

# Generalized Eilenberg Theorem: Varieties of Languages in a Category

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For finite automata as coalgebras in a category  $\mathcal{C}$  we study languages they accept, and varieties of such languages. This generalizes Eilenberg's concept of a variety of languages which corresponds to choosing as  $\mathcal{C}$  the category of boolean algebras. Eilenberg established a bijective correspondence between pseudovarieties of monoids and varieties of regular languages. In our generalization we work with a pair  $\mathcal{C}/\mathcal{D}$  of locally finite varieties of algebras that are pre-dual, i.e. dualize on the level of finite algebras, and we prove that pseudovarieties of  $\mathcal{D}$ -monoids bijectively correspond to varieties of regular languages in  $\mathcal{C}$ . As one instance, Eilenberg's result is recovered by choosing  $\mathcal{D} = \text{sets}$  and  $\mathcal{C} = \text{boolean algebras}$ . Another instance, Pin's result on pseudovarieties of ordered monoids, is covered by taking  $\mathcal{D} = \text{posets}$  and  $\mathcal{C} = \text{distributive lattices}$ . By choosing as  $\mathcal{C} = \mathcal{D}$  the self-pre-dual category of join-semilattices we obtain Polák's result on pseudovarieties of idempotent semirings. Similarly, using the self-pre-duality of vector spaces over a finite field  $K$ , our result covers that of Reutenauer on pseudovarieties of  $K$ -algebras. Several new variants of Eilenberg's theorem arise by taking other pre-dualities, e.g. between the categories of non-unital boolean rings and of pointed sets. In each of these cases we also prove a local variant of the bijection, where a fixed alphabet is assumed and one considers local varieties of regular languages over that alphabet in the category  $\mathcal{C}$ .

CCS Concepts: • **Theory of computation** → **Formal languages and automata theory**; **Regular languages**;

Additional Key Words and Phrases: algebraic automata theory, Eilenberg's theorem, category theory, coalgebra, bimonoids

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## 1 INTRODUCTION

Varieties of regular languages, as introduced by Eilenberg [14], are classes of regular languages closed under the set-theoretical boolean operations, derivatives, and preimages under monoid homomorphisms. In his monograph, Eilenberg proved that these varieties correspond bijectively to pseudovarieties of monoids (i.e. classes of finite monoids closed under finite products, submonoids

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and quotients.) In subsequent decades more than a dozen generalizations of this result were proved in the literature. Let us mention some examples:

- (1) Pin [21] introduced positive varieties of regular languages, where just union and intersection are used as set-theoretical operations (and complement is not considered), and proved a bijective correspondence to pseudovarieties of ordered monoids.
- (2) Polák’s disjunctive varieties [23] use just union, and they bijectively correspond to pseudovarieties of idempotent semirings.
- (3) Reutenauer [26] studied varieties of languages weighted over a finite field  $K$  and proved that they bijectively correspond to pseudovarieties of  $K$ -algebras.

In this paper a general categorical result is presented that covers Eilenberg’s theorem and all its above variants uniformly, and moreover exhibits several new correspondences. Our categorical approach to algebraic automata theory may be subsumed by the “equation”

$$\text{automata theory} = \text{duality} + \text{monoidal structure.}$$

The idea is to take a category  $\mathcal{C}$  (where languages and automata as coalgebras live) and a closed monoidal category  $\mathcal{D}$  (where automata as algebras and monoids live) that is pre-dual to  $\mathcal{C}$ , which means that  $\mathcal{C}$  is dual to  $\mathcal{D}$  when both are restricted to their finite objects. Specifically, in our setting  $\mathcal{C}$  and  $\mathcal{D}$  will be locally finite varieties of algebras or ordered algebras (i.e., all finitely generated algebras are finite), and pre-duality means that the full subcategories of finite algebras are dually equivalent. Moreover, the monoidal structure of  $\mathcal{D}$  is given by the usual tensor product of algebras.

All the Eilenberg-type correspondences mentioned above fit into this categorical framework. For example, the categories  $\mathcal{C}$  of boolean algebras and  $\mathcal{D}$  of sets are pre-dual via Stone duality, and  $\mathcal{D}$ -monoids are ordinary monoids: this is the setting of Eilenberg’s original result. The category  $\mathcal{C}$  of distributive lattices with 0 and 1 is pre-dual to the category  $\mathcal{D}$  of posets via Birkhoff duality, and  $\mathcal{D}$ -monoids are ordered monoids, which leads to Pin’s result on pseudovarieties of ordered monoids [21]. The category  $\mathcal{C}$  of join-semilattices with 0 is self-pre-dual (i.e., one takes  $\mathcal{D} = \mathcal{C}$ ), and  $\mathcal{D}$ -monoids are precisely idempotent semirings. This is the framework for Polák’s result [23]. For Reutenauer’s result [26] one takes the category  $\mathcal{C}$  of vector spaces over a finite field  $K$  which is also self-pre-dual (i.e.,  $\mathcal{D} = \mathcal{C}$ ), and observes that  $\mathcal{D}$ -monoids are precisely  $K$ -algebras. Additional instances arise by applying our framework to further pre-dualities; for example, we will derive an Eilenberg-type correspondence for pseudovarieties of monoids with 0, which takes as  $\mathcal{C}$  non-unital boolean rings and as  $\mathcal{D}$  pointed sets.

Apart from pre-duality, the heart of the matter is a coalgebraic characterization of the closure properties defining varieties of languages. We model deterministic  $\Sigma$ -automata in a locally finite variety  $\mathcal{C}$  as coalgebras  $Q \rightarrow T_{\Sigma}Q$  for the endofunctor

$$T_{\Sigma} : \mathcal{C} \rightarrow \mathcal{C}, \quad T_{\Sigma}Q = O_{\mathcal{C}} \times Q^{\Sigma},$$

where  $O_{\mathcal{C}}$  is a fixed object of  $\mathcal{C}$  representing the outputs. In particular, the set of all regular languages over  $\Sigma$  carries the structure of a  $T_{\Sigma}$ -coalgebra which is characterized as the *rational fixed point*  $qT_{\Sigma}$  of  $T_{\Sigma}$ , i.e., the terminal locally finite coalgebra. To get a grasp on *all* regular languages, independent of a particular alphabet, we introduce a category-theoretical concept of preimage of a language under a  $\mathcal{D}$ -monoid morphism.

In our general setting of two pre-dual categories  $\mathcal{C}$  and  $\mathcal{D}$ , for every finite alphabet  $\Sigma$  a *local variety of languages in  $\mathcal{C}$*  is a subcoalgebra of the rational fixed point closed under derivatives (and we characterize this closedness by the existence of certain coalgebra homomorphisms). A *variety of languages* is thus a collection  $V_{\Sigma}$  (for all alphabets  $\Sigma$ ) of local varieties closed under preimages of  $\mathcal{D}$ -monoid morphisms (which we also characterize by means of coalgebra homomorphisms).

Dualization of these concepts leads to the notion of a *pseudovariety of  $\mathcal{D}$ -monoids* as a class of finite  $\mathcal{D}$ -monoids closed under finite products, submonoids and homomorphic images. Our main result is the following

**GENERALIZED EILENBERG THEOREM.** *Varieties of languages in  $\mathcal{C}$  correspond bijectively to pseudovarieties of  $\mathcal{D}$ -monoids.*

All Eilenberg-type theorems mentioned above emerge as special cases by the corresponding choices of  $\mathcal{C}$  and  $\mathcal{D}$ . On our way to proving this theorem we will also establish four additional correspondence results:

- (1) *Object-finite* varieties of languages in  $\mathcal{C}$  (i.e., varieties with  $V_\Sigma$  finite for all  $\Sigma$ ) correspond to locally finite varieties (not just pseudovarieties!) of  $\mathcal{D}$ -monoids. In the case of (ordered) monoids this was shown by Klíma and Polák [19], and to the best of our knowledge this is a new result in all other cases.
- (2) *Simple* varieties of languages (i.e., those generated by a single alphabet) correspond to simple varieties of  $\mathcal{D}$ -monoids (which are those generated by a single finite monoid).
- (3) *Local* varieties of languages over a given alphabet  $\Sigma$  correspond to pseudovarieties of  $\Sigma$ -generated monoids.
- (4) *Fully invariant* local varieties of languages over  $\Sigma$  (where closedness under preimages w.r.t. endomorphisms of the free  $\mathcal{D}$ -monoid on  $\Sigma$  is required) correspond to pseudovarieties of fully invariant  $\Sigma$ -generated  $\mathcal{D}$ -monoids.

*Related Work.* This paper is based on our conference contributions to FOSSACS 2014 [1] and LICS 2015 [2]. Its purpose is to lay a common ground for Eilenberg's variety theorem [14] and its variants due to Pin [21], Polák [23] and Reutenauer [26].

Besides full proofs our paper also presents new examples: partial automata (as algebras in the category of pointed sets) and partial ordered automata (as algebras in the category of posets with bottom). Moreover, the correspondence result for fully invariant local varieties is new. A completely different proof of this result (and of other Eilenberg-type correspondences mentioned above) has appeared in the paper of Chen and Urbat [10] which has been worked out in parallel to our paper.

Our work is inspired by Gehrke, Grigorieff and Pin [16] who used Stone and Priestley duality to prove a bijective correspondence between local varieties of languages and classes of finite (ordered)  $\Sigma$ -generated monoids presented by profinite identities. This result provides a local view of Reiterman's theorem [25] which characterizes pseudovarieties of monoids in terms of profinite identities.

Somewhat surprisingly, it has only been in recent years that the fundamental role of duality in algebraic automata theory was fully recognized. Most of the work along these lines concerns the connection between regular languages and profinite algebras. Rhodes and Steinberg [27] view the regular languages over  $\Sigma$  as a comonoid (rather than just a coalgebra) in the category of boolean algebras, and this comonoid is shown to dualize to the free profinite semigroup on  $\Sigma$ . Similar results for free profinite monoids can be found in the aforementioned work of Gehrke et al. which built on previous work of Almeida [5] and Pippenger [22].

Bojańczyk [8] generalized Eilenberg's theorem in a direction orthogonal to our approach. Whereas we keep recognition of languages by monoids fixed and vary the base category, he proposes to study recognition of languages by finite algebras for a monad but keeps the base category of (sorted) sets fixed. He proves an Eilenberg-type correspondence in this setting. In this way he captures a number of instances that are not captured by our approach, e.g. languages of infinite words or trees. But his setting does not capture e.g. Polák's or Reutenauer's result. Working out a common roof over these two approaches is the goal of our recent preprint [30].

In connection with work on Eilenberg-type correspondences also work on generalizing Reiterman's theorem [25] is of interest. The classical Reiterman theorem establishes a bijection between pseudovarieties of finite monoids and classes of finite monoid specified by profinite equations. Recently, Chen and three of the current authors have provided a generalized Reiterman theorem for algebras for a monad [11].

Finally, while our focus in this paper is on *finite* automata and algebras, there has been some work on variety theorems with relaxed finiteness restrictions, e.g. Reutenauer's theorem [26] for weighted languages over arbitrary fields. This requires a generalization of our framework from varieties to  $(\mathcal{E}, \mathcal{M})$ -structured categories  $\mathcal{D}$  and has been worked out in [29] and, independently, in [28].

## 2 $\mathcal{D}$ -MONOIDS

Throughout the paper, let  $\mathcal{D}$  denote a variety of (finitary) algebras or ordered algebras. This means that  $\mathcal{D}$  is a category of algebras for a given signature, specified by a set of equations, or a category of ordered algebras for a signature specified by inequations. The forgetful functor from  $\mathcal{D}$  to **Set** is denoted by

$$|\cdot| : \mathcal{D} \rightarrow \mathbf{Set}.$$

We sometimes omit writing it; in particular we write  $h$  for the underlying map of a  $\mathcal{D}$ -morphism  $h$ .

**Assumptions 2.1.** We assume that  $\mathcal{D}$

- (a) is locally finite, i.e., every finitely generated algebra in  $\mathcal{D}$  is finite;
- (b) is entropic (a.k.a. commutative), i.e., for every pair  $A, B$  of objects the set  $\mathcal{D}(A, B)$  forms a subobject of  $B^{|A|}$  in  $\mathcal{D}$ ; we denote it by  $[A, B] \hookrightarrow B^{|A|}$ , and
- (c) has all epimorphisms surjective.

**Examples 2.2.** Each of the following categories  $\mathcal{D}$  satisfies our assumptions; see the Appendix for details. The object  $[A, B]$  is the set of all morphisms of  $\mathcal{D}$  with the structure defined pointwise in each case.

- (1) **Set**, sets and functions, and **Set** $_{\star}$ , pointed sets and point-preserving functions.
- (2) **Pos**, posets and monotone functions, and **Pos** $_{\perp}$ , posets with a least element  $\perp$  and monotone functions that are *strict*, i.e. preserve  $\perp$ .

Note, however, that the category of posets with a least and a largest element does *not* present an example. Indeed, a variety with two constants is never entropic (since the constants do not commute with each other).

- (3) **JSL**, join-semilattices (i.e., posets with binary joins) and their homomorphisms. And **JSL** $_{\perp}$ , join semilattices with a least element  $\perp$  and strict semilattice homomorphisms.
- (4) Recall that a *semiring* is a commutative monoid  $(K, +, 0)$  together with an associative binary operation  $\cdot$  having 0 as a zero element and satisfying the distributive laws. A *module* (more precisely, a left semimodule) for  $K$  is a commutative monoid  $(M, +, 0)$  together with a scalar multiplication (from  $K \times M$  to  $M$  and written as juxtaposition) satisfying the usual laws:

$$\begin{aligned} k(x + y) &= kx + ky \\ (k + l)x &= kx + lx \\ (kl)x &= k(lx) \\ k0 &= 0 = 0x \end{aligned}$$

Finally a *K-algebra* is a module which is also a unitary semiring w.r.t. the same addition.

The category  $\mathbf{Mod}_K$  of modules over a finite commutative semiring  $K$  and linear functions satisfies our assumptions. In our leading examples we restrict ourselves to the case where  $K$  is a finite field and denote the category by  $\mathbf{Vec}_K$ . The reason is that from Section 4 onwards we need an additional assumption on  $\mathcal{D}$ : its finite algebras form a category whose dual is, again, the category of finite algebras of a variety. This holds for  $\mathbf{Vec}_K$  but not for modules in general.

*Notation 2.3.* The free-algebra functor, which is the left-adjoint of the forgetful functor  $|\cdot| : \mathcal{D} \rightarrow \mathbf{Set}$ , is denoted by

$$\Psi : \mathbf{Set} \rightarrow \mathcal{D}.$$

The unit of the adjunction is given by the universal maps  $\eta_X : X \rightarrow |\Psi X|$ . We often omit the subscript  $X$  if it is clear from the context.

We assume, without loss of generality, that every set  $X$  is a subset of  $|\Psi X|$  and the universal map is the inclusion  $X \hookrightarrow |\Psi X|$ . This is substantiated by the following lemma (in which a variety is called *nontrivial* if it contains an algebra on more than one element). We put

$$\mathbb{1}_{\mathcal{D}} = \Psi 1.$$

**LEMMA 2.4.** *If the variety  $\mathcal{D}$  is nontrivial, then for every set  $X$  there exists a free object on  $X$  in  $\mathcal{D}$  containing  $X$  and such that the universal map is the set inclusion.*

**PROOF.** It is sufficient to prove that given a universal map  $\eta_X : X \rightarrow |\Psi X|$ , then the underlying function is monic in  $\mathbf{Set}$ . Then the uniqueness of free objects up to isomorphism makes the statement of the lemma obvious. Let  $D$  be an object of  $\mathcal{D}$  containing two distinct elements  $d_1, d_2$ . Then for arbitrary distinct elements  $x_1, x_2$  of  $X$  choose a function  $f : X \rightarrow |D|$  mapping  $x_i$  to  $d_i$  for  $i = 1, 2$ . The unique morphism  $f' : \Psi X \rightarrow D$  of  $\mathcal{D}$  with  $f = f' \cdot \eta$  proves  $\eta(x_1) \neq \eta(x_2)$ , as required.  $\square$

*Remark 2.5.* The category  $\mathcal{D}$  has the factorization system (epi, strong mono). Here epimorphisms are precisely the surjective morphisms, due to Assumption 2.1(c), and strong monomorphisms are precisely

- (a) the injective morphisms if  $\mathcal{D}$  is a variety of algebras, and
- (b) the order-reflecting morphisms if  $\mathcal{D}$  is a variety of ordered algebras.

*Remark 2.6.* Recall the concept of tensor product  $A \otimes B$  of objects of a variety of (possibly ordered) algebras. Given an object  $C$ , by a *bimorphism* from  $A$  and  $B$  to  $C$  is meant a function  $f : |A| \times |B| \rightarrow |C|$  such that each  $f(a, -)$  carries a morphism  $B \rightarrow C$  and each  $f(-, b)$  carries a morphism  $A \rightarrow C$ . The *tensor product* of  $A$  and  $B$  is a universal bimorphism  $u : |A| \times |B| \rightarrow |A \otimes B|$ . That is, for every bimorphism  $f : |A| \times |B| \rightarrow |C|$  there exists a unique morphism  $f' : A \otimes B \rightarrow C$  with  $f = f' \cdot u$ .

A variety  $\mathcal{D}$  is entropic iff  $(\mathcal{D}, \otimes, \mathbb{1}_{\mathcal{D}})$  is a closed monoidal category, that is, there is a bijection  $\mathcal{D}(A \otimes B, C) \cong \mathcal{D}(A, [B, C])$  natural in  $A, B, C \in \mathcal{D}$ ; see Banaschewski and Nelson [6]. The concept of a  $\mathcal{D}$ -monoid in this monoidal category can be spelled out as follows:

*Definition 2.7.* A  $\mathcal{D}$ -monoid  $(D, \circ, i)$  consists of an object  $D$  of  $\mathcal{D}$  and a monoid structure with multiplication  $\circ$  and unit  $i$  on its underlying set  $|D|$  such that the multiplication is a bimorphism, i.e., for every  $x \in |D|$  both  $x \circ -$  and  $- \circ x$  are endomorphisms of  $D$  in  $\mathcal{D}$ . A *morphism*  $h : (D, \circ, i) \rightarrow (D', \circ', i')$  of  $\mathcal{D}$ -monoids is a morphism  $h : D \rightarrow D'$  in  $\mathcal{D}$  preserving the monoid structure.  $\mathcal{D}$ -monoids and their morphisms form a category

$$\mathbf{Mon} \mathcal{D}.$$

*Remark 2.8.* (a) Observe that  $\mathbf{Mon} \mathcal{D}$  is a variety of (ordered) algebras: we can add  $\circ$  and  $i$  to the signature of  $\mathcal{D}$ , and add the monoid axioms and equations expressing that  $\circ$  is a bimorphism to the (in)equations presenting  $\mathcal{D}$ .

(b) Consequently,  $\text{Mon } \mathcal{D}$  is a complete category with limits preserved by the forgetful functor  $\text{Mon } \mathcal{D} \rightarrow \mathcal{D}$ .

(c) Since  $(\mathcal{D}, \otimes, \mathbb{1}_{\mathcal{D}})$  is a monoidal closed category, for every morphism  $f : D \rightarrow D'$  and every object  $A \in \mathcal{D}$  precomposition with  $f$  defines a morphism

$$(-) \cdot f : [D', A] \rightarrow [D, A]$$

and postcomposition with  $f$  defines a morphism

$$f \cdot (-) : [A, D] \rightarrow [A, D'].$$

*Example 2.9.* The object  $[A, A]$  of all endomorphisms of (an arbitrary) object  $A$  is a  $\mathcal{D}$ -monoid with the neutral element  $id_A$ , and the multiplication given by composition.

Indeed, denote by  $\pi_y : [A, A] \rightarrow A$  the restricted projection of the power of  $A$  for every  $y \in |A|$ . That is,  $\pi_y(f) = f(y)$  for all  $f : A \rightarrow A$ .

We conclude that an endofunction  $h : \mathcal{D}(A, A) \rightarrow \mathcal{D}(A, A)$  is an endomorphism on  $[A, A]$  iff all the composites  $\pi_y \cdot h$  are morphisms in  $\mathcal{D}$ . And that composition is a bimorphism from  $[A, B]$  and  $[B, C]$  to  $[A, C]$  for all objects  $A, B$  and  $C$ .

*Example 2.10.* We describe  $\mathcal{D}$ -monoids for our leading examples of  $\mathcal{D}$ :

- (1) **Set**-monoids are the usual monoids. **Set<sub>\*</sub>**-monoids are monoids with a *zero*, i.e., an element  $0$  satisfying the equations  $x \cdot 0 = 0 = 0 \cdot x$ .
- (2) **Pos**-monoids correspond to ordered monoids, i.e. monoids carrying a partial order with monotone multiplication. **Pos<sub>⊥</sub>**-monoids are ordered monoids with a zero element which is equal to  $\perp$ .
- (3) **JSL<sub>⊥</sub>**-monoids are precisely *idempotent semirings*, i.e., semirings  $(K, +, \cdot, 0)$  satisfying  $x + x = x$ . To see this, observe that an idempotent commutative semigroup is precisely a semilattice.. And the distributive laws state that the multiplication preserve the operations  $+$  and  $0$  in each variable, which means precisely that it is a bimorphism. Analogously for **JSL**: the only difference is that the existence of a unit,  $0$ , for addition is not required.
- (4) **Vec<sub>K</sub>**-monoids are precisely *K-algebras*.

*Remark 2.11.* For every alphabet  $\Sigma$  concatenation of words in  $\Sigma^*$  extends uniquely to a  $\mathcal{D}$ -monoid operation  $\bullet$  on  $\Psi\Sigma^*$  for which  $(\Psi\Sigma^*, \bullet, \varepsilon)$  is the free  $\mathcal{D}$ -monoid on  $\Sigma$ . More precisely, given a  $\mathcal{D}$ -monoid  $(D, \bullet, i)$  and a function  $f : \Sigma \rightarrow D$ , there exists a unique extension to a  $\mathcal{D}$ -monoid morphism  $\tilde{f} : (\Psi\Sigma^*, \bullet, \varepsilon) \rightarrow (D, \bullet, i)$ .

We present a detailed proof in Theorem 2.20 below. A short categorical proof is based on the observation that both the forgetful functor  $|-| : \mathcal{D} \rightarrow \mathbf{Set}$  and its left adjoint  $\Psi : \mathbf{Set} \rightarrow \mathcal{D}$  are monoidal functors. Thus, they extend to functors  $\text{Mon } \mathcal{D} \rightarrow \mathbf{Mon}$  and  $\mathbf{Mon} \rightarrow \text{Mon } \mathcal{D}$ , respectively, where  $\mathbf{Mon}$  denotes the category of ordinary monoids. Moreover, the extended functors form again an adjoint pair; see [12, Lemma 2.14/2.16] for a detailed discussion.

- Examples 2.12.* (1) **Set** and **Pos**: here  $\Psi\Sigma^* = \Sigma^*$  (with discrete ordering).  
 (2) **Set<sub>\*</sub>** and **Pos<sub>⊥</sub>**:  $\Psi\Sigma^* = \Sigma^* \cup \{\perp\}$ , where  $\perp$  is a zero element for word concatenation.  
 (3) **JSL<sub>⊥</sub>**:  $\Psi\Sigma^* = \mathcal{P}_f \Sigma^*$  is the semilattice of all finite languages over  $\Sigma$  ordered by inclusion, with multiplication given by language concatenation. For **JSL** we get the subsemilattice of all nonempty finite languages.  
 (4) **Vec<sub>K</sub>**:  $\Psi\Sigma^*$  is the vector space of all finite languages weighted in  $K$ . Vector addition is the usual union of weighted languages, and multiplication is their usual concatenation.

*Notation 2.13.* For all algebras  $A$  and  $B$  in  $\mathcal{D}$  and  $x \in A$  let  $ev_x$  (evaluation at  $x$ ) be the composite

$$ev_x : [A, B] \xrightarrow{\quad} B^A \xrightarrow{\pi_x} B \quad \text{with} \quad ev_x(f) = f(x) \quad \text{for all } f \in [A, B].$$

The assumption that  $\mathcal{D}$  is entropic closed gives rise to an inductive definition principle that we shall use extensively.

D:ind

*Definition 2.14 (Inductive Extension Principle).* Let  $(g_i : A \rightarrow B)_{i \in I}$  be a family of morphisms between two fixed  $\mathcal{D}$ -objects  $A$  and  $B$ . Its *inductive extension* is the family  $(g_x : A \rightarrow B)_{x \in |\Psi I|}$  defined as follows:

(1) Extend the function  $g : I \rightarrow \mathcal{D}(A, B)$ ,  $i \mapsto g_i$ , to a  $\mathcal{D}$ -morphism  $\bar{g} : \Psi I \rightarrow [A, B]$ :

$$\begin{array}{ccc} I & & \\ \eta \downarrow & \searrow g & \\ |\Psi I| & \xrightarrow{\bar{g}} & [[A, B]] \end{array}$$

(2) Put  $g_x := \bar{g}(x)$  for all  $x \in |\Psi I|$ .

R:ind

*Remark 2.15.* The above notation is consistent:  $g_x$  is, for  $x = i \in I$ , the original morphism  $g_i$ . This follows easily from the fact that the universal map  $\eta : I \rightarrow |\Psi I|$  is the inclusion map.

L:ind

**LEMMA 2.16.** *For all sets  $I$  and  $\mathcal{D}$ -objects  $A$ , the family  $(\text{ev}_x : [\Psi I, A] \rightarrow A)_{x \in |\Psi I|}$  is the inductive extension of  $(\text{ev}_i : [\Psi I, A] \rightarrow A)_{i \in I}$ .*

**PROOF.** Extend the function  $g : I \rightarrow [[\Psi I, A], A]$ ,  $i \mapsto \text{ev}_i$ , to a  $\mathcal{D}$ -morphism  $\bar{g}$ :

$$\begin{array}{ccc} I & & \\ \eta \downarrow & \searrow g & \\ |\Psi I| & \xrightarrow{\bar{g}} & [[[\Psi I, A], A]] \end{array}$$

We need to show that the extended family consists of  $g_x = \text{ev}_x$  for all  $x \in |\Psi I|$ . To this end, we first prove the equation

$$\text{ev}_f \cdot \bar{g} = f \quad \text{for all } f : \Psi I \rightarrow A. \quad (2.1)$$

It suffices to prove  $\text{ev}_f \cdot \bar{g} \cdot \eta = f \cdot \eta$ , and indeed we have for all  $i \in I$ :

$$\begin{aligned} \text{ev}_f \cdot \bar{g} \cdot \eta(i) &= \text{ev}_f \cdot g(i) && \text{def. } \bar{g} \\ &= \text{ev}_f(\text{ev}_i) && \text{def. } g \\ &= \text{ev}_i(f) && \text{def. ev} \\ &= f(i) = f \cdot \eta(i) && \text{def. ev.} \end{aligned}$$

Therefore, for all  $x \in \Psi I$  and  $f : \Psi I \rightarrow A$ ,

$$\begin{aligned} g_x(f) &= \text{ev}_f(g_x) && \text{def. ev} \\ &= \text{ev}_f \cdot \bar{g}(x) && \text{def. } g_x \\ &= f(x) && \text{by (2.1)} \\ &= \text{ev}_x(f) && \text{def. ev,} \end{aligned}$$

so  $g_x = \text{ev}_x$  as claimed.  $\square$

eq: above

**L:ind2** LEMMA 2.17 (INDUCTIVE PROOF PRINCIPLE). *Given the following collection of commutative diagrams*

$$\begin{array}{ccc}
 & A & \xrightarrow{g_i} B \\
 a \nearrow & & \searrow b \\
 A_0 & & B_0 \\
 a' \searrow & & \nearrow b' \\
 & A' & \xrightarrow{g'_i} B'
 \end{array} \quad (i \in I)$$

*then the corresponding inductive extensions  $(g_x)$  and  $(g'_x)$  make the following diagrams*

$$\begin{array}{ccc}
 & A & \xrightarrow{g_x} B \\
 a \nearrow & & \searrow b \\
 A_0 & & B_0 \\
 a' \searrow & & \nearrow b' \\
 & A' & \xrightarrow{g'_x} B'
 \end{array} \quad (x \in |\Psi I|)$$

*also commutative.*

PROOF. Recall from Remark 2.8(c) that we have the morphism

$$b \cdot (-) \cdot a : [A, B] \rightarrow [A_0, B_0]$$

in  $\mathcal{D}$ , and analogously  $b' \cdot (-) \cdot a'$ . The assumption of our lemma states that the square

$$\begin{array}{ccc}
 & [A, B] & \\
 g \nearrow & & \searrow b \cdot (-) \cdot a \\
 I & & [A_0, B_0] \\
 g' \searrow & & \nearrow b' \cdot (-) \cdot a' \\
 & [A', B'] &
 \end{array}$$

commutes in  $\mathbf{Set}$ , where  $g(i) = g_i$  and  $g'(i) = g'_i$  for  $i \in I$ . Let  $\bar{g} : \Psi I \rightarrow [A, B]$  and  $\bar{g}' : \Psi I \rightarrow [A', B']$  be the corresponding morphisms in  $\mathcal{D}$ . It follows from the universal property of  $\Psi I$  that the following square (in  $\mathcal{D}$ ) also commutes:

$$\begin{array}{ccc}
 & [A, B] & \\
 \bar{g} \nearrow & & \searrow b \cdot (-) \cdot a \\
 \Psi I & & [A_0, B_0] \\
 \bar{g}' \searrow & & \nearrow b' \cdot (-) \cdot a' \\
 & [A', B'] &
 \end{array}$$

This completes the proof. □

**E: free** Example 2.18. The collection of left translations on  $\Sigma^*$

$$w \cdot - : \Sigma^* \rightarrow \Sigma^* \hookrightarrow |\Psi \Sigma^*|, \text{ for } w \in \Sigma^*$$

inductively defines a collection of endomorphisms

$$l_x : \Psi \Sigma^* \rightarrow \Psi \Sigma^* \text{ for } x \in |\Psi \Sigma^*|.$$

Analogously, the collection of right-translations  $- \cdot w$  inductively defines

$$r_x : \Psi \Sigma^* \rightarrow \Psi \Sigma^* \text{ for } x \in |\Psi \Sigma^*|.$$



LEMMA 2.19. For all elements  $x, y$  of  $|\Psi\Sigma^*|$  the following equations hold:

- (a)  $r_x \cdot l_y = l_y \cdot r_x$ , and  
 (b)  $r_y(x) = l_x(y)$

PROOF. (a) Consider the diagrams below:

$$\begin{array}{ccc}
 \Psi\Sigma^* & \xrightarrow{r_w} & \Psi\Sigma^* \\
 l_v \downarrow & & \downarrow l_v \\
 \Psi\Sigma^* & \xrightarrow{r_w} & \Psi\Sigma^* \\
 \\
 \Psi\Sigma^* & \xrightarrow{r_x} & \Psi\Sigma^* \\
 l_v \downarrow & & \downarrow l_v \\
 \Psi\Sigma^* & \xrightarrow{r_x} & \Psi\Sigma^* \\
 \\
 \Psi\Sigma^* & \xrightarrow{r_x} & \Psi\Sigma^* \\
 l_y \downarrow & & \downarrow l_y \\
 \Psi\Sigma^* & \xrightarrow{r_x} & \Psi\Sigma^*
 \end{array}$$

The left square commutes for all  $v, w \in \Sigma^*$ : indeed, for all  $u \in \Sigma^*$  we have

$$l_v(r_w(\eta u)) = \eta(vuw) = r_w(l_v(\eta u))$$

by the definition of  $l_v$  and  $r_w$ . By induction it follows that the middle square commutes for all  $v \in \Sigma^*$  and  $x \in \Psi\Sigma^*$ . Using induction again we conclude that the right square commutes for all  $x, y \in \Psi\Sigma^*$ .

- (b) We first prove that the following diagram commutes for all  $y \in \Psi\Sigma^*$  (where  $\bar{l}$  is defined by extending the map  $w \mapsto l_w$  as shown in Definition 2.14(1), i.e., we have  $\bar{l}(x) = l_x$  for every  $x \in |\Psi\Sigma^*|$ ):

$$\begin{array}{ccc}
 \Psi\Sigma^* & \xrightarrow{\bar{l}} & [\Psi\Sigma^*, \Psi\Sigma^*] \\
 \parallel & & \downarrow \text{ev}_y \\
 \Psi\Sigma^* & \xrightarrow{r_y} & \Psi\Sigma^*
 \end{array} \tag{2.2}$$

diag:ev

By the induction proof principle, Lemma 2.16, it suffices to prove that it commutes for  $y = \eta w$  where  $w \in \Sigma^*$ . Indeed, we have for all  $v \in \Sigma^*$ :

$$\begin{aligned}
 \text{ev}_w \cdot \bar{l}(v) &= \text{ev}_w(l_v) && \text{def. } \bar{l} \\
 &= l_v(w) && \text{def. ev} \\
 &= vw && \text{def. } l_v \\
 &= r_w(v) && \text{def. } r_w,
 \end{aligned}$$

and therefore  $\text{ev}_{\eta w} \cdot \bar{l} = r_{\eta w}$ . It follows that, for all  $x, y \in |\Psi\Sigma^*|$ :

$$\begin{aligned}
 r_y(x) &= \text{ev}_y \cdot \bar{l}(x) && \text{by (2.2)} \\
 &= \text{ev}_y(l_x) && \text{def. } \bar{l} \\
 &= l_x(y) && \text{def. ev.}
 \end{aligned} \quad \square$$

THEOREM 2.20. A free  $\mathcal{D}$ -monoid on a set  $\Sigma$  is given by the object  $\Psi\Sigma^*$ , the multiplication

$$x \bullet y = r_x(y) = l_y(x)$$

and the neutral element  $\varepsilon$ .

PROOF. (a) We prove that  $(\Psi\Sigma^*, \bullet, \varepsilon)$  is a  $\mathcal{D}$ -monoid.

- (1)  $\bullet$  is associative: for all  $x, y, z \in |\Psi\Sigma^*|$  we have

$$x \bullet (y \bullet z) = l_x(r_z(y)) = r_z(l_x(y)) = (x \bullet y) \bullet z$$

using the definition of  $\bullet$  and Lemma 2.19(a).

(2)  $\varepsilon$  is the neutral element: for all  $x \in \Phi\Sigma^*$  we have

$$x \bullet \varepsilon = r_\varepsilon(x) = id(x) = x,$$

where the last but one equation holds since  $- \cdot \varepsilon$  extends to the identity on  $\Psi\Sigma^*$ ; and symmetrically  $\varepsilon \bullet x = x$ .

(3)  $\bullet$  is a  $\mathcal{D}$ -bimorphism since, for all  $x \in \Psi\Sigma^*$ , the functions  $l_x = x \bullet -$  and  $r_x = - \bullet x$  are  $\mathcal{D}$ -morphisms.

(b) Now we verify the universal property. Given  $f : \Sigma \rightarrow |A|$ , where  $(A, \circ, i)$  is a  $\mathcal{D}$ -monoid, one can first extend  $f$  to a monoid morphism  $f' : \Sigma^* \rightarrow A$  (using that  $\Sigma^*$  is the free monoid on  $\Sigma$ ) and then extend  $f'$  to a  $\mathcal{D}$ -morphism  $\bar{f} : \Psi\Sigma^* \rightarrow A$  with  $\bar{f} \cdot \eta = f'$  (by the universal property of  $\eta$ ). We only need to verify that  $\bar{f}$  is a monoid morphism. Firstly,  $\bar{f}$  preserves the unit:

$$\begin{aligned} \bar{f}(\varepsilon) &= f'(\varepsilon) && \text{def. } \bar{f} \\ &= i && f' \text{ monoid morphism.} \end{aligned}$$

To prove that  $\bar{f}$  also preserves the multiplication, consider the squares below where  $l'_x, r'_w : A \rightarrow A$  are the  $\mathcal{D}$ -morphisms  $l'_x = \bar{f}(x) \circ -$  and  $r'_w = - \circ \bar{f}(w)$ :

$$\begin{array}{ccc} \Psi\Sigma^* & \xrightarrow{r_w} & \Psi\Sigma^* \\ \bar{f} \downarrow & & \downarrow \bar{f} \\ A & \xrightarrow{r'_w} & A \end{array} \quad \begin{array}{ccc} \Psi\Sigma^* & \xrightarrow{l_x} & \Psi\Sigma^* \\ \bar{f} \downarrow & & \downarrow \bar{f} \\ A & \xrightarrow{l'_x} & A \end{array}$$

The left-hand square commutes for every  $w \in \Sigma^*$  because, for all  $v \in \Sigma^*$ , we have

$$\begin{aligned} \bar{f} \cdot r_w(v) &= \bar{f}(vw) && \text{def. } r_w \\ &= f'(vw) && \text{def. } \bar{f} \\ &= f'(v) \circ f'(w) && f' \text{ monoid morphism} \\ &= \bar{f}(v) \circ \bar{f}(w) && \text{def. } \bar{f} \\ &= r'_w \cdot \bar{f}(v) && \text{def. } r'_w. \end{aligned}$$

Hence, the left-hand square commutes for all  $x \in |\Psi\Sigma^*|$  due to the universal property of  $\eta : \Sigma^* \rightarrow |\Psi\Sigma^*|$ . Similarly, the right-hand square commutes for every  $x \in |\Psi\Sigma^*|$  because, for every  $w \in \Sigma^*$ ,

$$\begin{aligned} \bar{f} \cdot l_x(w) &= \bar{f} \cdot r_w(x) && \text{Lemma 2.19(b)} \\ &= r'_w \cdot \bar{f}(x) && \text{left-hand square} \\ &= \bar{f}(x) \circ \bar{f}(w) && \text{def. } r'_w \\ &= l'_x \cdot \bar{f}(w) && \text{def. } l'_x. \end{aligned}$$

Consequently, we get the desired equation:

$$\begin{aligned} \bar{f}(x \bullet y) &= \bar{f}(l_x(y)) && \text{def. } \bullet \\ &= l'_x(\bar{f}(y)) && \text{right-hand square} \\ &= \bar{f}(x) \circ \bar{f}(y) && \text{def. } l'_x. \quad \square \end{aligned}$$

**L: fact** LEMMA 2.21. *The (epi, strong mono)-factorization system of  $\mathcal{D}$  lifts to  $\text{Mon } \mathcal{D}$ , i.e. every  $\mathcal{D}$ -monoid morphism factorizes essentially uniquely into a surjective morphism followed by an injective (or order-reflecting, resp.) one.*

PROOF. This follows from epis in  $\mathcal{D}$  being the surjective morphisms, see Assumption 2.1(c), and from the fact that  $\text{Mon } \mathcal{D}$  is a variety of (possibly ordered) algebras, see Remark 2.8.  $\square$

- quotient** Remark 2.22. (1) When speaking about quotients and subobjects in the category of  $\mathcal{D}$ -monoids, we always refer to the above factorization system. Quotients of a  $\mathcal{D}$ -monoid  $D$  are often denoted by  $e : D \rightarrow D'$ .
- (2) The factorization system mentioned in (1) is, in general, *not* the (epi, strong mono)-factorization system of  $\text{Mon } \mathcal{D}$ . For example in  $\mathcal{D} = \text{Set}$  there exist monoid epimorphisms that are not surjective.
- (3) Since  $\text{Mon } \mathcal{D}$  is a variety, as explained in Remark 2.8, the Homomorphism Theorem holds: Given a surjective morphism  $e : A \rightarrow B$  of  $\mathcal{D}$ -monoids and a morphism  $h : A \rightarrow C$  of  $\mathcal{D}$ -monoids of the form  $h = k \cdot e$ , it follows that  $k : B \rightarrow C$  is also a morphism of  $\mathcal{D}$ -monoids.

In Section 6 we investigate, for a given alphabet  $\Sigma$ , local varieties of languages over  $\Sigma$ . The corresponding concept on the side of monoids are the  $\Sigma$ -generated ones. This means that a set of generators indexed by  $\Sigma$  is given. Or, equivalently, that a surjective  $\mathcal{D}$ -monoid homomorphism from the free  $\mathcal{D}$ -monoid is specified:

Definition 2.23. A  $\Sigma$ -generated  $\mathcal{D}$ -monoid is a quotient of  $\Psi\Sigma^*$  in  $\text{Mon } \mathcal{D}$ , i.e., a  $\mathcal{D}$ -monoid  $A$  together with an epimorphism  $e : \Psi\Sigma^* \rightarrow A$ . A morphism of  $\Sigma$ -generated  $\mathcal{D}$ -monoids from  $e : \Psi\Sigma^* \rightarrow A$  to  $e' : \Psi\Sigma^* \rightarrow A'$  is a  $\mathcal{D}$ -monoid morphism  $f : A \rightarrow A'$  with  $f \cdot e = e'$ .

**gen: mon** Remark 2.24. (a) Recall the partial order on quotients of an object  $D$  of a category via factorization: given quotients  $e_i : D \rightarrow D_i$  ( $i = 1, 2$ ) we put

$$e_1 \sqsubseteq e_2 \text{ iff } e_1 \text{ factorizes through } e_2,$$

i.e., there exists  $f : D_2 \rightarrow D_1$  with  $e_1 = f \cdot e_2$ .

- (b) In particular, the class of all  $\Sigma$ -generated  $\mathcal{D}$ -monoids is partially ordered. This partial order is a lattice in which the join of two members  $e_i : \Psi\Sigma^* \rightarrow D_i$ ,  $i = 1, 2$ , called their *subdirect product*, is obtained by factorizing the morphism  $\langle e_1, e_2 \rangle$  in  $\text{Mon } \mathcal{D}$ :

$$\begin{array}{c} \langle e_1, e_2 \rangle \\ \curvearrowright \\ D \xrightarrow{e} D' \xrightarrow{m} D_1 \times D_2 \end{array}$$

Here  $e$  represents the subdirect product.

**gen: mon** Examples 2.25. (a) In  $\mathcal{D} = \text{Set}$  a  $\Sigma$ -generated monoid is precisely a monoid together with a set of generators indexed by  $\Sigma$ . The relation  $e_1 \sqsubseteq e_2$  means that a monoid homomorphism from  $D_1$  to  $D_2$  exists respecting the labeled generators.

- (b) Analogously in  $\mathcal{D} = \text{Pos}, \text{JSL}_\perp$  etc.

### 3 AUTOMATA AS ALGEBRAS

Given a finite set  $\Sigma$  of inputs, an automaton in  $\mathcal{D}$  consists of an object  $Q$  (of states), a specified initial state in  $|Q|$ , represented by a morphism  $\alpha_{\text{in}} : \mathbb{1}_{\mathcal{D}} \rightarrow Q$  (see Notation 2.3), and a 'next-state' endomorphism  $\alpha_a : Q \rightarrow Q$  in  $\mathcal{D}$  for every input  $a \in \Sigma$ . We do not specify accepting states at this level. (However, they will be specified later, as in Section 5.) Thus, the structure of an automaton can be summarized by a morphism  $\alpha : \mathbb{1}_{\mathcal{D}} + \coprod_{\Sigma} Q \rightarrow Q$  of  $\mathcal{D}$ . In other words:

**D:ant** *Definition 3.1.* A  $\Sigma$ -automaton in  $\mathcal{D}$  is an algebra for the endofunctor  $L_\Sigma : \mathcal{D} \rightarrow \mathcal{D}$  defined by

$$L_\Sigma = \mathbb{1}_{\mathcal{D}} + \coprod_{\Sigma} (-).$$

Given  $\Sigma$ -automata  $(Q, (\alpha_a), \alpha_{\text{in}})$  and  $(Q', (\alpha'_a), \alpha'_{\text{in}})$ , an *automata morphism* is a morphism  $h : Q \rightarrow Q'$  of  $\mathcal{D}$  which preserves initial states and transitions, i.e., the following diagrams commute:

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{D}} & \xrightarrow{\alpha_{\text{in}}} & Q \\ & \searrow \alpha'_{\text{in}} & \downarrow h \\ & & Q' \end{array} \quad \text{and} \quad \begin{array}{ccc} Q & \xrightarrow{\alpha_a} & Q \\ h \downarrow & & \downarrow h \\ Q' & \xrightarrow{\alpha'_a} & Q' \end{array} \quad \text{for every input } a \in \Sigma.$$

**E:ant** *Examples 3.2.* (1) For  $\mathcal{D} = \mathbf{Set}$ , this is the usual concept of a deterministic automata without a specification of final states.

(2) An automaton in  $\mathbf{Set}_*$  on the state set  $Q \cup \{*\}$  is precisely a partial deterministic automaton with states given by  $Q$ . Transitions are partial self-maps, and the initial state need not be specified. Analogously for  $\mathbf{Pos}_\perp$ .

(3) All other examples of  $\mathcal{D}$  above work analogously: just an additional algebraic structure making  $Q$  an object of  $\mathcal{D}$  is given and every transition preserves this structure.

**N:words** *Notation 3.3.* We use the expected notation for the action of an automaton  $(A, (\alpha_a), \alpha_{\text{in}})$  on words in  $\Sigma^*$ :

$$\alpha_w = \alpha_{a_n} \cdots \alpha_{a_1} : A \rightarrow A \quad \text{for } w = a_1 \cdots a_n.$$

The Inductive Extension Principle 2.14 then yields endomorphisms

$$\alpha_x : A \rightarrow A \quad \text{for all } x \in |\Psi\Sigma^*|.$$

**R:ant** *Remark 3.4.* (a) Every  $\Sigma$ -generated  $\mathcal{D}$ -monoid

$$e : \Psi\Sigma^* \rightarrow (D, \circ, i)$$

can be considered as a  $\Sigma$ -automaton with the state object  $D$ , the initial state  $i$ , and the following transitions  $\alpha_a$ :

$$x \mapsto x \circ e(a) \quad \text{for all } a \in \Sigma.$$

Indeed,  $- \circ e(a)$  is an endomorphism in  $D$  since  $\circ$  is a bimorphism.

(b) Every morphism  $h$  of  $\Sigma$ -generated  $\mathcal{D}$ -monoids:

$$\begin{array}{ccc} & \Psi\Sigma^* & \\ e \swarrow & & \searrow e' \\ (D, \circ, i) & \xrightarrow{h} & (D', *, i') \end{array}$$

is also an automata morphism. Indeed,  $h(i) = i'$  since  $h$  preserves the monoid unit, and for all  $a \in \Sigma$  and  $x \in D$  we have

$$h(x \circ e(a)) = h(x) * h(e(a)) = h(x) * e'(a),$$

which means that  $h$  preserves transitions.

(c) In particular, the initial automaton is obtained from the free  $\mathcal{D}$ -monoid:

$$\Psi\Sigma^* = \mu L_\Sigma.$$

Indeed, for  $\mathcal{D} = \mathbf{Set}$  this is well known. And for general  $\mathcal{D}$  observe that  $L_\Sigma$  is a finitary functor which is a lifting of the corresponding set functor  $L_\Sigma^0 X = 1 + \Sigma \times X$ , i.e.  $\Psi L_\Sigma^0 = L_\Sigma \Psi$ . Since

the forgetful functor  $|-| : \mathcal{D} \rightarrow \mathbf{Set}$  creates colimits of  $\omega$ -chains and  $\mu L_\Sigma$  is the colimit of the canonical chain  $\Psi 0 \rightarrow L_\Sigma \Psi 0 \rightarrow L_\Sigma L_\Sigma \Psi 0 \rightarrow \dots$  we deduce  $\mu L_\Sigma = \Psi \Sigma^*$  in  $\mathcal{D}$ .

Alternatively, one can also observe that the left adjoint  $\Psi : \mathbf{Set} \rightarrow \mathcal{D}$  lifts to a left adjoint  $\bar{\Psi} : \text{Alg } L_\Sigma^0 \rightarrow \text{Alg } L_\Sigma$  between the categories of algebras for  $L_\Sigma^0$  and  $L_\Sigma$  mapping the automaton  $(A, (\alpha_a), \alpha_\epsilon)$  to  $(\Psi A, (\Psi \alpha_a), \Psi \alpha_\epsilon)$ ; see [17, Theorem 2.14]. Since left adjoints preserve initial objects, it follows that  $\mu L_\Sigma = \bar{\Psi}(\mu L_\Sigma^0) = \Psi \Sigma^*$ .

- (d) Given an automaton  $A$ , the unique homomorphism from  $\mu L_\Sigma = \Psi \Sigma^*$  into it computes the action of  $A$  on elements  $x$  of  $\Psi \Sigma^*$ :

$$e_A(x) = \alpha_x \cdot \alpha_{\text{in}} : \mathbb{1}_{\mathcal{D}} \rightarrow A.$$

inherit

*Remark 3.5.* The category  $\text{Alg } L_\Sigma$  of  $\Sigma$ -automata inherits the (epi, strong mono)-factorization from  $\mathcal{D}$ , see Remark 2.5.

Indeed, given a homomorphism  $h : (A, \alpha) \rightarrow (B, \beta)$  of automata, take its (epi, strong mono)-factorization  $h = m \cdot e$  in  $\mathcal{D}$ . Since  $L_\Sigma$  preserves epis, we obtain via diagonal fill-in an algebra structure on the domain of  $m$  such that  $m$  and  $e$  are  $L_\Sigma$ -algebra homomorphisms:

$$\begin{array}{ccc} L_\Sigma A & \xrightarrow{\alpha} & A \\ L_\Sigma e \downarrow & & \downarrow e \\ L_\Sigma C & \dashrightarrow & C \\ L_\Sigma m \downarrow & & \downarrow m \\ L_\Sigma B & \xrightarrow{\beta} & B \end{array}$$

Moreover, given a commutative square of  $L_\Sigma$ -algebra homomorphisms, where  $e$  is an epimorphism and  $m$  is a strong monomorphism in  $\mathcal{D}$ :

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{e} & (B, \beta) \\ f \downarrow & \dashrightarrow d & \downarrow g \\ (C, \gamma) & \xrightarrow{m} & (D, \delta) \end{array}$$

the unique diagonal  $d$  is easily seen to be a homomorphism of automata; this follows from Lemma 3.6.

L: homom

**LEMMA 3.6 (HOMOMORPHISM THEOREM).** *Let  $e : A \twoheadrightarrow B$  be a surjective homomorphism of  $\Sigma$ -automata. Given a  $\Sigma$ -automaton  $C$  and a morphism  $f : B \rightarrow C$  in  $\mathcal{D}$  such that  $f \cdot e$  carries a homomorphism of  $\Sigma$ -automata, then so does  $f$ .*

**PROOF.** Since  $L_\Sigma$  preserves epimorphisms, and the upper square and the outside of the following diagram

$$\begin{array}{ccc} L_\Sigma A & \xrightarrow{\alpha} & A \\ L_\Sigma e \downarrow & & \downarrow e \\ L_\Sigma B & \xrightarrow{\beta} & B \\ L_\Sigma f \downarrow & & \downarrow f \\ L_\Sigma C & \xrightarrow{\gamma} & C \end{array}$$

commute, so does the lower square since it does when precomposed by the epimorphism  $L_\Sigma e$ .  $\square$

We have seen that  $\Sigma$ -generated monoids yield  $\Sigma$ -automata. We now aim towards a characterization of those  $\Sigma$ -automata that are obtained in this way. The following concept is the crucial step:

**Definition 3.7.** Let  $A = (Q, (\alpha_a), \alpha_{in})$  be a  $\Sigma$ -automaton. The *delayed  $\Sigma$ -automaton*  $A_b$  for  $b \in \Sigma$  has the same state object  $Q$  and the same transitions  $\alpha_a$ , and its initial state is given by

$$\mathbb{1}_{\mathcal{D}} \xrightarrow{\alpha_{in}} Q \xrightarrow{\alpha_b} Q$$

**THEOREM 3.8.** A  $\Sigma$ -automaton  $A$  stems from a  $\Sigma$ -generated monoid iff

- (i)  $A$  is reachable, i.e., the unique automata morphism from  $\Psi\Sigma^*$  is surjective, and
- (ii) there is an automaton morphism from  $A$  into each of its delayed automata  $A_b$ ,  $b \in \Sigma$ .

**PROOF.** I. Necessity. Given a  $\Sigma$ -generated monoid

$$e : \Psi\Sigma^* \rightarrow (D, \circ, i)$$

then (i) is obvious. For (ii) we observe that for every  $b \in \Sigma$  the  $\mathcal{D}$ -morphism

$$e(b) \circ - : D \rightarrow D$$

is an automata morphism from  $D$ , our monoid viewed as automaton, to  $D_b$ . That means that the diagrams below commute:

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{D}} & \xrightarrow{i} & D \\ & \searrow \alpha_b \cdot i & \downarrow e(b) \circ - \\ & & D \end{array} \quad \begin{array}{ccc} D & \xrightarrow{\alpha_a} & D \\ e(b) \circ - \downarrow & & \downarrow e(b) \circ - \\ D & \xrightarrow{\alpha_a} & D \end{array}$$

Indeed, the left-hand one clearly commutes and for the right-hand one we have for every  $x \in D$ :

$$e(b) \circ \alpha_a(x) = e(b) \circ (x \circ e(a)) = (e(b) \circ x) \circ e(a) = \alpha_a(e(b) \circ x).$$

II. Sufficiency. Let  $A = (D, (\alpha_a), i)$  be a reachable  $\Sigma$ -automaton with automata morphisms

$$\beta_b : A \rightarrow A_b \quad (b \in \Sigma).$$

We need to provide a monoid multiplication  $\circ$  on  $D$  with unit  $i$  such that the unique automata morphism

$$e : \Psi\Sigma^* \rightarrow D$$

is a monoid homomorphism. This determines  $\circ$  as follows: given  $x, y \in D$  we can find their preimages  $x', y' \in |\Psi\Sigma^*|$  under  $e$  and the fact that  $e$  must be a monoid morphism implies that

$$x \circ y = e(x' \bullet y').$$

Thus the only fact we need to verify is that this operation is well defined, i.e., independent of the choice of  $x'$  and  $y'$ . It then follows from the Homomorphism Theorem (see 2.22(3)) that since  $\Psi\Sigma^*$  is a  $\mathcal{D}$ -monoid, so is  $(D, \circ, i)$ .

In order to prove that  $\circ$  above is well defined, we first introduce the following notation, in analogy to Notation 3.3:

$$\beta_w = \beta_{b_n} \cdot \dots \cdot \beta_{b_1}$$

for all words  $w = b_1 \dots b_n$  in  $\Sigma^*$ . The Inductive Extension Principle 2.14 yields a family  $\beta_x : D \rightarrow D$  for  $x \in |\Psi\Sigma^*|$ .

- (1) For all  $x \in |\Psi\Sigma^*|$ , let  $A_x = (D, (\alpha_a), \alpha_x \cdot i)$ . We claim that  $\beta_x : A \rightarrow A_x$  is an automata homomorphism, which means that the following squares commute:

$$\begin{array}{ccc} D & \xrightarrow{\beta_x} & D \\ \alpha_a \downarrow & & \downarrow \alpha_a \\ D & \xrightarrow{\beta_x} & D \end{array} \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{i} & D \\ & \searrow \alpha_x \cdot i & \downarrow \beta_x \\ & & D \end{array}$$

Indeed, they clearly commute for  $x = w \in \Sigma^*$  because  $\beta_w : A \rightarrow A_w$  is an  $L_\Sigma$ -algebra morphism. Therefore they commute for all  $x \in |\Psi\Sigma^*|$  by the Inductive Proof Principle 2.16.

- (2) We prove the equation

$$e(x \bullet y) = \alpha_y(e(x)) \quad \text{for all } x, y \in |\Psi\Sigma^*|.$$

Observe first that the following diagram commutes for all  $y \in |\Psi\Sigma^*|$ :

$$\begin{array}{ccc} \Psi\Sigma^* & \xrightarrow{r_y} & \Psi\Sigma^* \\ e \downarrow & & \downarrow e \\ D & \xrightarrow{\alpha_y} & D \end{array}$$

In fact, it commutes if  $y = w$  for some  $w \in \Sigma^*$  because  $e$  is an  $L_\Sigma$ -algebra homomorphism, so it commutes for all  $y$  by the Inductive Proof Principle 2.16. Therefore

$$e(x \bullet y) = e(r_y(x)) = \alpha_y(e(x)).$$

- (3) We prove the equation

$$e(x \bullet y) = \beta_x(e(y)) \quad \text{for all } x, y \in \Psi\Sigma^*.$$

First note that  $l_x$  of Example 2.18 is an automata morphism  $l_x : \mu L_\Sigma \rightarrow (\mu L_\Sigma)_x$ : it preserves transitions because we have  $l_x \cdot r_a = r_a \cdot l_x$  by Lemma 2.19(a), and it also preserves the initial state since  $l_x(\varepsilon) = x$  and the initial state of  $(\mu L_\Sigma)_x$  is  $x$ . By (1), the following square:

$$\begin{array}{ccc} \mu L_\Sigma & \xrightarrow{l_x} & (\mu L_\Sigma)_x \\ e \downarrow & & \downarrow e \\ D & \xrightarrow{\beta_x} & D_x \end{array}$$

commutes due to the initiality of  $\mu L_\Sigma$ . Therefore

$$e(x \bullet y) = e(l_x(y)) = \beta_x(e(y)).$$

- (4) We are ready to prove that  $\circ$  is a well-defined bimorphism. By (2) above,  $x \circ y$  is independent of the choice of  $x'$  and moreover  $- \circ y = \alpha_{y'}$ , for any  $y' \in |\Psi\Sigma^*|$  with  $e(y') = y$  is a morphism of  $\mathcal{D}$ . Analogously (3) states that  $x \circ y$  is independent of the choice of  $y'$  and that  $x \circ -$  is a morphism of  $\mathcal{D}$ . It follows that  $\circ : D \times D \rightarrow D$  is a well-defined bimorphism, and by definition we have  $e(x' \bullet y') = e(x') \circ e(y')$  for all  $x', y' \in \Psi\Sigma^*$ . The associative and unit laws for  $\circ$  hold in  $D$  because they hold in  $\Psi\Sigma^*$  and  $e$  is surjective. Moreover  $i = e(\varepsilon)$  is the unit for  $\circ$ , thus  $e$  is a monoid morphism

$$e : \Psi\Sigma^* \rightarrow (D, \circ, i).$$

Finally, the quotient algebra of  $\mu L_\Sigma$  it induces is precisely the given one  $(D, (\alpha_a), i)$ . For this we need to show  $x \circ e(a) = \alpha_a$  for all  $a \in \Sigma$ . Given  $x \in D$ , we choose  $x' \in |\Psi\Sigma^*|$  with  $e(x') = x$  and compute

$$x \circ e(a) = e(x') \circ e(a) = e(x' \bullet a) = e(r_a(x')) = \alpha_a(e(x')) = \alpha_a(x),$$

using the definitions of  $\circ$  and  $\bullet$  and the fact the  $e$  is an automata homomorphism.  $\square$

#### 4 AUTOMATA AS COALGEBRAS

In this section, we augment the algebraic perspective on automata with a dual perspective: we consider automata as coalgebras in a category  $\mathcal{C}$  that dualizes to  $\mathcal{D}$  on the level of finite objects. To this end, we work with the following

*Assumptions 4.1.* In addition to our Assumptions 2.1 we assume from now on that our category  $\mathcal{D}$  is *predual* to a locally finite variety  $\mathcal{C}$  of algebras. This means means that the full subcategories  $\mathcal{D}_f$  and  $\mathcal{C}_f$  of finite algebras are dually equivalent. The forgetful functor of  $\mathcal{C}$  is again denoted by  $|\cdot| : \mathcal{C} \rightarrow \mathbf{Set}$ .

**Notation 4.2.** (a) The equivalence functor between  $\mathcal{D}_f$  and  $\mathcal{C}_f^{op}$  is denoted by  $D \mapsto \hat{D}$  and  $f \mapsto \hat{f}$ .

(b) For the object of  $\mathcal{C}$  corresponding to  $\mathbb{1}_{\mathcal{D}}$  we use the notation

$$O_{\mathcal{C}} = \hat{\mathbb{1}}_{\mathcal{D}}.$$

We call  $O_{\mathcal{C}}$  the *output object* and its underlying set

$$O = |O_{\mathcal{C}}|$$

the *set of outputs*.

(c) The endofunctor  $L_\Sigma = \mathbb{1}_{\mathcal{D}} + \coprod_{\Sigma} \text{Id}$  on  $\mathcal{D}$  dualizes to the endofunctor on  $\mathcal{C}$  denoted by

$$T_\Sigma = O_{\mathcal{C}} \times \prod_{\Sigma} \text{Id}.$$

**Example 4.3.** (a) The following examples (see Appendix for details) have the property that  $O_{\mathcal{C}}$ , which is unique up to isomorphism, can be chosen so that its underlying set is  $O = \{0, 1\}$ .

$\mathcal{D}$	$\mathcal{C}$	for $D \in \mathcal{D}_f$ we have $\hat{D} =$
<b>Set</b>	<b>BA</b> (boolean algebras)	$\mathcal{P}D$ (power-set of $D$ )
<b>Set<math>_{\star}</math></b>	<b>BR</b> (non-unital boolean rings)	$\mathcal{P}(D - \{*\})$
<b>Pos</b>	<b>DL<math>_b</math></b> (bounded distributive lattices)	all $\downarrow$ -sets of $D$ , ordered by $\subseteq$
<b>Pos<math>_{\perp}</math></b>	<b>DL<math>_{\perp}</math></b> (distributive lattices with $\perp$ )	all $\downarrow$ -sets of $D \setminus \{\perp\}$
<b>JSL<math>_{\perp}</math></b>	<b>JSL<math>_{\perp}</math></b>	$D$ with dual ordering
<b>JSL</b>	<b>JSL<math>_b</math></b> (bounded join-semilattices)	$D \setminus \{\perp\}$ with dual ordering

(b) Finite ( $n$ -dimensional) vector spaces are self-dual, i.e. we have  $\mathcal{C} = \mathbf{Vec}_K$  for  $\mathcal{D} = \mathbf{Vec}_K$ . Here  $\hat{D} = [D, K]$  is the space of all linear transformations on  $D$  and  $O_{\mathcal{C}} = K$ .

**Remark 4.4.** (a) A *coalgebra* for  $T_\Sigma$  consists of an object  $Q$  (of states) and a morphism  $\gamma : Q \rightarrow T_\Sigma Q$ . This morphism is characterized by its components

$$\gamma_{\text{out}} : Q \rightarrow O_{\mathcal{C}} \quad (\text{output morphism}) \quad \text{and} \quad \gamma_a : Q \rightarrow Q \quad \text{for } a \in \Sigma \quad (\text{transitions}).$$

Thus, coalgebras for  $T_\Sigma$  are automata with outputs but no initial states. However, we reserve the term *automaton* for algebras for  $L_\Sigma$  in  $\mathcal{D}$ .



- (b) A *coalgebra homomorphism* from  $(Q, (\gamma_a), \gamma_{\text{out}})$  to  $(Q', (\gamma'_a), \gamma'_{\text{out}})$  is a morphism  $f : Q \rightarrow Q'$  of  $\mathcal{C}$  preserving the outputs and the transitions. That means that the following diagram commutes for every  $a \in \Sigma$ :

$$\begin{array}{ccc} Q & \xrightarrow{\gamma_a} & Q \\ f \downarrow & & \downarrow f \\ Q' & \xrightarrow{\gamma'_a} & Q' \end{array} \quad \begin{array}{c} \nearrow \gamma_{\text{out}} \\ O_{\mathcal{C}} \\ \nwarrow \gamma'_{\text{out}} \end{array}$$

**L:T** LEMMA 4.5. *The categories of finite  $\Sigma$ -automata and finite  $T_\Sigma$ -coalgebras are dually equivalent under the equivalence functor*

$$(Q, (\delta_a), i) \mapsto (\hat{Q}, (\hat{\delta}_a), \hat{i}).$$

PROOF. To say that the functor  $(-)^{\widehat{\phantom{x}}} : \mathcal{D}_f \rightarrow \mathcal{C}_f^{\text{op}}$  is an equivalence means that it is full, faithful and isomorphism-dense (i.e., every object of  $\mathcal{C}_f$  is isomorphic to  $\hat{D}$  for some  $D \in \mathcal{D}_f$ ).

It is obvious that  $(-)^{\widehat{\phantom{x}}}$  lifts to a functor from finite  $\Sigma$ -automata to finite  $T_\Sigma$ -coalgebras. The lifting is automatically faithful. It is full, since given a coalgebra homomorphism

$$\begin{array}{ccc} \hat{Q} & \xrightarrow{\hat{\gamma}_a} & \hat{Q} \\ f \downarrow & & \downarrow f \\ \hat{Q}' & \xrightarrow{\hat{\gamma}'_a} & \hat{Q}' \end{array} \quad \begin{array}{c} \nearrow \hat{\gamma}_{\text{out}} \\ O_{\mathcal{C}} \\ \nwarrow \hat{\gamma}'_{\text{out}} \end{array}$$

there exists a unique  $f_0 : Q' \rightarrow Q$  in  $\mathcal{D}$  with  $\hat{f}_0 = f$ , and from the above commutative diagram, since  $(-)^{\widehat{\phantom{x}}}$  is faithful, we conclude that  $f_0 : (Q, (\delta_a), i) \rightarrow (Q', (\delta'_a), i')$  is a morphism of  $\Sigma$ -automata.

Finally, every finite  $T_\Sigma$ -coalgebra  $(C, (\gamma_a), \gamma_{\text{out}})$  is isomorphic to the dual of a  $\Sigma$ -automaton as follows: choose  $Q$  in  $\mathcal{D}_f$  and an isomorphism  $f : C \rightarrow \hat{Q}$ . There exists a unique morphism  $i : \mathbb{1}_{\mathcal{D}} \rightarrow Q$  with  $\hat{i} \cdot f = \gamma_{\text{out}}$ . And for every  $a \in \Sigma$  there exists a unique  $\delta_a : Q \rightarrow Q$  with  $\hat{\delta}_a = f \cdot \gamma_a \cdot f^{-1}$ . Then  $(Q, (\delta_a), i)$  is a finite automaton with an isomorphism

$$f : (C, (\gamma_a), \gamma_{\text{out}}) \rightarrow (\hat{Q}, (\hat{\delta}_a), \hat{i}). \quad \square$$

**E: coalg** *Example 4.6.* (a) For all the categories  $\mathcal{C}$  in Example 4.3(a), where  $|O_{\mathcal{C}}| = \{0, 1\}$ , the output morphism  $\gamma_{\text{out}} : Q \rightarrow O_{\mathcal{C}}$  is determined by a subset of  $|Q|$ , the preimage of 1 under  $\gamma_{\text{out}}$ . Thus, finite  $T_\Sigma$ -coalgebras are precisely the classical finite deterministic automata with an additional structure (of an object of  $\mathcal{C}$ ) on the state set preserved by transitions and the final state predicate.

- (b) A finite  $T_\Sigma$ -coalgebra in  $\text{Vec}_K$  is a classical Moore automaton with the set  $K$  of outputs and with a vector space structure on  $Q$  such that all transitions are linear and the output function  $\gamma_{\text{out}} : Q \rightarrow K$  is also linear. These are the usual weighted automata with weights in  $K$ .

Every finite  $\Sigma$ -automaton  $(Q, (\delta_a), i)$  yields a  $T_\Sigma$ -coalgebra on the state space  $[Q, K]$  of linear forms with transitions  $f \mapsto f \cdot \delta_a$  (for  $f : Q \rightarrow K$ ) and output map  $f \mapsto f(i)$ . Every finite  $T_\Sigma$ -coalgebra is, up to isomorphism, of the form  $[Q, K]$ .

**language** *Definition 4.7.* Let  $\Sigma$  be an alphabet.

- (a) A *language* is a function from  $\Sigma^*$  to  $O$  (the output set).

(b) A state  $q \in |Q|$  of a  $T_\Sigma$ -coalgebra  $(Q, (\gamma_a), \gamma_{\text{out}})$  *accepts* the language

$$L_Q(q) : \Sigma^* \rightarrow O$$

assigning to a word  $w$  the output  $\gamma_{\text{out}} \cdot \gamma_w(q)$ , where we use the analogue of Notation 3.3.

(c) A language is called *regular* if it is accepted by a state of some finite  $T_\Sigma$ -coalgebra.

**Examples 4.8.** In all our examples except  $\mathbf{Vec}_K$  we have  $O = \{0, 1\}$ , thus, languages are precisely the subsets of  $\Sigma^*$  (represented by their characteristic functions). And “regular” has its usual meaning. For  $\mathbf{Vec}_K$  languages are weighted with weights in  $K$ .

**Remark 4.9.** The behaviours of finite coalgebras for a given endofunctor can be captured by its *rational fixed point* [3], which is constructed as the (filtered) colimit of all finite coalgebras. Thus, the rational fixed point

$$\varrho T_\Sigma$$

of the endofunctor  $T_\Sigma$  is the colimit of all finite coalgebras for  $T_\Sigma$ . Its coalgebra structure  $T_\Sigma \rightarrow T_\Sigma(\varrho T_\Sigma)$  is an isomorphism, see [3].

Milius [20] introduced the concept of a locally finitely presentable coalgebra: it is a coalgebra which is a filtered colimit of finite coalgebras. Equivalently, this is a coalgebra such that every element lies in a finite subcoalgebra. In our case of a locally finite variety  $\mathcal{C}$  we can speak about *locally finite* coalgebras. Milius proved the following

**PROPOSITION 4.10** ([20]). *The rational fixed point  $\varrho T_\Sigma$  is the terminal locally finite coalgebra.*

Recall that given a word  $w$  over  $\Sigma$ , the left derivative of a language  $L$  is the language

$$w^{-1}L = L(w-),$$

which is just the composite  $L \cdot l_w$  of  $L : \Sigma^* \rightarrow O$  and the left translation  $l_w : \Sigma^* \rightarrow \Sigma^*$ , sending a word  $u$  to  $wu$ . Analogously, the right derivative is the language

$$Lw^{-1} = L(-w),$$

which is the composite  $L \cdot r_w$  of  $L$  and the right translation  $r_w : \Sigma^* \rightarrow \Sigma^*$ . We extend this from words to all elements of  $\Psi\Sigma^*$ , using the notation of Example 2.18:

**Definition 4.11.** The *derivatives* of a language  $L : \Sigma^* \rightarrow O$  are defined, for all elements  $x$  of  $|\Psi\Sigma^*|$ , as follows:

$$x^{-1}L = L \cdot l_x \text{ (left derivative)} \quad \text{and} \quad Lx^{-1} = L \cdot r_x \text{ (right derivative).}$$

**Example 4.12** ([3]). The endofunctor  $T_\Sigma^0 X = \{0, 1\} \times X^\Sigma$  on  $\mathbf{Set}$  has the rational fixed point

$$\varrho T_\Sigma^0 = \mathbf{Reg} \Sigma \quad \text{of all regular languages over } \Sigma.$$

Its transitions are given by left derivatives

$$\gamma_a(L) = a^{-1}L \quad \text{for all regular languages } L \subseteq \Sigma^* \text{ and } a \in \Sigma,$$

and the final states are the languages containing the empty word.

This is a colimit of all finite  $\Sigma$ -automata  $Q$  (without initial states): the colimit injection

$$c_Q : Q \rightarrow \mathbf{Reg} \Sigma$$

assigns to every state  $q$  the language that  $q$  (as an initial state) accepts.

Example 4.12 generalizes immediately to our setting:

**PROPOSITION 4.13.** *The rational fixed point  $\varrho T_\Sigma$  is a lifting of the above automaton  $\mathbf{Reg} \Sigma$  of regular languages to  $\mathcal{C}$ .*

More precisely, we have  $\varrho T_\Sigma = (Q, (\gamma_a), \gamma_{out})$  such that

$$|Q| = \mathbf{Reg} \Sigma,$$

and the colimit cocone  $c_Q : Q \rightarrow \varrho T_\Sigma$  of the diagram of all finite  $T_\Sigma$ -coalgebras is given by

$$c_Q(q) = \text{the language accepted by } q.$$

PROOF. Our functor  $T_\Sigma = O_{\mathcal{C}} \times (-)^\Sigma$  is a lifting of the functor  $T_\Sigma^0 = O \times (-)^\Sigma$  from  $\mathbf{Set}$  to  $\mathcal{C}$  (since  $\mathcal{C}$ , being a variety, has products formed on the level of  $\mathbf{Set}$ ). Consider the forgetful functor

$$\mathbb{U} : \mathbf{Coalg} T_\Sigma \rightarrow \mathbf{Coalg} T_\Sigma^0,$$

along with its restriction

$$\mathbb{V} : (\mathbf{Coalg} T_\Sigma)_f \rightarrow (\mathbf{Coalg} T_\Sigma^0)_f$$

to the full subcategories given by all finite coalgebras. Clearly  $\mathbb{U}$  is finitary (i.e. preserves filtered colimits) and we have the commutative square

$$\begin{array}{ccc} \mathbf{Coalg} T_\Sigma & \xrightarrow{\mathbb{U}} & \mathbf{Coalg} T_\Sigma^0 \\ \uparrow I & & \uparrow I_0 \\ \mathbf{Coalg}_f T_\Sigma & \xrightarrow{\mathbb{V}} & \mathbf{Coalg}_f T_\Sigma^0 \end{array}$$

where  $I$  and  $I_0$  are the inclusion functors. We will prove below that the functor  $\mathbb{V}$  is cofinal<sup>1</sup>. This then implies the claim:

$$\begin{aligned} \mathbb{U}(\varrho T_\Sigma) &= \mathbb{U}(\text{colim } I) && \text{def. } \varrho T_\Sigma \\ &\cong \text{colim}(\mathbb{U}I) && \mathbb{U} \text{ finitary} \\ &\cong \text{colim}(I_0 \mathbb{V}) && \mathbb{U}I = I_0 \mathbb{V} \\ &\cong \text{colim}(I_0) && \mathbb{V} \text{ cofinal} \\ &= \varrho T_\Sigma^0 && \text{def. } \varrho T_\Sigma^0. \end{aligned}$$

The cofinality of  $\mathbb{V}$  requires to prove that for every finite  $T_\Sigma^0$ -coalgebra  $(Q, \gamma)$  the slice category  $(Q, \gamma) \downarrow \mathbb{V}$  is nonempty and connected, that is,

- (1) there exists a  $T_\Sigma^0$ -coalgebra homomorphism  $(Q, \gamma) \rightarrow \mathbb{V}(Q', \gamma')$  for some finite  $T_\Sigma$ -coalgebra  $(Q', \gamma')$ , and
- (2) any two coalgebra homomorphisms  $(Q, \gamma) \rightarrow \mathbb{V}(Q', \gamma')$  and  $(Q, \gamma) \rightarrow \mathbb{V}(Q'', \gamma'')$  are connected by a zig-zag of morphisms in  $(Q, \gamma) \downarrow \mathbb{V}$ .

Proof of (1). Let  $\Phi : \mathbf{Set} \rightarrow \mathcal{C}$  be the left adjoint of the forgetful functor  $U : \mathcal{C} \rightarrow \mathbf{Set}$ , and denote the unit and counit of the adjunction by  $\eta$  and  $\epsilon$ , respectively. Given a finite  $T_\Sigma^0$ -coalgebra  $\gamma : Q \rightarrow T_\Sigma^0 Q$  form the “free”  $T_\Sigma$ -coalgebra

$$\Phi Q \xrightarrow{\Phi \gamma} \Phi T_\Sigma^0 Q \xrightarrow{\Phi T_\Sigma^0 \eta_Q} \Phi T_\Sigma^0 U \Phi Q = \Phi U T_\Sigma \Phi Q \xrightarrow{\epsilon_{T_\Sigma \Phi Q}} T_\Sigma \Phi Q. \quad (4.1)$$

Note that  $\Phi Q$  is finite because  $\mathcal{C}$  is locally finite. Then

$$\eta_Q : (Q, \gamma) \rightarrow \mathbb{V}(\Phi Q, \epsilon_{T_\Sigma \Phi Q} \cdot \Phi T_\Sigma^0 \eta_Q \cdot \Phi \gamma)$$

<sup>1</sup>Recall that a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is *cofinal* if, for every  $B \in \mathcal{B}$ , the comma category  $B \downarrow F$  of all morphisms  $B \rightarrow FA$  ( $A \in \mathcal{A}$ ) is connected. Cofinality of  $F$  implies that for every diagram  $D : \mathcal{B} \rightarrow \mathcal{C}$  one has  $\text{colim } GF \cong \text{colim } G$ .

is a coalgebra homomorphism; indeed, the diagram below commutes by the naturality of  $\eta$  and the triangle identity  $U\epsilon \cdot \eta U = id$ :

$$\begin{array}{c}
 Q \xrightarrow{\gamma} T_{\Sigma}^0 Q \\
 \eta_Q \downarrow \quad \eta_{T_{\Sigma}^0 Q} \swarrow \quad T_{\Sigma}^0 \eta_Q \swarrow \quad T_{\Sigma}^0 \eta_Q \downarrow \\
 U\Phi Q \xrightarrow{U\Phi\gamma} U\Phi T_{\Sigma}^0 Q \xrightarrow{U\Phi T_{\Sigma}^0 \eta_Q} U\Phi T_{\Sigma}^0 U\Phi Q = U\Phi U T_{\Sigma} \Phi Q \xrightarrow{U\epsilon_{T_{\Sigma} \Phi Q}} U T_{\Sigma} \Phi Q = T_{\Sigma}^0 U\Phi Q \\
 \eta_{T_{\Sigma}^0 U\Phi Q} \swarrow \quad \eta_{T_{\Sigma}^0 U\Phi Q} \swarrow \quad \eta_{T_{\Sigma}^0 U\Phi Q} \swarrow \\
 T_{\Sigma}^0 U\Phi Q \xrightarrow{\eta_{T_{\Sigma}^0 U\Phi Q}} U T_{\Sigma} \Phi Q
 \end{array}$$

Proof of (2). Given any coalgebra homomorphism  $h : (Q, \gamma) \rightarrow \mathbb{V}(Q', \gamma')$  there exists a unique  $\mathcal{C}$ -morphism  $\bar{h} : \Phi Q \rightarrow Q'$  with  $U\bar{h} \cdot \eta_Q = h$  by the universal property of  $\eta$ . We claim that  $\bar{h}$  is a coalgebra homomorphism

$$\bar{h} : (\Phi Q, \epsilon_{T_{\Sigma} \Phi Q} \cdot \Phi T_{\Sigma}^0 \eta_Q \cdot \Phi \gamma) \rightarrow (Q', \gamma').$$

Indeed, the outside, left- and right-hand and upper square in the diagram below commute (for the upper square see (1)):

$$\begin{array}{ccc}
 Q & \xrightarrow{\gamma} & T_{\Sigma}^0 Q \\
 \eta_Q \downarrow & & \downarrow T_{\Sigma}^0 \eta_Q \\
 U\Phi Q & \xrightarrow{U(\epsilon_{T_{\Sigma} \Phi Q} \cdot \Phi T_{\Sigma}^0 \eta_Q \cdot \Phi \gamma)} & T_{\Sigma}^0 U\Phi Q \\
 U\bar{h} \downarrow & & \downarrow T_{\Sigma}^0 U\bar{h} \\
 UQ' & \xrightarrow{U\gamma'} & T_{\Sigma}^0 UQ'
 \end{array}$$

This, so does the lower square when precomposed with  $\eta_Q$ , from which the desired equation  $\gamma' \cdot \bar{h} = T_{\Sigma}^0 \bar{h} \cdot \epsilon_{T_{\Sigma} \Phi Q} \cdot \Phi T_{\Sigma}^0 \eta_Q \cdot \Phi \gamma$  follows. Now given two coalgebra homomorphisms  $h : (Q, \gamma) \rightarrow \mathbb{V}(Q', \gamma')$  and  $k : (Q, \gamma) \rightarrow \mathbb{V}(Q'', \gamma'')$ , the desired zig-zag in  $\text{Coalg}_f T_{\Sigma}$  is  $Q' \xleftarrow{\bar{h}} \Phi Q \xrightarrow{\bar{k}} Q''$ .  $\square$

*Remark 4.14.* In the previous proof, we established the cofinality of  $\mathbb{V}$  directly. For a more conceptual approach one can observe that the adjunction  $\Phi \dashv | \cdot | : \mathcal{C}_f \rightarrow \text{Set}_f$  lifts to an adjunction  $\bar{\Phi} \dashv \mathbb{V} : \text{Coalg}_f T_{\Sigma} \rightarrow \text{Coalg}_f T_{\Sigma}^0$ , where  $\bar{\Phi}$  maps a finite  $T_{\Sigma}^0$ -coalgebra  $Q \rightarrow T_{\Sigma}^0 Q$  to (4.1); see [17, Corollary 2.15]. Thus the functor  $\mathbb{V}$  is cofinal, being a right adjoint.

*Example 4.15.* In all our leading examples except  $\text{Vec}_K$  the rational fixed point  $\varrho T_{\Sigma}$  is the set of all regular languages  $L \subseteq \Sigma^*$  with operations of the variety  $\mathcal{C}$  given by the usual set-theoretical operations (e.g., finite union and finite intersection).

For  $\text{Vec}_K$ ,  $\varrho T_{\Sigma}$  is the vector space of all  $K$ -weighted regular languages  $L : \Sigma^* \rightarrow K$  with addition and scalar multiplication defined pointwise, e.g.

$$(L + L')(w) = L(w) + L'(w).$$

Recall that the set  $O$  of outputs carries the object  $O_{\mathcal{C}}$  of  $C$  corresponding to the free object of  $\mathcal{D}$  on one generator. The situation is completely symmetrical:

**L:O** LEMMA 4.16. *The output set  $O$  carries an object  $O_{\mathcal{D}}$  of  $\mathcal{D}$  such that  $\hat{O}_{\mathcal{D}}$  is a free object of  $\mathcal{C}$  on one generator.*

PROOF. Let  $\mathbb{1}_{\mathcal{C}}$  be free on one generator in  $\mathcal{C}$ . Since  $\mathcal{C}$  is locally finite this object lies in  $\mathcal{C}_f$  and is thus isomorphic to  $\hat{D}$  for some  $D \in \mathcal{D}_f$ . We observe that the sets  $|D|$  and  $O$  are isomorphic using the duality of  $\mathcal{D}_f$  and  $\mathcal{C}_f$ :

$$|D| \cong \mathcal{D}_f(\mathbb{1}_{\mathcal{D}}, D) \cong \mathcal{C}_f(\hat{D}, \hat{\mathbb{1}}_{\mathcal{D}}) \cong \mathcal{C}_f(\mathbb{1}_{\mathcal{C}}, O_{\mathcal{C}}) \cong O.$$

Since  $\mathcal{D}$  is a variety of (possibly ordered) algebras, there exists an object  $O_{\mathcal{D}}$  on the set  $O$  isomorphic to  $D$ . Then  $\hat{O}_{\mathcal{D}}$  is isomorphic to  $\hat{D} \cong \mathbb{1}_{\mathcal{C}}$ , thus,  $\hat{O}_{\mathcal{D}}$  is also free on one generator in  $\mathcal{C}$ .  $\square$

*Remark 4.17.* The objects  $O_{\mathcal{C}}$  and  $O_{\mathcal{D}}$  form *dualizing objects* for the dual equivalence between  $\mathcal{C}_f$  and  $\mathcal{D}_f$ : letting  $(\hat{\cdot}) : \mathcal{D}_f^{op} \rightarrow \mathcal{C}$  and  $(\tilde{\cdot}) : \mathcal{C}_f^{op} \rightarrow \mathcal{D}_f$  denote the two equivalence functors, one has

$$|\hat{D}| \cong \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \hat{D}) \cong \mathcal{D}(D, \tilde{\mathbb{1}}_{\mathcal{C}}) \cong \mathcal{D}(D, O_{\mathcal{D}})$$

for every  $D \in \mathcal{D}$ , and symmetrically  $|\tilde{C}| \cong \mathcal{C}(C, O_{\mathcal{C}})$  for every  $C \in \mathcal{C}$ . In other words, the equivalence functors are liftings of representable functors. See Johnstone [18] for a general discussion of dualizing objects in a setting of concrete categories.

**C:O** COROLLARY 4.18. *To give a language  $L$  over  $\Sigma$  is the same as to give a morphism*

$$\tilde{L} : \Psi\Sigma^* \rightarrow O_{\mathcal{D}} \text{ in } \mathcal{D}.$$

Indeed,  $L : \Sigma^* \rightarrow O = |O_{\mathcal{D}}|$  extends uniquely to a morphism  $\tilde{L} : \Psi\Sigma^* \rightarrow O_{\mathcal{D}}$ .

Recall that Eilenberg's definition of a variety of regular languages requires closedness under preimages w.r.t. morphisms  $h$  of free monoids. In **Set** the preimage of a language  $L \subseteq \Sigma^*$  is simply  $h^{-1}(L)$ . If  $L$  is identified with the characteristic function  $L : \Sigma^* \rightarrow \{0, 1\}$ , then the preimage is given by the composite  $\Delta^* \xrightarrow{h} \Sigma^* \xrightarrow{L} \{0, 1\}$ . This generalizes to our situation immediately:

**E:preim** *Definition 4.19.* Let  $h : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  be a  $\mathcal{D}$ -monoid morphism. The *preimage*  $h^{-1}(L)$  of a language  $L : \Sigma^* \rightarrow O$  under  $h$  is the language represented by the morphism  $\tilde{L} \cdot h$  of  $\mathcal{D}$ , or, equivalently, the following language in **Set**:

$$\Delta^* \hookrightarrow |\Psi\Delta^*| \xrightarrow{h} |\Psi\Sigma^*| \xrightarrow{\tilde{L}} O.$$

**E:preim** *Example 4.20.* (1) Let  $\mathcal{D} = \mathbf{Set}, \mathbf{Set}_*, \mathbf{Pos}$  or  $\mathbf{Pos}_{\perp}$ . Then the preimage of  $L \subseteq \Sigma^*$  is, as expected, simply the language  $h^{-1}(L)$ .

(2) For  $\mathcal{D} = \mathbf{JSL}_{\perp}$  we are given a semiring homomorphism  $h : \mathcal{P}_f\Delta^* \rightarrow \mathcal{P}_f\Sigma^*$ . A language  $L \subseteq \Sigma^*$  corresponds to the semilattice homomorphism from  $\mathcal{P}_f\Sigma^*$  to  $\{0, 1\}$  taking a finite language to 1 iff it meets  $L$ . Thus the preimage of  $L$  consists of words  $w$  over  $\Delta$  such that  $h(\{w\})$  meets  $L$ .

(3) For  $\mathcal{D} = \mathbf{Vec}_K$  we are given a linear map  $h : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ , where  $\Psi\Sigma^*$  consists of finite  $K$ -weighted languages over  $\Sigma$ . The preimage of a  $K$ -weighted language  $L$  over  $\Sigma$  assigns to a word  $w \in \Delta^*$  with  $h(w) = \sum_i k_i v_i$  the weight  $\sum_i k_i L(v_i)$ .

**ization** *Remark 4.21.* (a) Since  $\mathcal{C}$  is a variety, it has regular factorizations, and regular epimorphisms, being precisely the surjective ones, coincide with the strong epimorphisms. Thus, dually to the factorization system (epi, strong mono) on  $\mathcal{D}$  we use the factorization system (strong epi, mono) on  $\mathcal{C}$ .

(b) This factorization system lifts to the category of  $T_{\Sigma}$ -coalgebras. Indeed,  $T_{\Sigma}X = O_{\mathcal{C}} \times X^{\Sigma}$  preserves monomorphisms, so the proof is dual to that of Remark 3.5.

(c) When speaking about *subcoalgebras* we always regard them w.r.t. the above factorization system. More precisely, to give a subcoalgebra of a coalgebra  $(Q, (\gamma_a), \gamma_{out})$  means to give a subobject  $m : Q_0 \rightarrow Q$  of  $Q$  in  $\mathcal{C}$  closed under the transitions, i.e., such that each  $\gamma_a$  has a domain-codomain restriction to  $Q_0$ .

Observe that subobjects of  $Q$  are always representable by subsets of  $|Q|$  closed under the operations defining the variety  $\mathcal{C}$ .

**LEMMA 4.22 (HOMOMORPHISM THEOREM).** *Given a subcoalgebra  $m : A_0 \rightarrow A$  for  $T_\Sigma$ , then for every coalgebra  $B$  and every map  $h : |B| \rightarrow |A_0|$  such that  $m \cdot h : B \rightarrow A$  is a coalgebra homomorphism so is  $h : B \rightarrow A_0$ .*

**PROOF.** In the following diagram

$$\begin{array}{ccc}
 B & \xrightarrow{\beta} & T_\Sigma B \\
 h \downarrow & & \downarrow T_\Sigma h \\
 A_0 & \xrightarrow{\alpha_0} & T_\Sigma A_0 \\
 m \downarrow & & \downarrow T_\Sigma m \\
 A & \xrightarrow{\alpha} & T_\Sigma A
 \end{array}$$

the outside and the lower square commute. Since  $T_\Sigma m$  is a monomorphism, the upper square also commutes.  $\square$

sec:5

## 5 REGULAR LANGUAGES: COALGEBRAIC AND ALGEBRAIC ACCEPTANCE

In this section we compare the languages accepted by a finite automaton in  $\mathcal{D}$  with those accepted by its dual finite coalgebra in  $\mathcal{C}$ . Let  $(Q, \gamma)$  be a  $T_\Sigma$ -coalgebra. We have seen in Definition 4.7 which language is accepted by a state  $q$  of the coalgebra. This is the coalgebraic concept of acceptance of a language. Now in case our coalgebra is finite, then by Lemma 4.16 we can assume that it is the dual of a finite  $\Sigma$ -automaton  $(A, \alpha) = (A, (\alpha_a), \alpha_{in})$ , in symbols,  $Q = \hat{A}$ . The given state  $q$ , represented by a morphism  $q : \mathbb{1}_{\mathcal{C}} \rightarrow \hat{A}$  of  $\mathcal{C}_f$ , is then the dual of a morphism from  $A$  to  $O_{\mathcal{D}}$  that we denote by

$$\alpha_{out} : A \rightarrow O_{\mathcal{D}} \quad \text{where } q = \widehat{\alpha_{out}}.$$

This is the “missing” output morphism of the automaton! We thus have the natural algebraic concept of the state  $q$  accepting a language: this language assigns to every word  $w \in \Sigma^*$  the element of  $O = |O_{\mathcal{D}}|$  corresponding to the  $\mathcal{D}$ -morphism  $\alpha_{out} \cdot \alpha_w \cdot \alpha_{in} : \mathbb{1}_{\mathcal{D}} \rightarrow O_{\mathcal{D}}$ .

**Definition 5.1.** Let  $C = (Q, \gamma)$  be a finite coalgebra for  $T_\Sigma$  dual to a  $\Sigma$ -automaton  $(A, \alpha)$ . Then the state  $q : \mathbb{1}_{\mathcal{C}} \rightarrow Q$  of  $C$  *algebraically accepts* the language

$$L_A(q) : \Sigma^* \rightarrow O$$

assigning to every word  $w$  the element of  $O$  represented by the morphism  $\gamma_{out} \cdot \gamma_w \cdot q : \mathbb{1}_{\mathcal{C}} \rightarrow O_{\mathcal{C}}$ .

Are the two languages  $L_A(q)$  and  $L_Q(q)$  (see Definition 4.7) accepted by the given state equal? Almost, but not quite: the reason is that the dual equivalence between  $\mathcal{C}_f$  and  $\mathcal{D}_f$  reverses the order of composition and thus reverses the words in the accepted languages. To formalize this, we use the following concept:

D:rev

**Definition 5.2.** The map  $\Sigma^* \rightarrow \Sigma^*$  reversing words extends uniquely to a morphism of  $\mathcal{D}$

$$\text{rev}_\Sigma : \Psi\Sigma^* \rightarrow \Psi\Sigma^*.$$

The *reversal* of a language  $L : \Psi\Sigma^* \rightarrow \mathcal{O}_{\mathcal{D}}$  is the language  $L \cdot \text{rev}_{\Sigma}$ .

Observe that  $\text{rev}$  is a  $\mathcal{D}$ -monoid morphism  $\text{rev} : \Psi\Sigma^* \rightarrow (\Psi\Sigma^*)^{op}$ , where  $(\Psi\Sigma^*)^{op}$  is the reversed monoid of  $\Psi\Sigma^*$  with multiplication  $x \bullet^{op} y = y \bullet x$ .

L:rev

LEMMA 5.3. *Let  $(Q, \gamma)$  be a finite  $T_{\Sigma}$ -coalgebra dual to a  $\Sigma$ -automaton  $(A, \alpha)$ . Then the language accepted by a state  $q : \mathbb{1}_{\mathcal{C}} \rightarrow Q$  coalgebraically is the reversal of the language accepted by it algebraically:*

$$L_A(q) = L_Q(q) \cdot \text{rev}_{\Sigma}.$$

PROOF. Since  $\gamma_a$  is dual to  $\alpha_a$ , we deduce that  $\gamma_{a_2} \cdot \gamma_{a_1}$  is dual to  $\alpha_{a_1} \cdot \alpha_{a_2}$  etc. Thus for every word  $w$  with  $\text{rev}_{\Sigma}(w) = v$  we get

$$\gamma_w = \widehat{\alpha_v}.$$

A state  $q : \mathbb{1}_{\mathcal{C}} \rightarrow Q$  accepts coalgebraically the language  $L_Q(q)$  assigning to a word  $w$  the value  $\gamma_{\text{out}} \cdot \gamma_w \cdot q : \mathbb{1}_{\mathcal{C}} \rightarrow \mathcal{O}_{\mathcal{C}}$ . This is dual to  $\alpha_{\text{out}} \cdot \alpha_{\text{rev}_{\Sigma}(w)} \cdot \alpha_{\text{in}}$ , and this is the value of the reversed language accepted algebraically.  $\square$

For our Generalized Eilenberg Theorem below we need a coalgebraic characterization of the closure under preimages, see Definition 4.19. To this end we introduce the preimage of a  $\Sigma$ -automaton (and of a  $T_{\Sigma}$ -coalgebra) under a morphism of free  $\mathcal{D}$ -monoids.

def:af

Definition 5.4. Let  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  be a  $\mathcal{D}$ -monoid morphism. For every  $\Sigma$ -automaton  $(A, \alpha)$  we define its *preimage under  $f$*  as the  $\Delta$ -automaton  $(A, \alpha)^f = (A, \alpha^f)$  on the same states  $A$ , with the same initial state,  $\alpha_{\text{in}}^f = \alpha_{\text{in}}$ , and with transitions  $\alpha_b^f = \alpha_{f(b)}$  for all  $b \in \Delta$  (see Notation 3.3).

ex:af

- Example 5.5. (a) If  $\mathcal{D} = \mathbf{Set}$  or  $\mathbf{Pos}$ , we are given a monoid morphism  $f : \Delta^* \rightarrow \Sigma^*$ . Every  $\Sigma$ -automaton yields a  $\Delta$ -automaton with transitions  $\alpha_b^f = \alpha_{f(b)} = \alpha_{a_n} \cdots \alpha_{a_1}$  for  $f(b) = a_1 \cdots a_n$ .
- (b) If  $\mathcal{D} = \mathbf{Set}_{\star}$ , a zero-preserving monoid morphism  $f : \Delta^* + \{0\} \rightarrow \Sigma^* + \{0\}$  is given. The map  $\alpha_b^f$  is defined as in (a) if  $f(b) \neq 0$ , and otherwise  $\alpha_b^f(x) = \star_A$  for all  $x \in |A|$ , where  $\star_A$  is the point of  $A \in \mathbf{Set}_{\star}$ .
- (c) For  $\mathcal{D} = \mathbf{JSL}_{\perp}$ , we are given a semiring morphism  $f : \mathcal{P}_f\Delta^* \rightarrow \mathcal{P}_f\Sigma^*$ . If the value  $f(b)$  is a single word,  $f(b) = \{w\}$ , then the corresponding transition is again  $\alpha_b^f = \alpha_w$ . In general  $f(b) = \{w_1, \dots, w_k\}$ , and since  $\alpha_{(-)}$  is a semilattice homomorphism, we conclude that  $\alpha_b^f = \alpha_{w_1} \vee \cdots \vee \alpha_{w_k}$  (the join in  $[A, A]$ ).
- (d) Analogously for  $\mathcal{D} = \mathbf{Vec}_K$ : if  $f(b) = \{w_1, \dots, w_k\}$ , then  $\alpha_b^f = \alpha_{w_1} \oplus \cdots \oplus \alpha_{w_k}$  (vector addition in  $[A, A]$ ).

In the following lemma we denote, for every  $\Sigma$ -automaton  $A$ , by  $e_A : \Psi\Sigma^* \rightarrow A$  the unique morphism of  $\Sigma$ -automata, see Remark 3.4(c).

lem:pre

LEMMA 5.6. *Let  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  be a  $\mathcal{D}$ -monoid morphism.*

- (a)  *$f$  is also a morphism  $f : \Psi\Delta^* \rightarrow (\Psi\Sigma^*)^f$  of  $\Delta$ -automata.*
- (b) *Every homomorphism  $h : A \rightarrow A'$  of  $\Sigma$ -automata is also a homomorphism  $h : A^f \rightarrow (A')^f$  of  $\Delta$ -automata.*
- (c) *Given a  $\Sigma$ -automaton  $A$ , we have  $e_{A^f} = e_A \cdot f : \Psi\Delta^* \rightarrow A^f$ .*

PROOF. (a) Since  $f(\epsilon) = \epsilon$ , the initial state of  $\Psi\Delta^*$  is mapped to that of  $(\Psi\Sigma^*)^f$ . For any  $a \in \Delta$ , the  $a$ -transitions in  $\Psi\Delta^*$  and  $(\Psi\Sigma^*)^f$  are  $- \bullet a$  and  $- \bullet f(a)$ , respectively. Hence preservation of transitions amounts to the equation  $f(x \bullet a) = f(x) \bullet f(a)$  for all  $x \in \Psi\Delta^*$ , which holds because  $f$  is a  $\mathcal{D}$ -monoid morphism.

(b) We clearly have  $h \cdot \alpha'_{\text{in}} = h \cdot \alpha_{\text{in}} = \alpha'_{\text{in}} = (\alpha')_{\text{in}}^f$ . From  $h \cdot \alpha_w = \alpha'_w \cdot h$  for all  $w \in \Sigma^*$  we can conclude  $h \cdot \alpha_x = \alpha'_x \cdot h$  for all  $x \in |\Psi\Sigma^*|$  by induction. In particular, we have the desired equation  $h \cdot \alpha_{f(a)} = \alpha'_{f(a)} \cdot h$  for all  $a \in \Delta$ .

(c)  $\Psi\Delta^*$  is the initial  $\Delta$ -automaton, and by (a) and (b) both sides are  $\Delta$ -automata morphisms.  $\square$

C:pre

**COROLLARY 5.7.** *Let  $(A, \alpha)$  be a finite  $\Sigma$ -automaton and  $q$  a state of  $\hat{A}$ . Then  $q$  accepts in  $A^f$  the preimage of the language it accepts in  $A$ :*

$$L_{A^f}(q) = L_A(q) \cdot f : \Psi\Delta^* \rightarrow \mathcal{O}_{\mathcal{D}}.$$

Indeed, both sides are equal to  $\alpha_{\text{out}} \cdot e_A \cdot f$  for  $q = \widehat{\alpha_{\text{out}}}$ .

not: dag

**Notation 5.8.** For a  $\mathcal{D}$ -monoid morphism  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  we denote by  $f^\dagger$  the following  $\mathcal{D}$ -monoid morphism

$$f^\dagger = (\Psi\Delta^* \xrightarrow{\text{rev}_\Delta} (\Psi\Delta^*)^{op} \xrightarrow{f} (\Psi\Sigma^*)^{op} \xrightarrow{\text{rev}_\Sigma} \Psi\Sigma^*).$$

def: qf

**Definition 5.9.** Let  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  be a  $\mathcal{D}$ -monoid morphism. For every finite  $T_\Sigma$ -coalgebra  $(Q, \gamma)$  dual to a  $\Sigma$ -automaton  $(A, \alpha)$  we define the *preimage of  $(Q, \gamma)$  under  $f$*  as the  $T_\Delta$ -coalgebra  $(Q, \gamma)^f = (Q, \gamma^f)$  dual to the preimage of  $(A, \alpha)$  under  $f^\dagger$ . In other words

$$(Q, \gamma) = (\hat{A}, \hat{\alpha}) \quad \text{implies} \quad \gamma^f = \widehat{\alpha^{f^\dagger}}.$$

**Example 5.10.** The preimage of  $(Q, \gamma)$  with  $\gamma = \widehat{\alpha}$  under  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  has the same states and final states as  $(Q, \gamma)$ , and the transitions  $\gamma_b^f$  are given as follows:

(a) For  $\mathcal{C} = \mathbf{BA}$  or  $\mathbf{DL}_b$ ,

$$\gamma_b^f = \gamma_{a_n} \cdot \dots \cdot \gamma_{a_1} = \gamma_{f(b)}$$

where  $f(b) = a_1 \dots a_n$  (i.e.,  $f^\dagger(b) = a_n \dots a_1$ ). Indeed, this follows from  $\gamma_b^f = \widehat{\alpha_b^{f^\dagger}}$ .

(b) If  $\mathcal{C} = \mathbf{BR}$  and  $f : \Delta^* + \{\perp\} \rightarrow \Sigma^* + \{\perp\}$ , then  $\gamma_b^f = \gamma_{f(b)}$  if  $f(b) \neq \perp$ , and otherwise  $\gamma_b^f$  is the constant map of value  $\perp$ . The argument is similar to (a).

(c) Let  $\mathcal{C} = \mathbf{JSL}_\perp$  and  $f : \mathcal{P}_f\Delta^* \rightarrow \mathcal{P}_f\Sigma^*$ . We claim that

$$\gamma_b^f = \gamma_{w_1} \vee \dots \vee \gamma_{w_k}$$

where  $f(b) = \{w_1, \dots, w_k\}$  and the join is taken in  $[Q, Q]$  (i.e., pointwise). Indeed, observe that the map  $h \mapsto \widehat{h}$  gives a semilattice isomorphism  $[Q, Q] \cong [\widehat{Q}, \widehat{Q}]$ . This isomorphism maps  $\gamma_{w_i}$  to  $\alpha_{\text{rev}_\Sigma(w_i)}$ , and hence  $\gamma_{w_1} \vee \dots \vee \gamma_{w_k}$  to  $\alpha_{\text{rev}_\Sigma(w_1)} \vee \dots \vee \alpha_{\text{rev}_\Sigma(w_k)}$ . By Example 5.5(b) this morphism is  $\alpha_b^{f^\dagger}$  since  $f^\dagger(b) = \{\text{rev}_\Sigma(w_1), \dots, \text{rev}_\Sigma(w_k)\}$ .

(d) If  $\mathcal{C} = \mathbf{Vec}_K$  and  $f(b) = \sum_i k_i w_i$ , then

$$\gamma_b^f = \sum_i k_i \gamma_{w_i}.$$

Indeed, the map  $h \mapsto \widehat{h}$  gives an isomorphism of vector spaces  $[Q, Q] \cong [\widehat{Q}, \widehat{Q}]$ . This isomorphism maps  $\gamma_{w_i}$  to  $\sum_i \alpha_{\text{rev}_\Sigma(w_i)}$ . By Example 5.5(c) this morphism is  $\alpha_b^{f^\dagger}$ .

fpcoalg

**Remark 5.11.** (a) Recall that an object  $A$  of a category  $\mathcal{A}$  is called *finitely presentable* if its hom-functor  $\mathcal{A}(A, -) : \mathcal{A} \rightarrow \mathbf{Set}$  preserves filtered colimits. If  $\mathcal{A}$  is a locally finite variety of (ordered) algebras, the finitely presentable objects are precisely the finite algebras.



- (b) Every finite  $T_\Sigma$ -coalgebra is a finitely presentable object of  $\text{Coalg } T_\Sigma$ , see [4]. Hence, given a filtered colimit cocone  $c'_i : Q'_i \rightarrow Q'$  ( $i \in I$ ) in  $\text{Coalg } T_\Sigma$ , every coalgebra homomorphism  $h : Q \rightarrow Q'$  with finite domain  $Q$  factorizes (in  $\text{Coalg } T_\Sigma$ ) through some  $c'_i$ :

$$\begin{array}{ccc} Q & \xrightarrow{h} & Q' \\ & \searrow h' & \uparrow c'_i \\ & & Q'_i \end{array}$$

**Remark 5.12.** The forgetful functor  $\text{Coalg } T_\Sigma \rightarrow \mathcal{C}$  preserves and, in fact, creates colimits. The latter means that, given a diagram  $(Q_i, \gamma_i)$  ( $i \in I$ ) of  $T_\Sigma$ -coalgebras and a colimit cocone  $(c_i : Q_i \rightarrow Q)_{i \in I}$  in  $\mathcal{C}$ , there is a unique  $T_\Sigma$ -coalgebra structure  $\gamma$  on  $Q$  for which the maps  $c_i$  are  $T_\Sigma$ -coalgebra homomorphisms  $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$ . Moreover,  $(c_i)$  is a colimit cocone in  $\text{Coalg } T_\Sigma$ .

The uniqueness of  $\gamma$  gives rise to a useful proof principle: if two coalgebra structures  $\gamma$  and  $\gamma'$  on  $Q$  are given such that each  $c_i$  is a coalgebra homomorphism  $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$  and  $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma')$ , it follows that  $\gamma = \gamma'$ .

Next, we extend the construction  $(-)^f$  of Definition 5.9 from finite to locally finite coalgebras.

**Construction 5.13.** For every locally finite  $T_\Sigma$ -coalgebra  $(Q, \gamma)$  and  $\mathcal{D}$ -monoid morphism  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  we construct a  $T_\Delta$ -coalgebra  $(Q, \gamma)^f$  as follows:

- (1) Express  $(Q, \gamma)$  as a filtered colimit  $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$  ( $i \in I$ ) of finite  $T_\Sigma$ -coalgebras. Note that each connecting morphism  $c_{ij} : (Q_i, \gamma_i) \rightarrow (Q_j, \gamma_j)$  of the diagram is also a  $T_\Delta$ -coalgebra homomorphism  $c_{ij} : (Q_i, \gamma_i)^f \rightarrow (Q_j, \gamma_j)^f$  by the dual of Lemma 5.6(b).
- (2) Let  $\gamma^f$  be the unique  $T_\Delta$ -coalgebra structure on  $Q$  for which all  $c_i : (Q_i, \gamma_i)^f \rightarrow (Q, \gamma^f)$  are  $T_\Delta$ -coalgebra homomorphisms, see Remark 5.12. Put

$$(Q, \gamma)^f := (Q, \gamma^f).$$

The following lemma summarizes some important properties of this construction.

**LEMMA 5.14.** *Let  $(Q, \gamma)$  be a locally finite  $T_\Sigma$ -coalgebra and  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  a  $\mathcal{D}$ -monoid morphism.*

- (a) *The coalgebra structure of  $(Q, \gamma)^f$  is independent of the choice of the colimit cocone  $(c_i)$  in Construction 5.13.*
- (b) *If  $f = \Psi f_0^*$  for some  $f_0 : \Delta \rightarrow \Sigma$ , then  $(Q, \gamma)^f$  has the coalgebra structure*

$$Q \xrightarrow{\gamma} O_{\mathcal{C}} \times Q^\Sigma \xrightarrow{id \times Q^{f_0}} O_{\mathcal{C}} \times Q^\Delta. \quad (5.1)$$

- (c) *Every homomorphism  $h : Q \rightarrow Q'$  of locally finite  $T_\Sigma$ -coalgebras is also a homomorphism  $h : Q^f \rightarrow (Q')^f$  of  $T_\Delta$ -coalgebras.*
- (d) *For any  $\mathcal{D}$ -monoid morphism  $g : \Psi\Gamma^* \rightarrow \Psi\Delta^*$ ,*

$$(Q^f)^g = Q^{f \cdot g}.$$

- (e) *The construction  $(-)^f$  commutes with coproducts: given locally finite  $T_\Sigma$ -coalgebras  $Q_j$  ( $j \in J$ ), we have*

$$\left( \coprod_j Q_j \right)^f = \coprod_j Q_j^f.$$

**PROOF.** (a) Suppose  $\gamma^f$  has been defined by means of the cocone  $(c_i)$ , and another filtered colimit cocone

$$c'_j : (Q'_j, \gamma'_j) \rightarrow (Q, \gamma) \quad (j \in J)$$

with  $Q'_j$  finite is given. By Remark 5.12 it suffices to show that the maps  $c'_j$  are  $T_\Delta$ -coalgebra homomorphisms  $c'_j : (Q'_j, \gamma'_j)^f \rightarrow (Q, \gamma^f)$ .

Given  $j \in J$ , there exists by Remark 5.11 a  $T_\Sigma$ -coalgebra homomorphism  $g : (Q'_j, \gamma'_j) \rightarrow (Q_i, \gamma_i)$  with  $c_i \cdot g = c'_j$  for some  $i$ . It follows that  $c'_j$  is a  $T_\Delta$ -coalgebra homomorphism, being the composite of the  $T_\Delta$ -coalgebra homomorphisms

$$(Q'_j, \gamma'_j)^f \xrightarrow{g} (Q_i, \gamma_i)^f \xrightarrow{c_i} (Q, \gamma^f).$$

Indeed,  $g$  is a  $T_\Delta$ -coalgebra homomorphism using the definition of  $Q^f$  for finite  $Q$  (Definition 5.9) and Lemma 5.6(b), and  $c_i$  is one by the definition of  $\gamma^f$ .

(b) Given a finite  $L_\Sigma$ -algebra  $(A, \alpha)$ , the  $L_\Delta$ -algebra  $(A, \alpha)^f$  (see Definition 5.4) has the transitions  $\alpha_a^f = \alpha_{f_0(a)}$  for  $a \in \Delta$ . Hence, for the dual finite  $T_\Sigma$ -coalgebra  $(Q, \gamma)$ , the  $T_\Delta$ -coalgebra  $Q^f = \widehat{A}^{f^\dagger} = \widehat{A}^f$  has transitions  $\gamma_{f_0(a)} : Q \rightarrow Q$  for  $a \in \Delta$ , which proves (b) for finite coalgebras..

In the case where  $(Q, \gamma)$  is just locally finite, express  $(Q, \gamma)$  as a filtered colimit  $c_i : (Q_i, \gamma_i) \rightarrow (Q, \gamma)$  ( $i \in I$ ) of finite  $T_\Sigma$ -coalgebras, and consider the diagram below:

$$\begin{array}{ccccc} Q & \xrightarrow{\gamma} & O_{\mathcal{C}} \times Q^\Sigma & \xrightarrow{id \times Q^{f_0}} & O_{\mathcal{C}} \times Q^\Delta \\ \uparrow c_i & & \uparrow T_\Sigma c_i & & \uparrow T_\Delta c_i \\ Q_i & \xrightarrow{\gamma_i} & O_{\mathcal{C}} \times Q_i^\Sigma & \xrightarrow{id \times Q_i^{f_0}} & O_{\mathcal{C}} \times Q_i^\Delta \end{array}$$

This diagram commutes because  $c_i$  is a  $T_\Sigma$ -coalgebra homomorphism and by naturality. The lower row is the coalgebra structure of  $(Q_i, \gamma_i)^f$  by the first part of the proof. Hence  $c_i$  is a  $T_\Delta$ -coalgebra homomorphism from  $(Q_i, \gamma_i)^f$  to the coalgebra in the upper row, which implies that the upper row defines the coalgebra structure of  $(Q, \gamma)^f$ .

(c) Express  $Q$  and  $Q'$  as filtered colimits  $c_i : Q_i \rightarrow Q$  ( $i \in I$ ) and  $c'_j : Q'_j \rightarrow Q'$  ( $j \in J$ ) of finite  $T_\Sigma$ -coalgebras. By Remark 5.11 there exists for every  $i \in I$  some  $T_\Sigma$ -coalgebra homomorphism  $g : Q_i \rightarrow Q'_j$  for which the diagram below commutes:

$$\begin{array}{ccc} Q & \xrightarrow{h} & Q' \\ \uparrow c_i & & \uparrow c'_j \\ Q_i & \xrightarrow{g} & Q'_j \end{array}$$

It follows that  $h \cdot c_i : Q_i^f \rightarrow (Q')^f$  is a  $T_\Delta$ -coalgebra homomorphism, being the composite of the  $T_\Delta$ -coalgebra homomorphisms  $Q_i^f \xrightarrow{g} (Q'_j)^f \xrightarrow{c'_j} (Q')^f$  (one uses the same argument as in point (a)). Since the morphisms  $c_i$  are jointly epimorphic in  $\mathcal{C}$ , it follows that  $h$  is a  $T_\Delta$ -coalgebra homomorphism  $h : Q^f \rightarrow (Q')^f$ .

(d) (i) We first show that for all  $L_\Sigma$ -algebras  $(A, \alpha)$  we have

$$(A^f)^g = A^f \cdot g.$$

Indeed, the  $\Sigma$ -automata on both sides have states  $A$  and initial states  $\alpha_{in}$ . To see that they have the same transitions, consider the diagram below, cf. Notation 3.3 and Definition 5.4.

$$\begin{array}{ccc}
 \Psi\Sigma^* & \xrightarrow{x \mapsto \alpha_x} & [A, A] \\
 f \cdot g \uparrow & \swarrow f & \uparrow y \mapsto \alpha_y^f \\
 \Psi\Gamma^* & \xrightarrow{g} & \Psi\Delta^*
 \end{array} \tag{5.2}$$

The upper triangle commutes by the definition of  $\alpha^f$ . Hence, for all  $a \in \Gamma$ ,

$$\begin{aligned}
 (\alpha^{f \cdot g})_a &= \alpha_{fg(a)} && \text{(def. } \alpha^{f \cdot g}\text{)} \\
 &= (\alpha^f)_{ga} && \text{(diagram (5.2))} \\
 &= ((\alpha^f)^g)_a && \text{def. } (\alpha^f)^g.
 \end{aligned}$$

(ii) If  $Q$  is a finite  $T_\Sigma$ -coalgebra dual to a  $\Sigma$ -automaton  $A$ , we conclude from (i):

$$Q^{f \cdot g} = \widehat{A^{f \cdot g}^\dagger} = \widehat{A^{f^\dagger} \cdot g^\dagger} = \widehat{A^{f^\dagger}}^{g^\dagger} = (\widehat{A^{f^\dagger}})^g = (Q^f)^g.$$

(iii) Now let  $Q$  be locally finite, and express  $Q$  as a filtered colimit  $c_i : Q_i \rightarrow Q$  ( $i \in I$ ) of finite  $T_\Sigma$ -coalgebras. Hence by (c) we have  $T_\Gamma$ -coalgebra homomorphisms  $c_i : Q_i^{f \cdot g} \rightarrow Q^{f \cdot g}$  and  $c_i : (Q_i^f)^g \rightarrow (Q^f)^g$ , where  $Q_i^{f \cdot g} = (Q_i^f)^g$  by (ii) above. It follows that  $Q^{f \cdot g}$  and  $(Q^f)^g$  have the same coalgebra structure.

(e) (i) We first prove that, for each family of  $L_\Sigma$ -algebras  $(A_j, \alpha_j)$  ( $j \in J$ ),

$$\left(\prod_j A_j\right)^f = \left(\prod_j A_j^f\right).$$

Clearly the algebras on both sides of the equation have the same states  $\prod_j A_j$  and the same initial state. Concerning the transitions, consider the commutative diagram below:

$$\begin{array}{ccccc}
 \Psi\Delta^* & \xrightarrow{f} & \Psi\Sigma^* & \xrightarrow{a \mapsto \prod(\alpha_j)_a} & [\prod A_j, \prod A_j] \\
 & & \searrow \langle a \mapsto (\alpha_j)_a \rangle & & \uparrow (f_j) \mapsto \prod f_j \\
 & & & & \prod [A_j, A_j]
 \end{array}$$

The upper and lower path define the transitions of  $(\prod_j A_j)^f$  and  $\prod_j A_j^f$ , respectively. Hence they have the same transitions.

(ii) Suppose now that  $J$  is finite and finite  $T_\Sigma$ -coalgebras  $Q_j$ , where  $Q_j$  is dual to the  $\Sigma$ -automaton  $A_j$ . Thus the coproduct  $\coprod_j Q_j$  is dual to the product  $\prod_j A_j$ . Then we conclude from (i) and via duality:

$$\left(\coprod_j Q_j\right)^f = \left(\widehat{\prod_j A_j}\right)^{f^\dagger} = \widehat{\prod_j A_j^{f^\dagger}} = \prod_j \widehat{A_j^{f^\dagger}} = \coprod_j Q_j^f.$$

The statement for arbitrary  $J$  and locally finite coalgebras  $Q_j$  now follows from the fact that filtered colimits and coproducts commute in  $\text{Coalg } T_\Sigma$ , and every infinite coproduct is a filtered colimit of finite ones.  $\square$

E:pr

*Example 5.15.* If  $Q$  is a finite subcoalgebra of  ${}_{\varrho}T_\Sigma$ , then the languages accepted by  $Q^f$  are precisely the languages  $L \cdot f$  with  $L \in |Q|$ . This follows from Corollary 5.7 and the fact that every state  $L$  of  $Q$  accepts precisely the language  $L$ . Since  ${}_{\varrho}T_\Sigma$  is the filtered colimit of its finite subcoalgebras

$Q$ , an analogous description holds for  $(\varrho T_\Sigma)^f$ . Hence the unique  $T_\Delta$ -coalgebra homomorphism  $h : (\varrho T_\Sigma)^f \rightarrow \varrho T_\Delta$  maps every language  $L \in \varrho T_\Sigma$  to its preimage  $L \cdot f$ .

P:pre

**PROPOSITION 5.16.** *Let  $(Q, \gamma)$  be a finite  $T_\Sigma$ -coalgebra. The language accepted by a state  $q$  of the coalgebra  $Q^f$  is the preimage under  $f$  of the language  $q$  accepts in  $Q$ :*

$$L_{Q^f}(q) = L_Q(q) \cdot f.$$

**PROOF.** This is clear from the following computation, where we assume that our coalgebra is dual to a finite  $\Sigma$ -automaton  $A$ , thus  $Q^f$  is dual to  $A^{f^\dagger}$ :

$$\begin{aligned} L_{Q^f}(q) &= L_{A^{f^\dagger}(q)} \cdot \text{rev}_\Delta && \text{(Lemma 5.3)} \\ &= L_A(q) \cdot f^\dagger \cdot \text{rev}_\Delta && \text{(Corollary 5.7)} \\ &= L_A(q) \cdot \text{rev}_\Sigma \cdot f && \text{(definition of } f^\dagger) \\ &= L_Q(q) \cdot f && \text{(Lemma 5.3).} \quad \square \end{aligned}$$

E: subco

**Example 5.17.** Let  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  be a  $\mathcal{D}$ -monoid morphism, and let  $Q$  be a finite subcoalgebra of  $\varrho T_\Sigma$ .

- $Q^f$  is the  $T_\Delta$ -coalgebra of all languages  $L : \Psi\Delta^* \rightarrow O_{\mathcal{D}}$  in  $Q$  with transitions given by left derivatives  $\gamma_a(L) = f(a)^{-1}L$  for  $a \in \Delta$ .
- The unique coalgebra homomorphism  $h : Q^f \rightarrow \varrho T_\Sigma$  maps every language to its preimage under  $f$ . This follows from (a) and the fact that  $\varrho T_\Delta$  is the terminal locally finite coalgebra for  $T_\Delta$ .

## 6 LOCAL AND FULLY INVARIANT LOCAL VARIETIES

In the Introduction we mentioned the localization of Eilenberg's variety concept to a given alphabet  $\Sigma$ : a local variety is a set of regular languages over  $\Sigma$  closed under

- finite union, finite intersection, complement, and
- derivatives.

This concept stems from the work of Gehrke et al. [16]. We can also add the closure under preimages w.r.t. the endomorphisms of the free monoid on  $\Sigma$ , which leads to a concept we call *fully invariant local variety* of regular languages. In the present section we “localize” Eilenberg's theorem by characterizing the lattices of all local varieties and all fully invariant local varieties in terms of  $\mathcal{D}$ -monoids.

### 6.1 Local Varieties

Recall from Proposition 4.13 that the set  $\mathbf{Reg} \Sigma$  of all regular languages carries the rational fixed point  $\varrho T_\Sigma$  of the functor  $T_\Sigma = O_{\mathcal{L}} \times (-)^\Sigma$ . Thus, in the case  $\mathcal{L} = \mathbf{BA}$ , property (a) above states precisely that a local variety is a subobject of  $\varrho T_\Sigma$  in  $\mathbf{BA}$ . This motivates the following definition:

D: local

**Definition 6.1.** Let  $\Sigma$  be an alphabet. By a *local variety* of regular languages in  $\mathcal{L}$  is meant a subobject  $V$  of  $\varrho T_\Sigma$  closed under derivatives. That is, for every language  $L$  in  $|V|$  all derivatives  $Lw^{-1}$  and  $w^{-1}L$  with  $w \in \Sigma^*$  (see Definition 4.11) lie in  $|V|$ .

E: local

- Examples 6.2.** (1) The case  $\mathcal{L} = \mathbf{BA}$  gives precisely the concept of [16] discussed above.  
(2) For  $\mathcal{L} = \mathbf{DL}_b$  just delete “complement” from (a). This concept also appears in [16]  
(3) For  $\mathcal{L} = \mathbf{JSL}_\perp$  just delete “intersection” and “complement” from (a).  
(4) For  $\mathcal{L} = \mathbf{Vec}_K$ , a local variety is a set of  $K$ -weighted regular languages closed under addition, scalar multiplication, and derivatives.

**Remark 6.3.** Recall from Proposition 4.13 that the transitions  $\gamma_a$  of the coalgebra  $\varrho T_\Sigma$  are carried by left derivatives,  $\gamma_a(L) = a^{-1}L$ ,  $a \in \Sigma$ . Thus  $V$  being closed under left derivatives states precisely that  $V$  carries the structure of a subcoalgebra of  $\varrho T_\Sigma$ . (Indeed, being closed under left derivatives w.r.t. input symbols  $a \in \Sigma$  clearly implies being closed under left derivatives w.r.t. all words over  $\Sigma$ .) Subcoalgebras of  $\varrho T_\Sigma$  can be identified with subsets of  $\mathbf{Reg} \Sigma$  closed under the operations defining  $\mathcal{C}$  and under left derivatives – see Remark 4.21.

What about the closure under right derivatives? To express that condition in coalgebraic terms, we introduce the concept of a delay in a coalgebra for  $T_\Sigma$ . The delay changes only the output morphism:

**Definition 6.4.** Let  $C = (Q, (\gamma_a), \gamma_{\text{out}})$  be a coalgebra for  $T_\Sigma$ . The *delayed coalgebras*  $C_b$  for  $b \in \Sigma$  are defined by

$$C_b = (Q, (\gamma_a), \gamma_{\text{out}} \cdot \gamma_b).$$

**Example 6.5.** If  $C$  is the dual of a  $\Sigma$ -automaton  $A$  (see Lemma 4.5) then  $C_b$  is dual to the  $\Sigma$ -automaton  $A_b$  of Definition 3.7.

**Observation 6.6.** If a state  $q$  of  $C$  accepts the language  $L$ , then the same state accepts in  $C_b$  the right derivative  $Lb^{-1}$ . Indeed, the language accepted by  $q$  in  $C_b$  is defined, for  $w = a_1 \dots a_n$ , by

$$L_{C_b}(q)(w) = (\gamma_{\text{out}} \cdot \gamma_b) \cdot \gamma_{a_n} \cdot \dots \cdot \gamma_{a_1}(q) = \gamma_{\text{out}} \cdot (\gamma_b \cdot \gamma_{a_n} \cdot \dots \cdot \gamma_{a_1})(q) = L_C(q)(wb).$$

**PROPOSITION 6.7.** A finite subcoalgebra  $V$  of  $\varrho T_\Sigma$  is a local variety iff a coalgebra homomorphism from  $V_b$  to  $V$  exists for every input  $b \in \Sigma$ .

**PROOF.** (1) Suppose  $h_b : V_b \rightarrow V$  is a coalgebra homomorphism. We prove that  $L \in |V|$  implies  $Lb^{-1} \in |V|$ . Thus  $V$  (which is a subobject of  $\varrho T_\Sigma$  closed under left derivatives) is closed under right derivatives, i.e.,  $V$  is a local variety.

From Proposition 4.13 we know that since the inclusion map  $V \hookrightarrow \varrho T_\Sigma$  is a coalgebra homomorphism, every state  $L$  of  $V$  accepts the language  $L$ . Thus the colimit map  $c_V : V \rightarrow \varrho T_\Sigma$  is the inclusion. And analogously, the colimit map  $c_{V_b}$  assigns to  $L$  the language  $Lb^{-1}$  by the Observation 6.6. From Proposition 4.10 we know that the triangle below commutes:

$$\begin{array}{ccc} V_b & \xrightarrow{h_b} & V \\ & \searrow c_{V_b} & \swarrow c_V \\ & \varrho T_\Sigma & \end{array} \quad (6.1)$$

Applied to  $L \in |V_b|$  this states that  $h_b(L)$  accepts in  $V$  the language  $Lb^{-1}$ , thus,  $Lb^{-1} \in |V|$ .

(2) Suppose  $V$  is a local variety. For every  $b$  denote by  $h_b : |V| \rightarrow |V|$  the mapping  $h_b(L) = Lb^{-1}$ . Then in  $\mathbf{Set}$  we get a commutative triangle underlying the triangle (6.1). From the Homomorphism Theorem 4.22 we conclude that  $h_b$  is a coalgebra homomorphism.  $\square$

**Example 6.8.** If  $e : \Psi\Sigma^* \rightarrow D$  is a finite  $\Sigma$ -generated  $\mathcal{D}$ -monoid, then the dual coalgebra  $\widehat{D}$  of Lemma 4.5 is a subcoalgebra of  $\varrho T_\Sigma$ . Moreover, this is a local variety: the  $\Sigma$ -automata morphisms  $h_0 : D \rightarrow D_b$  of Theorem 3.8 yield coalgebra homomorphisms  $\widehat{h}_0 : \widehat{D}_b \rightarrow \widehat{D}$ .

**Remark 6.9.** Recall from Remark 2.24 the join-semilattice of all finite  $\Sigma$ -generated monoids. We are going to prove now that this is isomorphic to the poset of all finite local varieties of languages in  $\mathcal{C}$  (ordered by inclusion). In particular, the latter is also a join-semilattice.

**PROPOSITION 6.10.** For every finite alphabet  $\Sigma$  the following join semilattices are isomorphic:

- (a) all finite local varieties of languages in  $\mathcal{C}$ , and  
 (b) all finite  $\Sigma$ -generated  $\mathcal{D}$ -monoids.

PROOF. To every finite variety  $V$  assign a  $\Sigma$ -automaton  $\varphi(V)$  whose dual  $\widehat{\varphi(V)}$  is isomorphic to  $V$  in  $\text{Coalg } T_\Sigma$ , see Lemma 4.5. Let  $i_V : V \rightarrow \widehat{\varphi(V)}$  be an isomorphism. We apply Theorem 3.8.

- (i)  $\varphi(V)$  is reachable. Indeed, recall the description of  $\varrho T_\Sigma$  from Proposition 4.13. Every subautomaton  $m : A \rightarrow \varphi(V)$  (carried by a strong monomorphism in  $\mathcal{D}$ ) yields a quotient coalgebra of  $V$ :

$$\begin{array}{ccc} V & \xrightarrow{i_V} & \widehat{\varphi(V)} \xrightarrow{\widehat{m}} \widehat{A} \\ & \searrow c_V & \swarrow c_{\widehat{A}} \\ & & \varrho T_\Sigma \end{array}$$

This implies that  $\widehat{m}$  is monic (since  $c_V$  is), hence  $m$  is epic. Since  $m$  is also a strong monomorphism, it follows that  $m$  is an isomorphism. It follows that  $\varphi(V)$  has no proper subautomaton, which precisely means that it is reachable.

- (ii) The coalgebra homomorphisms  $h_b : V_b \rightarrow V$  of Proposition 6.7 yield automata homomorphisms, due to Lemma 4.5. Observe that for every  $\Sigma$ -automaton  $A$  we clearly have  $\widehat{A}_b = (\widehat{A})_b$ , thus, we get automata homomorphisms from  $\varphi(V)$  to  $\varphi(V)_b$  for all  $b \in \Sigma$ .

By Theorem 3.8 each  $\varphi(V)$  can be viewed as a  $\Sigma$ -generated  $\mathcal{D}$ -monoid. This yields the desired isomorphism of the above mentioned posets. Indeed:

- (1)  $\varphi$  is injective since given finite local varieties  $V$  and  $W$  with  $\varphi(V) \cong \varphi(W)$  it follows from  $\widehat{(-)}$  being an equivalence functor that  $V$  and  $W$  are isomorphic subcoalgebras of  $\varrho T_\Sigma$ , i.e., they represent the same subobject of  $\varrho T_\Sigma$ .
- (2)  $\varphi$  is surjective since every finite  $\Sigma$ -generated monoid  $e : \Psi\Sigma^* \rightarrow A$  defines a finite coalgebra  $\widehat{A}$  for  $T_\Sigma$  whose unique coalgebra morphism  $c_{\widehat{A}} : \widehat{A} \rightarrow \varrho T_\Sigma$  is monic (the argument is dual to that in (i) above). And the automata morphisms from  $A$  to  $A_b$  of Theorem 3.8 yield coalgebra morphisms from  $\widehat{A}_b$  to  $\widehat{A}$ . Thus,  $\widehat{A}$  is a local variety. Clearly  $\varphi(\widehat{A})$  is isomorphic to  $A$ , i.e., it represents the same  $\Sigma$ -generated monoid.
- (3) Given local varieties  $V$  and  $W$ , then  $V \subseteq W$  iff  $\varphi(V) \sqsubseteq \varphi(W)$  (see Remark 2.24). Indeed, given a coalgebra homomorphism  $h : V \rightarrow W$  for  $T_\Sigma$ , we know that  $c_W \cdot h = c_V$ . We also obtain a unique automata homomorphism  $k : \varphi(W) \rightarrow \varphi(V)$  for which the following square

$$\begin{array}{ccc} V & \xrightarrow{h} & W \\ i_V \downarrow & & \downarrow i_W \\ \widehat{\varphi(V)} & \xrightarrow{\widehat{k}} & \widehat{\varphi(W)} \end{array}$$

commutes. Let  $e_V$  and  $e_W$  be the unique  $\mathcal{D}$ -monoid morphisms dual to  $c_W$  and  $c_V$ . Then, by duality, the following triangle commutes:

$$\begin{array}{ccc} & \Psi\Sigma^* & \\ e_V \swarrow & & \searrow e_W \\ \varphi(V) & \xleftarrow{k} & \varphi(W) \end{array}$$

This proves  $e_V \sqsubseteq e_W$ . Conversely, assuming  $\varphi(V) \sqsubseteq \varphi(W)$ , more precisely, assuming that a commutative triangle as above is given in  $\text{Mon } \mathcal{D}$ , then  $i_W^{-1} \cdot \hat{k} \cdot i_V : V \rightarrow W$  is a coalgebra homomorphism which proves  $V \subseteq W$ .  $\square$

**LEMMA 6.11.** *Every finite subset of a local variety of languages in  $\mathcal{C}$  is contained in a finite local subvariety.*

**PROOF.** Since the intersection of local varieties of languages is again a local variety, it suffices to prove that every finite set of regular languages

$$S \hookrightarrow \mathbf{Reg } \Sigma = |\varrho T_\Sigma|$$

is contained in a finite local variety of languages in  $\mathcal{C}$ . Denote by  $\bar{S}$  the closure of  $S$  under left and right derivatives. The set  $\bar{S}$  is finite because every regular language has only finitely many derivatives. Therefore, since  $\mathcal{C}$  is a locally finite variety, the smallest subobject  $Q \hookrightarrow \varrho T_\Sigma$  in  $\mathcal{C}$  containing  $\bar{S}$  is also finite. Then  $Q$  is a local variety containing  $S$ . Indeed, the closure of  $Q$  under derivatives follows immediately from the fact that the maps  $L \mapsto a^{-1}L$  and  $L \mapsto La^{-1}$  ( $a \in \Sigma$ ) are endomorphisms on  $\varrho T_\Sigma$  in  $\mathcal{C}$ , see Remark 6.3 and Proposition 6.7.  $\square$

We are ready to prove the local variant of the Generalized Eilenberg Theorem: local varieties of languages correspond bijectively to pseudovarieties of finite  $\Sigma$ -generated  $\mathcal{D}$ -monoids. First we need the following concepts:

**Remark 6.12.** Recall that the *ideal completion*  $\text{Id}(A)$  of a join-semilattice  $A$  is the complete lattice of all ideals (i.e.  $\downarrow$ -sets closed under finite joins) of  $A$  ordered by inclusion.  $A$  is a subset of  $\text{Id}(A)$  when an element  $x \in A$  is identified with the ideal  $\downarrow x = \{y \mid y \leq x\}$ . Up to isomorphism  $\text{Id}(A)$  is characterized as a complete lattice containing  $A$  such that:

- (1) every element of  $\text{Id}(A)$  is a directed join of elements of  $A$ , and
- (2) the elements  $x$  of  $A$  are *compact* in  $\text{Id}(A)$ . This means that if  $x \leq \vee_{i \in I} y_i$  for a directed join of elements  $y_i$  of  $\text{Id}(A)$ , then  $x \leq y_i$  for some  $i$ .

**Definition 6.13.** A *pseudovariety of  $\Sigma$ -generated  $\mathcal{D}$ -monoids* is a set of finite  $\Sigma$ -generated  $\mathcal{D}$ -monoids closed under quotients and subdirect products, see Remark 2.24.

In other words, a pseudovariety is precisely an ideal in the poset of finite  $\Sigma$ -generated  $\mathcal{D}$ -monoids.

**Example 6.14.** Every finite  $\Sigma$ -generated  $\mathcal{D}$ -monoid  $e : \Psi\Sigma^* \rightarrow D$  yields the pseudovariety consisting of all quotients of  $e$ . We denote it by  $\downarrow e$ .

**Remark 6.15.** All pseudovarieties of  $\Sigma$ -generated  $\mathcal{D}$ -monoids (ordered by inclusion) form a complete lattice since they are clearly closed under arbitrary intersections. Also all local varieties of languages in  $\mathcal{C}$  form a complete lattice, being closed under intersections of subobjects of  $\varrho T_\Sigma$ . We now prove these lattices to be isomorphic:

**THEOREM 6.16 (LOCAL GENERALIZED EILENBERG THEOREM).** *For every finite alphabet  $\Sigma$ , the complete lattice of local varieties of languages in  $\mathcal{C}$  is isomorphic to the complete lattice of all pseudovarieties of  $\Sigma$ -generated  $\mathcal{D}$ -monoids.*

**PROOF.** (i) By definition, the poset of all pseudovarieties of  $\Sigma$ -generated  $\mathcal{D}$ -monoids (ordered by inclusion) is the ideal completion of the poset of all finite quotient-monoids  $e$  of  $\Psi\Sigma^*$ .

- (ii) Consequently, by using Proposition 6.10, it is sufficient to prove that the lattice of all local varieties of languages in  $\mathcal{C}$  is an ideal completion of the join semilattice of all finite ones. To this end, we verify the conditions (1) and (2) of Remark 6.12.

Observe first that directed joins of local varieties are formed on the level of **Set**, i.e., as directed unions. This follows from  $\mathcal{C}$  being a variety of (finitary) algebras and the fact that colimits in  $\text{Coalg } T_\Sigma$  are formed on the level of  $\mathcal{C}$ . Thus, condition (2) is trivial: every finite local variety  $V$  contained in a directed union  $\bigcup_{i \in I} V_i$  of local varieties is contained in  $V_i$  for some  $i$ .

Condition (1) follows from Lemma 6.11: every local variety  $V \hookrightarrow \varrho T_\Sigma$  is a directed union of all of its finite subsets, and therefore directed union of all of its finite subvarieties.  $\square$

cal:var

*Examples 6.17.* (a) In the following examples local varieties are subsets of **Reg**  $\Sigma$  and we get the following pairs of isomorphic complete lattices:

$\mathcal{C}$	local varieties closed under	$\mathcal{D}$	pseudovarieties of $\Sigma$ -gen.
<b>BA</b>	all boolean operations	<b>Set</b>	monoids
<b>BR</b>	disjoint union and $\emptyset$	<b>Set</b> $_\star$	monoids with zero
<b>DL</b> $_b$	finite unions & finite intersections	<b>Pos</b>	ordered monoids
<b>DL</b> $_\perp$	finite unions & non-empty fin. intersections	<b>Pos</b> $_\perp$	ordered monoids with $0 = \perp$
<b>JSL</b> $_\perp$	finite unions	<b>JSL</b> $_\perp$	idempotent semirings
<b>JSL</b> $_b$	finite unions and $\Sigma^*$	<b>JSL</b>	idempotent semirings with 1

(b) For  $\mathcal{C} = \mathcal{D} = \text{Vec}_K$  we get the isomorphism of the complete lattices of all sets of  $K$ -weighted languages closed under derivatives, addition and scalar multiplication and pseudovarieties of  $\Sigma$ -generated  $K$ -algebras.

## 6.2 Fully Invariant Local Varieties

Recall that **Mon**  $\mathcal{D}$  is a variety of algebras or ordered algebras. In General Algebra a congruence  $\sim$  on an algebra  $D$  is called fully invariant if every endomorphism of  $D$  induces an endomorphism of the quotient algebra  $D/\sim$ . We use the same terminology:

*Definition 6.18.* A  $\Sigma$ -generated  $\mathcal{D}$ -monoid  $e : \Psi\Sigma^* \rightarrow D$  is called *fully invariant* provided that for every  $\mathcal{D}$ -monoid endomorphism  $h$  of  $\Psi\Sigma^*$  there exists a  $\mathcal{D}$ -monoid endomorphism  $k$  of  $D$  making the square below commutative

$$\begin{array}{ccc}
 \Psi\Sigma^* & \xrightarrow{h} & \Psi\Sigma^* \\
 e \downarrow & & \downarrow e \\
 D & \xrightarrow{k} & D
 \end{array}$$

We are going to prove that all fully invariant  $\Sigma$ -generated  $\mathcal{D}$ -monoids bijectively correspond to those local varieties that are closed under preimages w.r.t.  $\mathcal{D}$ -monoid endomorphisms on  $\Psi\Sigma^*$ . We therefore call such varieties of languages fully invariant, too:

*Definition 6.19.* Let  $\Sigma$  be an alphabet. A *fully invariant local variety* of regular languages in  $\mathcal{C}$  is a local variety  $V$  closed under preimages w.r.t. all  $\mathcal{D}$ -monoid endomorphisms  $f$  of  $\Psi\Sigma^*$ ; that is,  $L \in |V|$  implies  $L \cdot f \in |V|$  (cf. Definition 4.19).

:strong

**PROPOSITION 6.20.** *A local variety  $V \hookrightarrow \varrho T_\Sigma$  is fully invariant iff for every  $\mathcal{D}$ -monoid endomorphism  $f$  of  $\Psi\Sigma^*$  there exists a coalgebra homomorphism from  $V^f$  (see Definition 5.9) to  $V$ .*

**PROOF.** Suppose that  $k : V^f \rightarrow V_\Sigma$  is a coalgebra homomorphism for  $T_\Sigma$ . Composed with the inclusion  $i : V \hookrightarrow \varrho T_\Sigma$  it yields the homomorphism  $h$  of Example 5.17 restricted to  $V^f$  – this follows from  $\varrho T_\Sigma$  being the terminal locally finite coalgebra. Thus  $i \cdot k$  takes every language  $L$  of  $V$  to  $L \cdot f$ , proving that  $L \cdot f$  lies in  $V$ .



For the converse, suppose that  $V$  is closed under preimages. Then the homomorphism  $h$  of Example 5.17 has a restriction  $h_0 : V^f \rightarrow V$ . That  $h_0$  is a coalgebra homomorphism is a consequence of the Homomorphism Theorem 4.22.  $\square$

**LEMMA 6.21.** *Every finite subset of a fully invariant local variety of languages in  $\mathcal{C}$  is contained in a finite fully invariant local subvariety.*

**PROOF.** In view of Lemma 6.11 it is sufficient to prove that given a fully invariant local variety  $j : V \rightarrow \rho T_\Sigma$  every finite local subvariety  $i : Q \rightarrow V$  is contained in a finite fully invariant local subvariety  $W \rightarrow V$ . Since  $Q$  is finite, there clearly exist endomorphisms  $h_1, \dots, h_n$  of  $\Psi\Sigma^*$  such that every coalgebra  $Q^h$ , for an endomorphism  $h$ , is equal to  $Q^{h_i}$  for some  $i = 1, \dots, n$ . Factorize the unique coalgebra homomorphism  $f : Q^{h_1} + \dots + Q^{h_n} \rightarrow \rho T_\Sigma$  as a surjective homomorphism  $e : Q^{h_1} + \dots + Q^{h_n} \rightarrow W$  followed by an injective one  $m : W \rightarrow \rho T_\Sigma$ , see Remark 4.21(b). Note that the coalgebra  $Q^{h_1} + \dots + Q^{h_n}$  is finite because  $Q$  is finite and  $\mathcal{C}$  is locally finite (so finite coproducts of finite objects in  $\mathcal{C}$  are finite). Consequently,  $W$  is finite. We now prove that  $W$  is a fully invariant local variety containing  $Q$ :

(a)  $Q$  is contained in  $W$  via the  $i$ -th coproduct injection for  $i$  satisfying  $Q^{id} = Q^{h_i}$ , since  $Q^{id} = Q$ .

(b)  $W$  is a local variety due to Propositions 6.7 and 6.20: For every input  $b \in \Sigma$  we have coalgebra homomorphisms  $f : V_b \rightarrow V$  and  $f_i : Q^{h_i} \rightarrow Q$ . We are to show that there is a coalgebra homomorphism from  $W_b$  to  $W$ . It is easy to see that the delay construction  $(-)_b$  commutes with coproducts (recalling that coproducts of coalgebras are formed on the level of the base category  $\mathcal{C}$ ). Moreover  $(Q^h)_b = (Q_b)^h$  since these coalgebras have the same transitions  $\gamma_{h(b)}$  and the same output  $\gamma_{out} \cdot \gamma_b$ . Finally, the above coalgebra homomorphisms  $f_i$  are also coalgebra homomorphisms

$$f_i : (Q_b)^{h_i} \rightarrow Q_b,$$

and they yield a coalgebra homomorphism

$$p : \coprod_i (Q_b)^{h_i} \rightarrow \coprod_i Q^{h_i}.$$

Analogously  $e$  and  $m$  are coalgebra homomorphisms for the delayed coalgebras.

The desired coalgebra homomorphism is obtained via diagonal fill-in in  $\text{Coalg } T_\Sigma$ :

$$\begin{array}{ccccc} \coprod_i (Q_b)^{h_i} & \xrightarrow{e} & W_b & \xrightarrow{m} & V_b \\ \downarrow p & & \downarrow f' & & \downarrow f \\ \coprod_i Q^{h_i} & \xrightarrow{e} & W & \xrightarrow{m} & V \end{array} \quad (6.2) \quad \text{eq:g'}$$

The outside of the diagram commutes because, for every  $i$ , the  $i$ -th components of the upper and lower path are coalgebra homomorphisms from the finite coalgebra  $(Q_b)^{h_i}$  to  $V$  which is a subcoalgebra of the terminal locally finite coalgebra  $\rho T_\Sigma$ . (Thus the monomorphism  $j : V \rightarrow \rho T_\Sigma$  merges both sides of the above square.)

(c)  $W$  is fully invariant due to Proposition 6.20. Indeed, for every endomorphism  $f$  of  $\Psi\Sigma^*$  we have a coalgebra homomorphism  $f^* : V^f \rightarrow V$ . Using (d) and (e) in Lemma 2.16, we can define a coalgebra homomorphism  $q : (\coprod_i Q^{h_i})^f \rightarrow (\coprod_i Q^{h_i})$  by choosing for every  $i$  an index  $j$  with  $Q^{h_i \cdot f} = Q^{h_j}$  and letting the  $i$ -th component of  $q$  be the coproduct injection of  $Q^{h_j}$ . It is easy to see that  $e$  and  $m$  are also coalgebra homomorphisms when  $(-)^f$  is applied. Therefore, the desired

coalgebra homomorphism from  $W^f$  to  $W$  is again obtained by the diagonal fill-in:

$$\begin{array}{ccccc} (\coprod_i Q^{h_i})^f & \xrightarrow{e} & W^f & \xrightarrow{m} & V^f \\ \downarrow q & & \downarrow f'' & & \downarrow f^* \\ \coprod_i Q^{h_i} & \xrightarrow{e} & W & \xrightarrow{m} & V \end{array}$$

□

**THEOREM 6.22 (FULLY INVARIANT LOCAL GENERALIZED EILENBERG THEOREM).** *For every finite alphabet  $\Sigma$  the following complete lattices are isomorphic:*

- (1) *all fully invariant local varieties of languages over  $\Sigma$  in  $\mathcal{C}$ , and*
- (2) *all pseudovarieties of fully invariant  $\Sigma$ -generated  $\mathcal{D}$ -monoids.*

**PROOF.** The argument is analogous to the proof of Theorem 6.16. We use ideal completions, and the fact that the lattice of all pseudovarieties of fully invariant  $\Sigma$ -generated  $\mathcal{D}$ -monoids is the ideal completion of its subposet of all finite members. And, due to the Lemma 6.21, the lattice of all fully invariant local varieties is the ideal completion of its subposet of all finite members.

Thus, it only remains to verify that two subposets above are isomorphic. This follows by the same argument as in Proposition 6.10. We just need to observe that given a finite  $\Sigma$ -generated  $\mathcal{D}$ -monoid  $e : \Psi\Sigma^* \rightarrow A$ , the dual coalgebra  $Q = \hat{A}$  is a fully invariant local variety iff  $e$  is fully invariant. To this end, we need to show that for any  $\mathcal{D}$ -monoid morphism  $h : \Psi\Sigma^* \rightarrow \Psi\Sigma^*$  and any morphism  $k : A \rightarrow A$  in  $\mathcal{D}$ , one has  $k \cdot e = e \cdot h$  if and only if  $\hat{k} : Q \rightarrow Q$  is coalgebra homomorphism from  $Q^h$  to  $Q$  – equivalently, that  $k$  is a  $\Sigma$ -algebra homomorphism from  $A$  to  $A^{h^\dagger}$ , see Proposition 6.20.

Recall from Remark 3.4 that the transitions  $\alpha_a$  of  $A$  as an automaton are defined by  $\alpha_a \cdot e = e \cdot r_a$ , where  $r_a(x) = x \bullet a$  denotes the right translation.

For the “if” direction, suppose that  $k : A \rightarrow A^{h^\dagger}$  is a  $\Sigma$ -automata morphism. Then for all inputs  $a$  we have  $k \cdot \alpha_a = \alpha_{h(a)} \cdot k$ , recalling the transitions from Definition 5.4 and the fact that  $h^\dagger(a) = h(a)$  for all  $a$  in  $\Sigma$ . By precomposing with  $e$  we obtain:

$$k \cdot e \cdot r_a = k \cdot \alpha_a \cdot e = \alpha_{h(a)} \cdot k \cdot e.$$

Thus  $k \cdot e : \Psi\Sigma^* \rightarrow A^{h^\dagger}$  is a  $\Sigma$ -automata morphism. Indeed, the above equation states that it preserves transitions, and it preserves the initial state since  $k$  does. Moreover,  $e \cdot h : \Psi\Sigma^* \rightarrow A^{h^\dagger}$  is a  $\Sigma$ -automata morphism because

$$e \cdot h \cdot r_a = e \cdot r_{h(a)} \cdot h = \alpha_{h(a)} \cdot e \cdot h,$$

where the first equation uses that  $h$  is a  $\mathcal{D}$ -monoid morphism, and the second one that  $e$  preserves transitions. Moreover, clearly  $e \cdot h$  preserves the initial state. Therefore the desired equality  $k \cdot e = e \cdot h$  follows from the initiality of  $\Psi\Sigma^*$ , see Remark 3.4(c).

For the “only if” direction, suppose that  $k \cdot e = e \cdot h$  holds. Then  $k : A \rightarrow A^h$  is a  $\Sigma$ -automata morphism. Indeed,  $k$  preserves the initial state  $e(\varepsilon)$  because  $h(\varepsilon) = \varepsilon$ . And it preserves transitions, i.e., we have  $k \cdot \alpha_a = \alpha_{h(a)} \cdot k$ , since by precomposing with  $e$  we get the equivalent equation

$$k \cdot \alpha_a \cdot e = k \cdot e \cdot r_a = e \cdot h \cdot r_a = e \cdot r_{h(a)} \cdot h = \alpha_{h(a)} \cdot e \cdot h = \alpha_{h(a)} \cdot k \cdot e.$$

This holds because  $h$ , being a  $\mathcal{D}$ -monoid endomorphism, fulfils  $r_{h(a)} \cdot h = h \cdot r_a$ .

□

## 7 OBJECT-FINITE AND SIMPLE VARIETIES

As the last preparation step to proving the full generalization of Eilenberg's theorem we study those varieties of languages which for every alphabet contain only finitely many languages. We call them object-finite. They are also interesting *per se* because, as we prove below, they bijectively correspond to *varieties* (not just pseudovarieties) of  $\mathcal{D}$ -monoids that are locally finite, i.e., whose finitely generated members are finite. This generalizes a result of Klíma and Polák [19] for the special cases  $\mathcal{C} = \mathbf{BA}$  or  $\mathcal{C} = \mathbf{DL}_b$ .

*Definition 7.1.* (1) By a *variety*  $V$  of regular languages is mean a collection of local varieties

$$V_\Sigma \hookrightarrow \varrho T_\Sigma$$

(indexed by all finite alphabets  $\Sigma$ ) closed under preimages, i.e., for every  $\mathcal{D}$ -monoid homomorphism  $h : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ , the preimage (see Definition 4.19) of every language in  $|V_\Sigma|$  lies in  $|V_\Delta|$ .

(2) The variety is called *object-finite* if  $V_\Sigma$  is finite for every alphabet  $\Sigma$ .

In order to give a coalgebraic characterization of closure under preimages, we extend Proposition 6.20; the proof is completely analogous:

L:pr

**PROPOSITION 7.2.** *A collection of local varieties  $V_\Sigma$  (for all finite alphabets  $\Sigma$ ) is closed under preimages iff for every  $\mathcal{D}$ -monoid morphism  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  there exists a  $T_\Delta$ -coalgebra homomorphism from  $V_\Sigma^f$  to  $V_\Delta$ .*

:simple

*Example 7.3.* Object-finite varieties in  $\mathcal{C}$  are ubiquitous: Given a variety  $V$  of languages and an alphabet  $\Sigma$ , then every finite local variety  $i : Q \rightarrow V_\Sigma$  defines an object-finite subvariety  $V'$  of  $V$  as follows. For every finite alphabet  $\Delta$  we define  $V'_\Delta$  by considering, for all  $\mathcal{D}$ -monoid morphisms  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ ,  $T_\Delta$ -coalgebra homomorphisms  $\hat{f} : (V_\Sigma)^f \rightarrow V_\Delta$  (cf. Proposition 7.2). Then  $V'_\Delta$  is the image of the coalgebra homomorphism from  $\coprod_{f:\Psi\Delta^*\rightarrow\Psi\Sigma^*} Q^f$  to  $V_\Delta$  whose components are  $\hat{f} \cdot i : Q^f \rightarrow V_\Delta$ . In other words, we have a factorization as in Remark 4.21:

$$[\hat{f} \cdot i] \equiv \coprod_{f:\Psi\Delta^*\rightarrow\Psi\Sigma^*} Q^f \xrightarrow{e_\Delta} V'_\Delta \xrightarrow{m_\Delta} V_\Delta . \quad (7.1)$$

eq:fact

The proof that  $V'$  is indeed an object-finite variety is presented in Proposition 7.12 below.

*Notation 7.4.* Let  $V$  be an object-finite variety of languages. For every alphabet  $\Sigma$  the finite local variety  $V_\Sigma$  corresponds, as we have seen in Proposition 6.10, to a  $\Sigma$ -generated  $\mathcal{D}$ -monoid that we denote by

$$e_\Sigma : \Psi\Sigma^* \rightarrow V_\Sigma^\circledast .$$

*Notation 7.5.* For every object-finite variety of languages  $V$  we denote by  $V^\circledast$  the class of all  $\mathcal{D}$ -monoids  $D$  such that for every alphabet  $\Sigma$  all morphisms  $f : \Psi\Sigma^* \rightarrow D$  of  $\mathcal{D}$ -monoids factorize through  $e_\Sigma$ .

:square

**LEMMA 7.6.** *Let  $Q \rightarrow \varrho T_\Sigma$  and  $Q' \rightarrow \varrho T_\Delta$  be finite local varieties of languages dual to finite automata  $A$  and  $A'$ , resp. For every  $\mathcal{D}$ -monoid morphism  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  the following statements are equivalent:*

(a) *there exists a  $T_\Delta$ -coalgebra homomorphism from  $h : Q^f \rightarrow Q'$ , and*

(b) there exists a  $\mathcal{D}$ -monoid morphism  $g$  making the following square commutative (recall  $f^\dagger$  from Notation 5.8):

$$\begin{array}{ccc} \Psi\Delta^* & \xrightarrow{f^\dagger} & \Psi\Sigma^* \\ e_{A'} \downarrow & & \downarrow e_A \\ A' & \xrightarrow{g} & A \end{array} \quad (7.2)$$

diag: sq

Moreover, we have  $h = \widehat{g}$ .

PROOF. We can assume that  $Q$  and  $Q'$  are dual to finite algebras  $A$  and  $A'$ , respectively.

(a)  $\Rightarrow$  (b) Given a  $T_\Delta$ -coalgebra homomorphism  $h : Q^f \rightarrow Q'$  we have, by Lemma 5.6(a), the following square of  $\Delta$ -automata morphisms

$$\begin{array}{ccc} \Psi\Delta^* & \xrightarrow{f^\dagger} & (\Psi\Sigma^*)^{f^\dagger} \\ e_{A'} \downarrow & & \downarrow e_A \\ A' & \xrightarrow{g} & A^{f^\dagger} \end{array} \quad (7.3)$$

diag: la

where  $h = \widehat{g}$ . This diagram commutes because  $\Psi\Delta^*$  is the initial  $\Delta$ -automaton. Therefore the square (7.2) commutes. That  $g$  is a  $\mathcal{D}$ -monoid morphism follows from Remark 2.22(3).

(b)  $\Rightarrow$  (a) Given a morphism  $g$  for which (7.2) commutes, then  $g : A' \rightarrow A^{f^\dagger}$  is a homomorphism of  $\Delta$ -automata by (7.3) and Lemma 3.6. Therefore, dually, we have the  $T_\Delta$ -coalgebra homomorphism  $h = \widehat{g} : Q^f \rightarrow Q'$ .  $\square$

-finite

PROPOSITION 7.7. If  $V$  is an object-finite variety of languages, then  $V^\circledast$  is a locally finite variety of  $\mathcal{D}$ -monoids. Its free algebra on a finite set  $\Sigma$  is  $V_\Sigma^\circledast$ .

PROOF. (1) We first verify that  $V^\circledast$  is a variety of  $\mathcal{D}$ -monoids.

- (a) Closure under products: let  $\pi_i : \prod_{i \in I} D_i \rightarrow D_i$  be a product of monoids  $D_i \in V^\circledast$ . Given a monoid morphism  $h : \Psi\Sigma^* \rightarrow \prod_{i \in I} D_i$  every morphism  $\pi_i \cdot h : \Psi\Sigma^* \rightarrow D_i$  factorizes as  $\pi_i \cdot h = k_i \cdot e_\Sigma$  for some  $k_i : V_\Sigma^\circledast \rightarrow D_i$ . Hence  $\langle k_i \rangle : V_\Sigma^\circledast \rightarrow \prod D_i$  is the desired morphism with  $h = \langle k_i \rangle \cdot e_\Sigma$ .
- (b) Closure under submonoids: given a  $\mathcal{D}$ -submonoid  $m : D \rightarrow D'$  with  $D' \in V^\circledast$  and a  $\mathcal{D}$ -monoid morphism  $h : \Psi\Sigma^* \rightarrow D$ , the  $\mathcal{D}$ -monoid morphism  $m \cdot h$  factorizes through  $e_\Sigma$ . Consequently  $h$  factorizes through  $e_\Sigma$  due to diagonal fill-in:

$$\begin{array}{ccc} \Psi\Sigma^* & \xrightarrow{e_\Sigma} & V_\Sigma^\circledast \\ h \downarrow & \swarrow & \downarrow k \\ D & \xrightarrow{m} & D' \end{array}$$

Thus  $D \in V^\circledast$ .

- (c) Closure under quotients: given a quotient  $\mathcal{D}$ -monoid  $e : D \twoheadrightarrow D'$  with  $D \in V^\circledast$  and a homomorphism  $h : \Psi\Sigma^* \rightarrow D'$ , choose a splitting of  $e$  in **Set**, i.e., a function  $u : |D'| \rightarrow |D|$  with  $e \cdot u = id$ . Recall the universal map  $\eta : \Sigma \hookrightarrow |\Psi\Sigma^*|$  of the free monoid (given by the inclusions map), and extend the map  $u \cdot h \cdot \eta : \Sigma \rightarrow |D|$  to a homomorphism  $k : \Psi\Sigma^* \rightarrow D$ , which then we

can factorize as  $k = k' \cdot e_\Sigma$  because  $D$  is in  $V^\circledast$ .

$$\begin{array}{ccccc}
 \Sigma & \xrightarrow{\eta} & \Psi\Sigma^* & \xrightarrow{e_\Sigma} & V_\Sigma^\circledast \\
 & & \downarrow k & \searrow h & \swarrow k' \\
 & & D & \xrightarrow[e]{u} & D'
 \end{array}$$

Then the  $\mathcal{D}$ -monoid morphism  $e \cdot k' : V_\Sigma^\circledast \rightarrow D'$  is the desired factorization of  $h$ ; indeed, using the universal property of the free  $\mathcal{D}$ -monoid  $\Psi\Sigma^*$  and the fact that  $u$  is injective, it suffices to prove that  $u \cdot h \cdot \eta = u \cdot e \cdot k' \cdot e_\Sigma \cdot \eta$  in  $\mathbf{Set}$ , which holds because

$$\begin{aligned}
 u \cdot h \cdot \eta &= u \cdot e \cdot u \cdot h \cdot \eta && (e \cdot u = id) \\
 &= u \cdot e \cdot k \cdot \eta && (\text{def. } k) \\
 &= u \cdot e \cdot k' \cdot e_\Sigma \cdot \eta && (\text{def. } k').
 \end{aligned}$$

(2) We establish that  $V_\Delta^\circledast$  lies in  $V^\circledast$ . Suppose that we are given a  $\mathcal{D}$ -monoid morphism  $h : \Psi\Sigma^* \rightarrow V_\Delta^\circledast$  we have to show that it factorizes through  $e_\Sigma : \Psi\Sigma^* \rightarrow V_\Sigma^\circledast$ . To this end choose a splitting of  $e_\Delta$  in  $\mathbf{Set}$  (i.e., a function  $u : |V_\Delta^\circledast| \rightarrow |\Psi\Delta^*|$  with  $e_\Delta \cdot u = id$ ), and extend  $\text{rev}_\Delta \cdot u \cdot h \cdot \eta$  to a  $\mathcal{D}$ -monoid morphism  $f : \Psi\Sigma^* \rightarrow \Psi\Delta^*$ . Since  $V$  is a variety of languages,  $f$  induces a coalgebra homomorphism for  $T_\Sigma$  from  $(V_\Delta)^f$  to  $V_\Sigma$  by Proposition 7.2. By Lemma 7.6, there is a  $\mathcal{D}$ -monoid morphism  $g : V_\Sigma^\circledast \rightarrow V_\Delta^\circledast$  such that  $e_\Delta \cdot f^\dagger = g \cdot e_\Sigma$ :

$$\begin{array}{ccccc}
 \Sigma & \xrightarrow{\eta} & \Psi\Sigma^* & \xrightarrow{f^\dagger} & \Psi\Delta^* \\
 & & \downarrow e_\Sigma & \searrow h & \uparrow u \\
 & & V_\Sigma^\circledast & \xrightarrow[g]{} & V_\Delta^\circledast
 \end{array}$$

We claim that  $g$  is the desired factorization of  $h$  in  $\mathbf{Mon} \mathcal{D}$ , i.e.  $g \cdot e_\Sigma = h$ . Using freeness of the  $\mathcal{D}$ -monoid  $\Psi\Sigma^*$  and since  $u$  is injective it suffices to prove  $u \cdot g \cdot e_\Sigma \cdot \eta = u \cdot h \cdot \eta$  in  $\mathbf{Set}$ , and indeed we have

$$\begin{aligned}
 u \cdot g \cdot e_\Sigma \cdot \eta &= u \cdot e_\Delta \cdot f^\dagger \cdot \eta && (\text{def. } g) \\
 &= u \cdot e_\Delta \cdot \text{rev}_\Delta \cdot f \cdot \text{rev}_\Sigma \cdot \eta && (\text{def. } f^\dagger) \\
 &= u \cdot e_\Delta \cdot \text{rev}_\Delta \cdot f \cdot \eta && (\text{rev}_\Sigma \cdot \eta = \eta) \\
 &= u \cdot e_\Delta \cdot \text{rev}_\Delta \cdot \text{rev}_\Delta \cdot u \cdot h \cdot \eta && (\text{def. } f) \\
 &= u \cdot e_\Delta \cdot u \cdot h \cdot \eta && (\text{rev}_\Delta \cdot \text{rev}_\Delta = id) \\
 &= u \cdot h \cdot \eta && (e_\Delta \cdot u = id).
 \end{aligned}$$

(3) From the definition of  $V^\circledast$  and (2) above we immediately conclude that  $V_\Delta^\circledast$  is the free monoid on a finite set  $\Delta$  in the variety  $V^\circledast$ . Hence, since  $V_\Delta^\circledast$  is finite,  $V^\circledast$  is a locally finite variety of  $\mathcal{D}$ -monoids.  $\square$

All object-finite varieties of languages form a complete lattice: every intersection of object-finite varieties is object-finite. In the following theorem we prove that the poset of all locally finite varieties of  $\mathcal{D}$ -monoids is isomorphic to that lattice, thus this is also a complete lattice.

**Eil:of** **THEOREM 7.8 (GENERALIZED EILENBERG THEOREM FOR OBJECT-FINITE VARIETIES).** *The complete lattice of all object-finite varieties of languages in  $\mathcal{C}$  is isomorphic to the complete lattice of all locally finite varieties of  $\mathcal{D}$ -monoids.*

**PROOF.** The above map  $V \mapsto V^\circledast$  defines the desired isomorphism. To see this, we describe its inverse  $W \mapsto W^\square$ . Let  $W$  be a locally finite variety of  $\mathcal{D}$ -monoids. Denote by  $D_\Sigma$  the free  $\mathcal{D}$ -monoid in  $W$  generated by the finite set  $\Sigma$ , and by  $e_\Sigma : \Psi\Sigma^* \twoheadrightarrow D_\Sigma$  the corresponding quotient, i.e., the unique  $\mathcal{D}$ -monoid morphism extending the universal map  $\Sigma \mapsto |D_\Sigma|$ . Define an object-finite variety  $W^\square$  of languages in  $\mathcal{C}$  by forming, for each finite  $\Sigma$ , the dual local variety  $W_\Sigma^\square \hookrightarrow \varrho T_\Sigma$  of  $e_\Sigma$ , i.e.,  $(W_\Sigma^\square)^\circledast \cong D_\Sigma$ .

(a) The collection  $(W_\Sigma^\square)$  is a variety of languages. Indeed, to verify closure under preimages, let  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  be a morphism of  $\mathcal{D}$ -monoids. By Proposition 7.2 it is our task to find a coalgebra homomorphism from  $(W_\Sigma^\square)^f$  to  $W_\Delta^\square$ . The latter coalgebra is dual to the  $\Delta$ -automaton corresponding to the  $\Delta$ -generated monoid  $e_\Delta : \Psi\Delta^* \rightarrow D_\Delta$ . Moreover, the universal property of  $e_\Delta$  implies that there is a morphism of  $\mathcal{D}$ -monoids

$$g : D_\Delta \rightarrow D_\Sigma \quad \text{with } g \cdot e_\Delta = e_\Sigma \cdot f^\dagger.$$

Then Lemma 7.6 implies the claim.

(b) The passages  $(-)^{\circledast}$  and  $(-)^{\square}$  are mutually inverse. Indeed, from Proposition 7.7 and the definition of  $(-)^{\circledast}$  and  $(-)^{\square}$  it is clear that  $(V^{\circledast})^{\square} = V$ . To show that  $(W^{\square})^{\circledast} = W$ , observe first that the varieties  $(W^{\square})^{\circledast}$  and  $W$  have by definition the same finitely generated free  $\mathcal{D}$ -monoids  $D_\Sigma$ , and hence contain the same finite  $\mathcal{D}$ -monoids. Moreover, both varieties are locally finite and hence form the closure (in the category  $\text{Mon } \mathcal{D}$ ) of their finite members under filtered colimits. It follows that  $(W^{\square})^{\circledast} = W$ , as claimed.

We conclude that  $V \mapsto V^{\circledast}$  defines a bijection between

- (i) the class of object-finite varieties of languages (ordered by objectwise inclusion), and
- (ii) the class of locally finite varieties of  $\mathcal{D}$ -monoids (ordered by inclusion).

Moreover, this bijection clearly preserves and reflects the order, so it is an isomorphism of complete lattices.  $\square$

**:simple** **Definition 7.9.** An object-finite variety  $V$  of languages is called *simple* if it is generated by a finite alphabet  $\Sigma$ . That is, every variety  $W$  with  $V_\Sigma \subseteq W_\Sigma$  contains all of  $V$ .

A (pseudo)variety of  $\mathcal{D}$ -monoids is called *simple* if it is generated by a single finite  $\mathcal{D}$ -monoid  $D$ , i.e., all members of the pseudovariety are quotients of submonoids of (finite) powers of  $D$ .

We are going to prove that simple varieties of languages bijectively correspond to simple varieties of  $\mathcal{D}$ -monoids. For the proof we need to extend the concept of delayed automaton of Definition 6.4.

**not:qx** **Notation 7.10.** Let  $A = (Q, \alpha)$  be a finite  $\Sigma$ -automaton and  $e_A : \Psi\Sigma^* \rightarrow A$  the initial homomorphism. For every element  $x$  of  $\Psi\Sigma^*$  let  $A_x$  be the  $\Sigma$ -automaton with the same states  $Q$ , the same transitions  $\alpha_a$ , but initial state  $e_A(x)$ .

Similarly, for any finite  $T_\Sigma$ -coalgebra  $C$  with dual  $\Sigma$ -automaton  $A$ , we define the  $T_\Sigma$ -coalgebra  $C_x$  as follows:

$$C_x = \widehat{A_{\text{rev}_\Sigma(x)}}.$$

**L:fx** **LEMMA 7.11.** *Let  $x \in |\Psi\Delta^*|$ .*

- (a) *Every homomorphism  $h : Q \rightarrow Q'$  between finite  $T_\Delta$ -coalgebras is also a homomorphism  $h : Q_x \rightarrow Q'_x$ .*
- (b) *If  $Q \mapsto \varrho T_\Delta$  is a finite local variety, then a  $T_\Delta$ -coalgebra homomorphism from  $Q_x$  to  $Q$  exists.*

(c) For every finite  $T_\Sigma$ -coalgebra  $Q$  and every  $\mathcal{D}$ -monoid morphism  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  we have

$$(Q^f)_x = (Q_{f_x})^f.$$

PROOF. For (a), let  $y := \text{rev}_\Sigma(x)$  and notice first that every homomorphism  $k : (A', \alpha') \rightarrow (A, \alpha)$  of  $\Delta$ -automata yields a homomorphism  $k : (A', \alpha')_y \rightarrow (A, \alpha)_y$  since by initiality of  $\Psi\Delta^*$  we have  $k \cdot e_{A'}(y) = e_A(y)$ . Now given a homomorphism  $h : Q \rightarrow Q'$  between finite  $T_\Delta$ -coalgebras, by Lemma 4.5 it is dual to a  $\Delta$ -automata homomorphism  $k : A' \rightarrow A$ . Hence, we have the homomorphism  $k : A'_y \rightarrow A_y$  whose dual is the desired coalgebra homomorphism.

For (b), apply Proposition 6.7 for  $x \in \Delta^*$ . The proof for the general case is completely analogous. It remains to prove (c). We first prove that

$$(A^f)_x = (A_{f_x})^f \quad (7.4)$$

for all  $\Sigma$ -automata  $A = (A, \alpha)$  and  $\mathcal{D}$ -monoid morphisms  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ . Indeed, both  $(A^f)_x$  and  $(A_{f_x})^f$  have states  $A$  and transitions  $\alpha_{fa}$  for  $a \in \Delta$ . Moreover, the initial state of  $(A^f)_x$  is  $e_{A'}(x)$ , and that of  $(A_{f_x})^f$  is  $e_A(fx)$ . Hence, by Lemma 5.6(c),  $(A^f)_x$  and  $(A_{f_x})^f$  have the same initial state.

Now let  $Q$  be a finite  $T_\Sigma$ -coalgebra, dual to  $A$ . Then due to the definition of  $Q^f$  and  $Q_x$  the above equation states that

$$(Q^{f^\dagger})_{\text{rev}_\Sigma(x)} = (Q_{\text{rev}_\Sigma(fx)})^{f^\dagger}.$$

This is precisely what we wanted to prove: use  $f$  in place of  $f^\dagger$  and  $\text{rev}_\Sigma(x)$  in place of  $x$ . Since  $f$  is a homomorphism, we see that  $\text{rev}_\Sigma(fx) = f \cdot \text{rev}_\Sigma(x)$  and we obtain the desired equality.  $\square$

PROPOSITION 7.12. Let  $V$  be a variety of languages in  $\mathcal{C}$  and let  $Q$  be a local subvariety of  $V_\Sigma$ . Then  $V'$  of Example 7.3 is a simple variety with  $Q \subseteq V'_\Sigma$ .

PROOF. (1) For every finite alphabet  $\Gamma$  and every  $\mathcal{D}$ -monoid morphism  $g : \Psi\Gamma^* \rightarrow \Psi\Delta^*$  we prove that there exists a  $T_\Gamma$ -coalgebra homomorphism

$$g' : (V'_\Delta)^g \rightarrow V'_\Gamma \quad \text{with} \quad m_\Gamma \cdot g' = \hat{g} \cdot m_\Delta. \quad (7.5)$$

Denote by  $p : \coprod_{f: \Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f \cdot g \rightarrow \coprod_{h: \Psi\Gamma^* \rightarrow \Psi\Sigma^*} Q^h$  the  $T_\Gamma$ -coalgebra homomorphism whose  $f$ -component is the coproduct injection of  $h = f \cdot g$ . Note that  $\coprod_f Q^{f \cdot g} = (\coprod_f Q^f)^g$  by Lemma 5.14. Hence we have the following diagram of  $T_\Gamma$ -coalgebra homomorphisms:

$$\begin{array}{ccccc} (\coprod_f Q^f)^g & \xrightarrow{e_\Delta} & (V'_\Delta)^g & \xrightarrow{m_\Delta} & V'_\Delta \\ p \downarrow & & \downarrow g' & & \downarrow \hat{g} \\ \coprod_h Q^h & \xrightarrow{e_\Gamma} & V'_\Gamma & \xrightarrow{m_\Gamma} & V'_\Gamma \end{array}$$

The outside of the diagram commutes because the  $f$ -components of the upper and lower path are  $T_\Gamma$ -coalgebra homomorphisms from the finite  $T_\Gamma$ -coalgebra  $Q^{f \cdot g}$  to  $V'_\Gamma \rightarrow \varrho T_\Gamma$ , and  $\varrho T_\Gamma$  is the terminal locally finite coalgebra. The desired  $T_\Gamma$ -coalgebra homomorphism  $g'$  is obtained via diagonal fill-in in  $\text{Coalg } T_\Gamma$ , see Remark 4.21.

(2)  $V'_\Delta$  is a local variety for every  $\Delta$ . Indeed, by definition it is a subcoalgebra of  $V_\Delta$  and hence of  $\varrho T_\Delta$ . To prove closure under right derivatives, use Proposition 6.7: since  $V_\Delta$  and  $Q$  are local varieties, we have  $T_\Delta$ -coalgebra homomorphisms  $h_a : (V_\Delta)_a \rightarrow V_\Delta$  for all  $a \in \Delta$  and  $T_\Sigma$ -coalgebra homomorphisms  $k_x : Q_x \rightarrow Q$  for all  $x \in \Psi\Sigma^*$ , see Lemma 7.11(b). Moreover,

$$\left( \coprod_f Q^f \right)_a = \coprod_f (Q^f)_a = \coprod_f (Q_{fa})^f$$

by Lemma 7.11(c) and since the construction  $(-)_a$  clearly commutes with coproducts. Hence we have the following diagram of  $T_\Delta$ -coalgebra homomorphisms

$$\begin{array}{ccccc}
 (\coprod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f)_a & \xrightarrow{e_\Delta} & (V'_\Delta)_a & \xrightarrow{m_\Delta} & (V_\Delta)_a \\
 \downarrow \coprod_f k_{f_a} & & \downarrow h'_a & & \downarrow h_a \\
 \coprod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f & \xrightarrow{e_\Delta} & V'_\Delta & \xrightarrow{m_\Delta} & V_\Delta
 \end{array}$$

whose outside commutes since  $V_\Delta$  is a subcoalgebra of the terminal locally finite coalgebra  $\varrho T_\Delta$ . Diagonal fill-in yields a  $T_\Delta$ -coalgebra homomorphism  $h'_a : (V'_\Delta)_a \rightarrow V'_\Delta$ , which shows that  $V'_\Delta$  is closed under right derivatives by Proposition 6.7.

(3)  $V'$  is a variety of languages. Indeed, apply Proposition 7.2 to conclude from (7.5) that  $V'$  is closed under preimages.

(4)  $Q \subseteq V'_\Sigma$ , due to the possibility of choosing  $f = id_{\Psi\Sigma^*}$  in (7.1) for the case  $\Delta = \Sigma$ .

(5)  $V'$  is object-finite. Note first that for every finite alphabet  $\Delta$  there exist only finitely many preimages  $Q^f$ , where  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  ranges over all  $\mathcal{D}$ -monoid morphisms: indeed,  $Q$  is finite and the coalgebra  $Q^f$  has the same set of states as  $Q$ . Choose  $f_1, \dots, f_n$  such that each  $Q^f$  is equal to  $Q^{f_i}$  for precisely one  $i$ . Then consider the  $T_\Delta$ -coalgebra homomorphism  $t : \coprod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f \rightarrow \coprod_{i=1}^n Q^{f_i}$  whose  $f$ -component is the coproduct injection of  $Q^{f_i}$  whenever  $Q^f = Q^{f_i}$ . Then  $e_\Delta = u \cdot t$  for the obvious morphism  $u : \coprod_{i=1}^n Q^{f_i} \rightarrow V'_\Delta$ . Thus,  $u$  is surjective since  $e_\Delta$  is, proving that  $V'_\Delta$  is finite (being a quotient of the finite coalgebra  $\coprod_{i=1}^n Q^{f_i}$ ).

(6)  $V'$  is simple. Indeed, let  $V''$  be a variety of languages with  $j : Q \rightarrow V''_\Sigma$  a local subvariety, then we prove that  $V'$  is a subvariety of  $V''$ . Denote by  $u'_\Delta : V'_\Delta \rightarrow \varrho T_\Delta$  and  $u''_\Delta : V''_\Delta \rightarrow \varrho T_\Delta$  the embeddings. By Proposition 7.2 we have a  $T_\Delta$ -coalgebra homomorphism  $\tilde{f} : (V''_\Sigma)^f \rightarrow V''_\Delta$ . Then the square

$$\begin{array}{ccc}
 \coprod_{f:\Psi\Delta^* \rightarrow \Psi\Sigma^*} Q^f & \xrightarrow{e_\Delta} & V'_\Delta \\
 \downarrow [\tilde{f} \cdot j] & \swarrow & \downarrow u'_\Delta \\
 V''_\Delta & \xrightarrow{u''_\Delta} & \varrho T_\Delta
 \end{array}$$

commutes due to  $\varrho T_\Delta$  being the terminal locally finite  $T_\Delta$ -coalgebra. Diagonal fill-in yields the desired embedding  $V'_\Delta \rightarrow V''_\Delta$ .  $\square$

**Remark 7.13.** (a) Let  $V$  be a simple variety of languages in  $\mathcal{C}$  generated by an alphabet  $\Sigma$ . Then  $V$  is also generated by any alphabet  $\Delta$  with  $|\Delta| \geq |\Sigma|$ .

Indeed, let  $V'$  be any variety with  $V_\Delta \subseteq V'_\Delta$ . Since  $V$  is generated by  $\Sigma$ , it suffices to show  $V_\Sigma \subseteq V'_\Sigma$ ; then  $V \subseteq V'$  follows. Observe first that there exist  $\mathcal{D}$ -monoid morphisms  $e : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  and  $m : \Psi\Sigma^* \rightarrow \Psi\Delta^*$  with  $e \cdot m = id$ . Indeed, if  $\Sigma \neq \emptyset$ , choose functions  $m_0 : \Sigma \hookrightarrow \Delta$  and  $e_0 : \Delta \rightarrow \Sigma$  with  $e_0 \cdot m_0 = id$  in  $\mathbf{Set}$  and put  $e = \Psi e_0$  and  $m = \Psi m_0$ . If  $\Sigma = \emptyset$ , consider the two (unique) monoid morphisms  $m' : \emptyset^* \rightarrow \Delta^*$  and  $e' : \Delta^* \rightarrow \emptyset^*$  (satisfying  $e' \cdot m' = id$ ), and put  $e = \Psi e'$  and  $m = \Psi m'$ . It is easy to see that  $e$  and  $m$  are indeed  $\mathcal{D}$ -monoid morphisms.

Now let  $L \in V_\Sigma$ . By closure of  $V$  under preimages we have  $L \cdot e \in V_\Delta \subseteq V'_\Delta$ . Since  $V'$  is also closed under preimages, we conclude  $L = L \cdot e \cdot m \in V'_\Sigma$ , thus  $V_\Sigma \subseteq V'_\Sigma$ .



- (b) All simple varieties of  $\mathcal{D}$ -monoids form a join-semilattice: if the variety  $V$  is generated by  $D$  and  $V'$  is generated by  $D'$ , then the variety  $W$  generated by  $D \times D'$  is the join of  $V$  and  $V'$ . Indeed, since  $D$  and  $D'$  are quotients of  $D \times D'$ , the variety  $W$  contains both  $V$  and  $V'$ . And, conversely, every variety containing  $V$  and  $V'$  contains the product  $D \times D'$ , and thus contains the whole variety  $W$ .

In the next theorem we prove that the poset of all simple varieties of  $\mathcal{D}$ -monoids is isomorphic to that of all simple varieties of languages. Hence, the latter is also a semilattice.

**THEOREM 7.14 (EILENBERG THEOREM FOR SIMPLE VARIETIES).** *The semilattices of simple varieties of languages in  $\mathcal{C}$ , simple varieties of  $\mathcal{D}$ -monoids and simple pseudovarieties of  $\mathcal{D}$ -monoids are isomorphic.*

**PROOF.** (1) For every variety  $W$  of  $\mathcal{D}$ -monoids the set  $W_f$  of all finite members of  $W$  is clearly a pseudovariety of  $\mathcal{D}$ -monoids. Conversely, for every pseudovariety  $W$  of  $\mathcal{D}$ -monoids, we denote by  $\langle W \rangle$  the variety generated by  $W$ , i.e., the closure of  $W$  under (arbitrary) products,  $\mathcal{D}$ -submonoids and quotient  $\mathcal{D}$ -monoids. We first show that the map  $W \mapsto W_f$  defines an isomorphism between the semilattices of simple pseudovarieties and simple varieties of  $\mathcal{D}$ -monoids, with inverse given by  $W \mapsto W_f$ .

(a) If  $W$  is a simple variety generated by the finite  $\mathcal{D}$ -monoid  $D$ , then  $W_f$  is a simple pseudovariety generated by  $D$ . Moreover, one has  $\langle W_f \rangle = W$ : the inclusion  $\subseteq$  holds because  $W_f \subseteq W$ , and  $\supseteq$  holds because every element of  $W$  lies in  $\langle D \rangle \subseteq \langle W_f \rangle$ .

(b) For every simple pseudovariety  $W$  of  $\mathcal{D}$ -monoids, we prove that the variety  $\langle W \rangle$  is locally finite and  $\langle W \rangle_f = W$ . In particular, the simple pseudovarieties of  $\mathcal{D}$ -monoids form a subposet of all locally finite varieties of  $\mathcal{D}$ -monoids via the order-embedding  $W \mapsto \langle W \rangle$ . Indeed, suppose that the pseudovariety  $W$  is generated by the finite  $\mathcal{D}$ -monoid  $D$ . Then the variety  $\langle W \rangle$  is also generated by  $D$ , i.e.,  $\langle W \rangle = \langle D \rangle$ . Indeed, one clearly has  $\langle D \rangle \subseteq \langle W \rangle$ . Conversely, every element  $M$  of  $\langle W \rangle$  is a quotient of a  $\mathcal{D}$ -submonoid of some product  $\prod D_i$  with  $D_i \in W$ , and every  $D_i$  is a quotient of a  $\mathcal{D}$ -submonoid of a (finite) power of  $D$ . Thus  $M$  lies in the closure of  $\{D\}$  under quotients,  $\mathcal{D}$ -submonoids and products, i.e.  $M \in \langle D \rangle$ .

Now fix a finite set  $\Sigma$  and consider all functions  $u : \Sigma \rightarrow |D|$ . They define a function  $\langle u \rangle : \Sigma \rightarrow |D|^{|D|^\Sigma}$  that extends uniquely to a  $\mathcal{D}$ -monoid morphism  $g : \Psi\Sigma^* \rightarrow D^{|D|^\Sigma}$ . Letting  $g = m \cdot e$  be its factorization in  $\text{Mon } \mathcal{D}$  (see Lemma 2.21), we get the commutative diagram below:

$$\begin{array}{ccccc}
 \Sigma & \xrightarrow{\eta} & \Psi\Sigma^* & \xrightarrow{e} & F\Sigma \\
 & \searrow \langle u \rangle & \downarrow g & \swarrow m & \\
 D & \xleftarrow{\pi_u} & D^{|D|^\Sigma} & & 
 \end{array}$$

From this we now prove that  $F\Sigma$  (with universal map  $e \cdot \eta$ ) is the free  $\Sigma$ -generated  $\mathcal{D}$ -monoid in  $\langle W \rangle$ .

(b1)  $F\Sigma$  has the universal property w.r.t.  $D$ : the above diagram shows that  $\pi_u \cdot m$  is a  $\mathcal{D}$ -monoid morphism extending  $u$ , and given any  $\mathcal{D}$ -monoid morphism  $h : F\Sigma \rightarrow D$  with  $h \cdot (e \cdot \eta) = u$ , we conclude

$$h \cdot e \cdot \eta = (\pi_u \cdot m) \cdot e \cdot \eta,$$

whence  $h = \pi_u \cdot m$  using the universal property of  $\eta$  and the fact that  $e$  is an epimorphism.

(b2) To show that  $F\Sigma$  has the universal property w.r.t. all  $\mathcal{D}$ -monoids in  $\langle W \rangle = \langle D \rangle$ , let  $V$  be the class of all  $\mathcal{D}$ -monoids  $M$  satisfying this universal property, i.e. such the every map  $v : \Sigma \rightarrow M$  with  $\Sigma$  a finite set factorizes uniquely through  $e \cdot \eta$ . In complete analogy to the first part of the

proof of Proposition 7.7, one shows that  $V$  forms a variety of  $\mathcal{D}$ -monoids. Since  $D \in V$  it follows that  $\langle D \rangle = \langle W \rangle \subseteq V$ , which proves the claim.

(b3) Finally,  $F\Sigma$  lies in  $W$ , being a submonoid of a finite power of  $D$ . This implies that every finite monoid in  $\langle W \rangle$  lies in  $W$ , since it is a quotient of some  $F\Sigma$ .

We conclude that  $\langle W \rangle$  is locally finite and  $\langle W \rangle_f = W$ , and this equation implies immediately that  $W \mapsto \langle W \rangle$  is an injective order-embedding.

This establishes the isomorphism between simple pseudovarieties and simple varieties of  $\mathcal{D}$ -monoids.

(2) Recall the isomorphism  $V \mapsto V^\circledast$  between object-finite varieties of languages in  $\mathcal{C}$  and locally finite varieties of  $\mathcal{D}$ -monoids from the proof of Theorem 7.8. In view of (1) above it suffices to show that this isomorphism restricts to one between simple varieties of languages and simple pseudovarieties of  $\mathcal{D}$ -monoids. Hence, we shall now prove that

$$V \text{ is simple iff } (V^\circledast)_f \text{ is simple.}$$

( $\Rightarrow$ ) If  $V$  is a simple variety of languages, generated by  $\Sigma$ , we prove that the pseudovariety  $(V^\circledast)_f$  is generated by the  $\mathcal{D}$ -monoid  $D$  with  $\hat{D} = V_\Sigma$ . First apply Example 7.3 to the finite local variety  $Q = V_\Sigma$ . Then for the resulting variety  $V'$  we have  $V' = V$ . Indeed,  $V' \subseteq V$  holds by construction of  $V'$ , and  $V \subseteq V'$  holds because  $V$  is generated by  $\Sigma$ . It follows that for every  $\Delta$  the morphism  $m_\Delta$  in Example 7.3 is an isomorphism, or equivalently, the family of morphisms  $\hat{f} : (V_\Sigma)^f \rightarrow V_\Delta$ , where  $f$  ranges over all  $\mathcal{D}$ -monoid morphisms  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$ , is collectively strongly epimorphic in  $\mathcal{C}$ . Since  $V = V'$  is object-finite, we can choose finitely many homomorphisms  $f_1, \dots, f_n : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  such that  $h = [\hat{f}_i] : \prod_{i=1}^n (V_\Sigma)^{f_i} \rightarrow V_\Delta$  is a strong epimorphism. By Lemma 7.6 we get a  $\mathcal{D}$ -submonoid  $D_0$  of  $D^n$  dual to  $V_\Delta$ . We conclude that every finitely generated free monoid  $D_0 = \widehat{V}_\Delta$  of the pseudovariety  $V^\circledast$  is a  $\mathcal{D}$ -submonoid of a finite power of  $D$ . Consequently  $(V^\circledast)_f$  is generated by  $D$ .

( $\Leftarrow$ ) If  $V$  is an object-finite variety of languages such that  $(V^\circledast)_f$  (and hence also  $V^\circledast$ ) is generated by a single finite  $\mathcal{D}$ -monoid  $D$ , we prove that  $V$  is simple. Put  $\Sigma = |D|$ , then since  $D$  is a quotient of the free  $\Sigma$ -generated monoid  $F$  in  $V^\circledast$  and  $F$  is finite, it follows that the pseudovariety  $(V^\circledast)_f$  is also generated by  $F$ . Thus, every  $\mathcal{D}$ -monoid in  $(V^\circledast)_f$  is a quotient of a submonoid of a finite power  $F^n$ . Consequently, every finitely generated free  $\mathcal{D}$ -monoid  $G$  of  $V^\circledast$  is a submonoid of a finite power  $F^n$ . (Indeed, given a quotient  $e : D' \twoheadrightarrow G$  and a submonoid  $i : D' \hookrightarrow F^n$ , choose a splitting  $u : |G| \rightarrow |D'|$ ,  $e \cdot u = id$ , in **Set**. Since  $G$  is free, we get a  $\mathcal{D}$ -monoid morphism  $h : G \rightarrow D'$  which on the generators coincides with  $u$ . Then  $e \cdot u = id$  implies  $e \cdot h = id$ , hence  $i \cdot h : G \rightarrow F^n$  is a submonoid.) Consequently, by composition with the projections  $F^n \rightarrow F$  we obtain a collectively monic collection  $g_1, \dots, g_n : G \rightarrow F$  of  $\mathcal{D}$ -monoid morphisms. Choose  $\mathcal{D}$ -monoid morphisms  $f_1, \dots, f_n : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  with  $e_\Sigma \cdot f_i = g_i \cdot e_\Delta$  (by starting with a splitting  $e_\Sigma \cdot v = id$  in **Set** and extending  $v \cdot g_i \cdot e_\Delta \cdot \eta : \Delta \rightarrow |\Psi\Sigma^*|$  to a  $\mathcal{D}$ -monoid morphism). By Lemma 7.6 we get a collection of  $T_\Delta$ -coalgebra homomorphisms  $h_i : (V_\Sigma)^{f_i} \rightarrow V_\Delta$  that is collectively strongly epic. Hence the corresponding homomorphism  $[h_i] : \prod_{i=1}^n (V_\Sigma)^{f_i} \rightarrow V_\Delta$  is a strong epimorphism in  $\mathcal{C}$ . We conclude that  $V$  is a simple variety generated by  $\Sigma$ : if  $V'$  is any variety such  $j : V_\Sigma \twoheadrightarrow V'_\Sigma$  is a local subvariety, then  $j$  is a  $T_\Delta$ -coalgebra homomorphism  $j : (V_\Sigma)^{f_i} \twoheadrightarrow (V'_\Sigma)^{f_i}$  for every  $i$  by Lemma 2.16. Thus we

have the following diagram of  $T_\Delta$ -coalgebra homomorphisms

$$\begin{array}{ccc}
 \coprod_i (V_\Sigma)^{f_i^\dagger} & \xrightarrow{[h_i]} & V_\Delta \\
 \Pi_i j \downarrow & & \downarrow \\
 \coprod_i (V'_\Sigma)^{f_i^\dagger} & \longrightarrow & V'_\Delta \xrightarrow{\gamma} \varrho T_\Delta
 \end{array}$$

where the morphism  $\coprod_i (V'_\Sigma)^{f_i^\dagger} \rightarrow V'_\Delta$  exists by closure of  $V'$  under preimages. Diagonal fill-in shows that  $V_\Delta \xrightarrow{\gamma} V'_\Delta$ .  $\square$

## 8 GENERALIZED EILENBERG THEOREM

Our main result now follows from Theorem 7.14 by a completion process, analogously to the local variant, Theorem 6.16.

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**THEOREM 8.1 (GENERALIZED EILENBERG THEOREM).** *The complete lattice of all varieties of languages in  $\mathcal{C}$  is isomorphic to that of all pseudovarieties of  $\mathcal{D}$ -monoids.*

**PROOF.** (a) Let  $\mathcal{L}_\mathcal{C}$  denote the poset of all varieties of languages in  $\mathcal{C}$  and  $\mathcal{L}_\mathcal{C}^0$  its subposet of all simple ones. We prove that  $\mathcal{L}_\mathcal{C}$  is the ideal completion of  $\mathcal{L}_\mathcal{C}^0$ . Note first that  $\mathcal{L}_\mathcal{C}$  is a complete lattice because a set-theoretical intersection (alphabet-wise) of varieties of languages  $V_i$  ( $i \in I$ ) is a variety  $V$ . Indeed, the functor  $T_\Sigma Q = O_\mathcal{C} \times Q^\Sigma$  clearly preserves (wide) intersections, thus an intersection of subcoalgebras of  $\varrho T_\Sigma$  in  $\mathcal{C}$  is again a subcoalgebra. And since  $\mathcal{C}$  is a variety of algebras, intersections in  $\mathcal{C}$  are formed on the level of **Set**. Now, from  $V_\Sigma = \bigcap_{i \in I} (V_i)_\Sigma$  it clearly follows that  $V_\Sigma$  is closed under derivatives. And closure under preimages is also clear: given  $L$  in  $|V_\Sigma|$  and  $f : \Psi\Delta^* \rightarrow \Psi\Sigma^*$  in **Mon**  $\mathcal{D}$ , we have  $L \cdot f$  in  $|(V_i)_\Delta|$  for all  $i$ , thus  $L \cdot f \in |V_\Delta|$ .

Observe that also an alphabet-wise directed union of varieties is a variety. The argument is the same: since  $\mathcal{C}$  is a variety of algebras, directed unions are formed on the level of **Set**.

It remains to verify the conditions (1) and (2) of Remark 6.12.

(1) Every simple variety  $V$  is compact in  $\mathcal{L}_\mathcal{C}$ . Indeed, suppose that  $V$  is generated by  $\Sigma$ , and let  $V' = \bigcup_{i \in I} V_i$  be a directed union with  $V \subseteq V'$ . Then  $V_\Sigma$  a local subvariety of  $V'_\Sigma$ , and since  $V_\Sigma$  is finite and directed unions in  $\mathcal{C}$  are formed on the level of **Set**, there exists  $i$  such that  $V_\Sigma$  is a local subvariety of  $(V_i)_\Sigma$ . Therefore  $V \subseteq V_i$  because  $V$  is generated by  $\Sigma$ .

(2) Every variety  $V$  of languages is the directed join (i.e. directed union) of its simple subvarieties. To show that the set of all simple subvarieties of  $V$  is indeed directed, suppose that two simple subvarieties  $W$  and  $W'$  of  $V$  are given. By Remark 7.13, they are generated by the same alphabet  $\Sigma$ . By Lemma 6.11 we know that the local variety  $V_\Sigma$  has a finite local subvariety  $Q$  containing  $W_\Sigma \cup W'_\Sigma$ . Now apply Proposition 7.12 to  $Q$  to get a simple variety  $V' \subseteq V$  containing  $W$  and  $W'$ .

Finally, using Lemma 6.11 and Proposition 7.12 again, we see that every language in  $V$  is contained in a simple subvariety of  $V$ . Hence  $V$  is the desired directed union.

(b) Let  $\mathcal{L}_\mathcal{D}$  denote the poset of all pseudovarieties of  $\mathcal{D}$ -monoids and  $\mathcal{L}_\mathcal{D}^0$  its subposet of all simple ones. We again prove that  $\mathcal{L}_\mathcal{D}$  is an ideal completion of  $\mathcal{L}_\mathcal{D}^0$ . First,  $\mathcal{L}_\mathcal{D}$  is a complete lattice because an intersection of pseudovarieties is a pseudovariety. Observe that also a directed union of pseudovarieties is a pseudovariety. It remains to verify (1) and (2) of Remark 6.12.

(1) Every simple pseudovariety  $W$ , generated by a finite  $\mathcal{D}$ -monoid  $D$ , is compact in  $\mathcal{L}_\mathcal{D}$ . Indeed, if a directed join (i.e. directed union) of pseudovarieties  $\bigcup_{i \in I} W_i$  contains  $W$  then some  $W_i$  contains  $D$  and hence  $W$ .

(2) Every pseudovariety  $W$  of  $\mathcal{D}$ -monoids is a directed union of simple ones. Indeed,  $W$  is the union of its simple subvarieties, and this union is directed by Remark 7.13(b).

(c) From Theorem 7.14 we know that  $\mathcal{L}_{\mathcal{C}}^0$  is isomorphic to  $\mathcal{L}_{\mathcal{D}}^0$ . Since  $\mathcal{L}_{\mathcal{C}}$  is the ideal completion of  $\mathcal{L}_{\mathcal{C}}^0$  by (1), and  $\mathcal{L}_{\mathcal{D}}$  is the ideal completion of  $\mathcal{L}_{\mathcal{D}}^0$  by (2), the uniqueness of completions gives the isomorphism  $\mathcal{L}_{\mathcal{C}} \cong \mathcal{L}_{\mathcal{D}}$ .  $\square$

For our predualties of Example 4.3 we thus obtain the concrete correspondences in the table below as special cases of the Generalized Eilenberg Theorem. The second column describes the  $\mathcal{C}$ -algebraic operations under which varieties of languages are closed (in addition to closure under derivatives and preimages), and the fourth column characterizes the  $\mathcal{D}$ -monoids.

$\mathcal{C}$	varieties closed under	$\mathcal{D}$	pseudovarieties of
<b>BA</b>	all boolean operations	<b>Set</b>	monoids
<b>BR</b>	disjoint union and $\emptyset$	<b>Set<math>_{\star}</math></b>	monoids with zero
<b>DL<math>_b</math></b>	finite unions & finite intersections	<b>Pos</b>	ordered monoids
<b>DL<math>_{\perp}</math></b>	finite unions & non-empty fin. intersections	<b>Pos<math>_{\perp}</math></b>	ordered monoids with $0 = \perp$
<b>JSL<math>_{\perp}</math></b>	finite unions	<b>JSL<math>_{\perp}</math></b>	idempotent semirings
<b>JSL<math>_b</math></b>	finite unions and $\Sigma^*$	<b>JSL</b>	idempotent semirings with 1
<b>Vec<math>_K</math></b>	addition and scalar multiplication	<b>Vec<math>_K</math></b>	$K$ -algebras

The cases  $\mathcal{C} = \mathbf{BA}, \mathbf{DL}_{01}, \mathbf{JSL}_b$  and  $\mathbf{Vec}_K$  are due to Eilenberg [14], Pin [21], Polák [23] and Reutenauer [26], respectively. The other three examples are new variants of Eilenberg's theorem.

## 9 CONCLUSIONS AND FUTURE WORK

In the present paper we demonstrated that Eilenberg's variety theorem, a central result of algebraic automata theory, holds at the level of an abstract duality between algebraic categories. Our result covers uniformly several known extensions and refinements of Eilenberg's theorem and also provides several new Eilenberg-type correspondences.

In the future we intend to classify all predual pairs  $(\mathcal{C}, \mathcal{D})$  satisfying our assumptions above. Using the *natural duality* framework of Clark and Davey [13], we expect to show that only finitely many Eilenberg theorems exist for every fixed finite output set  $O$ .

An important next step is a common roof over our present General Eilenberg Theorem and the one in Bojańczyk's work [8]. As mentioned in the introduction, this is the topic of our recent preprint [30].

We also mentioned that in a subsequent future work this should be generalized further so that also non-finitary Eilenberg-type correspondences are covered by our framework [29].

Finally, it should be interesting to see if it is possible to obtain a variety theorem for data languages based on nominal Stone duality [15]. On a similar note, we aim to investigate whether dualities modeling probabilistic phenomena (e.g. Gelfand or Kadison duality) lead to a meaningful algebraic language theory for probabilistic automata and languages.

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## A PREDUALITIES

In this appendix we verify that all our running examples of varieties  $\mathcal{D}$  of Example 2.2 satisfy the Assumptions 2.1:

- (a)  $\mathcal{D}$  is locally finite,
- (b)  $\mathcal{D}$  is entropic,
- (c) epimorphisms in  $\mathcal{D}$  are surjective.

Moreover, for the varieties  $\mathcal{C}$  of Example 4.3 we have that

- (d)  $\mathcal{C}$  is locally finite, and
- (e)  $\mathcal{D}$  is preducal to  $\mathcal{C}$ .

*Example A.1.*  $\mathcal{D} = \mathbf{Set}$  and  $\mathcal{C} = \mathbf{BA}$ .

Conditions (a)-(c) trivially hold.

(d)  $\mathbf{BA}$  is locally finite since the free boolean algebra on  $n$  generators has  $2^{2^n}$  elements.

(e) The preduality of  $\mathbf{Set}$  and  $\mathbf{BA}$  follows from the classical Stone Duality, since finite Stone spaces are discrete, thus, they form the category  $\mathbf{Set}_f$  of finite sets; see e.g. [18].

*Example A.2.*  $\mathcal{D} = \mathbf{Pos}$  and  $\mathcal{C} = \mathbf{DL}_b$ .

The category of posets is a variety of ordered algebras on the empty signature.

(a) It is clearly locally finite.

(b) It is entropic: all monotone maps from  $A$  to  $B$  ordered pointwise form a subposet of  $B^{|A|}$ .

(c) Epimorphisms are surjective. Indeed, let  $e : D \rightarrow E$  be an epimorphism in  $\mathbf{Pos}$ . Assuming an element  $x$  of  $E$  lies outside the image of  $e$ , define monotone maps  $f, g$  from  $E$  to the chain  $0 < 1$  by  $f(z) = 0$  iff  $z \leq x$ , and  $g(z) = 0$  iff there is  $y$  in  $D$  with  $z \leq e(y) < x$ . Then  $f \cdot e = g \cdot e$ , a contradiction.

(d) The category  $\mathbf{DL}_b$  is locally finite since  $\mathbf{BA}$  is.

(e) The preduality between posets and bounded distributive lattices follows from the classical Priestley Duality (between bounded distributive lattices and Priestley spaces, see [24]), since finite Priestley spaces have discrete topology, thus, the corresponding category is  $\mathbf{Pos}_f$ . However, the fact that  $\mathbf{Pos}_f$  is dually equivalent to  $(\mathbf{DL}_b)_f$  was already proved by Birkhoff [7].

This duality assigns to every finite bounded distributive lattice  $D$  the poset of all  $\downarrow$ -sets ordered by inclusion. And to every bounded lattice homomorphism the map given by preimages of  $\downarrow$ -sets.

*Example A.3.*  $\mathcal{D} = \mathbf{Set}_*$  and  $\mathcal{C} = \mathbf{BR}$ . By a *non-unital boolean ring* is meant the usual definition of a boolean ring except that the existence of a unit for multiplication is not required. The category  $\mathbf{BR}$  has as objects non-unitary boolean rings and as morphisms all maps preserving addition, multiplication and 0.<sup>2</sup>

The category  $\mathbf{Set}_*$  of pointed sets and point-preserving maps is clearly locally finite and entropic and its epimorphisms are clearly surjective. And the local finiteness of  $\mathbf{BR}$  follows from that of  $\mathbf{BA}$ . We now prove the preduality.

*Remark A.4.* (a) Every non-unital boolean ring is a distributive lattice with 0 where the meet is multiplication and the join is  $x \vee y = x + y + x \cdot y$ . Hence every *finite* non-unital boolean ring has a unit 1, the join of all its elements. However, homomorphisms between finite non-unital boolean rings need not preserve the unit.

(b) The category  $\mathbf{UBR}$  of *unital* boolean rings and unit-preserving ring homomorphisms is isomorphic to the category  $\mathbf{BA}$  of boolean algebras (and hence preducal to  $\mathbf{Set}$ ). Recall that under this isomorphism  $+$  corresponds to exclusive disjunction and  $\cdot$  to conjunction.

<sup>2</sup>Historical remark: in his fundamental monograph [9] Birkhoff defines the concept of boolean ring as the non-unital variant (and speaks about “unital boolean rings” in Stone duality).

**THEOREM A.5.**  $\mathbf{BR}$  and  $\mathbf{Set}_\star$  are preduial categories.

**PROOF.** Recall that  $(\mathbf{Set}_\star)_f$  is equivalent to the Kleisli category of the monad  $Z \mapsto Z + 1$  on  $\mathbf{Set}_f$  (whose morphisms from  $X$  to  $Y$  are the functions from  $X$  to  $Y + 1$ , representing the partial maps from  $X$  to  $Y$  in the usual sense). The dual comonad on  $\mathbf{UBR}_f \simeq \mathbf{Set}_f^{op}$  is  $MX = X \times 2$  (where  $2 = \{0, 1\}$  is the two-element boolean ring). Its counit is given by

$$X \times 2 \xrightarrow{\epsilon_X} X, \quad \epsilon_X(x, b) = x,$$

and its comultiplication by

$$X \times 2 \xrightarrow{\delta_X} X \times 2 \times 2, \quad \delta_X(x, b) = (x, b, b).$$

It suffices to show that the Co-Kleisli category  $\mathbf{Kl}(M)$  of this comonad is isomorphic to  $\mathbf{BR}_f$ . The desired isomorphism  $I : \mathbf{Kl}(M) \xrightarrow{\cong} \mathbf{BR}_f$  is identity on objects and takes a co-Kleisli morphism  $f : X \times 2 \rightarrow Y$  to the following morphism of  $\mathbf{BR}_f$

$$If : X \rightarrow Y, \quad If(x) = f(x, 0).$$

It remains to verify that  $I$  is (a) a well-defined functor which is (b) faithful and (c) full.

(a1)  $If$  is clearly a  $\mathbf{BR}_f$ -morphism since  $0 + 0 = 0$  and  $0 \cdot 0 = 0$ .

(a2)  $I$  preserves identities: the identity morphism of  $X \in \mathbf{Kl}(M)$  is  $\epsilon_X$ , and

$$I\epsilon_X(x) = \epsilon_X(x, 0) = x = id_X(x).$$

(a3)  $I$  preserves composition: the composition of co-Kleisli morphisms  $f : X \times 2 \rightarrow Y$  and  $g : Y \times 2 \rightarrow Z$  is  $g \bullet f : X \times 2 \rightarrow Z$  where

$$g \bullet f(x, b) = g \circ Mf \circ \delta_X(x, b) = g(f(x, b), b).$$

Therefore

$$\begin{aligned} I(g \bullet f)(x) &= g \bullet f(x, 0) \\ &= g(f(x, 0), 0) \\ &= g(If(x), 0) \\ &= Ig(If(x)) \\ &= Ig \circ If(x) \end{aligned}$$

(b)  $I$  is faithful: let  $f, g : X \times 2 \rightarrow Y$  be co-Kleisli morphisms with  $If = Ig$ , i.e.,  $f(x, 0) = g(x, 0)$  for all  $x$ . Then

$$\begin{aligned} f(x, 1) &= f(x, 0) + f(1, 1) + f(1, 0) \\ &= f(x, 0) + 1 + f(1, 0) \\ &= g(x, 0) + g(1, 1) + g(1, 0) \\ &= g(x, 1) \end{aligned}$$

which implies  $f = g$ .

(c)  $I$  is full: let  $f : X \rightarrow Y$  be a morphism of  $\mathbf{BR}_f$ . We claim that the map

$$g : X \times 2 \rightarrow Y, \quad g(x, b) = f(x) + b + b \cdot f(1),$$

is a morphism of  $\mathbf{UBR}_f$  with  $Ig = f$ . First, the equation  $Ig = f$  clearly holds:

$$\begin{aligned} Ig(x) &= g(x, 0) \\ &= f(x) + 0 + 0 \cdot f(1) \\ &= f(x). \end{aligned}$$

To show that  $g$  is a morphism of  $\mathbf{UBR}_f$ , we compute

$$g(0, 0) = f(0) + 0 + 0 \cdot f(1) = 0 + 0 + 0 = 0$$

and

$$\begin{aligned} g(1, 1) &= f(1) + 1 + 1 \cdot f(1) \\ &= f(1) + f(1) + 1 = 0 + 1 = 1. \end{aligned}$$

Further,

$$\begin{aligned} g(x, b) + g(x', b') &= (f(x) + b + b \cdot f(1)) + (f(x') + b' + b' \cdot f(1)) \\ &= f(x + x') + b + b' + (b + b') \cdot f(1) \\ &= g(x + x', b + b') \end{aligned}$$

and

$$\begin{aligned} g(x, b) \cdot g(x', b') &= (fx + b + b \cdot f(1)) \cdot (fx' + b' + b' \cdot f(1)) \\ &= fx \cdot fx' + (fx \cdot b' + fx \cdot b' \cdot f(1)) \\ &\quad + (b \cdot fx' + b \cdot fx' \cdot f(1)) \\ &\quad + (b \cdot b' \cdot f(1) + b \cdot b' \cdot f(1) \cdot f(1)) \\ &\quad + b \cdot b' + b \cdot b' \cdot f(1) \\ &= fx \cdot fx' + (fx \cdot b' + fx \cdot b') + (b \cdot fx' + b \cdot fx') \\ &\quad + (b \cdot b' \cdot f(1) + b \cdot b' \cdot f(1)) + b \cdot b' + b \cdot b' \cdot f(1) \\ &= f(x \cdot x') + b \cdot b' + b \cdot b' \cdot f(1) \\ &= g(x \cdot x', b \cdot b'). \end{aligned}$$

In the third step we use  $fx \cdot f(1) = fx$ ,  $fx' \cdot f(1) = fx'$  and  $f(1) \cdot f(1) = f(1)$ , and in the fourth one the equation  $u + u = 0$ .  $\square$

*Example A.6.*  $\mathcal{D} = \mathbf{Pos}_\perp$  and  $\mathcal{C} = \mathbf{DL}_\perp$ . The category of posets with a least element  $\perp$  and monotone functions preserving it satisfies (a)-(d), the verification is analogous to Example A.2. For (c) observe that both  $f$  and  $g$  map  $\perp$  to 0.  $\mathbf{DL}_\perp$  is locally finite since  $\mathbf{BA}$  is.

We now prove the preduality.

**THEOREM A.7.** *The categories  $\mathbf{DL}_\perp$  and  $\mathbf{Pos}_\perp$  are preducal.*

**PROOF.** Denote by  $(-)_\perp$  the monad on  $\mathbf{Pos}_f$  adding to every poset a new bottom element. It is clear that  $(\mathbf{Pos}_\perp)_f$  is equivalent to the Kleisli category of this monad

The dual comonad  $M$  on  $(\mathbf{DL}_b)_f \simeq (\mathbf{Pos}_f)^{op}$  also adds a new bottom,  $MX = X_\perp$ . Its counit

$$\varepsilon : X_\perp \rightarrow X$$



removes the bottom element and the comultiplication

$$\mu : X_{\perp} \rightarrow (X_{\perp})_{\perp}$$

takes the new bottom to the outward one. There is an obvious isomorphism  $I : Kl(M) \rightarrow (\mathbf{DL}_{\perp})_f$ : it is identity on objects, and it takes a morphism  $f : D_{\perp} \rightarrow D'$  to its domain restriction  $If : D \rightarrow D'$ .  $\square$

E: JSL

*Example A.8.*  $\mathcal{D} = \mathbf{JSL}$  and  $\mathcal{C} = \mathbf{JSL}_b$ . The category  $\mathbf{JSL}$  of join semilattices satisfies (a)-(c), the proof is analogous to that of Example A.2. For (c) observe that both  $f$  and  $g$  preserve finite joins.

The category of bounded join semilattices (i.e., those with a least and a largest element) and bounded homomorphisms is locally finite since  $\mathbf{BA}$  is. We now prove the desired preduality.

**THEOREM A.9.**  *$\mathbf{JSL}$  and  $\mathbf{JSL}_b$  are preduel categories.*

**PROOF.** The desired dual equivalence  $(\widehat{-}) : (\mathbf{JSL}_b)_f^{op} \xrightarrow{\sim} \mathbf{JSL}_f$  is defined on objects by

$$(Q, \vee, 0, 1) \mapsto \widehat{Q} = (Q \setminus \{1\}, \wedge)$$

and on morphisms  $h : Q \rightarrow R$  by

$$\widehat{h} : \widehat{R} \rightarrow \widehat{Q}, \quad \widehat{h}r = \bigvee_{hq \leq r} q,$$

where  $q$  ranges over  $Q$ . Here and in the following, the symbols  $\vee, \wedge, \leq, 0$  and  $1$  are always meant with respect to the order of  $Q$  or  $R$ . We need to verify that  $(\widehat{-})$  is (a) a well-defined functor, (b) essentially surjective (i.e., every object in  $(\mathbf{JSL})_f$  is isomorphic to one in the image of  $(\widehat{-})$ ), (c) faithful and (d) full.

(a1)  $\widehat{h}$  is well-defined as a function, that is, it maps the set  $\widehat{R} = R \setminus \{1\}$  to  $\widehat{Q} = Q \setminus \{1\}$ . Indeed, we have for all  $r \in R \setminus \{1\}$ :

$$h(\widehat{h}r) = h\left(\bigvee_{hq \leq r} q\right) = \bigvee_{hq \leq r} hq \leq r \quad (*)$$

which implies  $\widehat{h}r \neq 1$  because  $h1 = 1$ .

(a2)  $\widehat{h}$  is a  $\mathbf{JSL}$ -morphism, that is,  $\widehat{h}(r \wedge r') = \widehat{h}r \wedge \widehat{h}r'$  holds for all  $r, r' \in R \setminus \{1\}$ . Here “ $\leq$ ” follows from the (obvious) monotonicity of  $\widehat{h}$ . For the converse we compute

$$h(\widehat{h}r \wedge \widehat{h}r') \leq h(\widehat{h}r) \wedge h(\widehat{h}r') \leq r \wedge r'.$$

The first inequality uses monotonicity of  $h$  and the second one uses (\*). Hence  $\widehat{h}r \wedge \widehat{h}r'$  is among the elements  $q$  in the join  $\widehat{h}(r \wedge r') = \bigvee_{hq \leq r \wedge r'} q$ , which implies  $\widehat{h}r \wedge \widehat{h}r' \leq \widehat{h}(r \wedge r')$ .

(a3) The assignment  $h \mapsto \widehat{h}$  trivially preserves identity morphisms. For preservation of composition we consider  $h : Q \rightarrow R$  and  $k : R \rightarrow S$  in  $(\mathbf{JSL}_{01})_f$  and show  $(\widehat{kh})s = (\widehat{h} \cdot \widehat{k})s$  for all  $s \in S \setminus \{1\}$ , i.e.,

$$\bigvee_{(kh)q \leq s} q = \bigvee_{hq \leq \widehat{k}s} q.$$

This equation holds because, for all  $q \in Q$ , the inequalities  $kh(q) \leq s$  and  $hq \leq \widehat{k}s$  are equivalent. Indeed, if  $kh(q) \leq s$  then  $hq \leq \bigvee_{kr \leq s} r = \widehat{k}s$ . Conversely, if  $hq \leq \widehat{k}s$  then

$$k(hq) \leq k(\widehat{k}s) = k\left(\bigvee_{kr \leq s} r\right) = \bigvee_{kr \leq s} kr \leq s,$$

using that  $k$  is monotone and preserves joins.

(b) On the level of posets the construction  $Q \mapsto \widehat{Q}$  first removes the top element and then reverses the order. Conversely, first note that any nonempty finite semilattice necessarily has a top element, namely the (finite) join of all of its elements. Thus we can turn any semilattice  $P$  in  $(\mathbf{JSL})_f$  to a semilattice  $\overline{P}$  in  $(\mathbf{JSL}_b)_f$  by first adding a new bottom element and then reversing the order. Up to isomorphism these two constructions are clearly mutually inverse, so  $\widehat{\overline{P}} \cong P$  for all  $P$ . This proves that  $(\widehat{-})$  is essentially surjective.

(c) Given a morphism  $h : Q \rightarrow R$  in  $(\mathbf{JSL}_b)_f$ , we claim that, for all  $q \in Q$ ,

$$hq = \bigwedge_{q \leq \widehat{hr}} r$$

where  $r$  ranges over  $R \setminus \{1\}$ . This immediately implies that  $(\widehat{-})$  is faithful. To show “ $\leq$ ” let  $r \in R \setminus \{1\}$  with  $q \leq \widehat{hr}$ . Since  $h$  is monotone and preserves joins, we have

$$hq \leq h(\widehat{hr}) = h\left(\bigvee_{hq' \leq r} q'\right) = \bigvee_{hq' \leq r} hq' \leq r.$$

For the converse note that

$$q \leq \bigvee_{hq' \leq hq} q' = \widehat{h}(hq).$$

Hence  $hq$  is one of the elements  $r$  occurring in the meet above, which means that this meet is less or equal to  $hq$ .

(d) Given  $g : \widehat{R} \rightarrow \widehat{Q}$  in  $\mathbf{JSL}_f$  we need to find  $h : Q \rightarrow R$  in  $(\mathbf{JSL}_b)_f$  with  $g = \widehat{h}$ . First extend  $g : R \setminus \{1\} \rightarrow Q \setminus \{1\}$  to a map  $\overline{g} : R \rightarrow Q$  by setting  $\overline{g}1 = 1$ . Then  $\overline{g}$  preserves all meets of  $R$  because  $g$  preserves all non-empty meets of  $R \setminus \{1\}$ . Inspired by (c) we define

$$hq = \bigwedge_{q \leq \overline{g}r} r,$$

where  $r$  ranges over  $R$ . Let us show that  $h$  indeed defines a morphism of  $(\mathbf{JSL}_b)_f$ . First,  $h$  preserves 0 and 1 because

$$h0 = \bigwedge_{0 \leq \overline{g}r} r = \bigwedge_r r = 0,$$

and

$$h1 = \bigwedge_{1 \leq \overline{g}r} r = 1.$$

In the last equation we use that  $\overline{g}1 = 1$ , and no other element of  $R$  is mapped to 1 by  $\overline{g}$  since the codomain of  $g$  is  $Q \setminus \{1\}$ . For preservation of joins,  $h(q \vee q') = hq \vee hq'$  for all  $q, q' \in Q$ , first note that “ $\geq$ ” follows from the (obvious) monotonicity of  $h$ . For the other direction we compute

$$q \leq \bigwedge_{q \leq \overline{g}r} \overline{g}r = \overline{g}\left(\bigwedge_{q \leq \overline{g}r} r\right) = \overline{g}(hq)$$

and analogously  $q' \leq \overline{g}(hq')$ . Hence

$$q \vee q' \leq \overline{g}(hq) \vee \overline{g}(hq') \leq \overline{g}(hq \vee hq').$$

The last step uses the monotonicity of  $\bar{g}$ . So  $hq \vee hq'$  is among the elements  $r$  in the meet defining  $h(q \vee q') = \bigwedge_{hq \vee q' \leq \bar{g}r} r$ , which implies  $h(q \vee q') \leq hq \vee hq'$ .

Finally, we prove  $g = \widehat{h}$ , i.e.,

$$gr = \bigvee_{hq \leq r} q$$

for all  $r \in R \setminus \{1\}$ . To show “ $\geq$ ” take any  $q \in Q$  satisfying  $hq \leq r$ . Then

$$q \leq \bigwedge_{q \leq \bar{g}r'} \bar{g}r' = \bar{g}(\bigwedge_{q \leq \bar{g}r'} r') = \bar{g}(hq) \leq \bar{g}r = gr.$$

For “ $\leq$ ” note first that

$$h(gr) = \bigwedge_{gr \leq \bar{g}r'} r' \leq r.$$

The last step uses that  $r$  is one of the elements  $r'$  over which the meet is taken. Hence  $gr$  is one of the elements  $q$  in the join  $\bigvee_{hq \leq r} q$ , so  $gr \leq \bigvee_{hq \leq r} q$ .  $\square$

*Example A.10.*  $\mathcal{D} = \mathcal{C} = \mathbf{JSL}_\perp$ . This is completely analogous to the preceding example. The equivalence functor from  $(\mathbf{JSL}_\perp)^{op}$  to  $\mathbf{JSL}_\perp$  takes  $(Q, \vee, 0)$  to  $(Q, \wedge, 1)$ , where 0 and 1 are the least and largest elements of  $Q$ , resp.

*Example A.11.*  $\mathcal{D} = \mathcal{C} = \mathbf{Vec}_K$ . For every finite field  $K$  the category of vector spaces and linear maps

- (a) is locally finite, since a free object on  $n$  generators is  $K^n$ ,
- (b) is entropic:  $[A, B]$  is the vector space of all linear maps with operations defined pointwise,
- (c) has all epimorphism split, thus, surjective.

Finally,

(d) the self-duality of the category  $(\mathbf{Vec}_K)_f$  of finite-dimensional spaces is well-known: it assigns to every space  $Q$  the dual space  $Q^* = [Q, K]$ .

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