Generalized Eilenberg Theorem I: Local Varieties of Languages

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Abstract. We investigate the duality between algebraic and coalgebraic recognition of languages to derive a generalization of the local version of Eilenberg's theorem. This theorem states that the lattice of all boolean algebras of regular languages over an alphabet Σ closed under derivatives is isomorphic to the lattice of all pseudovarieties of Σ-generated monoids. By applying our method to different categories, we obtain three related results: one, due to Gehrke, Grigorieff and Pin, weakens boolean algebras to distributive lattices, one due to Polák weakens them to join-semilattices, and the last one considers vector spaces over $\mathbb{Z}_2$.

1 Introduction

Regular languages are precisely the behaviours of finite automata. A machine-independent characterization of regularity is the starting point of algebraic automata theory (see e.g. [10]): one defines recognition via preimages of monoid morphisms $f : \Sigma^* \to M$, where M is a finite monoid, and every regular language is recognized in this way by its syntactic monoid. A key result in this field is Eilenberg's variety theorem, which establishes a lattice isomorphism

$$\text{varieties of regular languages} \cong \text{pseudovarieties of monoids.}$$

Here a *variety of regular languages* is a family of sets $V_\Sigma \subseteq \mathsf{Reg}_\Sigma$, where Σ ranges over all finite alphabets and Reg_Σ are the regular languages over Σ, such that each V_Σ is closed under left and right derivatives[1] and boolean operations (union, intersection and complement), and moreover $\bigcup_\Sigma V_\Sigma$ is closed under preimages of monoid homomorphisms $\Sigma^* \to \Gamma^*$. And a *pseudovariety of monoids* is a set of finite monoids closed under finite products, submonoids and quotients (homomorphic images).

Recently Gehrke, Grigorieff and Pin [6, 7] proved a "local" version of Eilenberg's theorem where one works with a *fixed* alphabet Σ: there is a lattice isomorphism between *local varieties of regular languages* (sets of regular languages over Σ closed

[1] Recall that the left and right derivatives of a language $L \subseteq \Sigma^*$ are the languages $w^{-1}L = \{u : wu \in L\}$ and $Lw^{-1} = \{u : uw \in L\}$ for $w \in \Sigma^*$, respectively.

under boolean operations and derivatives) and *local pseudovarieties of monoids* (sets of Σ-generated finite monoids closed under quotients and subdirect products). At the heart of this result lies the use of Stone duality to relate the boolean algebra Reg_Σ, equipped with left and right derivatives, to the free Σ-generated profinite monoid.

In this paper we generalize the local Eilenberg theorem to the level of an abstract duality of categories. Our approach is based on the observation that deterministic automata are coalgebras for the functor $T_\Sigma X = \mathbb{2} \times X^\Sigma$ on sets, and that Reg_Σ can be captured categorically as the *rational fixpoint* ϱT_Σ of T_Σ, i.e., the terminal locally finite T_Σ-coalgebra [9]. The rational fixpoint ϱT exists more generally for every finitary endofunctor T on a locally finitely presentable category $\mathcal{C}$ [1]. In this paper we work with such a category $\mathcal{C}$ and its dual $\hat{\mathcal{D}} \cong \mathcal{C}^{op}$. The functor $T_\Sigma X = \mathbb{2} \times X^\Sigma$ on $\mathcal{C}$ (where $\mathbb{2}$ is a fixed $\mathcal{C}$-object) has the dual endofunctor $\hat{L}_\Sigma X = \mathbb{1} + \coprod_\Sigma X$ on $\hat{\mathcal{D}}$ (where $\mathbb{1}$ is dual to $\mathbb{2}$), so that T_Σ-coalgebras correspond to $\hat{L}_\Sigma$-algebras. This already gives an equivalent description of (possibly infinite) automata as algebras. However, we are mainly interested in *finite* automata, and so we will work with another category $\mathcal{D}$ – a "finitary approximation" of $\hat{\mathcal{D}}$ – and an endofunctor L_Σ on $\mathcal{D}$ induced by $\hat{L}_\Sigma$. Finite automata are then modeled either as T_Σ-coalgebras or L_Σ-algebras with finitely presentable carrier, shortly *fp-(co)algebras*. As a first approximation to the local Eilenberg theorem, we establish a lattice isomorphism

subcoalgebras of ϱT_Σ $\cong$ ideal completion of the poset of fp-quotient algebras of μL_Σ

where μL_Σ is L_Σ's initial algebra. This is "almost" the desired general local Eilenberg theorem. For the classical case one takes Stone duality ($\mathcal{C}$ = boolean algebras, $\hat{\mathcal{D}}$ = Stone spaces). Then $\mathcal{D}$ = sets, ϱT_Σ is the boolean algebra Reg_Σ, $L_\Sigma = 1 + \coprod_\Sigma \mathsf{Id}$ on sets and $\mu L_\Sigma = \Sigma^*$. The above isomorphism states that the boolean subalgebras of Reg_Σ closed under *left* derivatives correspond to sets of finite quotient algebras of Σ^* closed under quotients and subdirect products. What is missing is the closure under *right* derivatives on the coalgebra side, and quotient algebras of Σ^* which are *monoids* on the algebra side.

The final step is to prove that the above isomorphism restricts to one between local varieties of regular languages (= subcoalgebras of ϱT_Σ closed under right derivatives) and local pseudovarieties of monoids. For this purpose we introduce the concept of a bimonoid. If $\mathcal{D}$ is a concrete category with forgetful functor $|\cdot| : \mathcal{D} \to \mathsf{Set}$, then a *bimonoid* is a $\mathcal{D}$-object A equipped with a "bilinear" monoid multiplication $\circ$ on $|A|$, which means that the maps $a \circ -$ and $- \circ a$ carry $\mathcal{D}$-morphisms for all $a \in |A|$. For example, bimonoids in $\mathcal{D}$ = sets, posets, join-semilattices and vector spaces over $\mathbb{Z}_2$ are monoids, ordered monoids, idempotent semirings and $\mathbb{Z}_2$-algebras (in the sense of algebras over a field), respectively. Our General Local Eilenberg Theorem (Theorem 5.19) holds on this level of generality: if $\mathcal{C}$ and $\mathcal{D}$ are concrete categories satisfying some natural properties, there is a lattice isomorphism

local varieties of regular languages in $\mathcal{C}$ $\cong$ local pseudovarieties of bimonoids in $\mathcal{D}$.

This is the main result of our paper. By instantiating it to Stone duality ($\mathcal{C}$ = boolean algebras, $\hat{\mathcal{D}}$ = Stone spaces, $\mathcal{D}$ = sets) we recover the "classical" local Eilenberg theorem. Priestley duality ($\mathcal{C}$ = distributive lattices, $\hat{\mathcal{D}}$ = Priestley spaces, $\mathcal{D}$ = posets) gives another result of Gehrke et. al, namely a lattice isomorphism between *local lattice varieties of regular languages* (subsets of Reg_Σ closed under union, intersection and derivatives) and local pseudovarieties of ordered monoids. Finally, by taking $\mathcal{C}$ = join-semilattices and $\mathcal{C}$ = vector spaces over $\mathbb{Z}_2$, we obtain two new local Eilenberg theorems. The first one establishes a lattice isomorphism between *local semilattice varieties of regular languages* (subsets of Reg_Σ closed under union and derivatives) and local pseudovarieties of idempotent semirings, and the second one gives an isomorphism between *local linear varieties of regular languages* (subsets of Reg_Σ closed under symmetric difference and derivatives) and local pseudovarieties of $\mathbb{Z}_2$-algebras.

Related work. Our paper is inspired by the work of Gehrke, Grigorieff and Pin [6] who showed that the algebraic operation of the free profinite monoid on Σ dualizes to the derivative operations on the boolean algebra of regular languages (and similarly for the free ordered profinite monoid on Σ). Previously, the duality between the boolean algebra of regular languages and the Stone space of profinite words appeared (implicitly) in work by Almeida [3] and was formulated by Pippenger [11] in terms of Stone duality.

A categorical approach to the duality theory of regular languages has been suggested by Rhodes and Steinberg [14]. They introduce the notion of a boolean bialgebra, which is conceptually rather different from our bimonoids, and prove the equivalence of bialgebras and profinite semigroups. The precise connection to their work is yet to be understood.

Another related work is Polák [12]. He considered what we treat as the example of join-semilattices and obtained a (non-local) Eilenberg type theorem in this case. To the best of our knowledge the local version we prove does not follow from the global version, and so we believe that our result is new.

The origin of all the above work is, of course, Eilenberg's theorem [4]. Later Reiterman [13] proved another characterization of pseudovarieties of monoids in the spirit of Birkhoff's classical variety theorem. Reiterman's theorem states that any pseudovariety of monoids can be characterized by profinite equations (i.e., pairs of elements of a free profinite monoid). We do not treat profinite equations in the present paper.

2 The Rational Fixpoint

The aim of this section is to recall the rational fixpoint of a functor, which provides a coalgebraic view of the set of regular languages. As a prerequisite, we need a categorical notion of "finite automaton", and so we will work with categories where "finite" objects exist and are well-behaved – viz. *locally finitely presentable* categories [2].

Definition 2.1. (a) An object X of a category $\mathcal{C}$ is *finitely presentable* if the hom-functor $\mathcal{C}(X,-) : \mathcal{C} \to \mathsf{Set}$ is finitary (i.e., preserves filtered colimits). Let $\mathcal{C}_{fp}$ denote the full subcategory of all finitely presentable objects of $\mathcal{C}$.

(b) $\mathcal{C}$ is *locally finitely presentable* if it is cocomplete, $\mathcal{C}_{fp}$ is small up to isomorphism and every object of $\mathcal{C}$ is a filtered colimit of finitely presentable objects.

(c) $\mathcal{C}$ is *locally finitely super-presentable* if it is locally finitely presentable and $\mathcal{C}_{fp}$ is closed under finite products, subobjects (= monos) and quotients (= epis).

Example 2.2. The categories in the table below are locally finitely super-presentable. In each case, the finitely presentable objects are precisely the finite ones.

$\mathcal{C}$	objects	morphisms
Set	sets	functions
BA	boolean algebras	boolean morphisms
DL_{01}	distributive lattices with 0 and 1	lattice morphisms preserving 0 and 1
JSL_0	join-semilattices with 0	semilattice morphisms preserving 0
Vect $\mathbb{Z}_2$	vector spaces over the field $\mathbb{Z}_2$	linear maps
Pos	partially ordered sets	monotone functions

In contrast to DL_{01}, the category of lattices is not locally finitely super-presentable: a finitely generated lattice can have sublattices that are not finitely generated.

Definition 2.3. An endofunctor $T : \mathcal{C} \to \mathcal{C}$ is *strongly finitary* if it is finitary and preserves finitely presentable objects, i.e., $T[\mathcal{C}_{fp}] \subseteq \mathcal{C}_{fp}$.

Example 2.4. (a) If $\mathcal{C}$ is locally finitely super-presentable, then the functor

$$T_\Sigma = 2 \times \mathsf{Id}^\Sigma = 2 \times \mathsf{Id} \times \mathsf{Id} \times \ldots \times \mathsf{Id}$$

where Σ is a finite alphabet and 2 is a finitely presentable object of $\mathcal{C}$ is strongly finitary. T_Σ-coalgebras are deterministic automata, see e.g. [15]. Indeed, by the universal property of the product, to give a morphism $Q \to T_\Sigma Q = 2 \times Q^\Sigma$ means precisely to give an object Q (of states), morphisms $\delta_a : Q \to Q$ for every $a \in \Sigma$ (representing a-transitions) and a morphism $f : Q \to 2$ (representing final states). The usual concept of a deterministic automaton (without initial states) is captured as a coalgebra for T_Σ where $\mathcal{C}$ = Set and $2 = \{0, 1\}$. An important example of a T_Σ-coalgebra is the automaton Reg_Σ of regular languages. Its states are the regular languages over Σ, its transitions are

$$\delta_a(L) = a^{-1}L \qquad \text{for all } L \in \mathsf{Reg}_\Sigma \text{ and } a \in \Sigma,$$

and the final states are precisely the languages containing the empty word ε.

(b) Analogously, consider T_Σ as an endofunctor of $\mathcal{C}$ = BA with $2 = \{0, 1\}$ (the two-element boolean algebra). A coalgebra for T_Σ is a deterministic automaton with a boolean algebra structure on the state set Q. Moreover, the transition maps $\delta_a : Q \to Q$ are boolean homomorphisms, and the final states (given by the inverse image of 1 under $f : Q \to 2$) form an ultrafilter. The above automaton Reg_Σ is also a T_Σ-coalgebra in BA: the set of regular languages is a boolean algebra w.r.t. the usual set-theoretic operations, left derivatives preserve these operations, and the languages containing ε form a principal ultrafilter.

(c) Mealy automata with output object 2 are coalgebras for the strongly finitary functor $T = (2 \times \mathsf{Id})^\Sigma$.

(d) Nondeterministic automata in $\mathcal{C}$ = Set are coalgebras for the strongly finitary functor $TQ = 2 \times (\mathcal{P}_f Q)^\Sigma$ where $\mathcal{P}_f$ is the finite powerset functor and $2 = \{0, 1\}$.

Notation 2.5. $\mathsf{Coalg}\, T$ denotes the category of all T-coalgebras and their homomorphisms, and $\mathsf{Coalg}_{fp}\, T$ denotes the full subcategory of all *fp-coalgebras*, i.e., coalgebras $Q \to TQ$ with finitely presentable carrier Q.

Remark 2.6. If $\mathcal{C}$ is locally finitely presentable and $T : \mathcal{C} \to \mathcal{C}$ is finitary, let

$$r : \varrho T \to T(\varrho T)$$

be the filtered colimit of all fp-coalgebras, i.e., the colimit of the diagram $\mathsf{Coalg}_{fp} T \hookrightarrow \mathsf{Coalg}\, T$. As shown in [1], ϱT is a fixpoint of T, i.e. r is an isomorphism.

Definition 2.7. ϱT is called the *rational fixpoint* of T.

Example 2.8. The rational fixpoint of $T_\Sigma : \mathsf{Set} \to \mathsf{Set}$ is the automaton $\varrho T_\Sigma = \mathsf{Reg}_\Sigma$ of all regular languages over Σ, see Example 2.4(a). Analogously, the functor $T_\Sigma : \mathsf{BA} \to \mathsf{BA}$ has the rational fixpoint $\varrho T_\Sigma = \mathsf{Reg}_\Sigma$.

Definition 2.9 (see [9]). A coalgebra is called *locally finitely presentable* if it is a filtered colimit of fp-coalgebras. $\mathsf{Coalg}_{lfp} T$ denotes the full subcategory of $\mathsf{Coalg}\, T$ of all locally finitely presentable coalgebras. Hence $\mathsf{Coalg}_{fp} T \subseteq \mathsf{Coalg}_{lfp} T \subseteq \mathsf{Coalg}\, T$.

Example 2.10. A Σ-automaton in Set is locally finitely presentable iff, for every state q, the set of all states reachable from q is finite.

Remark 2.11. (a) Recall the *free completion* $\mathcal{A} \hookrightarrow \mathsf{Ind}\, \mathcal{A}$ of a small category $\mathcal{A}$ under filtered colimits: it is characterized up to equivalence by the property that $\mathsf{Ind}\, \mathcal{A}$ has filtered colimits and every functor $F : \mathcal{A} \to \mathcal{B}$ into a category $\mathcal{B}$ with filtered colimits has an essentially unique finitary extension $\overline{F} : \mathsf{Ind}\, \mathcal{A} \to \mathcal{B}$. If $\mathcal{A}$ has finite colimits then $\mathsf{Ind}\, \mathcal{A}$ is locally finitely presentable and $(\mathsf{Ind}\, \mathcal{A})_{fp} \cong \mathcal{A}$. Conversely, every locally finitely presentable category $\mathcal{C}$ arises in this way: $\mathcal{C} \cong \mathsf{Ind}(\mathcal{C}_{fp})$.
(b) If $\mathcal{A}$ is a join-semilattice then $\mathsf{Ind}\, \mathcal{A}$ is its ideal completion, see Remark 4.3.

Theorem 2.12. *Let $T : \mathcal{C} \to \mathcal{C}$ be a finitary endofunctor of a locally finitely presentable category $\mathcal{C}$.*

(a) ϱT is the terminal object of $\mathsf{Coalg}_{lfp} T$, i.e., the terminal locally finitely presentable T-coalgebra [9].
(b) $\mathsf{Coalg}_{lfp} T$ is the Ind -completion of $\mathsf{Coalg}_{fp} T$.

3 The Dual of the Rational Fixpoint

At the heart of our main results lies the investigation of a duality for our categories of interest (e.g. Stone duality for BA and Priestley duality for DL_{01}) and the induced algebra-coalgebra duality.

Assumptions 3.1. Throughout the rest of the paper we work with

(a) a locally finitely super-presentable category $\mathcal{C}$,
(b) a dual category $\hat{\mathcal{D}}$ with an equivalence functor $P : \hat{\mathcal{D}} \xrightarrow{\cong} \mathcal{C}^{op}$, such that the category

$$\mathcal{D} = \mathsf{Ind}\,(\mathcal{C}_{fp}^{op})$$

is locally finitely super-presentable, and

(c) a strongly finitary functor $T : \mathcal{C} \to \mathcal{C}$ preserving monomorphisms.

Example 3.2. In our applications we will work with the automata functor $T = T_\Sigma : \mathcal{C} \to \mathcal{C}$ from Example 2.4 and with the following categories:

$\mathcal{C}$	$\hat{\mathcal{D}}$	$\mathcal{D}$
BA	Stone	Set
DL_{01}	Priest	Pos
JSL_0	JSL_0 in Stone	JSL_0
$\mathsf{Vect}\,\mathbb{Z}_2$	$\mathsf{Vect}\,\mathbb{Z}_2$ in Stone	$\mathsf{Vect}\,\mathbb{Z}_2$

(a) For the category $\mathcal{C} = \mathsf{BA}$ we have the classical Stone duality: $\hat{\mathcal{D}}$ is the category Stone of Stone spaces (i.e., compact Hausdorff spaces with a base of clopen sets) and continuous maps. The equivalence functor $P : \mathsf{Stone} \to \mathsf{BA}^{op}$ assigns to each Stone space the boolean algebra of clopen sets, and its associated equivalence $P^{-1} : \mathsf{BA}^{op} \to \mathsf{Stone}$ assigns to each boolean algebra the Stone space of all ultrafilters. Since Stone duality restricts to a dual equivalence $\mathsf{BA}_{fp}^{op} \cong \mathsf{Set}_{fp}$, we have

$$\mathcal{D} = \mathsf{Ind}(\mathsf{BA}_{fp}^{op}) \cong \mathsf{Ind}(\mathsf{Set}_{fp}) \cong \mathsf{Set}.$$

(b) For the category $\mathcal{C} = \mathsf{DL}_{01}$ we have the classical Priestley duality: $\hat{\mathcal{D}}$ is the category Priest of Priestley spaces (i.e., ordered Stone spaces such that given $x \nleq y$ there is a clopen set containing x but not y) and continuous monotone maps. The equivalence functor $P : \mathsf{Priest} \to \mathsf{DL}_{01}^{op}$ assigns to each Priestley space the lattice of all clopen upsets, and its associated equivalence $P^{-1} : \mathsf{DL}_{01}^{op} \to \mathsf{Priest}$ assigns to each distributive lattice the Priestley space of all prime filters. Since Priestley duality restricts to a dual equivalence $(\mathsf{DL}_{01})_{fp}^{op} \cong \mathsf{Pos}_{fp}$, we have

$$\mathcal{D} = \mathsf{Ind}((\mathsf{DL}_{01})_{fp}^{op}) \cong \mathsf{Ind}(\mathsf{Pos}_{fp}) \cong \mathsf{Pos}.$$

(c) For $\mathcal{C} = \mathsf{JSL}_0$ the dual category $\hat{\mathcal{D}}$ is the category of join-semilattices in Stone, see [8]. Using the self-duality $(\mathsf{JSL}_0)_{fp}^{op} \cong (\mathsf{JSL}_0)_{fp}$ we obtain

$$\mathcal{D} = \mathsf{Ind}((\mathsf{JSL}_0)_{fp}^{op}) \cong \mathsf{Ind}((\mathsf{JSL}_0)_{fp}) \cong \mathsf{JSL}_0.$$

(d) For $\mathcal{C} = \mathsf{Vect}\,\mathbb{Z}_2$ the dual category $\hat{\mathcal{D}}$ is the category of $\mathbb{Z}_2$-vector spaces in Stone, see [8]. The self-duality $(\mathsf{Vect}\,\mathbb{Z}_2)_{fp}^{op} \cong (\mathsf{Vect}\,\mathbb{Z}_2)_{fp}$ yields

$$\mathcal{D} = \mathsf{Ind}((\mathsf{Vect}\,\mathbb{Z}_2)_{fp}^{op}) \cong \mathsf{Ind}((\mathsf{Vect}\,\mathbb{Z}_2)_{fp}) \cong \mathsf{Vect}\,\mathbb{Z}_2.$$

Remark 3.3. (a) Dually to Definition 2.1, an object X of $\hat{\mathcal{D}}$ is called *cofinitely presentable* if the hom-functor $\hat{\mathcal{D}}(-, X) : \hat{\mathcal{D}}^{op} \to \mathsf{Set}$ preserves filtered colimits. The full subcategory of all cofinitely presentable objects is denoted by $\hat{\mathcal{D}}_{cfp}$. Since $\mathcal{C}_{fp}^{op} \cong \hat{\mathcal{D}}_{cfp}$ we have $\mathcal{D} = \mathsf{Ind}\,(\mathcal{C}_{fp}^{op}) \cong \mathsf{Ind}\,(\hat{\mathcal{D}}_{cfp})$.

(b) The dual of Ind is denoted by Pro: if $\mathcal{A}$ is a small category, then $\mathsf{Pro}\,\mathcal{A}$ is its free completion under cofiltered limits. By duality, $\mathsf{Pro}\,\mathcal{A} \cong (\mathsf{Ind}\,\mathcal{A}^{op})^{op}$.

Example 3.4. (a) For the category Set_{fp} of finite sets, we have $\mathsf{Pro}(\mathsf{Set}_{fp}) \cong \mathsf{Stone}$. Indeed, Stone duality restricts to a duality between Set_{fp} (= finite Stone spaces) and BA_{fp}, so

$$\mathsf{Pro}(\mathsf{Set}_{fp}) \cong \mathsf{Pro}(\mathsf{BA}_{fp}^{op}) \cong (\mathsf{Ind}(\mathsf{BA}_{fp}))^{op} \cong \mathsf{BA}^{op} \cong \mathsf{Stone}.$$

(b) Analogously $\mathsf{Pro}(\mathsf{Pos}_{fp}) \cong \mathsf{Priest}$.

Definition 3.5. We denote by $\hat{L} : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ the dual of the functor $T : \mathcal{C} \to \mathcal{C}$, i.e., the essentially unique functor with $P\hat{L} = T^{op}P$.

Remark 3.6. The categories $\mathsf{Alg}\,\hat{L}$ and $\mathsf{Coalg}\,T$ are dually equivalent. Indeed, the equivalence functor $P : \hat{\mathcal{D}} \to \mathcal{C}^{op}$ induces an equivalence functor

$$\overline{P} : \mathsf{Alg}\,\hat{L} \to (\mathsf{Coalg}\,T)^{op}, \qquad (\hat{L}Z \xrightarrow{z} Z) \mapsto (PZ \xrightarrow{Pz} P\hat{L}Z = TPZ).$$

Example 3.7. The dual of $T_\Sigma X = \mathbb{2} \times X^\Sigma = \mathbb{2} \times \prod_\Sigma X : \mathcal{C} \to \mathcal{C}$, see Example 2.4, is the endofunctor of $\hat{\mathcal{D}}$

$$\hat{L}_\Sigma Z = \mathbb{1} + \coprod_\Sigma Z$$

where $\mathbb{1} = P^{-1}\mathbb{2}$. In $\hat{\mathcal{D}} = \mathsf{Stone}$ the object $\mathbb{1}$ is the one-element space. Hence, by the universal property of the coproduct, an $\hat{L}_\Sigma$-algebra $\hat{L}_\Sigma Z = \mathbb{1} + \coprod_\Sigma Z \to Z$ is a deterministic Σ-automaton (without final states) in Stone, given by a Stone space Z of states, continuous transition functions $\delta_a : Z \to Z$ for $a \in \Sigma$, and an initial state $\mathbb{1} \to Z$. Analogously for the other dualities of Example 3.2.

Remark 3.8. By the dual of Assumption 3.1(c), the functor $\hat{L}$ is *strongly cofinitary*, i.e., it preserves cofiltered limits and cofinitely presentable objects. In particular, $\hat{L}$ restricts to a functor

$$\hat{L}_{cfp} : \hat{\mathcal{D}}_{cfp} \to \hat{\mathcal{D}}_{cfp}.$$

Definition 3.9. The essentially unique finitary extension of the functor

$$\hat{\mathcal{D}}_{cfp} \xrightarrow{\hat{L}_{cfp}} \hat{\mathcal{D}}_{cfp} \hookrightarrow \mathsf{Ind}(\hat{\mathcal{D}}_{cfp}) = \mathcal{D}$$

is denoted by $L : \mathcal{D} \to \mathcal{D}$. It takes a formal filtered colimit to the actual colimit in $\mathcal{D}$.

Example 3.10. For $\hat{L}_\Sigma Z = \mathbb{1} + \coprod_\Sigma Z$ on $\hat{\mathcal{D}}$ (see Example 3.7) we get the endofunctor of $\mathcal{D}$

$$L_\Sigma Z = \mathbb{1} + \coprod_\Sigma Z.$$

Here $\mathbb{1} = P^{-1}\mathbb{2} \in \hat{\mathcal{D}}_{cfp}$ is an object of $\mathcal{D} = \mathsf{Ind}(\hat{\mathcal{D}}_{cfp})$. For $\hat{\mathcal{D}} = \mathsf{Stone}$ we have $\mathcal{D} = \mathsf{Set}$, so L_Σ-algebras are the classical deterministic automata without final states. Analogously for the other dualities of Example 3.2.

Notation 3.11. The category of all $\hat{L}$-algebras with a cofinitely presentable carrier (shortly *cfp-algebras*) is denoted by $\mathsf{Alg}_{cfp}\,\hat{L}$.

Example 3.12. (a) If $\hat{\mathcal{D}} = \mathsf{Stone}$, we have $\hat{\mathcal{D}}_{cfp} \cong \mathsf{BA}_{fp}^{op} \cong \mathsf{Set}_{fp}$, so cfp-algebras for $\hat{L}_\Sigma$ are the classical deterministic finite automata without final states.
(b) If $\hat{\mathcal{D}} = \mathsf{Priest}$, since $\hat{\mathcal{D}}_{cfp} = \mathsf{DL}_{01,fp}^{op} \cong \mathsf{Pos}_{fp}$, cfp-algebras for $\hat{L}_\Sigma$ are precisely the deterministic finite *ordered* automata without final states.

Definition 3.13. An $\hat{L}$-algebra is called *locally cofinitely presentable* if it is a cofiltered limit of cfp-algebras.

Remark 3.14. The category of all locally cofinitely presentable algebras is equivalent to $\mathsf{Pro}(\mathsf{Alg}_{cfp}\,\hat{L})$. This is the dual of Theorem 2.12. The initial object $\tau\hat{L}$ is what one can call the *dual of the rational fixpoint*. By the dual of Remark 2.6, one can construct $\tau\hat{L}$ as the limit of all cfp-algebras in $\mathsf{Alg}\,\hat{L}$, and $\tau\hat{L}$ is a fixpoint of $\hat{L}$.

Example 3.15. (a) For $\mathcal{C} = \mathsf{BA}$ and $\hat{\mathcal{D}} = \mathsf{Stone}$, we have $\tau\hat{L}_\Sigma$ = ultrafilters of regular languages.
(b) Analogously, for $\mathcal{C} = \mathsf{DL}_{01}$ and $\hat{\mathcal{D}} = \mathsf{Priest}$, we have $\tau\hat{L}_\Sigma$ = prime filters of regular languages.

Definition 3.16. We denote by $F : \mathcal{D} \to \hat{\mathcal{D}}$ the unique finitary functor for which

$$\begin{array}{ccc} & \hat{\mathcal{D}}_{cfp} & \\ \swarrow & & \searrow \\ \mathcal{D} = \mathsf{Ind}(\hat{\mathcal{D}}_{cfp}) & \xrightarrow[F]{} & \mathsf{Pro}(\hat{\mathcal{D}}_{cfp}) = \hat{\mathcal{D}} \end{array}$$

commutes, and by $U : \hat{\mathcal{D}} \to \mathcal{D}$ the unique cofinitary functor for which

$$\begin{array}{ccc} & \hat{\mathcal{D}}_{cfp} & \\ \swarrow & & \searrow \\ \hat{\mathcal{D}} = \mathsf{Pro}(\hat{\mathcal{D}}_{cfp}) & \xrightarrow[U]{} & \mathsf{Ind}(\hat{\mathcal{D}}_{cfp}) = \mathcal{D} \end{array}$$

commutes.

Lemma 3.17. *The functors F and U are well-defined and F is a left adjoint to U.*

Example 3.18. 1. If $\mathcal{C} = \mathsf{BA}$ then $\hat{\mathcal{D}} = \mathsf{Stone}$ and $\mathcal{D} = \mathsf{Set}$. Then $F : \mathsf{Set} \to \mathsf{Stone}$ is the Stone-Čech compactification and $U : \mathsf{Stone} \to \mathsf{Set}$ is the forgetful functor.
2. If $\mathcal{C} = \mathsf{DL}_{01}$ then $\hat{\mathcal{D}} = \mathsf{Priest}$ and $\mathcal{D} = \mathsf{Pos}$. Then $F : \mathsf{Pos} \to \mathsf{Priest}$ constructs the free Priestley space on a poset and $U : \mathsf{Priest} \to \mathsf{Pos}$ is the forgetful functor.

Notation 3.19. $\mathsf{Alg}_{fp}\,L$ is the full subcategory of $\mathsf{Alg}\,L$ of L-algebras with finitely presentable carrier, shortly *fp-algebras*. Note that $\mathsf{Alg}_{fp}\,L \cong \mathsf{Alg}_{cfp}\,\hat{L}$ because $\hat{\mathcal{D}}_{cfp} \cong \mathcal{D}_{fp}$.

Definition 3.20. $\hat{U} : \mathsf{Pro}(\mathsf{Alg}_{cfp}\,\hat{L}) \to \mathsf{Alg}\,L$ is the unique cofinitary functor that makes the triangle below commute:

$$\begin{array}{ccc} & \mathsf{Alg}_{cfp}\,\hat{L} \cong \mathsf{Alg}_{fp}\,L & \\ \swarrow & & \searrow \\ \mathsf{Pro}(\mathsf{Alg}_{cfp}\,\hat{L}) & \xrightarrow[\hat{U}]{} & \mathsf{Alg}\,L \end{array}$$

Example 3.21. For $T_\Sigma = \mathbb{2} \times \mathsf{Id}^\Sigma : \mathsf{BA} \to \mathsf{BA}$ we have $\hat{L}_\Sigma = \mathbb{1} + \coprod_\Sigma \mathsf{Id} : \mathsf{Stone} \to \mathsf{Stone}$ and $L_\Sigma = \mathbb{1} + \coprod_\Sigma \mathsf{Id} : \mathsf{Set} \to \mathsf{Set}$. The objects of $\mathsf{Pro}(\mathsf{Alg}_{cfp}\, \hat{L}_\Sigma)$ are the locally cofinitely presentable $\hat{L}_\Sigma$-algebras, and the functor $\hat{U} : \mathsf{Pro}(\mathsf{Alg}_{cfp}\, \hat{L}_\Sigma) \to \mathsf{Alg}\, L_\Sigma$ simply forgets the topology on the carrier of an L_Σ-algebra.

Proposition 3.22. *$\hat{U}$ is a right adjoint.*

Remark 3.23. It follows that the left adjoint $\hat{F}$ of $\hat{U}$ maps the initial L-algebra to the initial locally cofinitely presentable $\hat{L}$-algebra: $\hat{F}(\mu L) = \tau \hat{L}$. One can prove that $\hat{F}$ assigns to every L-algebra $\alpha : LA \to A$ the limit of the diagram of all its quotients in $\mathsf{Alg}_{fp}\, L = \mathsf{Alg}_{cfp}\, \hat{L}$. Thus, we see that $\tau \hat{L}$ can be constructed as the limit (taken in $\mathsf{Alg}\, \hat{L}$) of all finite quotient L-algebras of μL. This construction generalizes a similar one given by Gehrke [5]. See also Section 5.1.

4 Algebraic and Coalgebraic Recognition

We are ready to present our first take on the duality of algebraic and coalgebraic recognition and Eilenberg's theorem (see Proposition 4.2 and Theorem 4.4 below). At this stage our results are about subcoalgebras of the rational fixpoint ϱT and quotients of the initial L-algebra μL, and we obtain uniform proofs at the level of generality of the previous section. Recall that we have the following dualities:

Category	**Equivalently**	**Dual category**	**Equivalently**
$\mathcal{C}$		$\hat{\mathcal{D}}$	
$\mathsf{Coalg}\, T$		$\mathsf{Alg}\, \hat{L}$	
$\mathsf{Coalg}_{fp}\, T$		$\mathsf{Alg}_{cfp}\, \hat{L}$	$\mathsf{Alg}_{fp}\, L$
$\mathsf{Coalg}_{lfp}\, T$	$\mathsf{Ind}(\mathsf{Coalg}_{fp}\, T)$	$\mathsf{Pro}(\mathsf{Alg}_{cfp}\, T)$	

Definition 4.1. (a) By a *subcoalgebra* of a T-coalgebra (C, γ) is meant one represented by a homomorphism $m : (C', \gamma') \rightarrowtail (C, \gamma)$ with m monic in $\mathcal{C}$. Subcoalgebras are ordered as usual: $m \le \overline{m}$ iff m factorizes through $\overline{m}$ in $\mathsf{Coalg}\, T$. We denote by $\mathsf{Sub}(\varrho T)$ the poset of all subcoalgebras of ϱT, and by $\mathsf{Sub}_{fp}(\varrho T)$ the subposet of all *fp-subcoalgebras* of ϱT, i.e., those with finitely presentable carrier in $\mathcal{C}$.

(b) Likewise, a *quotient algebra* of an L-algebra (A, α) is one represented by an epi-carried homomorphism $e : (A, \alpha) \twoheadrightarrow (A', \alpha')$. Again the ordering is $e \le \overline{e}$ iff e factorizes through $\overline{e}$ in $\mathsf{Alg}\, L$ (so $id_{(A,\alpha)}$ is the largest quotient). We denote by $\mathsf{Quo}(\mu L)$ the poset of all quotient algebras of μL, and by $\mathsf{Quo}_{fp}(\mu L)$ the subposet of all *fp-quotient algebras*, i.e., those with finitely presentable carrier in $\mathcal{D}$.

Proposition 4.2. *The posets $\mathsf{Sub}_{fp}(\varrho T)$ and $\mathsf{Quo}_{fp}(\mu L)$ are isomorphic.*

Proof (Sketch). The inverse $\overline{P}^{-1} : (\mathsf{Coalg}\, T)^{op} \to \mathsf{Alg}\, \hat{L}$ of $\overline{P}$ in Remark 3.6 assigns to each T-coalgebra $C \xrightarrow{\gamma} TC$ the $\hat{L}$-algebra $\hat{L}(P^{-1}C)) = P^{-1}(TC) \xrightarrow{P^{-1}\gamma} P^{-1}C$. If (C, γ) is an fp-coalgebra, the L-algebra $\overline{P}^{-1}(C, \gamma)$ has a cofinitely presentable carrier in $\hat{\mathcal{D}}$ and (since $\hat{\mathcal{D}}_{cfp} = \mathcal{D}_{fp}$) can be viewed as an fp-algebra for L. We denote by

$$(C, \gamma) \xrightarrow{m} \varrho T \quad \text{and} \quad \mu L \xrightarrow{e} \overline{P}^{-1}(C, \gamma)$$

the unique homomorphisms, and prove that

$$m \text{ is monic (in } \mathcal{C}) \quad \text{iff} \quad e \text{ is epic (in } \mathcal{D}),$$

using the (epi, strong mono)- and (strong epi, mono)-factorization systems of $\mathcal{C}$ and $\mathcal{D}$, respectively. Therefore $m \mapsto e$ is an isomorphism $\mathsf{Sub}_{fp}(\varrho T) \cong \mathsf{Quo}_{fp}(\mu L)$.

Remark 4.3. Recall that the *ideal completion* $\mathsf{Ideal}(A)$ of a join-semilattice A is the complete lattice of all ideals (= join-closed downsets) of A ordered by inclusion. Up to isomorphism $\mathsf{Ideal}(A)$ is characterized as a complete lattice containing A such that:

(1) every element of $\mathsf{Ideal}(A)$ is a directed join of elements of A, and
(2) the elements of A are compact in $\mathsf{Ideal}(A)$: if $x \in A$ lies under a directed join of elements $y_i \in \mathsf{Ideal}(A)$, then $x \leq y_i$ for some i.

Theorem 4.4. *If T preserves preimages, then* $\mathsf{Sub}(\varrho T) \cong \mathsf{Ideal}(\mathsf{Quo}_{fp}(\mu L))$.

Proof (Sketch). Since $\mathsf{Sub}_{fp}(\varrho T) \cong \mathsf{Quo}_{fp}(\mu L)$ by Proposition 4.2, is suffices to prove that $\mathsf{Sub}(\varrho T)$ is the ideal completion of $\mathsf{Sub}_{fp}(\mu L)$. Firstly $\mathsf{Sub}(\varrho T)$ forms a complete lattice because $\mathsf{Coalg}\,T$ is cocomplete and has a factorization system carried by strong epis and monos in $\mathcal{C}$. Now one proves that $\mathsf{Sub}(\varrho T) \cong \mathsf{Ideal}(\mathsf{Sub}_{fp}(\varrho T))$ by establishing the properties (1) and (2) of Remark 4.3.

5 Local Eilenberg Theorem

The aim of this section is to prove our main result: a general local Eilenberg Theorem for deterministic automata, i.e., coalgebras for the functor $T_\Sigma = \mathbb{2} \times \mathsf{Id}^\Sigma : \mathcal{C} \to \mathcal{C}$. Here $\mathbb{2}$ is a fixed object of $\mathcal{C}_{fp}$, and we write $\mathbb{1} = P^{-1}\mathbb{2}$ for the corresponding $\hat{\mathcal{D}}$-object. Note that $\mathbb{1}$ lies in $\mathcal{D}_{cfp}$ und thus is also an object of $\mathcal{D} = \mathsf{Ind}\,(\hat{\mathcal{D}}_{cfp})$. T_Σ-coalgebras $Q \to \mathbb{2} \times Q^\Sigma$ and L_Σ-algebras $\mathbb{1} + \coprod_\Sigma A \to A$ are represented as triples

$$Q = (Q, \delta_a : Q \to Q, f : Q \to \mathbb{2}) \quad \text{and} \quad A = (A, \delta_a : A \to A, i : \mathbb{1} \to A).$$

Assumptions 5.1. We continue to work under the Assumptions 3.1 and make the following additional assumptions on $\mathcal{C}$ and $\mathcal{D}$:

(a) $\mathcal{C}$ and $\mathcal{D}$ are concrete categories, i.e., forgetful functors to Set are given (notation: $A \mapsto |A|$ for objects and $f \mapsto f$ for morphisms). We assume that these forgetful functors are strongly finitary right adjoints, and that $\mathcal{D}$'s forgetful functor preserves epimorphisms.
(b) An object $\mathbb{2} \in \mathcal{C}_{fp}$ is selected with underlying set $|\mathbb{2}| = \{0, 1\}$, and the corresponding object $\mathbb{1} = P^{-1}\mathbb{2}$ in $\mathcal{D}$ is free on one generator: $\mathbb{1} = \Psi 1$ for the left adjoint $\Psi : \mathsf{Set} \to \mathcal{D}$ of the forgetful functor.
(c) Every object C of $\mathcal{C}$ has, for a given subset $m : M \rightarrowtail |C|$, at most one subobject carried by m.
(d) $\mathcal{D}$ has hom-objects, i.e., for every pair of objects A and B the power $B^{|A|}$ has a subobject $[A, B] \rightarrowtail B^{|A|}$ carried by the set $\mathcal{D}(A, B)$ of all morphisms $A \to B$.

Example 5.2. All the categories in Example 3.2 meet these assumptions. In BA, JSL_0 and DL_{01} we choose $\mathbb{2}$ to be the chain $0 < 1$, and in $\mathsf{Vect}\,\mathbb{Z}_2$ we choose $\mathbb{2} = \mathbb{Z}_2$.

Remark 5.3. Since the forgetful functors are strongly finitary, the finitely presentable objects of $\mathcal{C}$ and $\mathcal{D}$ are carried by finite sets. Hence we will talk about *finite* objects rather than finitely presentable ones.

Proposition 5.4. *The rational fixpoint ϱT_Σ is carried by the automaton Reg_Σ of Example 2.8. Consequently, a subcoalgebra of ϱT_Σ is a set of regular languages closed under left derivatives and carrying a subobject of ϱT_Σ in $\mathcal{C}$.*

What about closure under *right* derivatives? Given a T_Σ-coalgebra $Q = (Q, \delta_a, f)$ and $w \in \Sigma^*$ we consider the coalgebra

$$Q_w = (Q, \delta_a, f \cdot \delta_w)$$

where, as usual, $\delta_w = \delta_{a_n} \cdots \delta_{a_1}$ for $w = a_1 \dots a_n$. Closure under right derivatives can be characterized coalgebraically as follows:

Proposition 5.5.
A subcoalgebra Q of ϱT_Σ is closed under right derivatives (i.e., $L \in |Q|$ implies $Lw^{-1} \in |Q|$ for each $w \in \Sigma^*$) iff there exists a coalgebra morphism from Q_w to Q for each $w \in \Sigma^*$.

Remark 5.6. Analogously, for an L_Σ-algebra $A = (A, \delta_a, i)$ and $w \in \Sigma^*$ we define

$$A_w = (A, \delta_a, \delta_w \cdot i).$$

Now let A be a finite quotient algebra μL_Σ. It corresponds to a finite right-derivative closed subcoalgebra of ϱT_Σ under the isomorphism of Proposition 4.2 iff an L_Σ-algebra morphism from A to A_w exists for every $w \in \Sigma^*$. Indeed, a coalgebra morphism $Q_w \to Q$ corresponds to an algebra morphism $A \to A_{w^r}$, where $A = \overline{P}^{-1} Q$ (see Remark 3.6) and w^r is the reversed word of w. Fortunately a better characterization is possible, using the concept of a bimonoid.

Definition 5.7. *A* bimonoid *in $\mathcal{D}$ is a triple $(A, \circ, i)$ where (i) A is a $\mathcal{D}$-object, (ii) $(|A|, \circ, i)$ is a monoid in* Set *and (iii) for all $a \in |A|$, the translations $a \circ -$ and $- \circ a$ carry endomorphisms of A. It is called* finite *if $A \in \mathcal{D}_{fp}$. A* bimonoid morphism *$h : (A, \circ, i) \to (A', \circ', i')$ is a $\mathcal{D}$-morphism $h : A \to A'$ that is also a monoid morphism.*

Example 5.8. Bimonoids in $\mathcal{D} = \mathsf{Set}$, Pos, JSL_0 and $\mathsf{Vect}\,\mathbb{Z}_2$ correspond to monoids, ordered monoids, idempotent semirings and $\mathbb{Z}_2$-algebras, respectively.

Construction 5.9. We define a monoid multiplication $\bullet$ on the free object $\Psi\Sigma^*$. For all $w \in \Sigma^*$, let $r_w : \Psi\Sigma^* \to \Psi\Sigma^*$ be the unique $\mathcal{D}$-morphism extending the map $- \cdot w$ on Σ^*. Let $\overline{r} : \Psi\Sigma^* \to [\Psi\Sigma^*, \Psi\Sigma^*]$ (see Assumptions 5.1(d)) be the unique $\mathcal{D}$-morphism extending the map $r : \Sigma^* \to \mathcal{D}(\Psi\Sigma^*, \Psi\Sigma^*)$, $w \mapsto r_w$.

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{-\cdot w} & \Sigma^* \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ |\Psi\Sigma^*| & \xrightarrow[r_w]{} & |\Psi\Sigma^*| \end{array} \qquad \begin{array}{ccc} & & \Sigma^* \\ & \swarrow^{\eta} & \downarrow{\scriptstyle r} \\ |\Psi\Sigma^*| & \xrightarrow[\overline{r}]{} & |[\Psi\Sigma^*, \Psi\Sigma^*]| \end{array}$$

Then define the multiplication $\bullet$ by

$$x \bullet y := [\overline{r}(y)](x) \quad \text{for all} x, y \in |\Psi\Sigma^*|.$$

Lemma 5.10. *$(\Psi\Sigma^*, \bullet, \eta\varepsilon)$ is a bimonoid, in fact the free bimonoid on Σ: for any bimonoid $(A, \circ, i)$ and any function $f : \Sigma \to |A|$, there is a unique extension to a bimonoid morphism $\overline{f} : \Psi\Sigma^* \to A$.*

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\eta} & |\Psi\Sigma^*| \\ \uparrow & & \downarrow \overline{f} \\ \Sigma & \xrightarrow[f]{} & |A| \end{array}$$

Example 5.11. (a) For $\mathcal{D}$ = Set and Pos we have $\Psi\Sigma^* = \Sigma^*$ (discretely ordered in the case $\mathcal{D}$ = Pos) with monoid multiplication = concatenation of words.
(b) For $\mathcal{D} = \mathsf{JSL}_0$ we have $\Psi\Sigma^* = \mathcal{P}_f\Sigma^*$ (finite languages over Σ) with join = union and monoid multiplication = concatenation of languages.
(c) For $\mathcal{D} = \mathsf{Vect}\,\mathbb{Z}_2$ we have $\Psi\Sigma^* = \mathcal{P}_f\Sigma^*$ with addition = symmetric difference and monoid multiplication = $\mathbb{Z}_2$-weighted concatenation of languages, i.e., $L \otimes L'$ consists of all words w having an odd number of factorizations $w = uu'$ with $u \in L$ and $u' \in L'$.

This motivates the following definition:

Definition 5.12. *(a) A Σ-*generated bimonoid *is a quotient bimonoid of $\Psi\Sigma^*$, represented by a bimonoid morphism $e : \Psi\Sigma^* \twoheadrightarrow A$ with e epic in $\mathcal{D}$.*
(b) We denote by Σ-$\mathsf{Bim}_{fp}(\mathcal{D})$ the poset of all Σ-generated finite bimonoids under the usual quotient ordering.

Remark 5.13. Σ-$\mathsf{Bim}_{fp}(\mathcal{D})$ is a join-semilattice. Indeed, it is easy to see that the category of finite bimonoids has finite limits, computed on the level of $\mathcal{D}$, and also inherits the (strong epi, mono)-factorization system from $\mathcal{D}$. Hence the join of two Σ-generated bimonoids $e : \Psi\Sigma^* \twoheadrightarrow A$ and $e' : \Psi\Sigma^* \twoheadrightarrow A'$ in Σ-$\mathsf{Bim}_{fp}(\mathcal{D})$ is their *subdirect product*, obtained by factorizing the product map $\langle e, e'\rangle : \Psi\Sigma^* \to A \times A'$.

Remark 5.14. Every Σ-generated bimonoid $e : \Psi\Sigma^* \twoheadrightarrow (A, \circ, i)$ induces an L_Σ-algebra $\widetilde{A} = (A, \delta_a, \overline{i})$ where $\delta_a : A \to A$ is the $\mathcal{D}$-morphism with $\delta_a(x) = x \circ e(\eta a)$ for all $x \in |A|$, and $\overline{i} : \mathbb{1} \to A$ is the free extension of $i : 1 \to |A|$. One can show that

$$\widetilde{\Psi\Sigma^*} = \mu L_\Sigma.$$

Proposition 5.15. *An finite quotient algebra A of μL_Σ is induced by a Σ-generated bimonoid iff L_Σ-algebra morphisms from A to A_w exist for all $w \in \Sigma^*$.*

Proof (Sketch). Every $e : \Psi\Sigma^* \twoheadrightarrow A$ in Σ-$\mathsf{Bim}_{fp}(\mathcal{D})$ yields a quotient algebra $e : \mu L_\Sigma \twoheadrightarrow \widetilde{A}$ of μL_Σ. For each $w \in \Sigma^*$, the desired L_Σ-algebra morphism $\widetilde{A} \to \widetilde{A}_w$ is the $\mathcal{D}$-morphism carried by $e(\eta w) \bullet -$. Conversely, let $e : \mu L_\Sigma \twoheadrightarrow (A, \delta_a, i)$ be any quotient algebra of μL_Σ such that L_Σ-algebra morphisms $A \to A_w$ exist. Define a monoid multiplication $\circ$ on $|A|$ as follows: given $x, y \in |A|$, choose $x' \in |\Psi\Sigma^*|$ and $y' \in |\Psi\Sigma^*|$ with $ex' = x$ and $ey' = y$ (using that e is surjective by Assumptions 5.1(a)), and put

$$x \circ y := e(x' \bullet y').$$

One then proves that $(A, \circ, i)$ is a well-defined bimonoid whose induced L_Σ-algebra is precisely (A, δ_a, i).

Definition 5.16. *By a* local variety of regular languages in $\mathcal{C}$ *is meant a subcoalgebra of ϱT_Σ closed under right derivatives.*

Example 5.17. Local varieties of regular languages in $\mathcal{C} = \mathsf{BA}$, DL_{01}, JSL_0 and $\mathsf{Vect}\,\mathbb{Z}_2$ are called *local varieties*, *local lattice varieties*, *local semilattice varieties* and *local linear varieties of regular languages*, respectively; see Introduction.

Definition 5.18. *By a* local pseudovariety of bimonoids in $\mathcal{D}$ *is meant a set of finite Σ-generated bimonoids in $\mathcal{D}$ closed under subdirect products and quotients, i.e., an ideal in the join-semilattice $\Sigma\text{-}\mathsf{Bim}_{fp}(\mathcal{D})$.*

Theorem 5.19 (General Local Eilenberg Theorem). *The lattice of local varieties of regular languages in $\mathcal{C}$ is isomorphic to the lattice of local pseudovarieties of bimonoids in $\mathcal{D}$.*

Proof (Sketch). Let $\mathsf{Sub}^r_{fp}(\varrho T_\Sigma)$ denote the poset of all finite local varieties of regular languages in $\mathcal{C}$, i.e., of all finite subcoalgebras of ϱT_Σ closed under right derivatives. From Remark 5.6 and Proposition 5.15 we get a join-semilattice isomorphism

$$\mathsf{Sub}^r_{fp}(\varrho T_\Sigma) \cong \Sigma\text{-}\mathsf{Bim}_{fp}(\mathcal{D}).$$

Taking ideal completions on both sides yields a complete lattice isomorphism

$$\mathsf{Ideal}(\mathsf{Sub}^r_{fp}(\varrho T_\Sigma)) \cong \mathsf{Ideal}(\Sigma\text{-}\mathsf{Bim}_{fp}(\mathcal{D})).$$

One then proves that $\mathsf{Ideal}(\mathsf{Sub}^r_{fp}(\varrho T_\Sigma))$ is isomorphic to the lattice of local varieties of regular languages in $\mathcal{C}$. Moreover, by definition $\mathsf{Ideal}(\Sigma\text{-}\mathsf{Bim}_{fp}(\mathcal{D}))$ is precisely the lattice of local pseudovarieties of bimonoids in $\mathcal{D}$.

Corollary 5.20. *By instantiating Theorem 5.19 to the categories of Example 3.2 we obtain the following lattice isomorphisms:*

$\mathcal{C}$	$\mathcal{D}$	local varieties of regular languages	$\cong$ local pseudovarieties of ...
BA	Set	local varieties	$\cong$ monoids
DL_{01}	Pos	local lattice varieties	$\cong$ ordered monoids
JSL_0	JSL_0	local semilattice varieties	$\cong$ idempotent semirings
$\mathsf{Vect}\,\mathbb{Z}_2$	$\mathsf{Vect}\,\mathbb{Z}_2$	local linear varieties	$\cong$ $\mathbb{Z}_2$-algebras

5.1 Profinite Monoids

As a consequence of Theorem 5.19 we obtain a generalization of the result of Gehrke, Grigorieff and Pin [6, 7] that Reg_Σ endowed with boolean operations and derivatives is dual to the free profinite monoid on Σ. To see this one observes first that the finite local varieties of languages form a cofinal subposet of $\mathsf{Sub}_{fp}(\varrho T_\Sigma)$ – in other words, every finite subcoalgebra of ϱT_Σ is contained in a finite local variety. Therefore the finite Σ-generated bimonoids form a cofinal subposet of $\mathsf{Quo}_{fp}(\mu L_\Sigma)$. Thus, the corresponding diagrams have the same limit in $\mathsf{Alg}\ \hat{L}_\Sigma$. Since the limit of all fp-quotients of μL_Σ is the initial L_Σ-algebra μL_Σ, we see that $\tau \hat{L}_\Sigma$ is also the limit of the directed diagram of all finite Σ-generated bimonoids. Hence, $\tau \hat{L}_\Sigma$ is a bimonoid and it is then easy to see that it is the free profinite bimonoid on Σ, where bimonoid now means bimonoid in $\hat{\mathcal{D}}$ w.r.t. $|U| = |-| \circ U : \hat{\mathcal{D}} \to \mathsf{Set}$ and "profinite" refers to the category $\mathsf{Pro}(\hat{\mathcal{D}}_{cfp}) = \hat{\mathcal{D}}$; in fact, (the carrier of) $\tau \hat{L}_\Sigma$ is $F(\Psi(\Sigma^*))$, where $F \cdot \Psi$ is the left adjoint of $|U|$.

Theorem 5.21. *$\tau \hat{L}_\Sigma$ is the free profinite bimonoid on Σ, and this structure is dual to the structure on ϱT_Σ given by its T_Σ-coalgebra structure and right derivatives.*

6 Conclusions and Future Work

Inspired by recent work of Gehrke, Grigorieff and Pin [6, 7] we have proved a generalized local Eilenberg theorem, parametric in a pair of dual categories $\mathcal{C}$ and $\hat{\mathcal{D}}$ and a type of coalgebras $T : \mathcal{C} \to \mathcal{C}$. By instantiating our framework to deterministic automata, i.e., the functor $T_\Sigma = 2 \times \mathsf{Id}^\Sigma$ on $\mathcal{C} = \mathsf{BA}$, DL_{01}, JSL_0 and $\mathsf{Vect}\,\mathbb{Z}_2$, we derived the local Eilenberg theorems for (ordered) monoids as in [6], as well as two new local Eilenberg theorems for idempotent semirings and $\mathbb{Z}_2$-algebras.

There remain a number of open points for further work. Firstly, our general approach should be extended to the ordinary (non-local) version of Eilenberg's theorem. Secondly, for different functors T on the categories we have considered our approach should provide the means to relate varieties of rational behaviours of T with varieties of appropriate algebras. In this way, we hope to obtain Eilenberg type theorems for systems such as Mealy and Moore automata, but also weighted or probabilistic automata – ideally, such results would be proved uniformly for a certain class of functors.

Another very interesting aspect we have not treated in this paper are profinite equations and syntactic presentations of varieties (of bimonoids or regular languages, resp.) as in the work of Gehrke, Grigorieff and Pin [6]. An important role in studying profinite equations will be played by the $\hat{L}$-algebra $\tau \hat{L}$, the dual of the rational fixpoint, that we identified as the free profinite bimonoid. A profinite equation is then a pair of elements of $\tau \hat{L}$. We intend to investigate this in future work.

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