr}} HA + A \xrightarrow{H\mathsf{inr}+A} H(HA + A) + A \xrightarrow{H(H\mathsf{inr}+A)+A} \cdots$$

That is, we form the ω-chain, W, in $[\mathcal{A}, \mathcal{A}]$ with

$$W_0 = Id \quad \text{and} \quad W_{n+1} = HW_n + Id$$

whose connecting maps $w_{n,k} : W_n \longrightarrow W_k$ are given by

$$w_{0,1} = \mathsf{inr} : Id \longrightarrow H + Id \quad \text{and} \quad w_{n+1,k+1} = Hw_{n,k} + Id$$

Denote by

$$(w_n : W_n \longrightarrow F)_{n\in\omega}$$

a colimit of this chain. Then also $(Hw_n : HW_n \longrightarrow HF)_{n\in\omega}$ is a colimit (since H is finitary) and we can define

$$\varphi : HF \longrightarrow F$$

by $\varphi \cdot Hw_n \equiv HW_n \xrightarrow{\mathsf{inl}} W_{n+1} \xrightarrow{w_{n+1}} F$ for all $n \in \omega$.

It follows that the H-algebra $\varphi_A : HFA \longrightarrow FA$ is free on A w.r.t. $(w_0)_A : A \longrightarrow FA$. And the corresponding free-algebra monad

$$\mathbb{F}_H = (F, w_0, \mu)$$

is a free monad on H, as proved by Barr. Furthermore $\mathbb{F}_H$ is a finitary monad: each W_n above is obviously finitary, therefore, so is $F = \operatorname{colim} W_n$. From Theorem 3.2 we thus obtain the following

COROLLARY 4.1. *The forgetful functor* $U : \mathrm{Tri}_f(\mathcal{A}) \longrightarrow \mathrm{Fin}[\mathcal{A},\mathcal{A}]$ *has a left adjoint assigning to every finitary endofunctor* H *the finitary Kleisli triple associated with the monad* $\mathbb{F}_H$.

REMARK 4.2. Explicitly, a free Kleisli triple on H is the triple $(F, \eta^F, (\_)^*)$ where for X finitely presentable we put

$FX = \operatorname{colim} W_n$ with $W_0 = X$ and $W_{n+1} = HW_n + X$

and

η^F_X is the first colimit injection.

For $s : X \longrightarrow FY$ we put

$s^* = \operatorname{colim} s_n$ where $s_0 = s$ and $s_{n+1} = [\varphi_Y \cdot Hs_n, s] : HW_n + X \longrightarrow FY$.

EXAMPLE 4.3. Let $H = H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ be the "polynomial" endofunctor of a signature $\Sigma = (\Sigma_n)_{n\in\omega}$:

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \dots$$

The FX is the algebra of all finite Σ-*trees* on X, i.e., trees labelled so that a node with $n > 0$ successors carries a label from Σ_n and a leaf carries a label from $X + \Sigma_0$. And s^* is the usual *tree-substitution*: $s^*(t)$ is the tree t in which every leaf labelled by $x \in X$ is substituted by the tree $s(x)$.

5. Free Iterative Kleisli Triples

In his attempt to describe algebraically potentially infinite computations, C. Elgot introduced iterative algebraic theories see [9]. A fundamental result of his group, see [10], was a description of a free iterative theory on any given signature Σ: it is the theory formed by all *rational* Σ-trees, i.e., finite and infinite Σ-trees which have, up to isomorphism, only finitely many subtrees. We have presented in [4] a proof that every finitary endofunctor H of Set generates a free iterative monad; this has been extended to all LFP categories in [5]. In fact, we obtain an adjoint situation between the category $\mathrm{Fin}[\mathcal{A},\mathcal{A}]$ of all finitary endofunctors of $\mathcal{A}$ and the category of all finitary iterative Kleisli triples on $\mathcal{A}$, which we shall introduce below.

The method used in [5] to obtain the general result is based on investigating iterative algebras rather than iterative theories:

DEFINITION 5.1. ([5].) An H-algebra $a : HA \longrightarrow A$ is called *iterative* if for every "flat equation" morphism

$$e : X \longrightarrow HX + A, \quad X \text{ finitely presentable,}$$

there exists a unique *solution*, i.e., a unique morphism $e^\dagger : X \longrightarrow A$ for which the following square

$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ HX+A & \xrightarrow[\ He^\dagger+A\ ]{} & HA+A \end{array}$$

commutes. We denote by $\mathsf{Alg}_{it}\, H$ the full subcategory of $\mathsf{Alg}\, H$ formed by all iterative algebras, and by $U_{it} : \mathsf{Alg}_{it}\, H \longrightarrow \mathcal{A}$ the natural forgetful functor.

It is easy to see that U_{it} has a left adjoint, i.e., every object X in $\mathcal{A}$ generates a free iterative H-algebra

$$\rho_X : H(RX) \longrightarrow RX$$

with a universal arrow $\eta_X : X \longrightarrow RX$.

DEFINITION 5.2. The *rational Kleisli triple* of H is the Kleisli triple $(R, \eta, (\_)^*)$ where for every finitely presentable object X we have the free iterative H-algebra $\eta_X : X \longrightarrow RX$ and for every morphism $s : X \longrightarrow RY$ (X and Y finitely presentable) we have the unique homomorphism $s^* : RX \longrightarrow RY$ of H-algebras with $s = s^* \cdot \eta_X$.

EXAMPLE 5.3. If $H = H_\Sigma$ then RX is the algebra of all rational Σ-trees on X (for every finite set X).

Coming back to Elgot's iterative monads, we shortly recall from [4] how these are defined in general locally finitely presentable categories $\mathcal{A}$. For the sake of convenience we assume from now on that $\mathcal{A}$ has monomorphic coproduct injections; this assumption can be omitted at the expense of a slightly more technical exposition, see [5].

A monad $\mathbb{T} = (T, \eta, \mu)$ is called *ideal* if there is a subfunctor $\tau : T' \hookrightarrow T$ such that

(i) $T = T' + Id$ with injections τ and η

and

(ii) $\mu : TT \longrightarrow T$ can be restricted to a natural transformation $\mu' : T'T \longrightarrow T'$, i.e., $\tau \cdot \mu' = \mu \cdot \tau T$.

By a *finitary equation morphism* for $\mathbb{T}$ is meant a morphism $e : X \longrightarrow T(X+Y)$ where X is finitely presentable. And e is called *guarded* if it factors through the first coproduct injection of $T(X+Y) = (HT(X+Y)+Y)+X$. A *solution* of e is a morphism $e^\dagger : X \longrightarrow TY$ such that the following square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ T(X+Y) & \xrightarrow[T[e^\dagger, \eta_Y]]{} & TTY \end{array}$$

An ideal monad is called *iterative* provided that every guarded finitary equation morphism has a unique solution.

We now translate all the above terms to the language of finitary Kleisli triples:

DEFINITION 5.4. By an *ideal finitary Kleisli triple* is understood a finitary Kleisli triple $(T, \eta, (\_)^*)$ together with a function assigning to every finitely presentable object X a subobject $\sigma_X : T'X \longrightarrow TX$ in such a way that for each morphism $s : X \longrightarrow TY$ (X and Y finitely presentable) a restriction $s^+ : T'X \longrightarrow T'Y$ of s^* exists:

$$\begin{array}{ccc} T'X & \xrightarrow{\quad s^+ \quad} & T'Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ TX & \xrightarrow[s^*]{} & TY \end{array}$$

EXAMPLES 5.5.

1. The free-algebra Kleisli triple $\mathbb{F}$ (see 4.2) is ideal: here $FX = HFX + X$, and we have the commutative squares

$$\begin{array}{ccc} HFX & \xrightarrow{\quad Hs^* \quad} & HFY \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\ FX & \xrightarrow[s^*]{} & FY \end{array}$$

   since s^* is a homomorphism of H-algebras, and the coproduct injections φ_X are monomorphic.

2. The rational Kleisli triple $\mathbb{R}$ (see 5.2) is ideal: again, $RX = HRX + X$, and we have the commutative squares

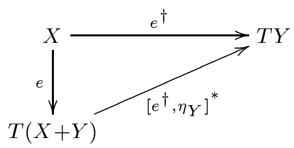
$$\begin{array}{ccc} HRX & \xrightarrow{\quad Hs^* \quad} & HRY \\ {\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ RX & \xrightarrow[s^*]{} & RY \end{array}$$

since s^* is a homomorphism of H-algebras.

3. The classical Kleisli triples are not ideal in general. For example, the Kleisli triple of the theory of groups is not ideal due to equations such as $xzz^{-1} = x$: consider $s : X \longrightarrow \mathbb{Z} = F(\{1\})$, where $X = \{x, z\}$ and $s(x) = s(z) = 1$, then for the term xzz^{-1} in the complement of $\eta_X : X \hookrightarrow FX$ we have $s^*(xzz^{-1}) = \eta(1)$.

DEFINITION 5.6. Let $(T, \eta, (\_)^*)$ with $\sigma : T' \hookrightarrow T$ be an ideal Kleisli triple.

1. A morphism $e : X \longrightarrow T(X + Y)$ with X finitely presentable is called a *finitary equation morphism*. It is called *guarded* provided that it factors through $[\sigma_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]$. A *solution* of e is a morphism $e^\dagger : X \longrightarrow TY$ such that the following triangle



commutes.

2. An ideal Kleisli triple $\mathbb{T}$ is called *iterative* provided that every guarded equation morphism has a unique solution.

THEOREM 5.7. ([5].) *For every finitary endofunctor H the rational Kleisli triple R is iterative.*

In fact, the above theorem was proved in the language of finitary monads, but the correspondence between the latter and iterative Kleisli triples (see Section 3) is obvious.

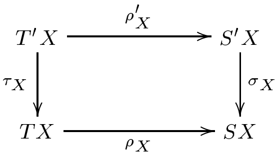
In fact, more has been proved in [5]: the rational monad is not only iterative but it is a free iterative monad on H. Let us translate this to the language of iterative triples. Given ideal finitary Kleisli triples

$$(T, \eta, (\_)^*) \quad \text{with} \quad \tau : T' \longrightarrow T$$

and

$$(S, \eta, (\_)^\sharp) \quad \text{with} \quad \sigma : S' \longrightarrow S$$

we call a morphism ρ of finitary triples (see 3.1) *ideal* provided that each ρ_X restricts to ρ'_X as follows:



NOTATION 5.8. We denote by

$$\mathrm{Tri}_{if}(\mathcal{A})$$

the category of all iterative, finitary Kleisli triples and ideal triple morphisms.

THEOREM 5.9. ([5].) *The rational triple is a free iterative finitary triple on the given finitary endofunctor H. More precisely, the forgetful functor*

$$U : \mathrm{Tri}_{if}(\mathcal{A}) \longrightarrow \mathrm{Fin}[\mathcal{A}, \mathcal{A}]$$

assigning to every iterative finitary triple S the subfunctor S', has a left adjoint, viz, the functor $H \mapsto R$ assigning to every finitary endofunctor its rational Kleisli triple.

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