

Varieties of Languages in a Category

Jiří Adámek, Robert S. R. Myers, Henning Urbat

Institut für Theoretische Informatik

Technische Universität Braunschweig, Germany

Stefan Milius

Lehrstuhl für Theoretische Informatik

Friedrich-Alexander Universität Erlangen-Nürnberg, Germany

Abstract—Eilenberg’s variety theorem, a centerpiece of algebraic automata theory, establishes a bijective correspondence between varieties of languages and pseudovarieties of monoids. In the present paper this result is generalized to an abstract pair of algebraic categories: we introduce varieties of languages in a category \mathcal{C} , and prove that they correspond to pseudovarieties of monoids in a closed monoidal category \mathcal{D} , provided that \mathcal{C} and \mathcal{D} are dual on the level of finite objects. By suitable choices of these categories our result uniformly covers Eilenberg’s theorem and three variants due to Pin, Polák and Reutenauer, respectively, and yields new Eilenberg-type correspondences.

Index Terms—Eilenberg’s theorem, varieties of languages, monoids, duality, automata, coalgebra, algebra.

I. INTRODUCTION

Algebraic automata theory investigates the relation between regular languages and algebraic structures like monoids, semigroups, or semirings. A major result concerns *varieties of languages*. These are classes of regular languages closed under

- (a) boolean operations (union, intersection and complement),
- (b) derivatives, i.e., with every language $L \subseteq \Sigma^*$ a variety contains its *left derivatives* $a^{-1}L = \{w \in \Sigma^* : aw \in L\}$ and *right derivatives* $La^{-1} = \{w \in \Sigma^* : wa \in L\}$ for all $a \in \Sigma$, and
- (c) preimages under monoid morphisms $f : \Delta^* \rightarrow \Sigma^*$.

Eilenberg proved in his monograph [12] that the lattice of all varieties of languages is isomorphic to the lattice of all *pseudovarieties of monoids*, these being classes of finite monoids closed under finite products, submonoids and homomorphic images. Several variants of Eilenberg’s theorem are known in the literature, altering the closure properties in the definition of a variety and replacing monoids by other algebraic structures. Pin [18] introduced *positive varieties of languages* where in (a) the closure under complement is omitted, and he proved a bijective correspondence to pseudovarieties of *ordered* monoids. Later Polák [20] further weakened (a) by also omitting closure under intersection, and the resulting *disjunctive varieties of languages* correspond to pseudovarieties of idempotent semirings. Reutenauer [22] studied a concept of variety where (a) is replaced by closure under symmetric difference, and obtained a correspondence to pseudovarieties of algebras over the binary field \mathbb{Z}_2 . (In fact Reutenauer considered algebras over arbitrary fields \mathbf{K} and varieties of formal power series

in lieu of languages). Finally, a new Eilenberg-type theorem we derive below deals with varieties of languages defined by closure under intersection and symmetric difference in lieu of (a), and relates them to pseudovarieties of monoids with 0.

In this paper a categorical result is presented that covers Eilenberg’s theorem and all its variants uniformly, and exhibits new applications. Our overall approach to algebraic automata theory may be subsumed by the “equation”

automata theory = duality + monoidal structure.

The idea is to take a category \mathcal{C} (where automata and languages live) and a closed monoidal category \mathcal{D} (where monoids live) with the property that \mathcal{C} and \mathcal{D} are *predual*. Specifically, in our setting \mathcal{C} and \mathcal{D} will be locally finite varieties of algebras or ordered algebras (i.e., all finitely generated algebras are finite), and preduality means that the full subcategories of finite algebras are dually equivalent. Moreover, the monoidal structure of \mathcal{D} is given by the usual tensor product of algebras.

All the Eilenberg-type correspondences mentioned above fit into this categorical framework. For example, the categories \mathcal{C} of boolean algebras and \mathcal{D} of sets are predual via Stone duality, and \mathcal{D} -monoids are ordinary monoids: this is the setting of Eilenberg’s original result. The category \mathcal{C} of distributive lattices with 0 and 1 is predual to the category \mathcal{D} of posets via Birkhoff duality [7], and \mathcal{D} -monoids are ordered monoids, which leads to Pin’s result for pseudovarieties of ordered monoids [18]. The category \mathcal{C} of join-semilattices with 0 is self-predual (i.e., one takes $\mathcal{D} = \mathcal{C}$), and \mathcal{D} -monoids are precisely idempotent semirings. This is the framework for Polák [20]. For Reutenauer’s result [22] one takes the category \mathcal{C} of vector spaces over a finite field \mathbf{K} which is also self-predual (i.e., $\mathcal{D} = \mathcal{C}$), and observes that \mathcal{D} -monoids are precisely \mathbf{K} -algebras. Lastly, our new example concerning pseudovarieties of monoids with 0 takes as \mathcal{C} non-unital boolean rings and as \mathcal{D} pointed sets.

Apart from preduality, the heart of the matter is a coalgebraic characterization of the closure properties defining varieties of languages. We model deterministic Σ -automata in a locally finite variety \mathcal{C} as coalgebras $Q \rightarrow T_\Sigma Q$ for the endofunctor

$$T_\Sigma : \mathcal{C} \rightarrow \mathcal{C}, \quad T_\Sigma Q = O_\mathcal{C} \times Q^\Sigma,$$

where $O_\mathcal{C}$ is a fixed two-element algebra in \mathcal{C} representing final and non-final states. In particular, the set of all regular languages over Σ carries the structure of a T_Σ -coalgebra whose

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transitions are given by left derivatives $L \rightarrow a^{-1}L$ for $a \in \Sigma$, and whose final states are the languages containing the empty word. This coalgebra admits an abstract characterization as the *rational fixpoint* ϱT_Σ of T_Σ , i.e., the terminal locally finite coalgebra. To get a grasp on *all* regular languages, independent of a particular alphabet, we introduce the functor

$$\varrho T : \mathbf{Set}_f^{op} \rightarrow \mathcal{C}$$

that maps each finite alphabet Σ to ϱT_Σ , and each letter substitution $h : \Delta \rightarrow \Sigma$ to the morphism $\varrho T_\Sigma \rightarrow \varrho T_\Delta$ taking preimages under the free monoid morphism $h^* : \Delta^* \rightarrow \Sigma^*$.

Consider now a variety V of languages in Eilenberg's sense (this is the case $\mathcal{C} = \text{boolean algebras}$), and denote by $V\Sigma$ the languages over the alphabet Σ contained in V . Then the closure condition (a) and the restriction of (c) to *length-preserving* monoid morphisms (those of the form h^*) state precisely that the map $\Sigma \mapsto V\Sigma$ defines a subfunctor

$$V \mapsto \varrho T.$$

Closure under left derivatives means that $V\Sigma$ is a subcoalgebra of ϱT_Σ . Finally, as we will demonstrate below, closure under right derivatives and preimages of arbitrary monoid morphisms amounts categorically to the existence of certain coalgebra homomorphisms. This expresses the closure properties of a variety of languages fully coalgebraically.

In our general setting of two preducal categories \mathcal{C} and \mathcal{D} a *variety of languages in \mathcal{C}* is thus a subfunctor $V \mapsto \varrho T$, subject to additional closure properties which are characterized by means of coalgebraic concepts. Dualizing these properties leads to the notion of a *pseudovariety of \mathcal{D} -monoids*: a class of finite \mathcal{D} -monoids closed under finite products, submonoids and homomorphic images. Our main result is the

Generalized Eilenberg Theorem. Varieties of languages in \mathcal{C} correspond bijectively to pseudovarieties of \mathcal{D} -monoids.

All Eilenberg-type theorems mentioned above emerge as special cases by the corresponding choices of \mathcal{C} and \mathcal{D} .

On our way to proving the Generalized Eilenberg Theorem we will also establish a bijective correspondence between *object-finite* varieties of languages in \mathcal{C} (those varieties with $V\Sigma$ finite for all Σ) and locally finite varieties (rather than pseudovarieties!) of \mathcal{D} -monoids. In the case of (ordered) monoids this has been shown by Klíma and Polák [16], and to the best of our knowledge it is a new result in all other cases.

Although our emphasis lies on varieties of *languages*, all our results hold more generally for Moore automata in lieu of acceptors, and hence for varieties of regular behaviors in lieu of regular languages – one simply replaces the two-element algebra $O_{\mathcal{C}}$ in the above definition of T_Σ by an arbitrary finite algebra in \mathcal{C} . We briefly explain this at the end of the paper.

Related Work. Our paper lays a common ground for Eilenberg's original variety theorem [12] and its variants due to Pin [18], Polák [20] and Reutenauer [22]. In our previous paper [1] we proved a *local* Eilenberg theorem where one considers classes of regular languages over a fixed alphabet Σ versus classes of

finite Σ -generated \mathcal{D} -monoids in our general setting of preducal categories \mathcal{C} and \mathcal{D} . The main technical achievement of [1] was the insight that finite subcoalgebras of ϱT_Σ closed under right derivatives dualize to finite Σ -generated \mathcal{D} -monoids. Our work was inspired by Gehrke, Grigorieff and Pin [13] who proved a bijective correspondence between local varieties of languages and classes of finite (ordered) Σ -generated monoids presented by profinite identities. This result provides a local view of Reiterman's theorem [21] characterizing pseudovarieties of monoids in terms of profinite identities.

Somewhat surprisingly, it has only been in recent years that the fundamental role of duality in algebraic automata theory was fully recognized. Most of the work along these lines concerns the connection between regular languages and profinite algebras. Rhodes and Steinberg [23] view the regular languages over Σ as a comonoid (rather than just a coalgebra) in the category of boolean algebras, and this comonoid is shown to dualize to the free profinite semigroup on Σ . Similar results for free profinite monoids can be found in the aforementioned work of Gehrke et al. which built on previous work of Almeida [3] and Pippenger [19].

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II. PREDUALITY, MONOIDS AND LANGUAGES

In this section we set the scene of preducal categories \mathcal{C} and \mathcal{D} , and introduce varieties of languages in \mathcal{C} and pseudovarieties of \mathcal{D} -monoids. The reader is assumed to be familiar with basic category theory and universal algebra.

A. Predual categories

Our categories of interest are varieties of algebras and varieties of ordered algebras. Given a finitary signature Γ , a *variety of Γ -algebras* is a full subcategory \mathcal{A} of $\mathbf{Alg} \Gamma$, the category of Γ -algebras and homomorphisms, closed under quotients (= homomorphic images), subalgebras and products. Equivalently, by Birkhoff's HSP theorem [6] a variety of algebras is a class of Γ -algebras specified by equations $t_1 = t_2$ between Γ -terms. The forgetful functor $\mathcal{A} \rightarrow \mathbf{Set}$ of a variety has a left adjoint assigning to every set X the free algebra over X .

Analogously, let $\mathbf{Alg}_{\leq} \Gamma$ be the category of all *ordered Γ -algebras*. These are Γ -algebras with a partial order on the underlying set such that all Γ -operations are order-preserving. Morphisms of $\mathbf{Alg}_{\leq} \Gamma$ are order-preserving homomorphisms. This category has a factorization system of surjective homomorphisms and injective order-embeddings. Thus the concept of a *subalgebra* of an ordered algebra A means that the order is inherited from A , whereas a *quotient* of A is represented by any surjective order-preserving homomorphism with domain A . A *variety of ordered Γ -algebras* is a full subcategory \mathcal{A} of $\mathbf{Alg}_{\leq} \Gamma$ closed under quotients, subalgebras and products. Equivalently, by the ordered version of Birkhoff's theorem due to Bloom [8], \mathcal{A} is specified by inequalities $t_1 \leq t_2$ between Γ -terms. (Recall that if t_1 and t_2 lie in $T_\Gamma X$, the

discretely ordered algebra of Γ -terms over variables X , then an ordered Γ -algebra A satisfies $t_1 \leq t_2$ provided that every homomorphism $h : T_\Gamma X \rightarrow A$ fulfils $h(t_1) \leq h(t_2)$. Again, the forgetful functor $\mathcal{A} \rightarrow \mathbf{Set}$ has a left adjoint constructing free algebras.

Definition II.1. Given a variety \mathcal{A} of (ordered) algebras we denote by \mathcal{A}_f the full subcategory of all finite algebras. \mathcal{A} is called *locally finite* if all free algebras on finitely many generators are finite.

In our applications we will encounter the locally finite varieties listed below. We view \mathbf{Pos} as a variety of ordered algebras (over the empty signature) and all other categories as varieties of non-ordered algebras.

| | |
|------------------------|--|
| Set | sets and functions |
| Set_* | pointed sets and point-preserving functions |
| Pos | partially ordered sets and order-preserving maps |
| BA | boolean algebras and boolean homomorphisms |
| BR | non-unital boolean rings (i.e., non-unital rings $(R, +, \cdot, 0)$ satisfying the equation $x \cdot x = x$) and ring homomorphisms |
| DL₀₁ | distributive lattices with 0 and 1 and lattice homomorphisms preserving 0 and 1 |
| JSL₀ | join-semilattices with 0 and semilattice homomorphisms preserving 0 |
| K-Vec | vector spaces over a finite field \mathbf{K} and linear maps |

Here is the central concept for our categorical approach to algebraic automata theory:

Definition II.2. Two locally finite varieties of (ordered) algebras \mathcal{C} and \mathcal{D} are called *predual* if their full subcategories \mathcal{C}_f and \mathcal{D}_f of finite algebras are dually equivalent.

In what follows \mathcal{C} will be a locally finite variety of algebras (where varieties of languages are formed), and \mathcal{D} will be a locally finite variety of algebras or ordered algebras (where varieties of monoids are formed). Let us establish some notation for this setting. The preduality of \mathcal{C} and \mathcal{D} is witnessed by an equivalence functor $\mathcal{C}_f^{op} \xrightarrow{\cong} \mathcal{D}_f$ whose action on objects and morphisms we denote by

$$Q \mapsto \widehat{Q} \quad \text{and} \quad h \mapsto \widehat{h}.$$

The varieties \mathcal{C} and \mathcal{D} come equipped with forgetful functors

$$|-| : \mathcal{C} \rightarrow \mathbf{Set} \quad \text{and} \quad |-| : \mathcal{D} \rightarrow \mathbf{Set}.$$

Finally, we write $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$ for the free algebras on one generator in \mathcal{C} and \mathcal{D} , respectively, and $O_{\mathcal{D}}$ and $O_{\mathcal{C}}$ for their dual algebras:

$$O_{\mathcal{D}} = \widehat{\mathbf{1}_{\mathcal{C}}} \quad \text{and} \quad \widehat{O_{\mathcal{C}}} \cong \mathbf{1}_{\mathcal{D}}.$$

Remark II.3. The last item is understood as a fixed choice of an algebra $O_{\mathcal{C}}$ in \mathcal{C}_f along with an isomorphism $i : \widehat{O_{\mathcal{C}}} \xrightarrow{\cong} \mathbf{1}_{\mathcal{D}}$. It follows that $|O_{\mathcal{C}}|$ and $|O_{\mathcal{D}}|$ are isomorphic:

$$\begin{aligned} |O_{\mathcal{C}}| &\cong \mathbf{Set}(1, |O_{\mathcal{C}}|) && \text{(canonically)} \\ &\cong \mathcal{C}(\mathbf{1}_{\mathcal{C}}, O_{\mathcal{C}}) && \text{(def. 1}_{\mathcal{C}}) \\ &\cong \mathcal{D}(\widehat{O_{\mathcal{C}}}, O_{\mathcal{D}}) && \text{(by duality)} \\ &\cong \mathcal{D}(\mathbf{1}_{\mathcal{D}}, O_{\mathcal{D}}) && \text{(composition with } i^{-1}) \\ &\cong \mathbf{Set}(1, |O_{\mathcal{D}}|) && \text{(def. 1}_{\mathcal{D}}) \\ &\cong |O_{\mathcal{D}}| && \text{(canonically)} \end{aligned}$$

To simplify the notation we identify the underlying sets of $O_{\mathcal{C}}$ and $O_{\mathcal{D}}$ via this isomorphism and thus assume $|O_{\mathcal{C}}| = |O_{\mathcal{D}}|$.

Assumptions II.4. For the rest of this paper we fix two predual locally finite varieties \mathcal{C} and \mathcal{D} with the following properties:

- (i) \mathcal{C} is a locally finite variety of algebras.
- (ii) \mathcal{D} is a locally finite variety of algebras or ordered algebras.
- (iii) Epimorphisms in \mathcal{D} are surjective.
- (iv) \mathcal{D} is *entropic*: for any two algebras A and B in \mathcal{D} , the set $[A, B]$ of homomorphisms from A to B is an algebra in \mathcal{D} with the pointwise algebraic structure, i.e., a subalgebra of the power $B^{|A|} = \prod_{a \in |A|} B$.
- (v) $|O_{\mathcal{C}}| = |O_{\mathcal{D}}| = \{0, 1\}$.

Condition (iv) means precisely that all operations of the variety \mathcal{D} commute, see e.g. [10, Theorem 3.10.3]: given an m -ary operation σ and an n -ary operation τ in the signature of \mathcal{D} and variables x_{ij} ($i = 1, \dots, m, j = 1, \dots, n$), the equation

$$\begin{aligned} &\sigma(\tau(x_{11}, \dots, x_{1n}), \dots, \tau(x_{m1}, \dots, x_{mn})) \\ &= \tau(\sigma(x_{11}, \dots, x_{m1}), \dots, \sigma(x_{1n}, \dots, x_{mn})) \end{aligned}$$

holds in \mathcal{D} . Condition (v) can be lifted to get a theory of regular behaviors in lieu of regular languages, cf. Section V.

Example II.5. The following pairs of categories \mathcal{C} and \mathcal{D} satisfy our assumptions:

| | | | | | |
|---------------|------------|------------------------|------------------------|--------------------------|------------------------|
| \mathcal{C} | BA | DL₀₁ | JSL₀ | Z₂-Vec | BR |
| \mathcal{D} | Set | Pos | JSL₀ | Z₂-Vec | Set_* |

(a) $\mathcal{C} = \mathbf{BA}$ is predual to $\mathcal{D} = \mathbf{Set}$ via Stone duality [24]: given a finite boolean algebra Q , the set \widehat{Q} consists of all atoms of Q , and given a homomorphism $h : Q \rightarrow R$ in \mathbf{BA}_f , the dual function $\widehat{h} : \widehat{R} \rightarrow \widehat{Q}$ in \mathbf{Set}_f is defined by

$$\widehat{h}(r) = \bigwedge \{q \in \widehat{Q} : h(q) \geq r\}. \quad (1)$$

We choose

$$\mathbf{1}_{\mathcal{C}} = 1 \quad \begin{array}{c} \top \\ \swarrow \quad \searrow \\ \quad \quad 0 \\ \downarrow \quad \uparrow \\ \perp \end{array} \quad \mathbf{1}_{\mathcal{D}} = \{1\} \quad O_{\mathcal{C}} = \begin{array}{c} 1 \\ | \\ 0 \end{array} \quad O_{\mathcal{D}} = \{0, 1\}$$

Here 1 is the generator of the one-generated free algebra $\mathbf{1}_{\mathcal{C}}$ and $\widehat{O_{\mathcal{C}}} = \{1\} = \mathbf{1}_{\mathcal{D}}$. Hence the isomorphisms $\widehat{O_{\mathcal{C}}} \cong \mathbf{1}_{\mathcal{D}}$ and $|O_{\mathcal{C}}| \cong |O_{\mathcal{D}}|$ of Remark II.3 are identity maps.

(b) $\mathcal{C} = \mathbf{DL}_{01}$ is pre-dual to $\mathcal{D} = \mathbf{Pos}$ via Birkhoff duality [7]: \widehat{Q} is the poset of all join-irreducible elements of a finite distributive lattice Q (ordered as in Q), and every homomorphism $h : Q \rightarrow R$ in $(\mathbf{DL}_{01})_f$ yields a monotone function $\widehat{h} : \widehat{R} \rightarrow \widehat{Q}$ by the formula (1) above. We choose

$$\mathbf{1}_{\mathcal{C}} = \begin{array}{c} 0 \\ | \\ 1 \\ | \\ \perp \end{array} \quad \mathbf{1}_{\mathcal{D}} = \{1\} \quad O_{\mathcal{C}} = \begin{array}{c} 1 \\ | \\ 0 \end{array} \quad O_{\mathcal{D}} = \begin{array}{c} 0 \\ | \\ 1 \end{array}$$

Note that 0 is the *top* element of $\mathbf{1}_{\mathcal{C}}$ and $O_{\mathcal{D}}$. Again the isomorphisms $\widehat{O_{\mathcal{C}}} \cong \mathbf{1}_{\mathcal{D}}$ and $|O_{\mathcal{C}}| \cong |O_{\mathcal{D}}|$ are identity maps.

(c) The category $\mathcal{C} = \mathbf{JSL}_0$ is self-pre-dual, see [15], so we can take $\mathcal{D} = \mathbf{JSL}_0$. The dual equivalence associates to each finite semilattice $Q = (X, \vee, 0)$ its opposite semilattice $\widehat{Q} = (X, \wedge, 1)$, and to each homomorphism $h : Q \rightarrow R$ in $(\mathbf{JSL}_0)_f$ the homomorphism $\widehat{h} : \widehat{R} \rightarrow \widehat{Q}$ with

$$\widehat{h}(r) = \bigvee \{q \in Q : h(q) \leq r\}$$

where the join is formed in Q . We choose

$$\mathbf{1}_{\mathcal{C}} = \begin{array}{c} 0 \\ | \\ 1 \end{array} \quad \mathbf{1}_{\mathcal{D}} = \begin{array}{c} 0 \\ | \\ 1 \end{array} \quad O_{\mathcal{C}} = \begin{array}{c} 1 \\ | \\ 0 \end{array} \quad O_{\mathcal{D}} = \begin{array}{c} 1 \\ | \\ 0 \end{array}$$

The isomorphisms $\widehat{O_{\mathcal{C}}} \cong \mathbf{1}_{\mathcal{D}}$ and $|O_{\mathcal{C}}| \cong |O_{\mathcal{D}}|$ are identity maps. Epimorphisms in \mathbf{JSL}_0 are surjective, as proved in [14].

(d) The category $\mathcal{C} = \mathbf{Z}_2\text{-Vec}$ of vector spaces over the binary field $\mathbf{Z}_2 = \{0, 1\}$ is also self-pre-dual, so $\mathcal{D} = \mathbf{Z}_2\text{-Vec}$. The dual equivalence assigns to each finite (i.e. finite-dimensional) \mathbf{Z}_2 -vector space Q its dual space $\widehat{Q} = [Q, \mathbf{Z}_2]$. For every linear map $h : Q \rightarrow R$ in $(\mathbf{Z}_2\text{-Vec})_f$ the dual map $\widehat{h} : \widehat{R} \rightarrow \widehat{Q}$ takes $u : R \rightarrow \mathbf{Z}_2$ to $u \cdot h$. We choose

$$\mathbf{1}_{\mathcal{C}} = O_{\mathcal{C}} = \mathbf{Z}_2 \quad \text{and} \quad \mathbf{1}_{\mathcal{D}} = O_{\mathcal{D}} = [\mathbf{Z}_2, \mathbf{Z}_2].$$

The isomorphism $\widehat{O_{\mathcal{C}}} \cong \mathbf{1}_{\mathcal{D}}$ is identity, and the isomorphism $|O_{\mathcal{C}}| \cong |O_{\mathcal{D}}|$ identifies the element $1 \in |O_{\mathcal{C}}| = |\mathbf{Z}_2|$ with $\text{id} \in |O_{\mathcal{D}}| = [\mathbf{Z}_2, \mathbf{Z}_2]$. Epimorphisms in $\mathbf{Z}_2\text{-Vec}$ split and hence are surjective.

(e) An interesting variation on (a) is the pre-duality of $\mathcal{C} = \mathbf{BR}$ and $\mathcal{D} = \mathbf{Set}_*$. Recall that we consider *non-unital* boolean rings (this is Birkhoff's original definition of a boolean ring, see [5]) and homomorphisms preserving $+$, \cdot and 0. Every *finite* non-unital boolean ring $(Q, +, \cdot, 0)$ can be viewed as a boolean algebra with $x \wedge y = x \cdot y$ and $x \vee y = x + y + x \cdot y$. The pre-duality of \mathbf{BR} and \mathbf{Set}_* takes Q to the pointed set $\widehat{Q} = \{\star, q_1, \dots, q_n\}$ where q_1, \dots, q_n are the atoms of Q . A homomorphism $h : Q \rightarrow R$ in \mathbf{BR}_f is mapped to the function $\widehat{h} : \widehat{R} \rightarrow \widehat{Q}$ with $\widehat{h}(r)$ defined by (1) if $r \neq \star$ and some $q \in Q$ with $h(q) \geq r$ exists, and otherwise $\widehat{h}(r) = \star$. We choose

$$\mathbf{1}_{\mathcal{C}} = O_{\mathcal{C}} = \{0, 1\} \quad \text{and} \quad \mathbf{1}_{\mathcal{D}} = O_{\mathcal{D}} = \{\star, 1\}.$$

The isomorphism $\widehat{O_{\mathcal{C}}} \cong \mathbf{1}_{\mathcal{D}}$ is identity, and the isomorphism $|O_{\mathcal{C}}| \cong |O_{\mathcal{D}}|$ identifies $0 \in |O_{\mathcal{C}}|$ with $\star \in |O_{\mathcal{D}}|$.

The pre-duality of \mathcal{C} and \mathcal{D} allows us to model deterministic automata both as coalgebras and as algebras for suitable endofunctors on \mathcal{C} and \mathcal{D} . Fix a finite input alphabet Σ and consider first the endofunctor on \mathcal{C}

$$T_{\Sigma}Q = O_{\mathcal{C}} \times Q^{\Sigma} = O_{\mathcal{C}} \times \prod_{\Sigma} Q.$$

A T_{Σ} -coalgebra (Q, γ) consists of an object Q in \mathcal{C} together with a morphism $\gamma : Q \rightarrow T_{\Sigma}Q$. By the universal property of the product, to give a T_{Σ} -coalgebra means to give morphisms $\gamma_a : Q \rightarrow Q$ for each $a \in \Sigma$, representing transitions, and a morphism $\gamma_{\text{out}} : Q \rightarrow O_{\mathcal{C}}$ representing final states. Hence T_{Σ} -coalgebras are deterministic Σ -automata in \mathcal{C} without an initial state, often denoted as triples

$$Q = (Q, \gamma_a, \gamma_{\text{out}}).$$

A *homomorphism* $h : (Q, \gamma') \rightarrow (Q', \gamma')$ of T_{Σ} -coalgebras is a morphism $h : Q \rightarrow Q'$ in \mathcal{C} with $\gamma' \cdot h = T_{\Sigma}h \cdot \gamma$, which means that the following diagram commutes for all $a \in \Sigma$:

$$\begin{array}{ccccc} Q & \xrightarrow{\gamma_a} & Q & & \\ h \downarrow & & \downarrow h & \searrow \gamma_{\text{out}} & \\ Q' & \xrightarrow{\gamma'_a} & Q' & \xrightarrow{\gamma'_{\text{out}}} & O_{\mathcal{C}} \end{array}$$

We write

$$\mathbf{Coalg} T_{\Sigma} \quad \text{and} \quad \mathbf{Coalg}_f T_{\Sigma}$$

for the categories of (finite) T_{Σ} -coalgebras and homomorphisms. The functor T_{Σ} has an associated endofunctor on \mathcal{D}

$$L_{\Sigma}A = \mathbf{1}_{\mathcal{D}} + \prod_{\Sigma} A$$

which is *pre-dual* to T_{Σ} in the sense that the restrictions

$$T_{\Sigma} : \mathcal{C}_f \rightarrow \mathcal{C}_f \quad \text{and} \quad L_{\Sigma} : \mathcal{D}_f \rightarrow \mathcal{D}_f,$$

to finite algebras are dual. That is, the diagram below commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{C}_f^{op} & \xrightarrow{T_{\Sigma}^{op}} & \mathcal{C}_f^{op} \\ \downarrow \widehat{(-)} & & \downarrow \widehat{(-)} \\ \mathcal{D}_f & \xrightarrow{L_{\Sigma}} & \mathcal{D}_f \end{array}$$

Dually to the concept of a T_{Σ} -coalgebra, an L_{Σ} -algebra (A, α) is an object A in \mathcal{D} together with a morphism $\alpha : L_{\Sigma}A \rightarrow A$. Equivalently, an L_{Σ} -algebra is given by morphisms $\alpha_a : A \rightarrow A$ for $a \in \Sigma$, representing transitions, and a morphism $\alpha_{\text{in}} : \mathbf{1}_{\mathcal{D}} \rightarrow A$ selecting an initial state. L_{Σ} -algebras are denoted as

$$A = (A, \alpha_a, \alpha_{\text{in}})$$

and can be viewed as deterministic Σ -automata in \mathcal{D} without final states. A *homomorphism* $h : (A, \alpha) \rightarrow (A', \alpha')$ of L_{Σ} -algebras is a \mathcal{D} -morphism $h : A \rightarrow A'$ with $h \cdot \alpha = \alpha' \cdot L_{\Sigma}h$. We write

$$\mathbf{Alg} L_{\Sigma} \quad \text{and} \quad \mathbf{Alg}_f L_{\Sigma}$$

for the categories of (finite) L_Σ -algebras and homomorphisms. From the preuality of T_Σ and L_Σ we immediately conclude:

Lemma II.6. *Coalg $_f T_\Sigma$ and Alg $_f L_\Sigma$ are dually equivalent categories. The dual equivalence is given on objects and morphisms by*

$$(Q, \gamma) \mapsto (\widehat{Q}, \widehat{\gamma}) \quad \text{and} \quad h \mapsto \widehat{h}.$$

In triple notation this dual equivalence maps a finite T_Σ -coalgebra $(Q, \gamma_a, \gamma_{\text{out}})$ to its *dual* L_Σ -algebra $(\widehat{Q}, \widehat{\gamma}_a, \widehat{\gamma}_{\text{out}})$. Hence finite T_Σ -coalgebras and finite L_Σ -algebras are essentially the same structures, and both can be understood as finite automata with additional (ordered) algebraic structure.

Example II.7. (a) Let $\mathcal{C} = \mathbf{BA}$ and $\mathcal{D} = \mathbf{Set}$. Then L_Σ -algebras are the usual concept of a deterministic automaton without final states. A T_Σ -coalgebra is a deterministic Σ -automaton with a boolean algebra structure on the state set Q such that (i) all transitions $\gamma_a : Q \rightarrow Q$ are boolean homomorphisms and (ii) the final states form an ultrafilter, determined by the preimage of 1 under the morphism $\gamma_{\text{out}} : Q \rightarrow O_{\mathcal{C}} = \{0, 1\}$. For *finite* automata (ii) means that precisely one atom $i \in Q$ is final and all final states form the upper set $\uparrow i = \{q \in Q : q \geq i\}$. The dual equivalence of the previous lemma takes a finite boolean automaton $Q = (Q, \gamma_a, \gamma_{\text{out}})$ to the automaton $\widehat{Q} = (\widehat{Q}, \widehat{\gamma}_a, \widehat{\gamma}_{\text{out}})$ in \mathbf{Set} whose states are the atoms of Q . The unique atomic final state $i \in Q$ is the initial state of \widehat{Q} , and there is a transition $x \xrightarrow{a} x'$ in \widehat{Q} iff

$$x' = \bigwedge \{y : y \xrightarrow{a} y' \text{ in } Q \text{ for some } y' \geq x\}.$$

(b) The case $\mathcal{C} = \mathbf{DL}_{01}$ and $\mathcal{D} = \mathbf{Pos}$ is analogous. Here L_Σ -algebras are *ordered* deterministic automata without final states. T_Σ -coalgebras carry a distributive lattice structure on Q , and again for finite Q the final states form an upper set $\uparrow i$.

Remark II.8. We frequently need to factorize (co-)algebra homomorphisms into a surjective and an injective part. For the variety \mathcal{C} we choose the factorization system

$$(\text{strong epi, mono}) = (\text{surjective, injective}).$$

Recall that an epimorphism e is called strong if it has the diagonal fill-in property w. r. t. all monomorphisms m : given morphisms u, r with $r \cdot e = m \cdot u$ there exists d with $u = d \cdot e$. Since the functor $T_\Sigma Q = O_{\mathcal{C}} \times Q^\Sigma$ preserves monomorphisms, this factorization system of \mathcal{C} lifts to $\mathbf{Coalg} T_\Sigma$: every coalgebra homomorphism $h : (Q, \gamma) \rightarrow (Q', \gamma')$ factorizes as a *quotient coalgebra* of (Q, γ) followed by a *subcoalgebra* of (Q', γ') , by which we mean that the underlying morphism in \mathcal{C} is a strong epimorphism (or a monomorphism, respectively).

Dually, in \mathcal{D} we consider the factorization system

$$(\text{epi, strong mono}).$$

If \mathcal{D} is a variety of algebras, this is just the (surjective, injective)-factorization system as epimorphisms in \mathcal{D} are surjective by Assumption II.4(iii). In case \mathcal{D} is a variety of ordered algebras, we get the (surjective, injective order-embedding)-factorization system, which is clearly the “right”

one for ordered algebras. Since $L_\Sigma = \mathbf{1}_{\mathcal{D}} + \coprod_{\Sigma} A$ preserves epimorphisms, the factorization system of \mathcal{D} lifts to $\mathbf{Alg} L_\Sigma$: every L_Σ -algebra homomorphism $h : (A, \alpha) \rightarrow (A', \alpha')$ factorizes as a *quotient algebra* of (A, α) followed by a *subalgebra* of (A', α') , i.e., the underlying morphism in \mathcal{D} is an epimorphism (or a strong monomorphism, respectively).

B. \mathcal{D} -Monoids

Our Assumption II.4(iv) that \mathcal{D} is entropic admits a more categorical interpretation. Given objects A, B and C in \mathcal{D} , a *bimorphism* is a function $f : |A| \times |B| \rightarrow |C|$ such that $f(a, -) : B \rightarrow C$ and $f(-, b) : A \rightarrow C$ are morphisms of \mathcal{D} for all $a \in A$ and $b \in B$. A *tensor product* of A and B is a universal bimorphism $t : |A| \times |B| \rightarrow |A \otimes B|$, i.e., for every bimorphism $f : |A| \times |B| \rightarrow |C|$ there is a unique morphism f' in \mathcal{D} making the diagram below commute.

$$\begin{array}{ccc} |A| \times |B| & \xrightarrow{t} & |A \otimes B| \\ & \searrow f & \downarrow |f'| \\ & & |C| \end{array}$$

As shown in [4], tensor products exist in any variety of (ordered) algebras, and “entropic” implies that $\mathcal{D} = (\mathcal{D}, \otimes, \mathbf{1}_{\mathcal{D}})$ is a symmetric monoidal closed category. In particular, as in any monoidal category, we have a notion of *monoid* in \mathcal{D} . In our setting it corresponds to the following algebraic concept:

Definition II.9. A \mathcal{D} -monoid (D, \circ, i) consists of an object D of \mathcal{D} and a monoid structure $(|D|, \circ, i)$ on its underlying set $|D|$ whose multiplication is a bimorphism, i.e., for every $x \in |D|$ both $x \circ -$ and $- \circ x$ are endomorphisms of D in \mathcal{D} . A *morphism* $h : (D, \circ, i) \rightarrow (D', \circ', i')$ of \mathcal{D} -monoids is a morphism $h : D \rightarrow D'$ in \mathcal{D} preserving the monoid structure. The \mathcal{D} -monoids and their morphisms form a category

\mathcal{D} -Mon.

Observe that \mathcal{D} -Mon is a variety of (ordered) algebras: add \circ and i to the signature of \mathcal{D} , and add the monoid axioms and equations expressing that \circ is a bimorphism to the (in)equalities presenting \mathcal{D} . The factorization system of \mathcal{D} , see Remark II.8, lifts to \mathcal{D} -Mon. Hence a *submonoid* of a \mathcal{D} -monoid D is a \mathcal{D} -monoid morphism into D carried by a strong monomorphism in \mathcal{D} , and a *quotient monoid* of D is a \mathcal{D} -monoid morphism with domain D carried by an epimorphism in \mathcal{D} .

Examples II.10. For our categories \mathcal{D} of Example II.5 the \mathcal{D} -monoids are characterized as follows:

- (a) $\mathcal{D} = \mathbf{Set}$: (ordinary) monoids.
- (b) $\mathcal{D} = \mathbf{Pos}$: ordered monoids.
- (c) $\mathcal{D} = \mathbf{JSL}_0$: idempotent semirings, i.e., semirings $(S, +, \cdot, 0)$ satisfying the equation $x+x = x$. Indeed, this means precisely that $(S, +, 0)$ is a semilattice, and the distributive laws of a semiring express that the multiplication preserves $+$ and 0 in each variable, so that it is a bimorphism.
- (d) $\mathcal{D} = \mathbf{Z}_2\text{-Vec}$: \mathbf{Z}_2 -algebras in the classical sense of algebras over a field, i.e., a \mathbf{Z}_2 -algebra is a vector space over

\mathbb{Z}_2 together with a monoid structure whose multiplication is distributive (= linear in each variable).

(e) $\mathcal{D} = \mathbf{Set}_*$: monoids with 0, i.e., monoids containing an element 0 such that $x \circ 0 = 0 \circ x = 0$ for all x . Morphisms of $\mathcal{D}\text{-Mon}$ are monoid morphisms preserving 0.

Definition II.11. A *pseudovariety of \mathcal{D} -monoids* is a class of finite \mathcal{D} -monoids closed under submonoids, quotients and finite products.

On our way to proving the generalized Eilenberg theorem we will encounter monoids with a specified set of generators. To this end we need to describe the free \mathcal{D} -monoids. We write

$$\Psi : \mathbf{Set} \rightarrow \mathcal{D}$$

for the free algebra functor, that is, the left adjoint of the forgetful functor $|-| : \mathcal{D} \rightarrow \mathbf{Set}$. For notational simplicity we assume that X is a subset of $|\Psi X|$, and the universal map is the inclusion $X \hookrightarrow |\Psi X|$. (For nontrivial varieties \mathcal{D} of (ordered) algebras such a choice of ΨX is always possible.)

Proposition II.12 (see [1], Prop. 4.22). *The free \mathcal{D} -monoid on a finite set Σ is the \mathcal{D} -monoid $(\Psi\Sigma^*, \bullet, \varepsilon)$ with multiplication \bullet extending the concatenation of words over Σ , and universal map $\Sigma \hookrightarrow \Sigma^* \hookrightarrow |\Psi\Sigma^*|$. That is, every map $f : \Sigma \rightarrow |D|$ into a \mathcal{D} -monoid D extends uniquely to a \mathcal{D} -monoid morphism \bar{f} :*

$$\begin{array}{ccc} \Sigma & \longrightarrow & |\Psi\Sigma^*| \\ & \searrow f & \downarrow |\bar{f}| \\ & & |D| \end{array}$$

Examples II.13. In our categories \mathcal{D} of Example II.5 the free \mathcal{D} -monoids $\Psi\Sigma^*$ on Σ are characterized as follows:

- (a) $\mathcal{D} = \mathbf{Set}$: the usual free monoid Σ^* .
- (b) $\mathcal{D} = \mathbf{Pos}$: the free ordered monoid Σ^* (discretely ordered).
- (c) $\mathcal{D} = \mathbf{JSL}_0$: the free idempotent semiring $\mathcal{P}_f\Sigma^*$, carried by the set of finite languages over Σ . The semilattice structure is union and \emptyset , the monoid multiplication is the concatenation $L_1 \bullet L_2 = L_1 L_2$ of languages, and the monoid unit is $\{\varepsilon\}$.
- (d) $\mathcal{D} = \mathbb{Z}_2\text{-Vec}$: the free \mathbb{Z}_2 -algebra $\mathcal{P}_f\Sigma^*$. Its vector addition is the symmetric difference $L_1 \oplus L_2 = (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$, and the zero vector is \emptyset . The monoid unit is again $\{\varepsilon\}$, and the monoid multiplication is \mathbb{Z}_2 -weighted concatenation of languages: $L_1 \bullet L_2$ consists of all words w having an odd number of decompositions $w = w_1 w_2$ with $w_i \in L_i$.
- (e) $\mathcal{D} = \mathbf{Set}_*$: the free monoid with 0. This is the monoid $\Sigma^* + \{0\}$ arising from Σ^* by adding a zero element.

Definition II.14. (a) A \mathcal{D} -monoid is called Σ -generated if a set of generators indexed by Σ is given in it. Equivalently: if it is a quotient monoid of the free \mathcal{D} -monoid $\Psi\Sigma^*$. Morphisms of Σ -generated \mathcal{D} -monoids are required to preserve the given generators. That is, given two Σ -generated \mathcal{D} -monoids $e_k : \Psi\Sigma^* \rightarrow (D_k, \circ_k, i_k)$, $k = 1, 2$, a morphism of Σ -generated \mathcal{D} -monoids is a \mathcal{D} -monoid morphism $f : (D_1, \circ_1, i_1) \rightarrow (D_2, \circ_2, i_2)$ with $e_2 = f \cdot e_1$.

(b) The *subdirect product* of two Σ -generated \mathcal{D} -monoids $e_k : \Psi\Sigma^* \rightarrow (D_k, \circ_k, i_k)$ is the \mathcal{D} -submonoid of their product which is the image of $\langle e_1, e_2 \rangle : \Psi\Sigma^* \rightarrow D_1 \times D_2$.

Definition II.15. By a *pseudovariety of Σ -generated \mathcal{D} -monoids* is meant a class of finite Σ -generated \mathcal{D} -monoids closed under subdirect products and quotients.

In other words, if \mathcal{L} denotes the poset of all finite quotients of $\Psi\Sigma^*$ in $\mathcal{D}\text{-Mon}$, then a pseudovariety is precisely an ideal \mathcal{L} : a subposet closed under finite joins (= subdirect products) and closed downwards (i.e., under quotients). Here we use the ordering of quotients $e : \Psi\Sigma^* \rightarrow D$ where $e_1 \leq e_2$ iff e_1 factorizes through e_2 .

Remark II.16. Every Σ -generated \mathcal{D} -monoid $e : \Psi\Sigma^* \rightarrow (D, \circ, i)$ defines the *associated L_Σ -algebra* $\alpha : L_\Sigma D \rightarrow D$ with the same object D of states, initial state $e(\varepsilon)$ and transitions given by right multiplication $\alpha_a(d) = d \circ e(a)$ for $a \in \Sigma$. In particular, the associated L_Σ -algebra of the free \mathcal{D} -monoid $\Psi\Sigma^*$ has the initial state ε and the transitions $-\bullet a$ for $a \in \Sigma$. This means that the above L_Σ -algebra structure of D is the unique one that makes e a homomorphism of L_Σ -algebras.

As shown in [1, Prop. 4.29], $\Psi\Sigma^*$ is the initial L_Σ -algebra: for every L_Σ -algebra (A, α) there exists a unique L_Σ -algebra homomorphism

$$e_A : \Psi\Sigma^* \rightarrow A.$$

Its restriction to Σ^* computes the action of A on words $w \in \Sigma^*$:

$$e_A(w) = \alpha_w \cdot \alpha_{in} : \mathbf{1}_{\mathcal{D}} \rightarrow A.$$

Here we use the notation

$$\alpha_w = \alpha_{a_n} \cdots \alpha_{a_1} : A \rightarrow A \quad \text{for } w = a_1 \cdots a_n.$$

Analogously, for coalgebras (Q, γ) we put $\gamma_w = \gamma_{a_n} \cdots \gamma_{a_1}$.

Since Σ -generated monoids are L_Σ -algebras one may ask for the converse: given an L_Σ -algebra, is it associated to some Σ -generated monoid? In the next subsection we will see that this question is, by duality, directly related to closure properties of classes of regular languages.

C. Languages

Our categorical approach to varieties of languages starts with a characterization of the regular languages over a fixed alphabet Σ by a universal property. Let us call a T_Σ -coalgebra Q *locally finite* if it is a filtered colimit of finite coalgebras. Or equivalently, if every state $q \in Q$ lies in a finite subcoalgebra of Q . As shown in [17], the terminal locally finite coalgebra ϱT_Σ – characterized by the property that every locally finite coalgebra has a unique homomorphism into it – is the filtered colimit of the diagram

$$\mathbf{Coalg}_f T_\Sigma \hookrightarrow \mathbf{Coalg} T_\Sigma$$

of *all* finite coalgebras. Its coalgebra structure is an isomorphism

$$\varrho T_\Sigma \cong T_\Sigma(\varrho T_\Sigma),$$

which is why ϱT_Σ is also called the *rational fixpoint* of T_Σ .

Proposition II.17 (see [1], Cor. 2.11). ϱT_Σ is carried by the set of all regular languages over Σ . The transition morphisms are carried by left derivatives $L \mapsto a^{-1}L$ for all $a \in \Sigma$, and the final states are precisely the languages containing the empty set. For any locally finite coalgebra (Q, γ) the unique homomorphism $L_Q : Q \rightarrow \varrho T_\Sigma$ maps a state $q \in Q$ to the language accepted by q :

$$L_Q(q) = \{w \in \Sigma^* : \gamma_{\text{out}} \cdot \gamma_w(q) = 1\}.$$

Example II.18. Continuing our Examples II.5, ϱT_Σ has the following algebraic structure as an object of the variety \mathcal{C} :

(a) For $\mathcal{C} = \mathbf{BA}$ the boolean algebra structure is $\cup, \cap, \overline{(-)}, \emptyset$ and Σ^* .

(b) For $\mathcal{C} = \mathbf{DL}_{01}$ the lattice structure is \cup, \cap, \emptyset and Σ^* .

(c) For $\mathcal{C} = \mathbf{JSL}_0$ the semilattice structure is \cup and \emptyset .

(d) For $\mathcal{C} = \mathbf{Z}_2\text{-Vec}$ the vector addition is symmetric difference $L \oplus M = (L \setminus M) \cup (M \setminus L)$ and the zero vector is \emptyset .

(e) For $\mathcal{C} = \mathbf{BR}$ the multiplication is \cap , the addition is symmetric difference \oplus , and the zero element is \emptyset .

Definition II.19. By a *local variety of languages* over Σ in \mathcal{C} is meant a subcoalgebra $Q \hookrightarrow \varrho T_\Sigma$ of the rational fixpoint closed under right derivatives, i.e., $L \in |Q|$ implies $La^{-1} \in |Q|$ for all $a \in \Sigma$.

Note that a local variety is closed under left derivatives automatically, being a subcoalgebra of ϱT_Σ . Closure under right derivatives also admits a fully coalgebraic description:

Notation II.20. Given a T_Σ -coalgebra (Q, γ) and an input $a \in \Sigma$, denote by $(Q, \gamma)_a$ the T_Σ -coalgebra with the same states and transitions, but whose final-state morphism is $\gamma_{\text{out}} \cdot \gamma_a : Q \rightarrow O_{\mathcal{C}}$.

Proposition II.21 (see [1], Prop. 4.3). A subcoalgebra (Q, γ) of ϱT_Σ is a local variety iff a T_Σ -coalgebra homomorphism from $(Q, \gamma)_a$ to (Q, γ) exists for every $a \in \Sigma$.

Proposition II.22 (see [1], Prop. 3.24 and 4.32). A finite T_Σ -coalgebra Q is a subcoalgebra of ϱT_Σ iff its dual L_Σ -algebra \widehat{Q} is a quotient algebra of $\Psi\Sigma^*$. In this case, Q is a local variety iff \widehat{Q} is the associated L_Σ -algebra of some finite Σ -generated \mathcal{D} -monoid.

In other words, for any finite coalgebra Q the unique T_Σ -coalgebra homomorphism $L_Q : Q \rightarrow \varrho T_\Sigma$ of Proposition II.17 is injective iff the unique L_Σ -algebra homomorphism $e_{\widehat{Q}} : \Psi\Sigma^* \rightarrow \widehat{Q}$ of Remark II.16 is surjective. Moreover, if $Q \hookrightarrow \varrho T_\Sigma$ is a local variety of languages, there exists a (unique) monoid structure on \widehat{Q} making $e_{\widehat{Q}}$ a \mathcal{D} -monoid morphism. In this case we call \widehat{Q} the *dual Σ -generated \mathcal{D} -monoid* of Q .

Proposition II.22 was the basis of the main result of [1]. Observe that the set of all local varieties of languages over Σ in \mathcal{C} forms a complete lattice whose meet is intersection. Analogously for the set of all pseudovarieties of Σ -generated \mathcal{D} -monoids.

Theorem II.23 (Local Eilenberg Theorem [1], Thm. 4.36). *The lattice of all local varieties of languages over Σ in \mathcal{C} is*

isomorphic to the lattice of all pseudovarieties of Σ -generated \mathcal{D} -monoids.

Local varieties of languages are local in the sense that a fixed alphabet Σ is considered. To get a global (alphabet-independent) view of all regular languages, we extend the map $\Sigma \mapsto \varrho T_\Sigma$ to a functor $\varrho T : \mathbf{Set}_f^{\text{op}} \rightarrow \mathcal{C}$. Observe first that for every $h : \Delta \rightarrow \Sigma$ in \mathbf{Set}_f there is a morphism $Q^h : Q^\Sigma \rightarrow Q^\Delta$ given by precomposition with h . Hence we can turn each T_Σ -coalgebra (Q, γ) into the T_Δ -coalgebra $(Q, \gamma)^h$ with the same states Q and coalgebra structure

$$Q \xrightarrow{\gamma} O_{\mathcal{C}} \times Q^\Sigma \xrightarrow{\text{id} \times Q^h} O_{\mathcal{C}} \times Q^\Delta.$$

This is a familiar construction for deterministic automata: if some state $q \in Q$ accepts the language $L \subseteq \Sigma^*$ in (Q, γ) , then it accepts the language $(h^*)^{-1}(L) \subseteq \Delta^*$ in $(Q, \gamma)^h$. Here $h^* : \Delta^* \rightarrow \Sigma^*$ denotes the free extension of h to a monoid morphism.

The coalgebra $(Q, \gamma)^h$ is locally finite if (Q, γ) is, as every subcoalgebra of (Q, γ) is also a subcoalgebra of $(Q, \gamma)^h$. In particular, $(\varrho T_\Sigma)^h$ is locally finite, so there is a unique T_Δ -coalgebra homomorphism $\varrho T_h : (\varrho T_\Sigma)^h \rightarrow \varrho T_\Delta$ into the terminal locally finite T_Δ -coalgebra. The morphism ϱT_h forms preimages under the monoid morphism h^* :

$$\varrho T_h(L) = (h^*)^{-1}(L) \quad \text{for all } L \in |\varrho T_\Sigma|.$$

Definition II.24. The *rational functor* $\varrho T : \mathbf{Set}_f^{\text{op}} \rightarrow \mathcal{C}$ assigns to every finite alphabet Σ the rational fixpoint ϱT_Σ and to every map $h : \Delta \rightarrow \Sigma$ the morphism $\varrho T_h : \varrho T_\Sigma \rightarrow \varrho T_\Delta$.

In the classical case $\mathcal{C} = \mathbf{BA}$, the rational functor maps each finite alphabet Σ to the boolean algebra of regular languages over Σ . A variety V of languages in Eilenberg's sense (see Introduction) is thus a subfunctor¹ $V \hookrightarrow \varrho T$ that assigns to every finite alphabet Σ a local variety $V\Sigma \hookrightarrow \varrho T_\Sigma$, and is closed under preimages of monoid morphisms $f : \Delta^* \rightarrow \Sigma^*$. To formulate the preimage condition categorically, we identify any language $L \subseteq \Sigma^*$ with its characteristic function $L : \Sigma^* \rightarrow \{0, 1\}$ in $\mathcal{D} = \mathbf{Set}$. Then the preimage of L under f is precisely the language represented by the composite function $L \cdot f : \Delta^* \rightarrow \{0, 1\}$. Thus the missing condition on our subfunctor V is the following: for every language $L \in V\Sigma$ we have $L \cdot f \in V\Delta$. Let us now extend these considerations to our general setting.

Notation II.25. Using the adjunction $\Psi \dashv |-| : \mathcal{D} \rightarrow \mathbf{Set}$, we identify any language $L : \Sigma^* \rightarrow \{0, 1\} = |O_{\mathcal{D}}|$ with the corresponding morphism $L : \Psi\Sigma^* \rightarrow O_{\mathcal{D}}$ of \mathcal{D} . The *preimage* of L under a \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ is the language $L \cdot f : \Psi\Delta^* \rightarrow O_{\mathcal{D}}$ over the alphabet Δ .

Definition II.26. (a) A subfunctor $V \hookrightarrow \varrho T$ of the rational functor is *closed under preimages* if, for every \mathcal{D} -monoid

¹Recall that a *subfunctor* of a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a natural transformation $m : F' \rightarrow F$ with monomorphic components $m_A : F'A \rightarrow FA$. To specify F' suffices to give the object map $A \mapsto F'A$ and monomorphisms m_A such that, for each $f : A \rightarrow A'$ in \mathcal{A} , the morphism $Ff \cdot m_A$ factorizes through $m_{A'}$. This uniquely determines the action of F' on morphisms.

morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ and every language $L : \Psi\Sigma^* \rightarrow O_{\mathcal{D}}$ in $V\Sigma$, the language $L \cdot f$ lies in $V\Delta$.

(b) By a *variety of languages* in \mathcal{C} is meant a subfunctor $V \mapsto \varrho T$ closed under preimages such that $V\Sigma$ is a local variety of languages for every alphabet $\Sigma \in \mathbf{Set}_f$.

In Theorem III.16 below we give a fully coalgebraic characterization of preimage closure.

Examples II.27. (a) The case $\mathcal{C} = \mathbf{BA}$ and $\mathcal{D} = \mathbf{Set}$ captures the original concept of Eilenberg [12]: a variety of languages in \mathbf{BA} forms boolean subalgebras $V\Sigma$ of ϱT_Σ , closed under derivatives and preimages of monoid morphisms $f : \Delta^* \rightarrow \Sigma^*$.

(b) In the case $\mathcal{C} = \mathbf{DL}_{01}$ and $\mathcal{D} = \mathbf{Pos}$ we just drop closure under complement: a variety of languages in \mathbf{DL}_{01} forms sublattices $V\Sigma$ of ϱT_Σ closed under derivatives and preimages of monoid morphisms $f : \Delta^* \rightarrow \Sigma^*$. This is the concept of a *positive variety of languages* studied by Pin [18].

(c) Let $\mathcal{C} = \mathcal{D} = \mathbf{JSL}_0$. Given a language $L \subseteq \Sigma^*$, the corresponding semilattice morphism $L : \mathcal{P}_f \Sigma^* \rightarrow \{0, 1\}$ takes a finite language $\{w_1, \dots, w_k\}$ to 1 iff $w_i \in L$ for some i (this follows from $\{w_1, \dots, w_k\} = \bigvee_{i=1}^k \{w_i\}$). The preimage of L under a semiring morphism $f : \mathcal{P}_f \Delta^* \rightarrow \mathcal{P}_f \Sigma^*$ corresponds to the language $M \subseteq \Delta^*$ of all words $u \in \Delta^*$ for which $f(u)$ contains some word of L . A variety of languages in \mathbf{JSL}_0 forms subsemilattices $V\Sigma$ of ϱT_Σ closed under derivatives and preimages of semiring morphisms $f : \mathcal{P}_f \Delta^* \rightarrow \mathcal{P}_f \Sigma^*$. This is the notion of variety introduced by Polák [20].

(d) If $\mathcal{C} = \mathcal{D} = \mathbf{Z}_2\text{-Vec}$, the linear map $L : \mathcal{P}_f \Sigma^* \rightarrow \{0, 1\}$ corresponding to $L \subseteq \Sigma^*$ takes $\{w_1, \dots, w_k\} \in \mathcal{P}_f \Sigma^*$ to 1 iff $w_i \in L$ for an odd number of $i = 1, \dots, k$. Thus the preimage of L under a \mathbf{Z}_2 -algebra morphism $f : \mathcal{P}_f \Delta^* \rightarrow \mathcal{P}_f \Sigma^*$ corresponds to the language $M \subseteq \Delta^*$ of all words $u \in \Delta^*$ for which $f(u)$ contains an odd number of words of L . A variety of languages in $\mathbf{Z}_2\text{-Vec}$ forms linear subspaces $V\Sigma$ of ϱT_Σ closed under derivatives and preimages of \mathbf{Z}_2 -algebra morphisms $f : \mathcal{P}_f \Delta^* \rightarrow \mathcal{P}_f \Sigma^*$. This notion of a variety was introduced by Reutenauer [22]; see also Section V.

(e) Finally, let $\mathcal{C} = \mathbf{BR}$ and $\mathcal{D} = \mathbf{Set}_*$. The preimage of $L \subseteq \Sigma^*$ under a zero-preserving monoid morphism $f : \Delta^* + \{0\} \rightarrow \Sigma^* + \{0\}$ consists of all words $w \in \Delta^*$ for which $f(w)$ lies in L . A variety of languages in \mathbf{BR} forms subrings $V\Sigma$ of ϱT_Σ closed under derivatives and preimages of zero-preserving monoid morphisms.

See the table in Section IV for a summary of our examples. The set of all varieties of languages in \mathcal{C} is a complete lattice since any intersection of varieties (formed objectwise) is a variety. The same holds for the set of all pseudovarieties of \mathcal{D} -monoids. Our main result, the *Generalized Eilenberg Theorem* (see Theorem IV.5), states that these two lattices are isomorphic. The rest of the paper is devoted to the proof.

III. COALGEBRAIC AND ALGEBRAIC LANGUAGE ACCEPTANCE

In this section we compare the languages accepted by a finite T_Σ -coalgebra in \mathcal{C} with those accepted by its dual finite L_Σ -algebra in \mathcal{D} .

Notation III.1. (a) Recall $|O_{\mathcal{C}}| = |O_{\mathcal{D}}| = \{0, 1\}$ from Assumption II.4(v), and let $\mathbf{1}_{\mathcal{C}} \xrightarrow{1_{O_{\mathcal{C}}}} O_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}} \xrightarrow{1_{O_{\mathcal{D}}}} O_{\mathcal{D}}$ denote the morphisms choosing the element 1. Note that $\widehat{1_{O_{\mathcal{C}}}} = 1_{O_{\mathcal{D}}}$ by Remark II.3.

(b) Recall from Proposition II.17 that a state $q : \mathbf{1}_{\mathcal{C}} \rightarrow Q$ of a finite T_Σ -coalgebra (Q, γ) accepts the language

$$L_Q(q) = \{w \in \Sigma^* : \gamma_{\text{out}} \cdot \gamma_w \cdot q = 1_{O_{\mathcal{C}}}\}$$

that we identify with the corresponding morphism of \mathcal{D}

$$L_Q(q) : \Psi\Sigma^* \rightarrow O_{\mathcal{D}},$$

see Notation II.25. Dually, for any L_Σ -algebra (A, α) equipped with a morphism $\alpha_{\text{out}} : A \rightarrow O_{\mathcal{D}}$ (representing a choice of final states), we define the *language accepted by α_{out}* by

$$L_A(\alpha_{\text{out}}) = \{w \in \Sigma^* : \alpha_{\text{out}} \cdot \alpha_w \cdot \alpha_{\text{in}} = 1_{O_{\mathcal{D}}}\}.$$

Using the L_Σ -algebra homomorphism $e_A : \Psi\Sigma^* \rightarrow A$ of Remark II.16, this language corresponds to the morphism

$$L_A(\alpha_{\text{out}}) = \alpha_{\text{out}} \cdot e_A : \Psi\Sigma^* \rightarrow O_{\mathcal{D}}.$$

Definition III.2. The map $\Sigma^* \rightarrow \Sigma^*$ reversing words extends uniquely to a morphism of \mathcal{D}

$$\text{rev}_\Sigma : \Psi\Sigma^* \rightarrow \Psi\Sigma^*.$$

The *reversal* of a language $L : \Psi\Sigma^* \rightarrow O_{\mathcal{D}}$ is the language $L \cdot \text{rev}_\Sigma$.

Observe that rev_Σ is a \mathcal{D} -monoid morphism

$$\text{rev}_\Sigma : \Psi\Sigma^* \rightarrow (\Psi\Sigma^*)^{op},$$

where $(\Psi\Sigma^*)^{op}$ is the reversed monoid of $\Psi\Sigma^*$ with multiplication $x \bullet^{op} y = y \bullet x$.

Lemma III.3. Let (Q, γ) be a finite T_Σ -coalgebra and $(\widehat{Q}, \widehat{\gamma})$ its dual L_Σ -algebra. Then the language accepted by a state $q : \mathbf{1}_{\mathcal{C}} \rightarrow Q$ is the reversal of the language accepted by $\widehat{q} : \widehat{Q} \rightarrow O_{\mathcal{D}}$:

$$L_{\widehat{Q}}(\widehat{q}) = L_Q(q) \cdot \text{rev}_\Sigma.$$

Proof. A state $q : \mathbf{1}_{\mathcal{C}} \rightarrow Q$ accepts a word $w = a_1 \cdots a_n$ iff

$$\gamma_{\text{out}} \cdot \gamma_{a_n} \cdots \gamma_{a_1} \cdot q = 1_{O_{\mathcal{C}}}.$$

This is dual to the equation

$$\widehat{q} \cdot \widehat{\gamma}_{a_1} \cdots \widehat{\gamma}_{a_n} \cdot \widehat{\gamma}_{\text{out}} = \widehat{1_{O_{\mathcal{C}}}} = 1_{O_{\mathcal{D}}},$$

which states precisely that \widehat{q} accepts the word $w^{rev} = a_n \cdots a_1$ in the L_Σ -algebra \widehat{Q} . It follows that the two morphisms $L_{\widehat{Q}}(\widehat{q})$ and $L_Q(q) \cdot \text{rev}_\Sigma$ agree on Σ^* , hence they are equal. \square

One of the cornerstones of our Generalized Eilenberg Theorem in Section IV is a coalgebraic characterization of the closure under preimages, see Definition II.26. To this end we introduce first the preimage A^f of an L_Σ -algebra A , and then the preimage Q^f of a locally finite T_Σ -coalgebra Q .

Notation III.4. (a) Let A be an object of \mathcal{D} . The object $[A, A]$ of endomorphisms, see Assumption II.4(iv), forms a \mathcal{D} -monoid with multiplication

$$[A, A] \times [A, A] \rightarrow [A, A], \quad (f, g) \mapsto g \cdot f,$$

given by functional composition and unit id_A .

(b) For an L_Σ -algebra (A, α) the notation $\alpha_w : A \rightarrow A$ for words $w \in \Sigma^*$ (see Remark II.16) is extended to $\alpha_x : A \rightarrow A$ for all $x \in |\Psi\Sigma^*|$ as follows: since $\Psi\Sigma^*$ is the free \mathcal{D} -monoid on Σ (see Proposition II.12), the function

$$\Sigma \rightarrow \mathcal{D}(A, A), \quad a \mapsto \alpha_a,$$

extends to a unique \mathcal{D} -monoid morphism $\Psi\Sigma^* \rightarrow [A, A]$ that we denote by $x \mapsto \alpha_x$.

Definition III.5. Let $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ be a \mathcal{D} -monoid morphism. For every L_Σ -algebra (A, α) we define its *preimage under f* as the L_Δ -algebra $(A, \alpha)^f = (A, \alpha^f)$ on the same states A , with the same initial state, $\alpha_{\text{in}}^f = \alpha_{\text{in}}$, and with transitions $\alpha_b^f = \alpha_{f(b)}$ for all $b \in \Delta$.

Example III.6. (a) If $\mathcal{D} = \mathbf{Set}$ or \mathbf{Pos} , we are given a monoid morphism $f : \Delta^* \rightarrow \Sigma^*$. Every L_Σ -algebra A yields an L_Δ -algebra with transitions $\alpha_b^f = \alpha_{f(b)} = \alpha_{a_n} \cdot \dots \cdot \alpha_{a_1}$ for $f(b) = a_1 \cdot \dots \cdot a_n$.

(b) If $\mathcal{D} = \mathbf{JSL}_0$, we are given a semiring morphism $f : \mathcal{P}_f\Delta^* \rightarrow \mathcal{P}_f\Sigma^*$. If the value $f(b)$ is a single word, $f(b) = \{w\}$, then the corresponding transition is again $\alpha_b^f = \alpha_w$. In general $f(b) = \{w_1, \dots, w_k\}$, and since $\alpha_{(-)}$ is a semilattice homomorphism, we conclude that $\alpha_b^f = \alpha_{w_1} \vee \dots \vee \alpha_{w_k}$ (the join in $[A, A]$).

(c) Analogously for $\mathcal{D} = \mathbf{Z}_2\text{-Vec}$: if $f(b) = \{w_1, \dots, w_k\}$, then $\alpha_b^f = \alpha_{w_1} \oplus \dots \oplus \alpha_{w_k}$ (vector addition in $[A, A]$).

(d) If $\mathcal{D} = \mathbf{Set}_*$ a zero-preserving monoid morphism $f : \Delta^* + \{0\} \rightarrow \Sigma^* + \{0\}$ is given. The map α_b^f is defined as in (a) if $f(b) \neq 0$, and otherwise $\alpha_b^f(x) = \star_A$ for all $x \in |A|$, where \star_A is the point of $A \in \mathbf{Set}_*$.

Lemma III.7. Let $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ be a \mathcal{D} -monoid morphism.

(a) f is also an L_Δ -algebra homomorphism $f : \Psi\Delta^* \rightarrow (\Psi\Sigma^*)^f$.

(b) Every L_Σ -algebra homomorphism $h : A \rightarrow A'$ is also an L_Δ -algebra homomorphism $h : A^f \rightarrow (A')^f$.

(c) Given an L_Σ -algebra A we have, in Remark II.16,

$$e_{A^f} = e_A \cdot f : \Psi\Delta^* \rightarrow A^f.$$

Proof. (a) Since $f(\varepsilon) = \varepsilon$ the initial state of $\Psi\Delta^*$ is mapped to the one of $(\Psi\Sigma^*)^f$. For any $a \in \Delta$, the a -transitions in $\Psi\Delta^*$ and $(\Psi\Sigma^*)^f$ are $-\bullet a$ and $-\bullet f(a)$, respectively. Hence preservation of transitions amounts to the equation $f(x \bullet a) = f(x) \bullet f(a)$ for all $x \in \Psi\Delta^*$, which holds because f is a \mathcal{D} -monoid morphism.

(b) We clearly have $h \cdot \alpha_{\text{in}}^f = h \cdot \alpha_{\text{in}} = \alpha_{\text{in}}' = (\alpha')_{\text{in}}^f$. From $h \cdot \alpha_w = \alpha'_w \cdot h$ for all $w \in \Sigma^*$ we can conclude $h \cdot \alpha_x = \alpha'_x \cdot h$ for all $x \in |\Psi\Sigma^*|$. Indeed, both sides define \mathcal{D} -morphisms $\Psi\Sigma^* \rightarrow [A, A']$ in the variable x which agree on Σ^* . Thus

they are equal. In particular, we have the desired equation $h \cdot \alpha_{f(a)} = \alpha'_{f(a)} \cdot h$ for all $a \in \Delta$.

(c) $\Psi\Delta^*$ is the initial L_Δ -algebra, and by (a) and (b) both sides are L_Δ -algebra homomorphisms. \square

Corollary III.8. Let A be an L_Σ -algebra and $\alpha_{\text{out}} : A \rightarrow O_{\mathcal{D}}$ an output morphism. Then α_{out} accepts in A^f the preimage of the language it accepts in A :

$$L_{A^f}(\alpha_{\text{out}}) = L_A(\alpha_{\text{out}}) \cdot f : \Psi\Delta^* \rightarrow O_{\mathcal{D}}.$$

Proof. Both sides are equal to $\alpha_{\text{out}} \cdot e_{A^f} \cdot f$. \square

Notation III.9. For any \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ we denote by f^\dagger the \mathcal{D} -monoid morphism

$$\Psi\Delta^* \xrightarrow{\text{rev}\Delta} (\Psi\Delta^*)^{\text{op}} \xrightarrow{f} (\Psi\Sigma^*)^{\text{op}} \xrightarrow{\text{rev}\Sigma} \Psi\Sigma^*.$$

Definition III.10. Let $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ be a \mathcal{D} -monoid morphism. For every finite T_Σ -coalgebra (Q, γ) we define its *preimage under f* as the T_Δ -coalgebra $(Q, \gamma)^f = (Q, \widehat{\gamma}^f)$ whose dual is the preimage of the dual L_Σ -algebra $(\widehat{Q}, \widehat{\gamma})$ under f^\dagger . Shortly:

$$\widehat{Q}^f = \widehat{Q}^{f^\dagger}.$$

If $f = \Psi h^*$ for a function $h : \Delta \rightarrow \Sigma$, it is easy to see that the coalgebra Q^f of the previous definition coincides with the coalgebra Q^h introduced for the definition of qT (see II.24).

Example III.11. The preimage of (Q, γ) under $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ has the same states and final states, and the transitions are given as follows:

(a) Let $\mathcal{C} = \mathbf{BA}, \mathbf{DL}_{01}$ and $f : \Delta^* \rightarrow \Sigma^*$. Letting $\alpha_a = \widehat{\gamma}_a$ we get, by Example III.6(a), the formula $\alpha_b^{f^\dagger} = \alpha_{a_1} \cdot \dots \cdot \alpha_{a_n}$ where $f(b) = a_1 \cdot \dots \cdot a_n$ (i.e., $f^\dagger(b) = a_n \cdot \dots \cdot a_1$). Since $\widehat{\gamma}_b^f = \alpha_b^{f^\dagger}$ it follows that $\gamma_b^f = \gamma_{a_n} \cdot \dots \cdot \gamma_{a_1} = \gamma_{f(b)}$.

(b) Let $\mathcal{C} = \mathbf{JSL}_0$ and $f : \mathcal{P}_f\Delta^* \rightarrow \mathcal{P}_f\Sigma^*$. We claim that $\gamma_b^f = \gamma_{w_1} \vee \dots \vee \gamma_{w_k}$ where $f(b) = \{w_1, \dots, w_k\}$ and the join is taken in $[Q, Q]$ (i.e., pointwise). Indeed, observe that the map $h \mapsto \widehat{h}$ gives a semilattice isomorphism $[Q, Q] \cong [\widehat{Q}, \widehat{Q}]$. Letting $\alpha_a = \widehat{\gamma}_a$, this isomorphism maps γ_{w_i} to $\alpha_{w_i}^{rev}$, and hence $\gamma_{w_1} \vee \dots \vee \gamma_{w_k}$ to $\alpha_{w_1}^{rev} \vee \dots \vee \alpha_{w_k}^{rev}$. By Example III.6(b) this morphism is $\alpha_b^{f^\dagger}$ since $f^\dagger(b) = \{w_1^{rev}, \dots, w_k^{rev}\}$.

(c) If $\mathcal{C} = \mathbf{Z}_2\text{-Vec}$ and $f : \mathcal{P}_f\Delta^* \rightarrow \mathcal{P}_f\Sigma^*$, then $\gamma_b^f = \gamma_{w_1} \oplus \dots \oplus \gamma_{w_k}$ where $f(b) = \{w_1, \dots, w_k\}$. Indeed, the map $h \mapsto \widehat{h}$ gives an isomorphism of vector spaces $[Q, Q] \cong [\widehat{Q}, \widehat{Q}]$. Letting $\alpha_a = \widehat{\gamma}_a$ this isomorphism maps γ_{w_i} to $\alpha_{w_i}^{rev}$, and hence $\gamma_{w_1} \oplus \dots \oplus \gamma_{w_k}$ to $\alpha_{w_1}^{rev} \oplus \dots \oplus \alpha_{w_k}^{rev}$. By Example III.6(c) this morphism is $\alpha_b^{f^\dagger}$.

(d) If $\mathcal{C} = \mathbf{BR}$ and $f : \Delta^* + \{0\} \rightarrow \Sigma^* + \{0\}$, then $\gamma_b^f = \gamma_{f(b)}$ if $f(b) \neq 0$, and otherwise γ_b^f is the zero map. The argument is similar to (a).

Proposition III.12. The language accepted by a state q of the coalgebra Q^f is the preimage under f of the language q accepts in Q :

$$L_{Q^f}(q) = L_Q(q) \cdot f.$$

Proof. This follows from the computation

$$\begin{aligned}
L_{Q^f}(q) &= L_{\widehat{Q}^\dagger}(\widehat{q}) \cdot \text{rev}_\Delta && \text{(Lemma III.3)} \\
&= L_{\widehat{Q}^\dagger}(\widehat{q}) \cdot \text{rev}_\Delta && \text{(def. } Q^f\text{)} \\
&= L_{\widehat{Q}}(\widehat{q}) \cdot f^\dagger \cdot \text{rev}_\Delta && \text{(Corollary III.8)} \\
&= L_{\widehat{Q}}(\widehat{q}) \cdot \text{rev}_\Sigma \cdot f \cdot \text{rev}_\Delta \cdot \text{rev}_\Delta && \text{(def. } f^\dagger\text{)} \\
&= L_{\widehat{Q}}(\widehat{q}) \cdot \text{rev}_\Sigma \cdot f \\
&= L_Q(q) \cdot f && \text{(Lemma III.3).} \quad \square
\end{aligned}$$

Example III.13. If Q is finite subcoalgebra of ϱT_Σ then Q^f is the T_Δ -coalgebra of all languages $L : \Psi\Sigma^* \rightarrow \mathcal{O}_\mathcal{D}$ in Q with transitions given by left derivatives $\gamma_a(L) = f(a)^{-1}L$ for $a \in \Delta$. Here we extend the notation $w^{-1}L$ for left derivatives from words $w \in \Sigma^*$ to all elements x of $\Psi\Sigma^*$ as follows: let $l_x : \Psi\Sigma^* \rightarrow \Psi\Sigma^*$ be the left translation, $l_x(y) = x \bullet y$, then the left derivative $x^{-1}L$ of a language $L : \Psi\Sigma^* \rightarrow \mathcal{O}_\mathcal{D}$ is $L \cdot l_x$.

We now extend the preimage concept from finite coalgebras to locally finite ones. A T_Σ -coalgebra Q is locally finite iff it is the filtered colimit of the diagram of all its finite subcoalgebras $Q_i \hookrightarrow Q$ (whose connecting T_Σ -coalgebra homomorphisms $d_{i,j} : Q_i \rightarrow Q_j$ are inclusion maps). Given any \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$, every $d_{i,j}$ is also a T_Δ -coalgebra homomorphism $d_{i,j} : Q_i^f \rightarrow Q_j^f$ by the dual of Lemma III.7(b). Hence the coalgebras Q_i^f and homomorphisms $d_{i,j}$ form a filtered diagram in $\mathbf{Coalg} T_\Delta$.

Definition III.14. For every \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ and every locally finite T_Σ -coalgebra Q we denote by Q^f the filtered colimit of the diagram of all Q_i^f , where Q_i ranges over all finite subcoalgebras of Q .

Example III.15. If Q is a finite subcoalgebra of ϱT_Σ then the languages accepted by Q^f are precisely the languages $L \cdot f$ with $L \in |Q|$. This follows from the Proposition III.12 and the fact that every state L of Q accepts precisely the language L . Since ϱT_Σ is the filtered colimit of its finite subcoalgebras Q , an analogous description holds for $(\varrho T_\Sigma)^f$. Hence the unique T_Δ -coalgebra homomorphism $h : (\varrho T_\Sigma)^f \rightarrow \varrho T_\Delta$ maps every language L in ϱT_Σ to its preimage $L \cdot f$.

Recall from Definition II.26 the concept of closure under preimages. This can now be formulated coalgebraically, much in the spirit of Proposition II.21.

Theorem III.16. A subfunctor V of the rational functor ϱT is closed under preimages iff for every \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ there exists a T_Δ -coalgebra homomorphism from $(V\Sigma)^f$ to $V\Delta$.

Proof. Suppose that $k : (V\Sigma)^f \rightarrow V\Delta$ is a T_Δ -coalgebra homomorphism. Composed with the inclusion $i : V\Delta \hookrightarrow \varrho T_\Delta$ it yields the homomorphism h of Example III.15 restricted to $(V\Sigma)^f$ – this follows from ϱT_Δ being the terminal locally finite T_Δ -coalgebra. Thus $i \cdot k$ takes every language L of $|V\Sigma|$ to $L \cdot f$, proving that $L \cdot f$ lies in $V\Delta$.

For the converse, suppose that V is closed under preimages. Then the homomorphism h of Example III.15 has a restriction

$h_0 : (V\Sigma)^f \rightarrow V\Delta$. That h_0 is a coalgebra homomorphism is a consequence of the following homomorphism theorem for coalgebras: if $g : Q \rightarrow R$ and $i : R' \hookrightarrow R$ are T_Σ -coalgebra homomorphisms such that i is injective and $g = i \cdot k$ for some morphism k in \mathcal{C} , then k is a coalgebra homomorphism. This follows easily from the observation that T_Σ preserves monomorphisms. \square

IV. GENERALIZED EILENBERG THEOREM

In this section we present our main result, the Generalized Eilenberg Theorem. First we consider two “finite” versions of this theorem, proved by Klíma and Polák [16] for the cases $\mathcal{C} = \mathbf{BA}$ and $\mathcal{C} = \mathbf{DL}_{01}$.

Definition IV.1. A variety V of languages in \mathcal{C} is called *object-finite* if $V\Sigma$ is finite for every alphabet Σ .

In the next theorem we will consider locally finite varieties of \mathcal{D} -monoids, see Definition II.1. Recall that, in comparison to the *pseudovarieties* of Definition II.11, varieties of \mathcal{D} -monoids may contain infinite monoids and are closed under finite and infinite products. All locally finite varieties of \mathcal{D} -monoids form a lattice whose meet is intersection. The same holds for all object-finite varieties of languages where the intersection is taken objectwise.

Theorem IV.2 (Generalized Eilenberg Theorem for Object-Finite Varieties). *The lattice of all object-finite varieties of languages in \mathcal{C} is isomorphic to the lattice of all locally finite varieties of \mathcal{D} -monoids.*

Proof sketch. Given a variety V of languages in \mathcal{C} , denote by $e_\Sigma : \Psi\Sigma^* \rightarrow \widehat{V\Sigma}$ the dual finite Σ -generated monoid of the local variety $V\Sigma \hookrightarrow \varrho T_\Sigma$, see Theorem II.22. Let V° be the class of all \mathcal{D} -monoids D such that every \mathcal{D} -monoid morphism $h : \Psi\Sigma^* \rightarrow D$, where Σ is any finite alphabet, factorizes (necessarily uniquely) through e_Σ :

$$h = (\Psi\Sigma^* \xrightarrow{e_\Sigma} \widehat{V\Sigma} \xrightarrow{h'} D).$$

A routine calculation shows that V° is a variety of \mathcal{D} -monoids whose free monoid on Σ is $\widehat{V\Sigma}$. Since $\widehat{V\Sigma}$ is finite, V° is locally finite.

Conversely, for a locally finite variety W of \mathcal{D} -monoids (with free monoids $e_\Sigma : \Psi\Sigma^* \twoheadrightarrow D_\Sigma$), we obtain an object-finite variety of languages $W^\square \hookrightarrow \varrho T$ with $W^\square \Sigma$ defined by $\widehat{W^\square \Sigma} = D_\Sigma$ for all Σ .

The constructions $V \mapsto V^\circ$ and $W \mapsto W^\square$ are mutually inverse and hence define the desired lattice isomorphism. \square

Definition IV.3. An object-finite variety V of languages in \mathcal{C} is called *simple* if it is generated by a single alphabet Σ . That is, given any variety V' such that $V\Sigma$ is a local subvariety of $V'\Sigma$, then V is a subfunctor of V' . A pseudovariety of \mathcal{D} -monoids is called *simple* if it is generated by a single finite \mathcal{D} -monoid D , i.e., all members of the pseudovariety are quotients of submonoids of finite powers D^n ($n < \omega$).

Theorem IV.4 (Generalized Eilenberg Theorem for Simple Varieties). *The poset of all simple varieties of languages in \mathcal{C} is isomorphic to the poset of all simple pseudovarieties of \mathcal{D} -monoids.*

Proof sketch. For every locally finite variety W of \mathcal{D} -monoids the class W_f of finite members of W forms a pseudovariety of \mathcal{D} -monoids. The isomorphism $V \mapsto V^\circledast$ in the proof of Theorem IV.2 restricts to one between simple varieties of languages and simple pseudovarieties of \mathcal{D} -monoids, that is, V is simple iff $(V^\circledast)_f$ is simple. \square

Our main result now follows from Theorem IV.4 by a completion process. Recall that a *cpo* is a poset with directed joins. By a *free cpo-completion* of a poset P^0 is meant a cpo P containing P^0 as a subposet such that

(C1) every element x of P^0 is compact in P (that is, whenever x lies under a directed join $\bigvee p_i$ of elements of P , then $x \leq p_i$ for some i), and

(C2) the closure of P^0 under directed joins is all of P .

These two properties determine P uniquely up to isomorphism. Concretely P can be constructed as the set of all ideals (= directed down-sets) of P^0 , ordered by inclusion.

Theorem IV.5 (Generalized Eilenberg Theorem). *The lattice of all varieties of languages in \mathcal{C} is isomorphic to the lattice of all pseudovarieties of \mathcal{D} -monoids.*

Proof sketch. One proves that

(1) the lattice $\mathcal{L}_\mathcal{C}$ of all varieties of languages in \mathcal{C} is a free cpo-completion of the poset $\mathcal{L}_\mathcal{C}^0$ of all simple varieties of languages, and

(2) the lattice $\mathcal{L}_\mathcal{D}$ of all pseudovarieties of \mathcal{D} -monoids is a free cpo-completion of the poset $\mathcal{L}_\mathcal{D}^0$ of all simple pseudovarieties.

This requires a verification of the properties (C1) and (C2) above. Since $\mathcal{L}_\mathcal{C}^0 \cong \mathcal{L}_\mathcal{D}^0$ by Theorem IV.4, and free cpo-completions are unique up to isomorphism, it follows that $\mathcal{L}_\mathcal{C} \cong \mathcal{L}_\mathcal{D}$. \square

For our five predualitys of Example II.5 we thus obtain the concrete correspondences in the table below as special cases of the Generalized Eilenberg Theorem. The second column describes the \mathcal{C} -algebraic operations under which varieties of languages are closed (in addition to closure under derivatives and preimages), and the fourth column characterizes the \mathcal{D} -monoids.

| CATEGORY \mathcal{C} | var. of languages are closed under | CATEGORY \mathcal{D} | \mathcal{D} -monoids |
|--------------------------|---|--------------------------|--------------------------|
| BA | $\cup, \cap, \overline{(-)}, \emptyset, \Sigma^*$ | Set | monoids |
| DL₀₁ | $\cup, \cap, \emptyset, \Sigma^*$ | Pos | ordered monoids |
| JSL₀ | \cup, \emptyset | JSL₀ | idempotent semirings |
| Z₂-Vec | \oplus, \emptyset | Z₂-Vec | Z ₂ -algebras |
| BR | \oplus, \cap, \emptyset | Set_* | monoids with 0 |

The cases $\mathcal{C} = \mathbf{BA}$, **DL₀₁** and **Z₂-Vec** are due to Eilenberg [12], Pin [18] and Reutenauer [22], respectively. The case $\mathcal{C} = \mathbf{JSL}_0$ is “almost” the result of Polák [20]: his *disjunctive varieties of languages* are required to contain Σ^* for every Σ , and he considers semirings without 0. In our setting this would mean to take the preduality categories $\mathcal{C} = \mathbf{JSL}_{01}$ (join-semilattices with 0 and 1) and $\mathcal{D} = \mathbf{JSL}$ (join-semilattices). We opted for the more symmetric preduality $\mathcal{C} = \mathcal{D} = \mathbf{JSL}_0$ as semirings are usually considered with a zero element. The last example, $\mathcal{C} = \mathbf{BR}$, is a new variant of Eilenberg’s theorem.

V. VARIETIES OF REGULAR BEHAVIORS

Although all our results so far concerned acceptors and varieties of languages they accept, we can with little effort generalize the whole theory to Moore automata, where the output morphism γ_{out} has, in lieu of $\{0, 1\}$, any finite set O (of outputs) as codomain. The role of languages over Σ is now taken over by functions $\beta : \Sigma^* \rightarrow O$, and the role of regular languages by *regular behaviors*, i.e., those β realized by a state of some finite Moore automaton.

Given a fixed finite set O of outputs, all we need to change in the previous text is Assumption II.4(v) which is replaced by $|O_\mathcal{C}| = O$, where $O_\mathcal{C}$ is the object dual to $\mathbf{1}_\mathcal{D}$. For the object $O_\mathcal{D}$ dual to $\mathbf{1}_\mathcal{C}$ we can assume, as in Remark II.3, that $|O_\mathcal{C}| = |O_\mathcal{D}|$. Moore automata are modeled as coalgebras for the endofunctor on \mathcal{C}

$$T_\Sigma Q = O_\mathcal{C} \times Q^\Sigma.$$

Example V.1 (Linear weighted automata). Let $\mathcal{C} = \mathcal{D}$ be the category **K-Vec** of vector spaces over a finite field **K**. Here $O_\mathcal{C} = \mathbf{K}$, the one-dimensional space. Thus, a T_Σ -coalgebra is a *linear weighted automaton*: it consists of a vector space Q of states, a linear output function $\gamma_{\text{out}} : Q \rightarrow \mathbf{K}$ and linear transitions $\gamma_a : Q \rightarrow Q$ for $a \in \Sigma$.

The rational fixpoint ϱT_Σ of the functor T_Σ is carried by the set of all regular behaviors $\beta : \Sigma^* \rightarrow O$ (that we identify with the corresponding morphisms $\beta : \Psi\Sigma^* \rightarrow O_\mathcal{D}$ in \mathcal{D}). Its output map assigns the value at the empty word, $\gamma_{\text{out}}(\beta) = \beta(\varepsilon)$, and its transitions are given by *left derivatives* $\gamma_a(\beta) = \beta(a \cdot -)$ for all $a \in \Sigma$. Symmetrically, the *right derivatives* of β are the functions $\beta(- \cdot a)$ for $a \in \Sigma$. We define the *rational functor* $\varrho T : \mathbf{Set}_f^{\text{op}} \rightarrow \mathcal{C}$ in complete analogy to Definition II.24.

Definition V.2. A subfunctor $V \hookrightarrow \varrho T$ is *closed under preimages* if for every \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ and every behavior $\beta : \Psi\Sigma^* \rightarrow O_\mathcal{D}$ in $V\Sigma$ the behavior $\beta \cdot f$ lies in $V\Delta$. A *variety of behaviors in \mathcal{C}* is a subfunctor $V \hookrightarrow \varrho T$ closed under preimages and (left and right) derivatives.

Example V.3 (Varieties of rational power series). If $\mathcal{C} = \mathcal{D} = \mathbf{K-Vec}$, the coalgebra ϱT_Σ consists of all rational power series $\beta : \Sigma^* \rightarrow \mathbf{K}$, i.e., behaviors of finite linear weighted automata. Moreover, \mathcal{D} -monoids are precisely algebras over the field **K**, and the free **K**-algebra $\Psi\Sigma^*$ is carried by the set of all functions $\Sigma^* \rightarrow \mathbf{K}$ with finite support. Given a morphism

$f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ of free \mathbf{K} -algebras, the preimage of a power series $\beta : \Sigma^* \rightarrow \mathbf{K}$ under f is

$$\beta' : \Delta^* \rightarrow \mathbf{K}, \quad \beta'(w) = \sum_{v \in \Sigma^*} f(w)(v) \cdot \beta(v).$$

(This sum is well-defined because $f(w) : \Sigma^* \rightarrow \mathbf{K}$ has finite support.) A variety of behaviors in $\mathbf{K}\text{-Vec}$ forms linear subspaces $V\Sigma$ of ρT_Σ closed under derivatives and preimages of \mathbf{K} -algebra morphisms. This coincides with the concept of a *variety of rational power series* introduced by Reutenauer [22]. His Theorem III.1.1 is therefore a special case of

Theorem V.4 (Generalized Eilenberg Theorem for Regular Behaviors). *The lattice of all varieties of behaviors in \mathcal{C} is isomorphic to the lattice of all pseudovarieties of \mathcal{D} -monoids.*

VI. CONCLUSIONS AND FUTURE WORK

In the present paper we demonstrated that Eilenberg’s variety theorem, a central result of algebraic automata theory, holds at the level of an abstract duality between (algebraic) categories. Our result covers uniformly several known extensions and refinements of Eilenberg’s theorem and also provides a new Eilenberg-type correspondence for pseudovarieties of monoids with 0.

In the future we intend to classify all predual pairs $(\mathcal{C}, \mathcal{D})$ satisfying our Assumptions II.4 above. Using the *natural duality* framework of Clark and Davey [11], we expect to show that only finitely many Eilenberg theorems exist for every fixed finite output set O .

We also believe that more can be said about the relationship between local varieties of languages and varieties of languages. By studying languages and monoids in a setting of Grothendieck fibrations, it seems possible that the global Eilenberg theorem turns out to be an instance of the local one – or that both theorems are instances of one and the same result. This might lead to independent proofs of our results, and at the same time to a more abstract and hence illuminating view of the concepts involved.

Another important direction is the connection of regular languages to profinite algebras. All results mentioned in the Related Work can be interpreted in the setting $(\mathcal{C}, \mathcal{D}) = (\mathbf{BA}, \mathbf{Set})$ or $(\mathbf{DL}_{01}, \mathbf{Pos})$, and we aim to extend them to general pairs of predual categories. This will require the generalization of two core concepts of algebraic automata theory, the syntactic (ordered) monoid associated to a regular language and the free profinite (ordered) monoid on an alphabet Σ , to a notion of *syntactic \mathcal{D} -monoid* and *free profinite \mathcal{D} -monoid*, respectively. We conjecture that the free profinite \mathcal{D} -monoid on Σ arises as the limit of all finite quotient monoids of $\Psi\Sigma^*$, and hope to derive a generalized Reiterman theorem. In the classical setting the theorems of Eilenberg and Reiterman are two key ingredients in a great success of algebraic automata theory: these results allow to specify classes of regular languages (e.g. the star-free languages) by profinite identities, which leads to decidability results for such classes. Perhaps, new such results are enabled through subsequent work in our generalized setting.

Finally, we are interested in extending our results from regular languages and Moore behaviors to other notions of rational behavior, such as ω -regular languages or regular tree languages. Here the role of monoids is taken over by two-sorted algebras called *Wilke algebras* (or *right binoids*, see Wilke [25]) and *forest algebras* (introduced by Bojanczyk and Walukiewicz [9]), respectively. The main challenge will be to identify the proper categorical model for the corresponding acceptors, Büchi automata and tree automata.

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This appendix provides all proofs we omitted due to space limitations, along with some technical lemmas required for these proofs.

Remark A.1. (a) Recall that an object A of a category \mathcal{A} is called *finitely presentable* if its hom-functor $\mathcal{A}(A, -) : \mathcal{A} \rightarrow \mathbf{Set}$ preserves filtered colimits. If \mathcal{A} is a locally finite variety of (ordered) algebras, the finitely presentable objects are precisely the finite algebras.

(b) Every finite T_Σ -coalgebra is a finitely presentable object of $\mathbf{Coalg} T_\Sigma$, see [2]. Hence, given a filtered colimit cocone $c'_i : Q'_i \rightarrow Q'$ ($i \in I$) in $\mathbf{Coalg} T_\Sigma$, every coalgebra homomorphism $h : Q \rightarrow Q'$ with finite domain Q factorizes (in $\mathbf{Coalg} T_\Sigma$) through some c'_i :

$$\begin{array}{ccc} Q & \xrightarrow{h} & Q' \\ & \searrow h' & \uparrow c'_i \\ & & Q'_i \end{array}$$

Remark A.2. The forgetful functor $\mathbf{Coalg} T_\Sigma \rightarrow \mathcal{C}$ preserves and creates colimits. The latter means that, given a diagram (Q_i, γ_i) ($i \in I$) of T_Σ -coalgebras and a colimit cocone $(c_i : Q_i \rightarrow Q)_{i \in I}$ in \mathcal{C} , there is a unique T_Σ -coalgebra structure γ on Q for which the maps c_i are T_Σ -coalgebra homomorphisms $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$. Moreover, (c_i) is a colimit cocone in $\mathbf{Coalg} T_\Sigma$.

The uniqueness of γ gives rise to a useful proof principle: if two coalgebra structures γ and γ' on Q are given such that each c_i is a coalgebra homomorphism $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$ and $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma')$, it follows that $\gamma = \gamma'$.

The preimage construction Q^f , see Definition III.14, has the following equivalent formulation:

Construction A.3. For every locally finite T_Σ -coalgebra (Q, γ) and \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ we construct a T_Δ -coalgebra $(Q, \gamma)^f$ as follows:

(1) Express (Q, γ) as a filtered colimit $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$ ($i \in I$) of finite T_Σ -coalgebras.

(2) Let γ^f be the unique T_Δ -coalgebra structure on Q for which all $c_i : (Q_i, \gamma_i)^f \rightarrow (Q, \gamma^f)$ are T_Δ -coalgebra homomorphism, see Remark A.2. Put

$$(Q, \gamma)^f := (Q, \gamma^f).$$

The following lemma summarizes some important properties of this construction.

Lemma A.4. *Let (Q, γ) be a locally finite T_Σ -coalgebra and $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ a \mathcal{D} -monoid morphism.*

(a) *The coalgebra structure of $(Q, \gamma)^f$ is independent of the choice of the colimit cocone (c_i) in Construction A.3.*

(b) *If $f = \Psi f_0^*$ for some $f_0 : \Delta \rightarrow \Sigma$, then $(Q, \gamma)^f$ has the coalgebra structure*

$$Q \xrightarrow{\gamma} O_{\mathcal{C}} \times Q^\Sigma \xrightarrow{\text{id} \times Q^{f_0}} O_{\mathcal{C}} \times Q^\Delta, \quad (2)$$

c.f. the definition of the rational functor Q^f (Definition II.24).
(c) *Every homomorphism $h : Q \rightarrow Q'$ of locally finite T_Σ -coalgebras is also a homomorphism $h : Q^f \rightarrow (Q')^f$ of T_Δ -coalgebras.*

(d) *For any \mathcal{D} -monoid morphism $g : \Psi\Gamma^* \rightarrow \Psi\Delta^*$,*

$$(Q^f)^g = Q^{f \circ g}.$$

(e) *The construction $(-)^f$ commutes with coproducts: given locally finite T_Σ -coalgebras Q_j ($j \in J$), we have*

$$\left(\coprod_j Q_j \right)^f = \coprod_j Q_j^f.$$

Proof. (a) Suppose γ^f has been defined by means of the cocone (c_i) , and another filtered colimit cocone

$$c'_j : (Q'_j, \gamma'_j) \rightarrow (Q, \gamma) \quad (j \in J)$$

with Q'_j finite is given. By Remark A.2 it suffices to show that the maps c'_j are T_Δ -coalgebra homomorphisms $c'_j : (Q'_j, \gamma'_j)^f \rightarrow (Q, \gamma^f)$.

Given $j \in J$, there exists by Remark A.1 a T_Σ -coalgebra homomorphism $g : (Q'_j, \gamma'_j) \rightarrow (Q_i, \gamma_i)$ with $c_i \circ g = c'_j$ for some i . It follows that c'_j is a T_Δ -coalgebra homomorphism, being the composite of the T_Δ -coalgebra homomorphisms

$$(Q'_j, \gamma'_j)^f \xrightarrow{g} (Q_i, \gamma_i)^f \xrightarrow{c_i} (Q, \gamma^f).$$

Indeed, g is a T_Δ -coalgebra homomorphism using the definition of Q^f for finite Q (Definition III.10) and Lemma III.7(b), and c_i is one by the definition of γ^f .

(b) Given an L_Σ -algebra (A, α) the L_Δ -algebra $(A, \alpha)^f$ (see Definition III.5) has the transitions $\alpha_a^f = \alpha_{f_0(a)}$ for $a \in \Delta$. Hence, for a finite T_Σ -coalgebra (Q, γ) , the L_Δ -algebra $\widehat{Q^f} = \widehat{Q}^{f^\dagger} = \widehat{Q}^f$ has transitions $\widehat{\gamma}_{f_0(a)} : \widehat{Q} \rightarrow \widehat{Q}$ for $a \in \Delta$. Dually Q^f has the transitions $\gamma_{f_0(a)} : Q \rightarrow Q$, which are precisely the transitions corresponding to the coalgebra structure (2).

In the case where (Q, γ) is just locally finite, express (Q, γ) as a filtered colimit $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$ ($i \in I$) of finite T_Σ -coalgebras, and consider the diagram below:

$$\begin{array}{ccccc} Q & \xrightarrow{\gamma} & O_{\mathcal{C}} \times Q^\Sigma & \xrightarrow{\text{id} \times Q^{f_0}} & O_{\mathcal{C}} \times Q^\Delta \\ \uparrow c_i & & \uparrow T_\Sigma c_i & & \uparrow T_\Delta c_i \\ Q_i & \xrightarrow{\gamma_i} & O_{\mathcal{C}} \times Q_i^\Sigma & \xrightarrow{\text{id} \times Q_i^{f_0}} & O_{\mathcal{C}} \times Q_i^\Delta \end{array}$$

This diagram commutes because c_i is a T_Σ -coalgebra homomorphism and by naturality. The lower row is the coalgebra structure of $(Q_i, \gamma_i)^f$ by the first part of the proof. Hence c_i is a T_Δ -coalgebra homomorphism from $(Q_i, \gamma_i)^f$ to the coalgebra in the upper row, which implies that the upper row defines the coalgebra structure of $(Q, \gamma)^f$.

(c) Express Q and Q' as filtered colimits $c_i : Q_i \rightarrow Q$ ($i \in I$) and $c'_j : Q'_j \rightarrow Q'$ ($j \in J$) of finite T_Σ -coalgebras. By Remark A.1 there exists for every $i \in I$ some T_Σ -coalgebra

homomorphism $g : Q_i \rightarrow Q'_j$ for which the diagram below commutes:

$$\begin{array}{ccc} Q & \xrightarrow{h} & Q' \\ \uparrow c_i & & \uparrow c'_j \\ Q_i & \xrightarrow{g} & Q'_j \end{array}$$

It follows that $h \cdot c_i : Q_i^f \rightarrow (Q')^f$ is a T_Δ -coalgebra homomorphism, being the composite of the T_Δ -coalgebra homomorphisms $Q_i^f \xrightarrow{g} (Q'_j)^f \xrightarrow{c'_j} (Q')^f$ (one uses the same argument as in point (a)). Since the morphisms c_i are jointly epimorphic in \mathcal{C} , it follows that h is a T_Δ -coalgebra homomorphism $h : Q^f \rightarrow (Q')^f$.

(d) (i) We first show that for all L_Σ -algebras (A, α) we have

$$(A^f)^g = A^{f \cdot g}.$$

Indeed, both algebras have states A and initial states α_{in} . To see that they have the same transitions, consider the diagram below, cf. Notation III.4 and Definition III.5.

$$\begin{array}{ccc} \Psi\Sigma^* & \xrightarrow{x \mapsto \alpha_x} & [A, A] \\ \uparrow f \cdot g & \swarrow f & \uparrow y \mapsto \alpha_y^f \\ \Psi\Gamma^* & \xrightarrow{g} & \Psi\Delta^* \end{array} \quad (3)$$

The upper triangle commutes by the definition of α^f . Hence, for all $a \in \Gamma$,

$$\begin{aligned} (\alpha^{f \cdot g})_a &= \alpha_{fg(a)} && \text{(def. } \alpha^{f \cdot g}) \\ &= (\alpha^f)_{ga} && \text{(diagram (3))} \\ &= ((\alpha^f)^g)_a && \text{def. } (\alpha^f)^g \end{aligned}$$

(ii) If Q is a finite T_Σ -coalgebra we conclude from (i):

$$\widehat{Q^{f \cdot g}} = \widehat{Q^{(f \cdot g)^\dagger}} = \widehat{Q^{f^\dagger \cdot g^\dagger}} = (\widehat{Q^{f^\dagger}})^{g^\dagger} = (\widehat{Q^f})^g,$$

so $Q^{f \cdot g} = (Q^f)^g$.

(iii) Now let Q be locally finite, and express Q as a filtered colimit $c_i : Q_i \rightarrow Q$ ($i \in I$) of finite T_Σ -coalgebras. Hence by (c) we have T_Γ -coalgebra homomorphisms $c_i : Q_i^{f \cdot g} \rightarrow Q^{f \cdot g}$ and $c_i : (Q_i^f)^g \rightarrow (Q^f)^g$, where $Q_i^{f \cdot g} = (Q_i^f)^g$ by (ii) above. It follows that $Q^{f \cdot g}$ and $(Q^f)^g$ have the same coalgebra structure.

(e) (i) We first prove that, for each family of L_Σ -algebras (A_j, α_j) ($j \in J$),

$$\left(\prod_j A_j \right)^f = \left(\prod_j A_j^f \right).$$

Clearly the algebras on both sides of the equation have the same states $\prod_j A_j$ and the same initial state. Concerning the transitions, consider the commutative diagram below:

$$\begin{array}{ccc} \Psi\Delta^* & \xrightarrow{f} & \Psi\Sigma^* \xrightarrow{a \mapsto \prod(\alpha_j)_a} \left[\prod A_j, \prod A_j \right] \\ & \searrow \langle a \mapsto (\alpha_j)_a \rangle & \downarrow (f_j) \mapsto \prod f_j \\ & & \prod [A_j, A_j] \end{array}$$

The upper and lower path define the transitions of $(\prod_j A_j)^f$ and $\prod_j A_j^f$, respectively. Hence they have the same transitions.

(ii) Suppose now that J is finite and finite T_Σ -coalgebras Q_j are given. Then we conclude from (i) and duality:

$$\left(\prod_j \widehat{Q_j} \right)^f = \left(\widehat{\prod_j Q_j} \right)^{f^\dagger} = \left(\prod_j \widehat{Q_j} \right)^{f^\dagger} = \prod_j \widehat{Q_j}^{f^\dagger} = \widehat{\prod_j Q_j^f}.$$

The statement for arbitrary J and locally finite coalgebras Q_j now follows from the fact that filtered colimits and coproducts commute in $\mathbf{Coalg} T_\Sigma$, and every infinite coproduct is a filtered colimit of finite ones. \square

Remark A.5. Recall the *homomorphism theorem* for coalgebras: if a T_Σ -coalgebra homomorphism $g : Q \rightarrow R$ factorizes in \mathcal{C} through a subcoalgebra $i : R' \hookrightarrow R$, then the factorizing morphism $g' : Q \rightarrow R'$ is a coalgebra homomorphism.

$$\begin{array}{ccc} & R & \\ g \nearrow & & \nwarrow i \\ Q & \xrightarrow{g'} & R' \end{array}$$

This follows easily from the observation that T_Σ preserves monomorphisms.

Remark A.6. Note that locally finite T_Σ -coalgebras are closed under subcoalgebras. Indeed, suppose that $i : Q' \hookrightarrow Q$ is a T_Σ -coalgebra homomorphism, where Q is locally finite. Now for every $q \in |Q'|$ we have a finite subcoalgebra Q_0 of Q containing q . Since T_Σ preserves intersections we can conclude that $Q_0 \cap Q'$ is a subcoalgebra of Q' containing q .

Lemma A.7. A family of subcoalgebras $m_\Sigma : V\Sigma \hookrightarrow \varrho T_\Sigma$ ($\Sigma \in \mathbf{Set}_f$) forms a subfunctor of ϱT iff, for every function $f_0 : \Delta \rightarrow \Sigma$ in \mathbf{Set}_f , a T_Σ -coalgebra homomorphism from $(V\Sigma)^f$ to $V\Delta$ exists, where $f = \Psi f_0^*$.

Proof. Suppose that the $V\Sigma$ form a subfunctor $V \hookrightarrow \varrho T$, and let $f_0 : \Delta \rightarrow \Sigma$. Then we have the commutative diagram below with $g = V f_0$:

$$\begin{array}{ccc} \varrho T_\Sigma & \xrightarrow{\varrho T_{f_0}} & \varrho T_\Delta \\ \uparrow m_\Sigma & & \uparrow m_\Delta \\ V\Sigma & \xrightarrow{g} & V\Delta \end{array}$$

By definition, ϱT_{f_0} is a T_Δ -coalgebra homomorphism $\varrho T_{f_0} : (\varrho T_\Sigma)^f \rightarrow \varrho T_\Delta$, and by Lemma A.4(b), m_Σ is a T_Δ -coalgebra homomorphism $m_\Sigma : (V\Sigma)^f \rightarrow (\varrho T_\Sigma)^f$. The homomorphism theorem then implies that g is a T_Δ -coalgebra homomorphism from $(V\Sigma)^f$ to $V\Delta$.

Conversely, given a T_Δ -coalgebra homomorphism $g : (V\Sigma)^f \rightarrow V\Delta$, the above square commutes because both $\varrho T_{f_0} \cdot m_\Sigma$ and $m_\Delta \cdot g$ are T_Δ -coalgebra homomorphisms from $(V\Sigma)^f$ into the terminal locally finite T_Δ -coalgebra ϱT_Δ . Note that $(V\Sigma)^f$ is locally finite because $V\Sigma$ is locally finite, being a subcoalgebra of the locally finite coalgebra ϱT_Σ , see Remark A.6. \square

Remark A.8. Dually to Remark A.5, we have the following homomorphism theorem for L_Σ -algebras: if an L_Σ -algebra homomorphism $f : A \rightarrow C$ factorizes in \mathcal{D} through an L_Σ -quotient algebra $e : A \twoheadrightarrow B$, then the factorizing morphism f' is an L_Σ -algebra homomorphism.

$$\begin{array}{ccc} & A & \\ e \swarrow & & \searrow f \\ B & \dashrightarrow & C \\ & f' & \end{array}$$

The analogous statement holds for \mathcal{D} -monoid morphisms. Since **Alg** L_Σ and **\mathcal{D} -Mon** are varieties of (ordered) algebras, this is a special case of the well-known homomorphism theorem of universal algebra.

Lemma A.9. Let $Q \twoheadrightarrow \varrho T_\Sigma$ and $Q' \twoheadrightarrow \varrho T_\Delta$ be finite local varieties of languages, and $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ a \mathcal{D} -monoid morphism. Then there exists a T_Δ -coalgebra homomorphism from $h : Q^f \rightarrow Q'$ iff there exists a \mathcal{D} -monoid morphism g making the following square commute:

$$\begin{array}{ccc} \Psi\Delta^* & \xrightarrow{f^\dagger} & \Psi\Sigma^* \\ e_{\widehat{Q}'} \downarrow & & \downarrow e_{\widehat{Q}} \\ \widehat{Q}' & \xrightarrow{g} & \widehat{Q} \end{array} \quad (4)$$

In this case, we have $g = \widehat{h}$.

Proof. Given a T_Δ -coalgebra homomorphism $h : Q^f \rightarrow Q'$ we have, by Lemma III.7, the following square of L_Δ -algebra homomorphisms

$$\begin{array}{ccc} \Psi\Delta^* & \xrightarrow{f^\dagger} & (\Psi\Sigma^*)^{f^\dagger} \\ e_{\widehat{Q}'} \downarrow & & \downarrow e_{\widehat{Q}} \\ \widehat{Q}' & \xrightarrow{g} & \widehat{Q}^{f^\dagger} \end{array} \quad (5)$$

where $g = \widehat{h}$. This diagram commutes because $\Psi\Delta^*$ is the initial L_Δ -algebra. Therefore the square (4) commutes. That g is a \mathcal{D} -monoid morphism follows from Remark A.8.

Conversely, given a morphism g for which (4) commutes, then $g : \widehat{Q}' \rightarrow \widehat{Q}^{f^\dagger} = \widehat{Q}^f$ is an L_Δ -algebra homomorphism by (5) and Remark A.8, so dually a T_Δ -coalgebra homomorphism $Q^f \rightarrow Q'$ exists. \square

Notation A.10. Let V be an object-finite variety of languages in \mathcal{C} .

(a) We denote by $e_\Sigma : \Psi\Sigma^* \twoheadrightarrow \widehat{V\Sigma}$ the Σ -generated \mathcal{D} -monoid corresponding to the local variety $V\Sigma$.

(b) Since V is closed under preimages, we have a (unique) T_Δ -coalgebra homomorphism $\overline{V}f : (V\Sigma)^f \rightarrow V\Delta$ for every \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$, see Theorem III.16. This notation is in good harmony with the functor notation for V : for a function $f_0 : \Delta \rightarrow \Sigma$ and the corresponding \mathcal{D} -monoid morphism $f = \Psi f_0^* : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ we have $Vf_0 = \overline{V}f$: indeed, both maps are T_Δ -coalgebra homomorphisms

from $(V\Sigma)^f$ to $V\Delta$ and for the T_Δ -coalgebra homomorphism $i : V\Delta \twoheadrightarrow \varrho T_\Delta$ we have that $i \cdot Vf_0 = i \cdot \overline{V}f : (V\Sigma)^f \rightarrow \varrho T_\Delta$ agree by the finality of ϱT_Δ ; now use that i is monomorphic to conclude $Vf_0 = \overline{V}f$.

(c) We denote by V^\circledast the class of all \mathcal{D} -monoids D such that every \mathcal{D} -monoid homomorphism $h : \Psi\Sigma^* \rightarrow D$, where Σ is any finite set, factorizes (necessarily uniquely) through e_Σ .

$$\begin{array}{ccc} \Psi\Sigma^* & & \\ e_\Sigma \downarrow & \searrow h & \\ \widehat{V\Sigma} & \dashrightarrow & D \end{array}$$

Proposition A.11. V^\circledast is a locally finite variety of \mathcal{D} -monoids whose free \mathcal{D} -monoid on Σ is $\widehat{V\Sigma}$ for every finite set Σ .

Proof. (1) We first verify that V^\circledast is a variety of \mathcal{D} -monoids.

(a) Closure under products: let $\pi_i : \prod_{i \in I} D_i \rightarrow D_i$ be a product of monoids $D_i \in V^\circledast$. Given a monoid morphism $h : \Psi\Sigma^* \rightarrow \prod_{i \in I} D_i$ every morphism $\pi_i \cdot h : \Psi\Sigma^* \rightarrow D_i$ factorizes as $\pi_i \cdot h = k_i \cdot e_\Sigma$ for some $k_i : \widehat{V\Sigma} \rightarrow D_i$. Hence $\langle k_i \rangle : \widehat{V\Sigma} \rightarrow \prod D_i$ is the desired morphism with $h = \langle k_i \rangle \cdot e_\Sigma$.

(b) Closure under submonoids: given $m : D \twoheadrightarrow D'$ with $D' \in V^\circledast$ and a monoid morphism $h : \Psi\Sigma^* \rightarrow D$, the homomorphism $m \cdot h$ factorizes through e_Σ in **\mathcal{D} -Mon**. Consequently h factorizes through e_Σ due to diagonal fill-in:

$$\begin{array}{ccc} \Psi\Sigma^* & \xrightarrow{e_\Sigma} & \widehat{V\Sigma} \\ h \downarrow & \swarrow & \downarrow k \\ D & \xrightarrow{m} & D' \end{array}$$

Thus $D \in V^\circledast$.

(c) Closure under quotients: given $e : D \twoheadrightarrow D'$ with $D \in V^\circledast$ and a homomorphism $h : \Psi\Sigma^* \rightarrow D'$, choose a splitting of e in **Set**, i.e., a function $u : |D'| \rightarrow |D|$ with $e \cdot u = \text{id}$. Let $\eta : \Sigma \twoheadrightarrow |\Psi\Sigma^*|$ denote the universal map of the free monoid $\Psi\Sigma^*$, and extend the map $u \cdot h \cdot \eta : \Sigma \rightarrow |D|$ to a homomorphism $k : \Psi\Sigma^* \rightarrow D$, which then factorizes as $k = k' \cdot e_\Sigma$ because D is in V^\circledast . Then the \mathcal{D} -monoid morphism $e \cdot k' : \widehat{V\Sigma} \rightarrow D'$ is the desired factorization of h .

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\eta} & \Psi\Sigma^* & \xrightarrow{e_\Sigma} & \widehat{V\Sigma} \\ & & \downarrow k & \swarrow h & \downarrow k' \\ & & D & \xrightarrow{e} & D' \\ & & & \swarrow u & \downarrow \\ & & & & D' \end{array}$$

Indeed, using freeness of the \mathcal{D} -monoid $\Psi\Sigma^*$ and since u is injective it suffices to prove that $u \cdot h \cdot \eta = u \cdot e \cdot k' \cdot e_\Sigma \cdot \eta$ in **Set**, which holds because

$$\begin{aligned} u \cdot h \cdot \eta &= u \cdot e \cdot u \cdot h \cdot \eta && (e \cdot u = \text{id}) \\ &= u \cdot e \cdot k \cdot \eta && (\text{def. } k) \\ &= u \cdot e \cdot k' \cdot e_\Sigma \cdot \eta && (\text{def. } k'). \end{aligned}$$

(2) The \mathcal{D} -monoid $\widehat{V\Delta}$ lies in V^\circledast for all finite Δ . Indeed, given a \mathcal{D} -monoid morphism $h : \Psi\Sigma^* \rightarrow \widehat{V\Delta}$ where Σ is

finite, we can choose a splitting in **Set** (a function $u : |\widehat{V\Delta}| \rightarrow |\Psi\Delta^*|$ with $e_\Delta \cdot u = \text{id}$), and extend $u \cdot h \cdot \eta$ to a \mathcal{D} -monoid morphism $f : \Psi\Sigma^* \rightarrow \Psi\Delta^*$. By Lemma A.9 there is a \mathcal{D} -monoid morphism $g : \widehat{V\Sigma} \rightarrow \widehat{V\Delta}$ such that $g \cdot e_\Sigma = e_\Delta \cdot f$.

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\eta} & \Psi\Sigma^* & \xrightarrow{f} & \Psi\Delta^* \\ & & \downarrow e_\Sigma & \searrow h & \uparrow u \\ & & \widehat{V\Sigma} & \xrightarrow{g} & \widehat{V\Delta} \\ & & & & \downarrow e_\Delta \end{array}$$

We claim that g is the desired factorization, i.e., $g \cdot e_\Sigma = h$. Using freeness of the \mathcal{D} -monoid $\Psi\Sigma^*$ and since u is injective it suffices to prove $u \cdot g \cdot e_\Sigma \cdot \eta = u \cdot h \cdot \eta$ in **Set**, and indeed we have

$$\begin{aligned} u \cdot g \cdot e_\Sigma \cdot \eta &= u \cdot e_\Delta \cdot f \cdot \eta && \text{(def. } g) \\ &= u \cdot e_\Delta \cdot u \cdot h \cdot \eta && \text{(def. } f) \\ &= u \cdot h \cdot \eta && (e_\Delta \cdot u = \text{id}). \end{aligned}$$

(3) From the definition of V^\circledast and (2) above we immediately conclude that $\widehat{V\Delta}$ is the free monoid on a finite set Δ in the variety V^\circledast . Hence, since $\widehat{V\Delta}$ is finite, V^\circledast is a locally finite variety of \mathcal{D} -monoids. \square

Proof of Theorem IV.2. The above map $V \mapsto V^\circledast$ defines the desired isomorphism. To see this, we describe its inverse $W \mapsto W^\square$. Let W be a locally finite variety of \mathcal{D} -monoids with free monoids $e_\Sigma : \Psi\Sigma^* \rightarrow D_\Sigma$ in W . Define an object-finite variety W^\square of languages in \mathcal{C} by forming, for each finite Σ , the dual local variety $W^\square_\Sigma \hookrightarrow \varrho T_\Sigma$ of e_Σ , i.e., $\widehat{W^\square_\Sigma} \cong D_\Sigma$. To verify that W^\square is a subfunctor of ϱT , consider a function $f_0 : \Delta \rightarrow \Sigma$ in **Set**, and the corresponding \mathcal{D} -monoid morphism $f = \Psi f_0^* : \Psi\Delta^* \rightarrow \Psi\Sigma^*$. Since D_Δ is the free monoid on Δ in W , we get a unique \mathcal{D} -monoid morphism $g : V_\Delta \rightarrow V_\Sigma$ with $g \cdot e_\Delta = e_\Sigma \cdot f$. By Lemma A.9 we dually get a T_Δ -coalgebra homomorphism $(W^\square_\Sigma)^f \rightarrow W^\square_\Delta$, which implies that W^\square is a subfunctor of ϱT by Lemma A.7.

From Proposition A.11 and the definition of $(-)^{\circledast}$ and $(-)^{\square}$ it is clear that $(V^\circledast)^\square = V$. To show that $(W^\square)^{\circledast} = W$, observe first that the varieties $(W^\square)^{\circledast}$ and W have by definition the same finitely generated free \mathcal{D} -monoids D_Σ , and hence contain the same finite \mathcal{D} -monoids. Moreover, both varieties are locally finite and hence form the closure (in the category $\mathcal{D}\text{-Mon}$) of their finite members under filtered colimits. It follows that $(W^\square)^{\circledast} = W$, as claimed.

We conclude that $V \mapsto V^\circledast$ defines a bijection between the lattices of object-finite varieties of languages (ordered by objectwise inclusion) and locally finite varieties of \mathcal{D} -monoids (ordered by inclusion). Moreover, clearly this bijection preserves and reflects the order, so it is a lattice isomorphism. \square

We now turn to the Eilenberg theorem for simple varieties. For the proof we need to extend the right-derivative construction of Notation II.20.

Notation A.12. Let (A, α) be an L_Δ -algebra and $e_A : \Psi\Delta^* \rightarrow A$ the initial homomorphism, see Remark II.16. For $x \in |\Psi\Delta^*|$

let $(A, \alpha)_x$ be the L_Δ -algebra with the same states A , the same transitions α_a , but initial state $e_A(x)$. For a finite T_Δ -coalgebra Q we define the T_Δ -coalgebra Q_x by

$$\widehat{Q}_x = \widehat{Q}_{\text{rev}_\Delta(x)}.$$

Lemma A.13. Let $x \in |\Psi\Delta^*|$.

(a) Every homomorphism $h : Q \rightarrow Q'$ between finite T_Δ -coalgebras is also a homomorphism $h : Q_x \rightarrow Q'_x$.

(b) If $Q \hookrightarrow \varrho T_\Delta$ is a finite local variety, then a T_Δ -coalgebra homomorphism from Q_x to Q exists for every $x \in \Psi\Delta^*$.

(c) For every finite T_Σ -coalgebra Q and \mathcal{D} -monoid morphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ we have

$$(Q^f)_x = (Q_{f_x})^f.$$

Proof. For (a) notice first that every homomorphism $h : (A, \alpha) \rightarrow (A', \alpha')$ of L_Δ -algebras yields a homomorphism $h : (A, \alpha)_x \rightarrow (A', \alpha')_x$ since by initiality of $\Psi\Delta^*$ we have $h \cdot e_A(x) = e_{A'}(x)$. Now given a homomorphism $h : Q \rightarrow Q'$ between finite T_Δ -coalgebras, we have its dual L_Δ -algebra homomorphism $\widehat{h} : \widehat{Q}' \rightarrow \widehat{Q}$. Hence, we have the L_Δ -algebra homomorphism $\widehat{h} : \widehat{Q}'_x = \widehat{Q}'_{\text{rev}_\Delta(x)} \rightarrow \widehat{Q}_{\text{rev}_\Delta(x)} = \widehat{Q}_x$ and by duality the desired T_Δ -coalgebra homomorphism $h : Q_x \rightarrow Q'_x$.

For (b) see [1, Prop. 4.31] and its proof. It remains to prove (c). We first prove that

$$(A^f)_x = (A_{f_x})^f \quad (6)$$

for all L_Σ -algebras $A = (A, \alpha)$ and \mathcal{D} -monoid morphisms $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$. Indeed, both $(A^f)_x$ and $(A_{f_x})^f$ have states A and transitions α_{f_a} for $a \in \Delta$. Moreover, the initial state of $(A^f)_x$ is $e_{A^f}(x)$, and the initial state of $(A_{f_x})^f$ is $e_A(fx)$. Hence, by Lemma III.7(c), $(A^f)_x$ and $(A_{f_x})^f$ have the same initial state.

Now let Q be a finite T_Σ -coalgebra. Then

$$\begin{aligned} (\widehat{Q^f})_x &= (\widehat{Q^f})_{\text{rev}_\Delta(x)} && \text{(def. } (-)_x) \\ &= (\widehat{Q^{f^\dagger}})_{\text{rev}_\Delta(x)} && \text{(def. } Q^f) \\ &= (\widehat{Q_{f^\dagger \cdot \text{rev}_\Delta(x)}})^{f^\dagger} && \text{(by (6))} \\ &= (\widehat{Q_{\text{rev}_\Sigma \cdot f(x)}})^{f^\dagger} && \text{(def } f^\dagger) \\ &= (\widehat{Q_{f_x}})^{f^\dagger} && \text{(def. } Q_{f_x}) \\ &= (\widehat{Q_{f_x}})^f && \text{(def. } (-)^f). \end{aligned}$$

Hence $(Q^f)_x = (Q_{f_x})^f$, as claimed. \square

Construction A.14. Let V be a variety of languages in \mathcal{C} and Σ a finite alphabet. Given a finite local variety $i : Q \hookrightarrow V\Sigma$ we define a subvariety $V' \hookrightarrow V$ as follows. To define $V'\Delta$ for all finite Δ , consider, for every \mathcal{D} -monoid homomorphism $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$, the T_Δ -coalgebra homomorphism $\overline{V}f : (V\Sigma)^f \rightarrow V\Delta$, see Notation A.10. Then factorize the coalgebra homomorphism $[\overline{V}f \cdot i] : \coprod_{f : \Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f \rightarrow V\Delta$ as in Remark II.8:

$$[\overline{V}f \cdot i] \equiv \coprod_{f : \Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f \xrightarrow{e_\Delta} V'\Delta \xrightarrow{m_\Delta} V\Delta. \quad (7)$$

Lemma A.15. V' is a subfunctor of V (via the m_Δ 's) and forms a simple variety with $Q \subseteq V'\Sigma$.

Proof. (1) For every finite alphabet Γ and every \mathcal{D} -monoid morphism $g : \Psi\Gamma^* \rightarrow \Psi\Delta^*$ we prove that there exists a T_Γ -coalgebra homomorphism

$$g' : (V'\Delta)^g \rightarrow V'\Gamma \quad \text{with} \quad m_\Gamma \cdot g' = \bar{V}g \cdot m_\Delta. \quad (8)$$

By Lemma A.7, this implies in particular that V' is a subfunctor of V . Denote by $p : \prod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^{f \cdot g} \rightarrow \prod_{h:\Psi\Gamma^* \rightarrow \Psi\Sigma^*} Q^h$ the T_Γ -coalgebra homomorphism whose f -component is the coproduct injection of $h = f \cdot g$. Note that $\prod_f Q^{f \cdot g} = (\prod_f Q^f)^g$ by Lemma A.4. Hence we have the following diagram of T_Γ -coalgebra homomorphisms:

$$\begin{array}{ccccc} (\prod_f Q^f)^g & \xrightarrow{e_\Delta} & (V'\Delta)^g & \xrightarrow{m_\Delta} & (V\Delta)^g \\ p \downarrow & & \downarrow g' & & \downarrow \bar{V}g \\ \prod_h Q^h & \xrightarrow{e_\Gamma} & V'\Gamma & \xrightarrow{m_\Gamma} & V\Gamma \end{array} \quad (9)$$

The outside of the diagram commutes because the f -components of the upper and lower path are T_Γ -coalgebra homomorphisms from the finite T_Γ -coalgebra $Q^{f \cdot g}$ to $V\Gamma \rightarrow \varrho T_\Gamma$, and ϱT_Γ is the terminal locally finite coalgebra. The desired T_Γ -coalgebra homomorphism g' is obtained via diagonal fill-in in $\mathbf{Coalg} T_\Gamma$, see Remark II.8.

(2) $V'\Delta$ is a local variety for every Δ . Indeed, by definition $V'\Delta$ it is a subcoalgebra of $V\Delta$ and hence of ϱT_Δ . To prove closure under right derivatives, use Proposition II.21: since $V\Delta$ and Q are local varieties, we have T_Δ -coalgebra homomorphisms $h_a : (V\Delta)_a \rightarrow V\Delta$ for all $a \in \Delta$ and T_Σ -coalgebra homomorphisms $k_x : Q_x \rightarrow Q$ for all $x \in \Psi\Sigma^*$, see Lemma A.13(b). Moreover,

$$\left(\prod_f Q^f\right)_a = \prod_f (Q^f)_a = \prod_f (Q_{f_a})^f$$

by Lemma A.13(c) and since the construction $(-)_a$ clearly commutes with coproducts. Hence we have the following diagram of T_Δ -coalgebra homomorphisms

$$\begin{array}{ccccc} \left(\prod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f\right)_a & \xrightarrow{e_\Delta} & (V'\Delta)_a & \xrightarrow{m_\Delta} & (V\Delta)_a \\ \prod_f k_{f_a} \downarrow & & \downarrow h'_a & & \downarrow h_a \\ \prod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f & \xrightarrow{e_\Delta} & V'\Delta & \xrightarrow{m_\Delta} & V\Delta \end{array}$$

whose outside commutes by a finality argument analogous to (1). Diagonal fill-in yields a T_Δ -coalgebra homomorphism $h'_a : (V'\Delta)_a \rightarrow V'\Delta$, which shows that $V'\Delta$ is closed under right derivatives by Proposition II.21.

(3) V' is a variety of languages. Indeed, apply Theorem III.16 to conclude from (8) that V' is closed under preimages. Moreover $Q \subseteq V'\Sigma$, due to the possibility of choosing $f = \text{id}_{\Psi\Sigma^*}$ in (7) for the case $\Delta = \Sigma$.

(4) V' is object-finite. Note first that for every finite alphabet Δ there exist only finitely many preimages Q^f , where $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ ranges over all \mathcal{D} -monoid morphisms: indeed, Q is finite and the coalgebra Q^f has the same set of states Q . Choose f_1, \dots, f_n such that each Q^f is equal to Q^{f_i} for some i . Then consider the T_Δ -coalgebra homomorphism $t : \prod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f \rightarrow \prod_{i=1}^n Q^{f_i}$ whose f -component is the coproduct injection of Q^{f_i} whenever $Q^f = Q^{f_i}$. Then $e_\Delta = u \cdot t$ for the obvious morphism $u : \prod_{i=1}^n Q^{f_i} \rightarrow V'\Delta$. Thus, u is surjective since e_Δ is, proving that $V'\Delta$ is finite (being a quotient of the finite coalgebra $\prod_{i=1}^n Q^{f_i}$).

(5) V' is simple. Indeed, given a variety V'' of languages with $j : Q \rightarrow V''\Sigma$ a local subvariety, we prove $V' \rightarrow V''$. Denote by $u'_\Delta : V'\Delta \rightarrow \varrho T_\Delta$ and $u''_\Delta : V''\Delta \rightarrow \varrho T_\Delta$ the embeddings and by $\bar{V}''f : (V''\Sigma)^f \rightarrow V''\Delta$ the T_Δ -coalgebra homomorphism of Theorem III.16. Then the square

$$\begin{array}{ccc} \prod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f & \xrightarrow{e_\Delta} & V'\Delta \\ \bar{V}''f \cdot j \downarrow & \swarrow & \downarrow u'_\Delta \\ V''\Delta & \xrightarrow{u''_\Delta} & \varrho T_\Delta \end{array}$$

commutes due to ϱT_Δ being the terminal locally finite T_Δ -coalgebra. Diagonal fill-in yields the desired embedding $V'\Delta \rightarrow V''\Delta$. \square

Remark A.16. For every locally finite variety W of \mathcal{D} -monoids the set W_f of all finite members of W is clearly a pseudovariety of \mathcal{D} -monoids. Conversely, for every pseudovariety W , we denote by $\langle W \rangle$ the variety generated by W , i.e., the closure of W under (infinite) products, submonoids and quotient monoids.

Lemma A.17. For every simple pseudovariety W of \mathcal{D} -monoids, the variety $\langle W \rangle$ is locally finite and $\langle W \rangle_f = W$. In particular, the simple pseudovarieties of \mathcal{D} -monoids form a (full) subposet of all locally finite varieties of \mathcal{D} -monoids via the order-embedding $W \mapsto \langle W \rangle$.

Proof. Suppose that the pseudovariety W is generated by the finite \mathcal{D} -monoid D . Then it is easy to see that also the variety $\langle W \rangle$ is generated by D , i.e., $\langle W \rangle = \langle D \rangle$. Fix a finite set Σ and consider all functions $u : \Sigma \rightarrow |D|$. They define a function $\langle u \rangle : \Sigma \rightarrow |D|^{|D|^\Sigma}$ that extends uniquely to a \mathcal{D} -monoid morphism $g : \Psi\Sigma^* \rightarrow D^{|D|^\Sigma}$. Letting $g = m \cdot e$ be its factorization in $\mathcal{D}\text{-Mon}$, we get the commutative diagram below:

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\eta} & \Psi\Sigma^* & \xrightarrow{e} & F\Sigma \\ u \downarrow & \searrow \langle u \rangle & \downarrow g & \swarrow m & \\ D & \xleftarrow{\pi_u} & D^{|D|^\Sigma} & & \end{array}$$

This shows that $F\Sigma$ (with universal map $e \cdot \eta$) is the free Σ -generated \mathcal{D} -monoid in $\langle W \rangle$: it has the universal property w.r.t. D by the above diagram, and hence it has the universal property w.r.t. all monoids in $\langle W \rangle = \langle D \rangle$. Moreover $F\Sigma$ lies

in W , being a submonoid of a finite power of D . This implies that every finite monoid in $\langle W \rangle$ lies in W , since it is a quotient of a finitely generated free monoid.

We conclude that $\langle W \rangle$ is locally finite and $\langle W \rangle_f = W$, and this equation implies immediately that $W \mapsto \langle W \rangle$ is an injective order-embedding \square

Proof of Theorem IV.4. Recall the isomorphism $V \mapsto V^\circledast$ between object-finite varieties of languages in \mathcal{C} and locally finite varieties of \mathcal{D} -monoids from the proof of Theorem IV.2. In view of Lemma A.17 it suffices to show that this isomorphism restricts to one between simple varieties of languages and simple pseudovarieties of \mathcal{D} -monoids, that is,

$$V \text{ is simple iff } (V^\circledast)_f \text{ is simple.}$$

(\Rightarrow) If V is a simple variety of languages, generated by Σ , we prove that the pseudovariety $(V^\circledast)_f$ is generated by the \mathcal{D} -monoid $\widehat{V\Sigma}$. First apply Construction A.14 to the finite local variety $Q = V\Sigma$. Then for the resulting variety V' we have $V' = V$. Indeed, $V' \subseteq V$ follows from Lemma A.15, and $V \subseteq V'$ holds because V is generated by Σ . It follows that for every Δ the morphism m_Δ in Construction A.14 is an isomorphism, or equivalently, the family of morphisms $\overline{V}f : (V\Sigma)^f \rightarrow V\Delta$, where f ranges over all \mathcal{D} -monoid morphisms $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ is collectively strongly epimorphic in \mathcal{C} . As in the proof of Lemma A.15 we choose finitely many homomorphisms $f_1, \dots, f_n : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ with $h = [\overline{V}f_i] : \prod_{i=1}^n (V\Sigma)^{f_i} \twoheadrightarrow V\Delta$ a strong epimorphism. Dually, by Lemma A.9 we get a \mathcal{D} -submonoid $\widehat{h} : \widehat{V\Delta} \hookrightarrow \prod_{i=1}^n \widehat{V\Sigma}$. We conclude that every finitely generated free monoid $V\Delta$ of the pseudovariety V^\circledast is a \mathcal{D} -submonoid of a finite power of $\widehat{V\Sigma}$. Consequently $(V^\circledast)_f$ is generated by $\widehat{V\Sigma}$.

(\Leftarrow) If V is an object-finite variety of languages such that $(V^\circledast)_f$ (and hence also V^\circledast) is generated by a single finite \mathcal{D} -monoid D , we prove that V is simple. Put $\Sigma = |D|$, then since D is a quotient of the free Σ -generated monoid $\widehat{V\Sigma}$ in V^\circledast , it follows that the pseudovariety $(V^\circledast)_f$ is also generated by $\widehat{V\Sigma}$. Thus, every \mathcal{D} -monoid in $(V^\circledast)_f$ is a quotient of a submonoid of a finite power $\widehat{V\Sigma}^n$. Consequently, every free algebra $\widehat{V\Delta}$ of V^\circledast is a submonoid of a finite power $\widehat{V\Sigma}^n$. (Indeed, given a quotient $e : D' \twoheadrightarrow \widehat{V\Delta}$ and a submonoid $i : D' \hookrightarrow \widehat{V\Sigma}^n$, choose a splitting $u : |\widehat{V\Delta}| \rightarrow |D'|$, $e \cdot u = \text{id}$, in **Set**. Since $\widehat{V\Delta}$ is free, we get a \mathcal{D} -monoid morphism $h : \widehat{V\Delta} \rightarrow D'$ which on the generators coincides with u . Then $e \cdot u = \text{id}$ implies $e \cdot h = \text{id}$, hence $i \cdot h : \widehat{V\Delta} \rightarrow \widehat{V\Sigma}^n$ is a submonoid.) Consequently, by composition with the projections $\widehat{V\Sigma}^n \rightarrow \widehat{V\Sigma}$ we obtain a collectively monic collection $g_1, \dots, g_n : \widehat{V\Delta} \rightarrow \widehat{V\Sigma}$ of \mathcal{D} -monoid morphisms. Choose \mathcal{D} -monoid morphisms $f_1, \dots, f_n : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ with $e_\Sigma \cdot f_i = g_i \cdot e_\Delta$ (by starting with a splitting $e_\Sigma \cdot v = \text{id}$ in **Set** and extending $v \cdot g_i \cdot e_\Delta \cdot \eta : \Delta \rightarrow |\Psi\Sigma^*|$ to a \mathcal{D} -monoid morphism). By Lemma A.9 we get a collection of T_Δ -coalgebra homomorphisms $h_i : (V\Sigma)^{f_i} \rightarrow V\Delta$ that is collectively strongly epic. Hence the corresponding homomorphism $[h_i] : \prod_{i=1}^n (V\Sigma)^{f_i} \twoheadrightarrow V\Delta$ is a strong

epimorphism in \mathcal{C} . We conclude that V is a simple variety generated by Σ : if V' is any variety such $j : V\Sigma \twoheadrightarrow V'\Sigma$ is a local subvariety, then j is a T_Δ -coalgebra homomorphism $j : (V\Sigma)^{f_i} \twoheadrightarrow (V'\Sigma)^{f_i}$ for every i by Lemma A.4. Thus we have the diagram of T_Δ -coalgebra homomorphisms

$$\begin{array}{ccc} \prod_i (V\Sigma)^{f_i} & \xrightarrow{[h_i]} & V\Delta \\ \prod j \downarrow & & \downarrow \\ \prod_i (V'\Sigma)^{f_i} & \longrightarrow & V'\Delta \longrightarrow \varrho T_\Delta \end{array}$$

where the morphism $\prod_i (V'\Sigma)^{f_i} \rightarrow V'\Delta$ exists by closure of V' under preimages. Diagonal fill-in shows that $V\Delta \twoheadrightarrow V'\Delta$. \square

Lemma A.18. *Let V be a simple variety of languages in \mathcal{C} . If V is generated by an alphabet Σ , then V is generated by any alphabet Δ with $|\Delta| \geq |\Sigma|$.*

Proof. Let V' be any variety with $V\Delta \subseteq V'\Delta$. Since V is generated by Σ , it suffices to show $V\Sigma \subseteq V'\Sigma$ – then $V \subseteq V'$ follows. Observe first that there exist \mathcal{D} -monoid morphisms $e : \Psi\Delta^* \twoheadrightarrow \Psi\Sigma^*$ and $m : \Psi\Sigma^* \hookrightarrow \Psi\Delta^*$ with $e \cdot m = \text{id}$. Indeed, if $\Sigma \neq \emptyset$, choose functions $m_0 : \Sigma \hookrightarrow \Delta$ and $e_0 : \Delta \rightarrow \Sigma$ with $e_0 \cdot m_0 = \text{id}$ in **Set** and put $e = \Psi e_0$ and $m = \Psi m_0$. If $\Sigma = \emptyset$, consider the two (unique) monoid morphisms $m' : \emptyset^* \rightarrow \Delta^*$ and $e' : \Delta^* \rightarrow \emptyset^*$ (satisfying $e' \cdot m' = \text{id}$), and put $e = \Psi e'$ and $m = \Psi m'$. It is easy to see that e and m are indeed \mathcal{D} -monoid morphisms.

Now let $L \in V\Sigma$. By closure of V under preimages we have $L \cdot e \in V\Delta \subseteq V'\Delta$. Since V' is also closed under preimages, we conclude $L = L \cdot e \cdot m \in V'\Sigma$ and thus $V\Sigma \subseteq V'\Sigma$. \square

Proof of Theorem IV.5. (1) Let $\mathcal{L}_\mathcal{C}$ denote the poset of all varieties of languages in \mathcal{C} and $\mathcal{L}_\mathcal{C}^0$ its subposet of all simple ones. We prove that $\mathcal{L}_\mathcal{C}$ is the free cpo-completion of $\mathcal{L}_\mathcal{C}^0$. Note first that $\mathcal{L}_\mathcal{C}$ is a complete lattice (in particular, a cpo) because an objectwise intersection of varieties of languages V_i ($i \in I$) is a variety V . Indeed, the functor $T_\Sigma Q = O_\mathcal{C} \times Q^\Sigma$ clearly preserves (wide) intersections, thus an intersection of subcoalgebras of ϱT_Σ in \mathcal{C} is again a subcoalgebra. And since \mathcal{C} is a variety of algebras, intersections in \mathcal{C} are formed on the level of **Set**. Now, from $V\Sigma = \bigcap_{i \in I} V_i\Sigma$ it clearly follows that $V\Sigma$ is closed under derivatives. And closure under preimages is also clear: given L in $|V\Sigma|$ and $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ in $\mathcal{D}\text{-Mon}$, we have $L \cdot f$ in $|V_i\Delta|$ for all i , thus $L \cdot f \in |V\Delta|$.

Observe that also an objectwise directed union of varieties is a variety. The argument is the same: since \mathcal{C} is a variety of algebras, directed unions are formed on the level of **Set**.

It remains to verify the conditions (C1) and (C2) of a free cpo-completion.

(C1) Every simple variety V is compact in $\mathcal{L}_\mathcal{C}$. Indeed, suppose that V is generated by Σ , and let $V' = \bigcup_{i \in I} V_i$ be a directed union with $V \subseteq V'$. Then $V\Sigma$ a local subvariety of $V'\Sigma$, and since $V\Sigma$ is finite and $|V'\Sigma| = \bigcup_{i \in I} |V_i\Sigma|$ in **Set**, there exists i with $V\Sigma$ a local subvariety of $V_i\Sigma$. Therefore $V \subseteq V_i$ because V is simple.

(C2) Every variety V of languages is the directed join (i.e. directed union) of its simple subvarieties. To show that the set of all simple subvarieties of V is indeed directed, suppose that two simple subvarieties V_0 and V_1 of V are given. By Lemma A.18 we can assume that both varieties are generated by the same alphabet Σ . In [1, Corollary 4.5] we proved that any finite subset of a local variety $Q' \rightsquigarrow \varrho T_\Sigma$ is contained in a finite local subvariety $Q \rightsquigarrow Q'$. Letting $Q' = V\Sigma$, this implies that $V\Sigma$ has a finite local subvariety Q containing $V_0\Sigma \cup V_1\Sigma$. Now apply Construction A.14 to Q to get a simple variety $V' \subseteq V$ containing V_0 and V_1 .

Finally, by [1, Corollary 4.5] and Construction A.14 again, every language in V is contained in a simple subvariety of V . Hence V is the desired directed union.

(2) Let $\mathcal{L}_\mathcal{D}$ denote the poset of all pseudovarieties of \mathcal{D} -monoids and $\mathcal{L}_\mathcal{D}^0$ its subposet of all simple ones. We again prove that $\mathcal{L}_\mathcal{D}$ is a free cpo-completion of $\mathcal{L}_\mathcal{D}^0$. First, $\mathcal{L}_\mathcal{D}$ is a complete lattice (in particular, a cpo) because an intersection of pseudovarieties is a pseudovariety. Observe that also a directed union of pseudovarieties is a pseudovariety. It remains to verify (C1) and (C2).

(C1) Every simple pseudovariety W , generated by a finite \mathcal{D} -monoid D , is compact in $\mathcal{L}_\mathcal{D}$. Indeed, if a directed join (i.e. directed union) of pseudovarieties $\bigcup_{i \in I} W_i$ contains W then some W_i contains D and hence W .

(C2) Every pseudovariety W of \mathcal{D} -monoids is a directed union of simple ones. Indeed, clearly W is the union of its simple subvarieties, and this union is directed because two simple subvarieties $W_1, W_2 \subseteq W$ (generated by D_1 and D_2 , respectively) are contained in the simple subvariety of W generated by $D_1 \times D_2$.

(3) From Theorem IV.4 we know that $\mathcal{L}_\mathcal{E}^0 \cong \mathcal{L}_\mathcal{D}^0$. Since $\mathcal{L}_\mathcal{E}$ is the free cpo-completion of $\mathcal{L}_\mathcal{E}^0$ by (1), and $\mathcal{L}_\mathcal{D}$ is the free cpo-completion of $\mathcal{L}_\mathcal{D}^0$ by (2), the uniqueness of completions gives $\mathcal{L}_\mathcal{E} \cong \mathcal{L}_\mathcal{D}$. \square

APPENDIX B PREDUALITIES

In this appendix we prove in detail the preduality of \mathbf{JSL}_{01} and \mathbf{JSL} (the basis of Poláks original Eilenberg-type correspondence [20]) and the preduality of \mathbf{BR} and \mathbf{Set}_* (the basis of our new Eilenberg-type correspondence).

Theorem B.1. \mathbf{JSL}_{01} and \mathbf{JSL} are predual.

Proof. The desired dual equivalence $\widehat{(-)} : (\mathbf{JSL}_{01})_f^{op} \xrightarrow{\cong} \mathbf{JSL}_f$ is defined on objects by

$$Q = (Q, \vee, 0, 1) \mapsto \widehat{Q} = (Q \setminus \{1\}, \wedge)$$

and on morphisms $h : Q \rightarrow R$ by

$$\widehat{h} : \widehat{R} \rightarrow \widehat{Q}, \quad \widehat{hr} = \bigvee_{hq \leq r} q,$$

where q ranges over Q . Here and in the following, the symbols $\vee, \wedge, \leq, 0$ and 1 are always meant with respect to the order of Q or R . We need to verify that $\widehat{(-)}$ is (a) a well-defined

functor, (b) essentially surjective, (c) faithful and (d) full.

(a1) \widehat{h} is well-defined as a function, that is, it maps the set $\widehat{R} = R \setminus \{1\}$ to $\widehat{Q} = Q \setminus \{1\}$. Indeed, we have for all $r \in R \setminus \{1\}$:

$$h(\widehat{hr}) = h\left(\bigvee_{hq \leq r} q\right) = \bigvee_{hq \leq r} hq \leq r \quad (*)$$

which implies $\widehat{hr} \neq 1$ because $h1 = 1$.

(a2) \widehat{h} is a **JSL**-morphism, that is, $\widehat{h}(r \wedge r') = \widehat{hr} \wedge \widehat{hr}'$ holds for all $r, r' \in R \setminus \{1\}$. Here “ \leq ” follows from the (obvious) monotonicity of h . For the converse we compute

$$h(\widehat{hr} \wedge \widehat{hr}') \leq h(\widehat{hr}) \wedge h(\widehat{hr}') \leq r \wedge r'.$$

The first inequality uses monotonicity of h and the second one uses (*). Hence $\widehat{hr} \wedge \widehat{hr}'$ is among the elements q in the join $\widehat{h}(r \wedge r') = \bigvee_{hq \leq r \wedge r'} q$, which implies $\widehat{hr} \wedge \widehat{hr}' \leq \widehat{h}(r \wedge r')$.

(a3) The assignment $h \mapsto \widehat{h}$ trivially preserves identity morphisms. For preservation of composition we consider $h : Q \rightarrow R$ and $k : R \rightarrow S$ in $(\mathbf{JSL}_{01})_f$ and show $\widehat{kh}(s) = \widehat{h} \cdot \widehat{k}(s)$ for all $s \in S \setminus \{1\}$, i.e.,

$$\bigvee_{kh(q) \leq s} q = \bigvee_{hq \leq \widehat{k}s} q.$$

This equation holds because, for all $q \in Q$, the inequalities $kh(q) \leq s$ and $hq \leq \widehat{k}s$ are equivalent. Indeed, if $kh(q) \leq s$ then $hq \leq \bigvee_{kr \leq s} r = \widehat{k}s$. Conversely, if $hq \leq \widehat{k}s$ then

$$k(hq) \leq k(\widehat{k}s) = k\left(\bigvee_{kr \leq s} r\right) = \bigvee_{kr \leq s} kr \leq s,$$

using that k is monotone and preserves joins.

(b) On the level of posets the construction $Q \mapsto \widehat{Q}$ first removes the top element and then reverses the order. Conversely, we can turn any semilattice² P in \mathbf{JSL}_f to a semilattice \overline{P} in $(\mathbf{JSL}_{01})_f$ by first adding a new bottom element and then reversing the order. Up to isomorphism these two constructions are clearly mutually inverse, so $\overline{\widehat{P}} \cong P$ for all P . This proves that $\widehat{(-)}$ is essentially surjective.

(c) Given $h : Q \rightarrow R$ in $(\mathbf{JSL}_{01})_f$ we claim that, for all $q \in Q$,

$$hq = \bigwedge_{q \leq \widehat{hr}} r \quad (**)$$

where r ranges over $R \setminus \{1\}$. This immediately implies that $\widehat{(-)}$ is faithful. To show “ \leq ” let $r \in R \setminus \{1\}$ with $q \leq \widehat{hr}$. Since h is monotone and preserves joins, we have

$$hq \leq h(\widehat{hr}) = h\left(\bigvee_{hq' \leq r} q'\right) = \bigvee_{hq' \leq r} hq' \leq r.$$

²Note that any nonempty finite semilattice necessarily has a top element, namely the (finite) join of all of its elements.

For the converse note that

$$q \leq \bigvee_{hq' \leq hq} q' = \widehat{h}(hq).$$

Hence hq is one of the elements r occurring in the meet (**), which means that this meet is $\leq hq$.

(d) Given $g : \widehat{R} \rightarrow \widehat{Q}$ in \mathbf{JSL}_f we need to find $h : Q \rightarrow R$ in $(\mathbf{JSL}_{01})_f$ with $g = \widehat{h}$. First extend $g : R \setminus \{1\} \rightarrow Q \setminus \{1\}$ to a map $\bar{g} : R \rightarrow Q$ by setting $\bar{g}1 = 1$. Then \bar{g} preserves all meets of R because g preserves all non-empty meets of $R \setminus \{1\}$. Inspired by (c) we define

$$hq = \bigwedge_{q \leq \bar{g}r} r,$$

where r ranges over R . Let us show that h indeed defines a \mathbf{JSL}_{01} -morphism. First, h preserves 0 and 1 because

$$h0 = \bigwedge_{0 \leq \bar{g}r} r = \bigwedge_r r = 0,$$

and

$$h1 = \bigwedge_{1 \leq \bar{g}r} r = 1$$

In the last equation we use that $\bar{g}1 = 1$, and no other element of R is mapped to 1 by \bar{g} since the codomain of g is $Q \setminus \{1\}$. For preservation of joins, $h(q \vee q') = hq \vee hq'$ for all $q, q' \in Q$, first note that “ \geq ” follows from the (obvious) monotonicity of h . For the other direction we compute

$$q \leq \bigwedge_{q \leq \bar{g}r} \bar{g}r = \bar{g}(\bigwedge_{q \leq \bar{g}r} r) = \bar{g}(hq)$$

and analogously $q' \leq \bar{g}(hq')$. Hence

$$q \vee q' \leq \bar{g}(hq) \vee \bar{g}(hq') \leq \bar{g}(hq \vee hq').$$

The last step uses the monotonicity of \bar{g} . So $hq \vee hq'$ is among the elements r in the meet defining $h(q \vee q') = \bigwedge_{q \vee q' \leq \bar{g}r} r$, which implies $h(q \vee q') \leq hq \vee hq'$.

Finally, we prove $g = \widehat{h}$, i.e.,

$$gr = \bigvee_{hq \leq r} q$$

for all $r \in R \setminus \{1\}$. To show “ \geq ” take any $q \in Q$ satisfying $hq \leq r$. Then

$$q \leq \bigwedge_{q \leq \bar{g}r'} \bar{g}r' = \bar{g}(\bigwedge_{q \leq \bar{g}r'} r') = \bar{g}(hq) \leq \bar{g}r = gr.$$

For “ \leq ” note first that

$$h(gr) = \bigwedge_{gr \leq \bar{g}r'} r' \leq r.$$

The last step uses that r is one of the elements r' over which the meet is taken. Hence gr is one of the elements q in the join $\bigvee_{hq \leq r} q$, so $gr \leq \bigvee_{hq \leq r} q$. \square

Remark B.2. (a) Every non-unital boolean ring is a distributive lattice with 0 where the meet is multiplication and the join is $x \vee$

$y = x + y + x \cdot y$. Hence every *finite* non-unital boolean ring has a unit 1, the join of all its elements. However, homomorphisms between finite non-unital boolean rings need not preserve the unit.

(b) The category \mathbf{UBR} of *unital* boolean rings and unit-preserving ring homomorphisms is isomorphic to the category \mathbf{BA} of boolean algebras (and hence preduel to \mathbf{Set}). Recall that under this isomorphism $+$ corresponds to exclusive disjunction and \cdot to conjunction.

Theorem B.3. \mathbf{BR} and \mathbf{Set}_* are preduel.

Proof. Recall that $(\mathbf{Set}_*)_f$ is equivalent to the Kleisli category of the monad $Z \mapsto Z + 1$ on \mathbf{Set}_f . The dual comonad on $\mathbf{UBR}_f \simeq \mathbf{Set}_f^{op}$ is $MX = X \times \mathbf{2}$ (where $\mathbf{2} = \{0, 1\}$ is the two-element boolean ring) with counit

$$X \times \mathbf{2} \xrightarrow{\varepsilon_X} X, \quad \varepsilon_X(x, b) = x,$$

and comultiplication

$$X \times \mathbf{2} \xrightarrow{\delta_X} X \times \mathbf{2} \times \mathbf{2}, \quad \delta_X(x, b) = (x, b, b),$$

and it suffices to show that the Co-Kleisli category $\mathbf{Kl}(M)$ of this comonad is isomorphic to \mathbf{BR}_f . The desired isomorphism $I : \mathbf{Kl}(M) \xrightarrow{\cong} \mathbf{BR}_f$ is identity on objects and takes a Kleisli morphism $f : X \times \mathbf{2} \rightarrow Y$ to the \mathbf{BR}_f -morphism

$$If : X \rightarrow Y, \quad If(x) = f(x, 0).$$

It remains to verify that I is a well-defined full and faithful functor.

(1) If is clearly a \mathbf{BR}_f -morphism since $0 + 0 = 0$ and $0 \cdot 0 = 0$.

(2) I preserves identities: the identity morphism of $X \in \mathbf{Kl}(M)$ is ε_X , and

$$I\varepsilon_X(x) = \varepsilon_X(x, 0) = x = \text{id}_X(x).$$

(3) I preserves composition: the composition of Kleisli morphisms $f : X \times \mathbf{2} \rightarrow Y$ and $g : Y \times \mathbf{2} \rightarrow Z$ is $g \bullet f : X \times \mathbf{2} \rightarrow Z$ where

$$g \bullet f(x, b) = g \circ Mf \circ \delta_X(x, b) = g(f(x, b), b).$$

Therefore

$$\begin{aligned} I(g \bullet f)(x) &= g \bullet f(x, 0) \\ &= g(f(x, 0), 0) \\ &= g(If(x), 0) \\ &= Ig(If(x)) \\ &= Ig \circ If(x) \end{aligned}$$

(4) I is faithful: let $f, g : X \times \mathbf{2} \rightarrow Y$ be Kleisli morphisms with $If = Ig$, i.e., $f(x, 0) = g(x, 0)$ for all x . Then

$$\begin{aligned} f(x, 1) &= f(x, 0) + f(1, 1) + f(1, 0) \\ &= f(x, 0) + 1 + f(1, 0) \\ &= g(x, 0) + g(1, 1) + g(1, 0) \\ &= g(x, 1) \end{aligned}$$

which implies $f = g$.

(5) I is full: let $f : X \rightarrow Y$ be a \mathbf{BR}_f -morphism. We claim that the map

$$g : X \times \mathbf{2} \rightarrow Y, \quad g(x, b) = f(x) + b + b \cdot f(1),$$

is a \mathbf{UBR}_f -morphism with $Ig = f$. First, the equation $Ig = f$ clearly holds:

$$\begin{aligned} Ig(x) &= g(x, 0) \\ &= f(x) + 0 + 0 \cdot f(1) \\ &= f(x). \end{aligned}$$

To show that g is a \mathbf{UBR}_f -morphism we compute

$$g(0, 0) = f(0) + 0 + 0 \cdot f(1) = 0 + 0 + 0 = 0$$

and

$$\begin{aligned} g(1, 1) &= f(1) + 1 + 1 \cdot f(1) \\ &= f(1) + f(1) + 1 = 0 + 1 = 1. \end{aligned}$$

Further,

$$\begin{aligned} g(x, b) + g(x', b') &= (f(x) + b + b \cdot f(1)) + (f(x') + b' + b' \cdot f(1)) \\ &= f(x + x') + b + b' + (b + b') \cdot f(1) \\ &= g(x + x', b + b') \end{aligned}$$

and

$$\begin{aligned} g(x, b) \cdot g(x', b') &= (fx + b + b \cdot f(1)) \cdot (fx' + b' + b' \cdot f(1)) \\ &= fx \cdot fx' + (fx \cdot b' + fx \cdot b' \cdot f(1)) \\ &\quad + (b \cdot fx' + b \cdot fx' \cdot f(1)) \\ &\quad + (b \cdot b' \cdot f(1) + b \cdot b' \cdot f(1) \cdot f(1)) \\ &\quad + b \cdot b' + b \cdot b' \cdot f(1) \\ &= fx \cdot fx' + (fx \cdot b' + fx \cdot b') + (b \cdot fx' + b \cdot fx') \\ &\quad + (b \cdot b' \cdot f(1) + b \cdot b' \cdot f(1)) + b \cdot b' + b \cdot b' \cdot f(1) \\ &= f(x \cdot x') + b \cdot b' + b \cdot b' \cdot f(1) \\ &= g(x \cdot x', b \cdot b'). \end{aligned}$$

In the third step we use $fx \cdot f(1) = fx$, $fx' \cdot f(1) = fx'$ and $f(1) \cdot f(1) = f(1)$, and in the fourth step the equation $u + u = 0$. \square

Remark B.4. By composing the equivalences

$$\mathbf{BR}_f \simeq \mathbf{Kl}(M) \simeq (\mathbf{Kl}(Z \mapsto Z + 1))^{op} \simeq (\mathbf{Set}_*)_f^{op}$$

of the above proof, one obtains the explicit description of the preduality between \mathbf{BR} and \mathbf{Set}_* in Example II.5(e).