Terminal Coalgebras for Finitary Functors

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Abstract

We present a result that implies that an endofunctor on a category has a terminal coalgebra obtainable as a countable limit of its terminal-coalgebra sequence. It holds for finitary endofunctors preserving nonempty binary intersections on locally finitely presentable categories, assuming that the posets of strong quotients and subobjects of every finitely presentable object satisfy the descending chain condition. This allows one to adapt finiteness arguments that were originally advanced by Worrell concerning terminal coalgebras for finitary set functors. Examples include the categories of sets, posets, vector spaces, graphs, and nominal sets. A similar argument is presented for the category of metric spaces (although it is not locally finitely presentable).

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1 Introduction

Coalgebras capture various types of state-based systems in a uniform way by encapsulating the type of transitions as an endofunctor on a suitable base category. Coalgebras also come with a canonical behaviour domain given by the notion of a terminal coalgebra. So results on the existence and construction of terminal coalgebras for endofunctors are at the heart of the theory of universal coalgebra. The topic is treated in our monograph [5]. A well-known construction of the terminal coalgebra for an endofunctor was first presented by Adámek [2] (in dual form) and independently by Barr [11]. The idea is to iterate a given endofunctor F on the unique morphism $F1 \to 1$ to obtain the following ω^{op}-chain

$$1 \xleftarrow{!} F1 \xleftarrow{F!} FF1 \xleftarrow{FF!} FFF1 \xleftarrow{FFF!} \cdots \tag{1.1}$$

and then continue transfinitely. For each ordinal i, we write V_i for the ith iterate. So we have

$$V_0 = 1, \qquad V_{i+1} = FV_i, \qquad \text{and } V_i = \lim_{j<i} V_j \text{ when } i \text{ is a limit ordinal;} \tag{1.2}$$

the connecting morphisms are as expected. In particular, for every ordinal i, we have a morphism $V_{i+1} \to V_i$. If the transfinite chain converges in the sense that this morphism is an isomorphism for some i, then its inverse is the structure of a terminal coalgebra for F [2, dual of second prop.]. This happens for a limit ordinal i iff F preserves the limit V_i. However,

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in general, this transfinite chain does not converge at all (e.g. for the power-set functor), and moreover, if it does converge, then the number of iterations needed to obtain the terminal coalgebra can be arbitrarily large. For example, the set functor $\mathcal{P}_\alpha$, which assigns to a set the set of all subsets of cardinality smaller than α, requires $\alpha + \omega$ iterations [4].

A famous result by Worrell [20] states that a finitary set functor needs at most $\omega + \omega$ iterations to converge. We generalize this result to other base categories by isolating properties of the category of sets and endofunctors on it that entail it:

1. The *descending chain condition* (DCC), which states that for every finitely presentable object (a category-theoretic generalization of the notion of a finite set) every strictly decreasing chain of subobjects or strong quotient objects is finite.
2. The preservation of *nonempty binary intersections*, that is, pullbacks of two monomorphisms such that the domain is not a strict initial object (cf. Definition 2.5).

The first condition is inspired by the descending chain condition in algebra and more specifically by the Noetherian condition introduced by Urbat and Schröder [19]. Regarding the second one, it was shown by Trnková that every set functor preserves nonempty finite intersections [18]. In addition, every finitary set functor preserves *all* nonempty intersections [5, Thm. 4.4.3].

Our main result (Theorem 5.1) holds for locally finitely presentable categories satisfying the DCC: every finitary endofunctor preserving nonempty binary intersections has a terminal coalgebra obtained in $\omega + \omega$ steps. We also show that the DCC is satisfied by a large number of categories of interest, such as sets, posets, graphs, vector spaces, boolean algebras, and nominal sets.

The category of metric spaces and non-expanding maps is not locally finitely presentable, and so our main result is not applicable to it. Nevertheless, we provide in Theorem 6.5 a sufficient condition for an endofunctor to have a terminal coalgebra obtained in $\omega + \omega$ steps: the endofunctor should be finitary and preserve nonempty binary intersections (as in our Theorem 5.1), and it also should preserve isometric embeddings.

Related work. As we have mentioned above, our DCC condition was inspired by Urbat and Schröder [19]. However, the results here are disjoint from the ones in op. cit.

The most closely related paper to this one is our previous work [1]. That paper also contains results on endofunctors having terminal coalgebras in $\omega + \omega$ steps. But those results pertain to endofunctors belonging to one of several inductively defined classes. For example, for sets it studies the *Kripke polynomial functors*. This is the smallest class of endofunctors containing the constant functors and the finite power-set functor, and closed under products, coproducts, and composition. We have shown [1, Thm. 3.5] that every Kripke polynomial set functor has a terminal coalgebra in $\omega + \omega$ steps. Similarly, we have shown [1, Thm. 5.9] a result for the category of metric spaces and non-expanding maps, replacing the finite power-set functor with the Hausdorff functor: this assigns to a space the set of its compact subsets, using a well-known metric $\bar{d}$ (cf. Example 6.6). Again, every *Hausdorff polynomial functor* has a terminal coalgebra in $\omega + \omega$ steps. However, since the Hausdorff functor is not finitary on the category of all metric spaces, these results cannot be inferred from Theorem 5.1. In contrast, the results in this paper apply to categories not mentioned in op. cit: see Example 4.3 and Proposition 4.4. Both the theory and the specific results that follow are new.

A slightly stronger condition than our DCC was introduced in previous work [9]. The relationship of the two condition is dicussed in Section 4.

Another related result concerns the category of complete metric spaces: locally contracting endofunctors F on this category satisfying $F\emptyset \neq \emptyset$ have a terminal coalgebra obtained in ω steps [6] (see also [5, Cor. 5.2.18]). Moreover, this is then also an initial algebra.

2 Preliminaries

We review a few preliminary points. We assume that readers are familiar with algebras and coalgebras for an endofunctor, as well as with locally finitely presentable categories and (strong-epi, mono)-factorizations.

First we set up some notation. We write $S \rightarrowtail X$ for monomorphisms and $X \twoheadrightarrow E$ for strong epimorphisms. Given an endofunctor F, we write νF for its terminal coalgebra, if it exists.

Regarding the ω^{op}-chain in (1.1), let $\ell_n \colon V_\omega \to F^n 1$ $(n < \omega)$ be the limit cone. We obtain a unique morphism $m \colon FV_\omega \to V_\omega$ such that for all $n \in \omega^{\mathsf{op}}$, we have

$$\begin{array}{ccc} FV_\omega & \xrightarrow{\quad m \quad} & V_\omega \\ & {\scriptstyle F\ell_n} \searrow \quad \swarrow {\scriptstyle \ell_{n+1}} & \\ & F^{n+1}1 & \end{array} \tag{2.1}$$

This is the connecting morphism from $V_{\omega+1} = FV_\omega$ to V_ω in the transfinite chain (1.2).

If F preserves the limit V_ω, then m is an isomorphism (and conversely). Therefore, its inverse yields the terminal coalgebra $m^{-1} \colon V_\omega \to FV_\omega$ [2, dual of second prop.]; shortly $\nu F = V_\omega$. Then we say that the terminal coalgebra is *obtained in ω steps*.

This technique of *finitary iteration* is the most basic and prominent construction of terminal coalgebras. However, it does *not* apply to the finite power-set functor $\mathcal{P}_{\mathsf{f}}$. For that functor $FV_\omega \not\cong V_\omega$ [3, Ex. 3(b)]. However, a modification of finitary iteration does apply, as shown by Worrell [20, Th. 11]. One needs a *second infinite iteration*, iterating F on the morphism $m \colon FV_\omega \to V_\omega$ rather than on $! \colon F1 \to 1$, obtaining the ω^{op}-chain

$$V_\omega \xleftarrow{\;m\;} V_{\omega+1} \xleftarrow{\;Fm\;} V_{\omega+2} \xleftarrow{\;FFm\;} \cdots . \tag{2.2}$$

Its limit is denoted by

$$V_{\omega+\omega} = \lim_{n<\omega} V_{\omega+n}. \tag{2.3}$$

Worrell proved that when F is a finitary set functor, it preserves this limit. Therefore, we obtain that $V_{\omega+\omega}$ carries a terminal coalgebra; shortly $\nu F = V_{\omega+\omega}$, and we say that the terminal coalgebra is *obtained in $\omega + \omega$ steps*.

Limits of ω^{op}-chains. We recall the following characterization of limits of ω^{op}-chains.

▶ **Remark 2.1.** Consider an ω^{op}-chain

$$X_0 \xleftarrow{\;f_0\;} X_1 \xleftarrow{\;f_1\;} X_2 \xleftarrow{\;f_2\;} \cdots . \tag{2.4}$$

In Set, Top, Met, and K-Vec, the limit L is carried by the set of all sequences $(x_n)_{n<\omega}$, $x_n \in X_n$ that are *compatible*: $f_n(x_{n+1}) = x_n$ for every n. The limit projections are the functions $\ell_n \colon L \to X_n$ defined by $\ell_n((x_i)) = x_n$.

1. In Top, the topology on L has as a base the sets $\ell_n^{-1}(U)$, for U open in X_n.
2. In Met, the metric on L is defined by $d((x_n), (y_n)) = \sup_{n<\omega} d(x_n, y_n)$.
3. In K-Vec, the limit L is a subspace of $\Pi_i X_i$.

Locally finitely presentable categories. We continue with a terse review of locally finitely presentable categories; see [7] for background. A diagram $\mathcal{D} \to \mathcal{A}$ is *directed* if its domain $\mathcal{D}$ is a directed poset (i.e. nonempty and such that every pair of elements has an upper bound). A functor is *finitary* if it preserves directed colimits. An object A of a category $\mathcal{A}$ is *finitely presentable* if its hom-functor $\mathcal{A}(A, -)\colon \mathcal{A} \to \mathsf{Set}$ preserves directed colimits. A category is *locally finitely presentable* (lfp, for short) if it is cocomplete and has a set of finitely presentable objects such that every object is a directed colimit of objects from that set.

▶ **Example 2.2.** We list a number of examples of lfp categories.

1. The category Set of all sets and $\mathsf{Set_p}$ of pointed sets; the finitely presentable objects are precisely the finite sets.
2. The category Gra of graph and their homomorphisms as well as Pos of posets and monotone maps; finitely presentable objects are precisely the finite graphs or posets, respectively.
3. Every finitary variety, that is, any category of algebras specified by operations of finite arity and equations; the finitely presentable objects are precisely those algebras which have a presentation by finitely many generators and relations (in the usual sense of universal algebra). The following three items are instances of this one.
4. The category Bool of Boolean algebras and their homomorphisms; the finitely presentable objects are precisely the finite Boolean algebras. The same holds for every locally finite variety, e.g. join-semilattices or distributive lattices.
5. The category M-Set of sets with an action of a monoid M, and equivariant maps; the finitely presentable objects are precisely the orbit-finite M-sets (i.e. those having finitely many orbits).
6. The category K-Vec of vector spaces over a field K and linear maps; the finitely presentable objects are precisely the finite-dimensional vector spaces.
   More generally, given a semiring $\mathbb{S}$, the category $\mathbb{S}$-Mod of all $\mathbb{S}$-semimodules is lfp.
7. The category Nom of nominal sets and equivariant maps; the finitely presentable objects are precisely the orbit-finite nominal sets.
8. A poset, considered as a category, is lfp iff it is an algebraic lattice: a complete lattice in which every element is a join of compact ones. (An element x is *compact* if for every subset S, $x \le \bigvee S$ implies that $x \le \bigvee S'$ for some finite $S' \subseteq S$.)

▶ **Remark 2.3.** We next recall definitions concerning subobjects.

1. For a fixed object A, the monomorphisms with codomain A have a natural preorder: given $c\colon C \rightarrowtail A$ and $c'\colon C' \rightarrowtail A$, we say that $c \le c'$ iff $c = c' \cdot m$ for some monomorphism $m\colon C \to C'$. A *subobject* of A is an equivalence class of monomorphisms under the induced equivalence relation. We write representatives to denote subobjects.
2. A subobject (represented by) $c\colon C \rightarrowtail A$ is *finitely presentable* if its domain C is a finitely presentable object.

▶ **Remark 2.4.** We recall properties of an lfp category $\mathcal{A}$ used in the proof of Theorem 5.1:

1. $\mathcal{A}$ is complete [7, Rem 1.56] (and cocomplete by definition).
2. $\mathcal{A}$ has a (strong-epi, mono)-factorization system [7, Rem. 1.62].
3. Every morphism from a finitely presentable object to a directed colimit factorizes through one of the colimit maps.
4. Every object is the colimit of the canonical directed diagram of all of its finitely presentable subobjects [9, Lemma 3.1]. Moreover, given any finitely presentable subobject $c\colon C \rightarrowtail A$, it is easy to see that the object A is the colimit of the diagram of all its finitely presentable subobjects $s\colon S \rightarrowtail A$ such that $c \le s$.

5. The collection of all finitely presentable objects, up to isomorphism, is a set. It is a *generator* of $\mathcal{A}$; it follows that a morphism $m\colon X \to Y$ is monic iff for every pair $u, v\colon U \to X$ of morphisms with a finitely presentable domain U, we have that $m\cdot u = m\cdot v$ implies $u = v$.

▶ **Definition 2.5** [14]. An initial object 0 is *strict* if every morphism with codomain 0 is an isomorphism. A monomorphism $A \rightarrowtail B$ is *empty* if its domain is a strict initial object; it is *nonempty* if it is not empty.

An intersection (a wide pullback of monomorphisms) is *empty* if its domain is a strict initial object, that is, the limit cone is formed by empty monomorphisms; the intersection is nonempty if it is not empty.[1]

An endofunctor $F\colon \mathcal{A} \to \mathcal{A}$ *preserves nonempty intersections* if F takes a nonempty intersection to a (not necessarily nonempty) wide pullback.

▶ **Remark 2.6.** Every endofunctor preserving nonempty binary intersections preserves nonempty monomorphisms. This holds since a morphism is monic iff the pullback along itself is formed by a pair of identity morphisms.

▶ **Example 2.7.** **1.** In Set, the initial object $\emptyset$ is strict. A nonempty intersection is an intersection of subsets having a common element. Trnková [18] proved that every set functor preserves nonempty binary intersections.
It follows that every finitary set functor preserves nonempty intersections [5, Thm. 4.4.3].

2. The initial object $\{0\}$ in K-Vec is not strict. Thus all subobjects are nonempty. Every endofunctor on K-Vec preserves finite intersections [9, Ex. 4.3].

3. In Gra and Pos nonempty intersections are, as in Set, intersections of subobjects having a common element.

▶ **Remark 2.8.** **1.** Unlike on Set and K-Vec, on most everyday categories finitary endofunctors may fail to preserve nonempty intersections. For example, consider the category Gra of graphs. We exhibit a finitary endofunctor not preserving nonempty binary intersections. We denote by 1 the terminal graph, a single loop, and by S a single node which has no loop. Let F be the extension of the identity functor with $FX = X$ if X has no loop, else $FX = X + 1$. The graph $1 + 1$ has subobjects $S + 1$ and $1 + S$ with the nonempty intersection $S + S$, but F does not preserve it.

2. The colection of all finitary endofunctors on lfp categories preserving non-empty intersections is, nevertheless, large. It contains constant functors, finite power-functors $(-)^n$ ($n \in \mathbb{N}$), and it is closed under finite products and composites. It is also closed under coproducts provided that they commute with pullbacks (which holds in categories such as Pos, Gra, and Nom).

3. On the category Nom, the abstraction functor (cf. [16, Thm. 4.12]) and the finite power-set functor preserve intersections.

▶ **Remark 2.9.** Let A be an object of a locally finitely presentable category.

1. If A is not strictly initial, then it has a nonempty finitely presentable subobject. To see this, let $c_i\colon C_i \rightarrowtail A$ ($i \in I$) be the colimit cocone of the diagram in Remark 2.4.4. If each C_i is strictly initial, then so is the colimit A. Indeed, the colimit of any diagram of strict initial objects is itself strict initial.

[1] There is no condition on the (non-)emptiness of the *family* of monomorphisms which is intersected here.

2. Moreover, if A is not strictly initial, then it is the directed colimit of the canonical diagram of all its *nonempty* finitely presentable subobjects. To see this, combine Remark 2.4.4 with the previous item.
3. If for some ordinal $i \le \omega + \omega$ the object V_i is strictly initial, then νF is obtained in $\omega + \omega$ steps by default. Indeed, recall the transfinite chain V_j from (1.2). The connecting morphism from $V_{i+1} = FV_i$ to V_i is an isomorphism, whence $\nu F = V_i$.

3 A Sufficient Condition for $\nu F = V_{\omega+\omega}$

We first present a simple result that holds for all endofunctors of all categories. It will then be used twice in the sequel. In it, we recall the notation $V_{\omega+\omega}$ from (2.2). Following this, we introduce DCC-categories and prove a generalization of Worrell's result for them (Theorem 5.1).

▶ **Proposition 3.1.** *Let $F\colon \mathcal{A} \to \mathcal{A}$ be an endofunctor such that the following hold:*

1. *The limit V_ω of the ω^{op}-chain* (2.3) *exists, and the morphism $m\colon FV_\omega \to V_\omega$ is monic.*
2. *F preserves monomorphisms.*
3. *$\mathcal{A}$ has and F preserves nonempty intersections of ω^{op}-chains of monomorphisms.*

Then the limit $V_{\omega+\omega}$ is preserved by F; therefore $\nu F = V_{\omega+\omega}$.

Proof. Let V_i be defined for all ordinals i by $V_0 = 1$, $V_{i+1} = FV_i$, and $V_i = \lim_{j<i} V_j$ for limit ordinals i. The ω^{op}-chain (1.1) is its beginning, (2.1) defines the connecting morphism $m\colon V_{\omega+1} \to V_\omega$, and the ω^{op}-chain (2.2), repeated below, is the continuation of the chain in (1.1) up to $V_{\omega+\omega} = \lim_{i<\omega+\omega} V_i$:

$$V_\omega \xleftarrow{\ m\ } V_{\omega+1} \xleftarrow{\ Fm\ } V_{\omega+2} \xleftarrow{\ FFm\ } \cdots .$$

From Items 1 and 2, the morphisms $F^i m$ are monic. By Remark 2.9.3, we may assume without loss of generality that $V_{\omega+\omega}$ is not strictly initial. So by Item 3, F preserves this limit. It follows that $\nu F = V_{\omega+\omega}$, as explained at the beginning of Section 2. ◀

4 DCC-Categories

We introduce lfp categories satisfying a descending chain condition, shortly DCC-categories. Examples are presented and the related condition of graduatedness is discussed. We prove that ω^{op}-limits in DCC-categories are finitary. In Section 5, we prove that $\nu F = V_{\omega+\omega}$ for all finitary endofunctors on DCC-categories preserving nonempty binary intersections.

We have already seen the order of subobjects of a fixed object A (cf. Remark 2.3). (This corresponds to the preordered collection in the slice category $\mathcal{A} \downarrow A$.) Dually, we use the order on strong quotients, represented by strong epimorphisms $e\colon A \twoheadrightarrow E$: given $e'\colon A \twoheadrightarrow E'$, we have $e \le e'$ iff $e' = u \cdot e$ for some $u\colon E \to E'$. This corresponds to the preordered collection in the slice category $A \downarrow \mathcal{A}$. In the literature, the opposite order on quotients is also used. For example, Urbat and Schröder [19], whose work has inspired our next definition, use that opposite order. So readers of papers in this area should be careful.

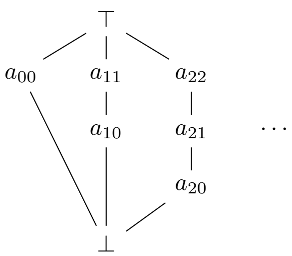
▶ **Definition 4.1.** A locally finitely presentable category $\mathcal{A}$ is a *DCC-category* if every finitely presentable object A satisfies the following *descending chain condition*: Every strictly descending chain of subobjects or strong quotients of A is finite.

Our notion is also related to the stronger notion of graduatedness [9]: an lfp category is *graduated* if to every every finitely presentable object A a natural number n is assigned, called the *grade of* A, such that every (proper) subobject and every (proper) strong quotient of A is finitely presentable, and with a grade at most (smaller than, respectively) the grade of A.

Clearly, every graduated lfp category is DCC. But not conversely:

▶ **Example 4.2.** Here is a DCC-category which is not graduated. Consider the poset A with top element $\top$, bottom element $\bot$, and elements a_{nm} $(n \le m < \omega)$ ordered as follows:

$a_{ij} \le a_{nm}$ iff $i = n$ and $j \le m$.



This is a complete lattice with all elements compact (i.e. finitely presentable). Thus, it is an lfp category. The DCC condition is obvious. But $\top$ cannot have a (finite) grade: its grade would have to be at least 2, due to $\bot < a_0 < \top$, and at least 3 due to $\bot < a_{10} < a_{11} < \top$, and so on.

▶ **Example 4.3** [9]**.** The following categories are graduated (cf. Example 2.2): Set, $\mathsf{Set_p}$, Bool, $\mathbb{S}$-Mod for a finite semiring $\mathbb{S}$, M-Set for a finite monoid M, Gra, K-Vec, and Pos. In the first four categories, the grade is the cardinality of the underlying set. The grade of a graph having n vertices and k edges is $n + k$, and the grade of a vector space is its dimension. Finally, the grade of a poset is described as follows. Let $\mathbb{N} \times \mathbb{N}$ be the poset of pairs of natural numbers ordered lexicographically, and let $\mathbb{P}$ be the subposet of pairs (n, k) with $k \le n^2$. There is an order-isomorphism $\varphi\colon \mathbb{P} \to \mathbb{N}$. The grade of a poset with n elements which contains k comparable pairs is $\varphi(n, k)$.

An important example of a graduated category not included in op. cit. is Nom, the category of nominal sets and equivariant maps. We present a proof based on ideas by Urbat and Schröder [19]. We assume that readers are familiar with basic notions (like orbit and support) from the theory of nominal sets, see Pitts [16].

▶ **Proposition 4.4.** *The category* Nom *is a graduated lfp category.*

Proof. Our proof proceeds in several steps.

1. The finitely presentable objects of Nom are precisely the orbit-finite nominal sets [15, Prop. 2.3.7]. By equivariance, a subobject of a nominal set X is given by a number of orbits. So the DCC on subobjects of an orbit-finite nominal set clearly holds.
2. For the descending chain condition for strong quotients, first recall that in Nom all quotients are strong, and they are represented by the surjective equivariant maps. We first consider single-orbit nominal sets and recall that the supports of elements of an orbit all have the same cardinality. We also recall the standard fact [16, Exercise 5.1] that every single-orbit nominal set X whose elements have supports of cardinality n (this is the *degree* of X) is a quotient of the nominal set $\mathbb{A}^{\#n} = \{(a_1, \ldots, a_n) : |\{a_1, \ldots, a_n\}| = n\}$, where $\mathbb{A}$ denotes the set of names (or atoms). Now observe that a quotient of $\mathbb{A}^{\#n}$ having degree n

is determined by a subgroup G of the symmetric group S_n. More specifically, the quotient determined by G identifies $(a_1, \dots, a_n)$ and $(a_{\pi(1)}, \dots, a_{\pi(n)})$ for every $(a_1, \dots, a_n) \in \mathbb{A}^{\#n}$ and every $\pi \in G$. Conversely, given a quotient $e\colon \mathbb{A}^{\#n} \twoheadrightarrow X$ we obtain G as consisting of all those π for which e identifies the above two n-tuples for every $a_1, \dots, a_n$ in $\mathbb{A}$. We conclude that every strictly descending chain of quotients of $\mathbb{A}^{\#n}$ all having degree n corresponds to a strictly descending chain of subgroups of S_n; the same holds of course for every single-orbit nominal set of degree n. Such a chain must be finite, since so is S_n. In fact, more can be said: For $n \geq 2$, such a chain of subgroups of S_n has length at most $2n - 3$ [10] (and for $n = 1$, S_n is trivial, of course, so chains of subgroups have length 0).

3. For general orbit-finite sets we now conclude that for every proper strong quotient of a nominal set X one of the following three numbers strictly decreases while the other two do not increase: the number of orbits, the degree of some orbit of X, or the maximum length of the above chain of subgroups of S_n for some orbit. Thus, Nom is DCC.
4. To see that Nom is even graduated, observe that we can assign to each orbit-finite nominal set X the sum of the three numbers mentioned in point 3 above. It is then clear that for every proper nominal subset or proper quotient of X the grade is strictly smaller. ◀

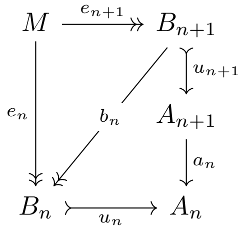
▶ **Example 4.5.** The category Ab of abelian groups is not DCC. The group $\mathbb{Z}$ of integers is finitely presentable, but it has the following descending sequence of proper subgroups

$$\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \cdots .$$

▶ **Definition 4.6.** A category has *finitary* ω^{op}*-limits* if for every limit $\ell_n\colon L \to A_n$ $(n < \omega)$ of an ω^{op}-chain and every finitely presentable subobject $m\colon M \rightarrowtail L$, some morphism $\ell_k \cdot m\colon M \to A_k$ is monic.

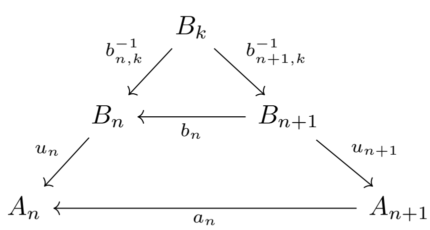
▶ **Proposition 4.7.** *Every DCC-category has finitary* ω^{op}*-limits.*

Proof. Let $\mathcal{A}$ be a DCC-category. Let $\ell_n\colon L \to A_n$ be a limit cone of an ω^{op}-chain $D = (A_n)$ with connecting morphisms $a_{n+1}\colon A_{n+1} \to A_n$. Given a finitely presentable object M and a monomorphism $m\colon M \rightarrowtail L$, factorize $\ell_n \cdot m$ as a strong epimorphism $e_n\colon M \twoheadrightarrow B_n$ followed by a monomorphism $u_n\colon B_n \rightarrowtail A_n$ (Remark 2.4.2). We obtain a subchain (B_n) of (A_n) with connecting maps b_n given by the diagonal fill-ins, as shown below:



(4.1)

Notice that b_n is a strong epimorphism, since so is e_n. We thus have a descending chain (B_n) of strong quotients of the finitely presentable object M: $e_0 \geq e_1 \geq e_2 \geq \cdots$. By the DCC condition, there is some k such that for $n \geq k$, b_n is an isomorphism. For $n \geq k$, let $b_{n,k}\colon B_n \to B_k$ be the composites recursively defined by $b_{n+1,k} = b_{n,k} \cdot b_n$. Thus, for every $n \geq k$, the triangle below commutes, where the lower part commutes by (4.1):

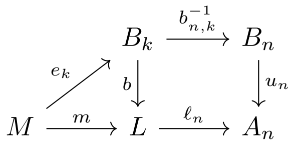


(4.2)

Let D' be the ω^{op} chain $(A_n)_{n\geq k}$. This is a shortening of our original ω^{op}-chain D, and so its limit is $\ell_n\colon L \to A_n$ $(n \geq k)$. The commutativity of all diagrams (4.2) shows that we have a cone $(u_n \cdot b_{n,k}^{-1})_{n\geq k}$ of D'. Thus, there exists $b\colon B_k \to L$ such that

$$\ell_n \cdot b = u_n \cdot b_{n,k}^{-1} \qquad (n \geq k).$$

Consider the following diagram for $n \geq k$:



(4.3)

The square commutes, and we now prove that so does the outside. We show that for all $n \geq k$ and all $0 \leq i \leq n-k$,

$$u_n \cdot b_{n,n-i}^{-1} \cdot e_{n-i} = \ell_n \cdot m. \tag{4.4}$$

We argue by induction on i. For $i = 0$, this holds using $b_{n,n} = \mathsf{id}$ and the factorization $u_n \cdot e_n = \ell_n \cdot m$. Assume (4.4) for i. Fix $n \geq k$ such that $n - k \geq i + 1$. Then

$$\begin{array}{lll} & u_n \cdot b_{n,n-(i+1)}^{-1} \cdot e_{n-(i+1)} & \\ = & u_n \cdot b_{n,n-i}^{-1} \cdot b_{n-i-1}^{-1} \cdot e_{n-i-1} & \text{since } b_{n,n-i-1} = b_{n-i-1} \cdot b_{n,n-i} \\ = & u_n \cdot b_{n,n-i}^{-1} \cdot e_{n-i} & \text{since } e_{n-i-1} = b_{n-i-1} \cdot e_{n-i} \\ = & \ell_n \cdot m & \text{by induction hypothesis} \end{array}$$

The induction completed, we take $i = n - k$ in (4.4) to see the commutativity of the outside of (4.3) for all n. Since the limit cone $(\ell_n)_{n\geq k}$ is collectively monic, the triangle in (4.3) commutes: $m = b \cdot e_k$. As m is monic, so is e_k. Thus, $\ell_k \cdot m = u_k \cdot e_k$ is also monic. ◀

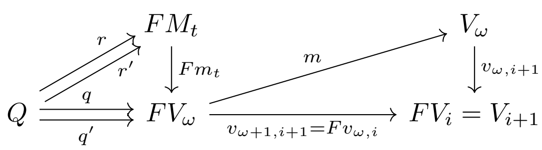
5 Terminal Coalgebras in $\omega + \omega$ Steps

We are ready to state and prove the main theorem of this paper.

▶ **Theorem 5.1.** *Let $\mathcal{A}$ be a DCC-category. Every finitary endofunctor $F\colon \mathcal{A} \to \mathcal{A}$ which preserves nonempty binary intersections has a terminal coalgebra obtained in $\omega + \omega$ steps.*

Proof. We will apply Proposition 3.1. Due to Remark 2.9.3 we can assume without loss of generality that V_i is not strictly initial for any $i \leq \omega + \omega$.

1. We first show that the canonical morphism $m\colon V_{\omega+1} \to V_\omega$ is monic. Consider a parallel pair $q, q'\colon Q \rightrightarrows FV_\omega$ such that $m \cdot q = m \cdot q'$. We prove that $q = q'$. By Remark 2.4.5, we may assume that Q is a finitely presentable object. Using that V_ω can be assumed not to be strictly initial and Remark 2.9.2, we may express V_ω as a directed colimit of nonempty finitely presentable subobjects, say $m_t\colon M_t \rightarrowtail V_\omega$ $(t \in T)$. Since F is finitary, $Fm_t\colon FM_t \to FV_\omega$ is also a directed colimit. Hence, q and q' factorize through Fm_t for some t. We denote the factorizing morphisms by r and r', respectively. It is sufficient to show that they are equal. To this end consider the following diagram:



(5.1)

The limit $v_{\omega,i}\colon V_\omega \to V_i$ is finitary (Proposition 4.7). Thus, there is some i so that $v_{\omega,i} \cdot m_t \colon M_t \to V_i$ is monic, and this monomorphism is nonempty. Since F preserves nonempty binary intersections, it preserves nonempty monomorphisms (Remark 2.6). Hence, the following morphism is monic:

$$\ell = \big(FM_t \xrightarrow{Fm_t} FV_\omega \xrightarrow{Fv_{\omega,i}} FV_i\big).$$

It is enough to show that ℓ merges r and r'. The triangle on the right in (5.1) commutes. Thus we obtain

$$\ell = Fv_{\omega,i} \cdot Fm_t = v_{\omega,i+1} \cdot m \cdot Fm_t.$$

Using that m merges q and q', we see that ℓ merges r and r':

$$\begin{aligned} \ell \cdot r &= v_{\omega,i+1} \cdot m \cdot Fm_t \cdot r \\ &= v_{\omega,i+1} \cdot m \cdot q \\ &= v_{\omega,i+1} \cdot m \cdot q' \\ &= v_{\omega,i+1} \cdot m \cdot Fm_t \cdot r' \\ &= \ell \cdot r'. \end{aligned}$$

Since ℓ is monic, we have $r = r'$ whence $q = q'$, as desired.

2. Next, we prove that F preserves nonempty intersections of ω^{op}-chains of subobjects. Consider such a chain $a_i\colon A_{i+1} \rightarrowtail A_i$, and let its limit cone be $\ell_i\colon L \rightarrowtail A_i$, where L is not strictly initial. It follows that neither is any of the A_i. Take a cone

$$q_i\colon Q \to FA_i \quad (i < \omega).$$

Our task is to find a morphism $q\colon Q \to FL$ such that $q_i = F\ell_i \cdot q$ for all i. (This is unique: all maps ℓ_i are nonempty monic, whence all $F\ell_i$ are monic.)
Using Remark 2.4.4, we can assume, without loss of generality, that Q is finitely presentable: for a general object Q, express it as a colimit of finitely presentable subobjects Q_t, and use the result which we prove for each Q_t.
Choose a nonempty, finitely presentable subobject $c\colon C \rightarrowtail L$ (Remark 2.9.1). Note that this gives nonempty, finitely presentable subobjects

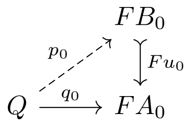
$$c_i = \big(C \overset{c}{\rightarrowtail} L \overset{\ell_i}{\rightarrowtail} A_i\big) \qquad \text{for every } i < \omega,$$

which, moreover, form a cone: $c_i = a_i \cdot c_{i+1}$ for every $i < \omega$.
By recursion on i we define a subchain (B_i) of (A_i) given by intersections

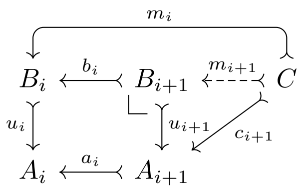
$$\begin{array}{ccccccc} B_0 & \overset{b_0}{\leftarrowtail} & B_1 & \overset{b_1}{\leftarrowtail} & B_2 & \overset{b_2}{\leftarrowtail} & \cdots \\ {\scriptstyle u_0}\downarrow & \lrcorner & \downarrow{\scriptstyle u_1} & \lrcorner & \downarrow{\scriptstyle u_2} & & \\ A_0 & \underset{a_0}{\leftarrowtail} & A_1 & \underset{a_1}{\leftarrowtail} & A_2 & \underset{a_2}{\leftarrowtail} & \cdots \end{array}$$

together with a cone $p_i\colon Q \to FB_i$ such that $Fu_i \cdot p_i = q_i$ and a cone $m_i\colon C \rightarrowtail B_i$ such that $c_i = u_i \cdot m_i$; this shows that all the intersections are nonempty.
To define B_0 and u_0, express A_0 as a directed colimit of all its finitely presentable subobjects $u\colon B \rightarrowtail A_0$ that contain c_0 (Remark 2.4.4). Then use that F preserves

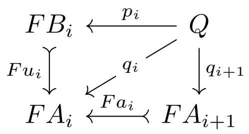
this colimit: for the morphism $q_0 : Q \to FA_0$ we may find a subobject $u_0 : B_0 \rightarrowtail A_0$ containing c_0 such that q_0 factorizes through Fu_0 via some $p_0 : Q \to FB_0$, say:



Since u_0 contains the subobject c_0, we have a monomorphism $m_0 : C_0 \rightarrowtail B_0$ such that $c_0 = u_0 \cdot m_0$.
In the induction step we are given B_i, u_i, p_i, and m_i. Form the intersection of u_i and a_i to obtain B_{i+1}, b_i, and u_{i+1} as shown in the left-hand square below:



The outside commutes by induction hypothesis: $u_i \cdot m_i = c_i = a_i \cdot c_{i+1}$. Hence, we obtain the monomorphism m_{i+1} as indicated such that the upper part and right-hand triangle commute, as desired. Since C is not strictly initial, neither is B_{i+1}, whence the intersection of a_i and u_i is nonempty.
So by hypothesis, F preserves the above pullback. Since the square below commutes



there is a unique morphism $p_{i+1} : Q \to FB_{i+1}$ such that

$$p_i = Fb_i \cdot p_{i+1} \qquad \text{and} \qquad q_{i+1} = Fu_{i+1} \cdot p_{i+1}.$$

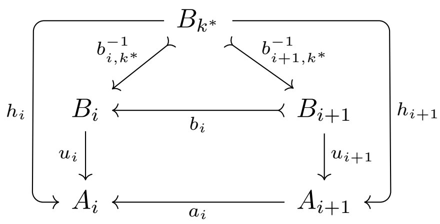
For all $i \le j < \omega$, we form the composite morphism

$$b_{j,i} = \left(B_j \xrightarrow{b_{j-1}} B_{j-1} \xrightarrow{b_{j-2}} \cdots \xrightarrow{b_{i+1}} B_{i+1} \xrightarrow{b_i} B_i\right).$$

We obtain a descending chain of subobjects $b_{j,0} : B_j \rightarrowtail B_0$ $(j < \omega)$ of the finitely presentable object B_0. Since $\mathcal{A}$ is DCC, there is some $k^* < \omega$ such that $b_{k^*,0}$ represents the same subobject as $b_{j,0}$ for every $j \ge k^*$. Hence, the morphism b_{j,k^*} is an isomorphism. The shortened ω^{op}-chain $(A_i)_{i \ge k^*}$ has the limit cone $(\ell_i)_{i \ge k^*}$. The morphisms

$$h_i = \left(B_{k^*} \xrightarrow{b_{i,k^*}^{-1}} B_i \xrightarrow{u_i} A_i\right) \qquad (i \ge k^*)$$

form a cone: we see that $h_i = a_i \cdot h_{i+1}$ from the commutativity of the diagram below:



So there is a unique morphism $h\colon B_{k^*} \to L$ such that $\ell_i \cdot h = u_i \cdot b_{i,k^*}^{-1}$ for $i \geq k^*$.
The desired morphism is

$$q = \big(Q \xrightarrow{p_{k^*}} FB_{k^*} \xrightarrow{Fh} FL\big).$$

In order to verify that $q_i = F\ell_i \cdot q$, it is sufficient to show this for $i \geq k^*$; it then follows also for all $i < k^*$, since the q_i and ℓ_i form cocones:

$$q_i = Fa_{k^*,i} \cdot q_{k^*} = Fa_{k^*,i} \cdot F\ell_{k^*} \cdot q = F\ell_i \cdot q \qquad \text{for } i < k^*.$$

Now observe first that since (p_i) form a cone of (FB_i), we have

$$Fb_{i,k^*} \cdot p_i = p_{k^*}.$$

By definition of h, we also have $u_i = \ell_i \cdot h \cdot b_{i,k^*}$. Therefore for all $i \geq k^*$, we obtain

$$q_i = Fu_i \cdot p_i = F\ell_i \cdot Fh \cdot Fb_{i,k^*} \cdot p_i = F\ell_i \cdot Fh \cdot p_{k^*} = F\ell_i \cdot q.$$

This extends to all $i < k^*$, the argument is as above.
Having checked all the conditions in Proposition 3.1, we are done. ◀

▶ **Corollary 5.2.** *Every finitary endofunctor on* Set *or K*-Vec *has a terminal coalgebra obtained in $\omega + \omega$ steps.*

Indeed, every set functor preserves nonempty binary intersections [18, Prop. 2.1], and every endofunctors on K-Vec preserves finite intersections [9, Ex. 4.3].

The following example demonstrates that without extra conditions there is no uniform bound on the convergence of the terminal-coalgebra chain for finitary functors on locally finitely presentable categories.

▶ **Example 5.3.** For every ordinal n, we present a locally finitely presentable category and a finitary endofunctor which needs n steps for its terminal-coalgebra chain to converge. The category is the complete lattice of all subsets of n (considered as the set of all ordinals $i < n$). The functor is the monotone map F preserving the empty set, and on all other sets $X \subseteq n$,

$$FX = X \setminus \{\min X\}.$$

This is monotone, since given $X \subseteq Y$, if X contains $\min Y$, then $\min X = \min Y$. The only coalgebra for F is empty; thus $\nu F = \emptyset$.

The functor F is finitary because for every directed union $X = \bigcup X_t$ of nonempty subsets, $\min X$ lies in some X_t. Since the union is directed, X is also a union of all X_s where $s \geq t$. Then $\min X$ is contained in all X_s. It follows that $\min X_s = \min X$, thus $FX_s = X_s \setminus \{\min X\}$ for all $s \geq t$. Consequently,

$$\mathsf{colim}\, FX_s = \mathsf{colim}\, X_s \setminus \{\min X\} = X \setminus \{\min X\} = FX.$$

The terminal-coalgebra chain V_i is given by $V_0 = n$ and $V_i = n \setminus i$ for all $0 < i < n$, which is easy to prove by transfinite induction. Thus, that chain takes precisely n steps to converge to the empty set, the terminal coalgebra for F.

6 Finitary Endofunctors on Metric Spaces

We denote by Met the category of *(extended) metric spaces*, where 'extended' means that we might have $d(x, y) = \infty$. The morphisms are non-expanding maps: the functions $f : X \to Y$ where $d(f(x), f(x')) \leq d(x, x')$ holds for every pair $x, x' \in X$. (Note that this class of morphisms is smaller than the class of continuous functions between metric spaces.)

We have seen in Theorem 5.1 a sufficient condition for an endofunctor to have a terminal coalgebra in $\omega + \omega$ steps. This result does not apply to Met, since that category is not locally finitely presentable; in fact, the empty space is the only finitely presentable object [8, Rem. 2.7]. However, for finitary endofunctors on Met, we are able to prove an analogous result. To do this, we work with finite spaces in lieu of finitely presentable objects. Recall that a *subspace* S of a metric space M is a subset $S \subseteq M$ equipped with the metric inherited from M. Moreover, note that there is a bijective correspondence between subobjects of M represented by isometric embeddings and subspaces of M: indeed, for every subspace $S \subseteq M$, the inclusion $S \hookrightarrow M$ is an isometric embedding, and conversely, if $f : M' \to M$ is any isometric embedding, then it is monic and represents the same subobject of M as the inclusion map $f[M'] \hookrightarrow M$ of the subspace on the image of f. We need the following fact.

▶ **Lemma 6.1.** *Every metric space is a directed colimit of the diagram of all its finite subspaces.*

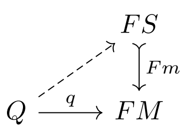
Proof. Fix a metric space M. Let $f_S : S \to A$ be a cocone of the diagram of all finite subspaces $m_S : S \hookrightarrow M$ of M. Then there is a unique map $f : M \to A$ which restricts to f_S for each finite subspace: $f \cdot m_S = f_S$. This map is non-expanding: given elements $x, y \in M$, let S be the subspace of M given by $\{x, y\}$. Since $f_S : S \hookrightarrow A$ is non-expanding, the distance of $f(x)$ and $f(y)$ in A is at most equal to the distance of x and y in M, that is, in S. ◀

▶ **Remark 6.2.** One easily derives that, given a metric space M and a finite subspace $S \hookrightarrow M$, the space M is the directed colimit of the diagram of all its finite subspaces containing S (cf. Remark 2.4.4).

▶ **Proposition 6.3.** *The category* Met *has finitary ω^{op}-limits in the following sense: for every limit $l_n : L \to A_n$ $(n < \omega)$ of an ω^{op}-chain and every finite subobject $m : M \rightarrowtail L$, some morphism $l_k \cdot m : M \to A_k$ is monic.*

Proof. This follows since Set has finitary ω^{op}-limits (Proposition 4.7) because the forgetful functor into Set (1) preserves limits and (2) preserves and reflects monomorphisms. ◀

▶ **Lemma 6.4.** *Let F be a finitary endofunctor on* Met *preserving isometric embeddings. For every non-expanding map $q : Q \to FM$ where Q is finite, there exists a factorization through Fm for some finite subspace $m : S \hookrightarrow M$:*



Proof. 1. Given a directed diagram D in Met of metric spaces A_i $(i \in I)$ and subspace embeddings $a_{i,j} : A_i \hookrightarrow A_j$ $(i \leq j)$, the colimit C is the union $\bigcup_{i \in I} A_i$ with the metric inherited from the subspaces: for $x, y \in \bigcup_{i \in I} A_i$, the distance $d(x, y)$ in C is their distance in A_i for any $i \in I$ such that $x, y \in A_i$.
An analogous statement holds for a directed diagram whose connecting morphisms are isometric embeddings.

2. Given $q\colon Q \to FM$, let D_M be the directed diagram of all finite subspaces of M and all inclusion maps. Its colimit is M, using Item 1. Since F is finitary, FM is the colimit of FD_M, which is a directed diagram of isometric embeddings. The image $q[Q]$ is a finite subspace of FM. By Item 1, there exists a finite subspace $m\colon S \hookrightarrow M$ such that the colimit injection Fm of $FC = \mathsf{colim}\, FD_M$ satisfies $q[Q] \subseteq Fm[FS]$. Let $q'\colon Q \to FS$ be the unique map such that $q = Fm \cdot q'$. Then q' is non-expanding because so is q and because Fm is an isometric embedding. ◀

The following theorem has a proof analogous to that of Theorem 5.1. Recall that a functor preserving nonempty binary intersections also preserves nonempty monomorphisms (Remark 2.6). This time, we need an extra condition that isometric embeddings are preserved:

▶ **Theorem 6.5.** *Let F be a finitary endofunctor on* Met *preserving nonempty binary intersections and isometric embeddings. Then it has a terminal coalgebra obtained in $\omega + \omega$ steps.*

Proof. We again use Proposition 3.1. By Remark 2.9.3, we may assume without loss of generality that all V_i, $i \le \omega + \omega$ are nonempty.

1. The morphism $m\colon V_{\omega+1} \to V_\omega$ is monic: given a non-empty space Q and $q, q'\colon Q \to FV_\omega$ such that $m \cdot q = m \cdot q'$, we prove that $q = q'$. By Lemma 6.1, we may assume that Q is finite. Thus, there exists a nonempty finite subspace $m_t\colon M_t \hookrightarrow Q$ such that both q and q' factorize through Fm_t: we have morphisms $r, r'\colon Q \to FM_t$ such that $q = Fm_t \cdot r$ and $q' = Fm_t \cdot r'$. As in Item 1 of the proof of Theorem 5.1, we derive $r = r'$. Since Fm_t is monic (because F preserves nonempty binary intersections), this proves $q = q'$.
2. We prove that F preserves nonempty limits of ω^{op}-chains of monomorphisms

$$a_i\colon A_{i+1} \rightarrowtail A_i \qquad (i < \omega).$$

Let $\ell_i\colon L \to A_i$ be the limit cone. Given a cone $q_i\colon Q \to FA_i$ $(i < \omega)$, we only need to find a morphism $q\colon Q \to FL$ such that $q_i = F\ell_i \cdot q$ $(i < \omega)$.
Using Lemma 6.1, we may assume that Q is finite. We define a subchain (B_i) of (A_i) carried by nonempty binary subspaces $u_i\colon B_i \hookrightarrow A_i$, together with cones $p_i\colon Q \to FB_i$ and $m_i\colon C \to B_i$ such that $Fu_i \cdot p_i = q_i$ and $c_i = u_i \cdot m_i$. We use recursion analogous to that in Item 2 of the proof of Theorem 5.1. In order to define B_0, u_0, and p_0 use Lemma 6.4: there is a nonempty binary subspace $u_0\colon B_0 \hookrightarrow A_0$ and a morphism $p_0\colon Q \to FB_0$ such that $q_0 = Fu_0 \cdot p_0$. The induction step and the rest of the proof is as in Theorem 5.1. ◀

▶ **Example 6.6.** The finite power-set functor has a lifting $\mathcal{P}_\mathsf{f}\colon \mathsf{Met} \to \mathsf{Met}$ that maps a metric space X to the space $\mathcal{P}_\mathsf{f}X$ of all finite subsets of X equipped with the Hausdorff distance[2]

$$\bar{d}(S,T) = \max\left(\sup\nolimits_{x\in S} d(x,T), \sup\nolimits_{y\in T} d(y,S)\right), \qquad \text{for } S, T \subseteq X \text{ compact},$$

where $d(x,S) = \inf_{y\in S} d(x,y)$. In particular $\bar{d}(\emptyset, T) = \infty$ for nonempty compact sets T. For a non-expanding map $f\colon X \to Y$ we have $\mathcal{P}_\mathsf{f} f\colon S \mapsto f[S]$.

This functor is clearly finitary; in fact, it is the free algebra monad for the variety of quantitative semilattices [13, Sec. 9].

We now show that it preserves isometric embeddings. Indeed, if $m\colon X \hookrightarrow Y$ is the inclusion of a subspace, then $\mathcal{P}_\mathsf{f} m$ preserves distances: given finite subsets S and T of the

[2] The definition goes back to Pompeiu [17] and was popularized by Hausdorff [12, p. 293].

metric space X, then for every $x \in S$, we have that the distance $d(x,T)$ is the same in X and Y. By symmetry, the Hausdorff distance of S and T is also the same in $\mathcal{P}_\mathsf{f} X$ and in $\mathcal{P}_\mathsf{f} Y$.

Finally, $\mathcal{P}_\mathsf{f}$ preserves nonempty binary intersections because it is a lifting of a set functor and since intersections of metric spaces are formed on the level of sets.

▶ **Corollary 6.7.** *The lifted functor* $\mathcal{P}_\mathsf{f} \colon \mathsf{Met} \to \mathsf{Met}$ *has a terminal coalgebra* $\nu\mathcal{P}_\mathsf{f} = V_{\omega+\omega}$.

7 Conclusions and Future Work

This paper gives a sufficient condition for the terminal coalgebra νF for an endofunctor to be obtained in $\omega + \omega$ steps of the well-known iterative construction. This generalizes Worrell's theorem that states that finitary endofunctors on Set have that property. Our generalization concerns DCC-categories; examples include sets, vector spaces, posets, nominal sets, and many others. Finitary endofunctors preserving nonempty binary intersections are proved to have terminal coalgebras in $\omega + \omega$ steps.

The category of abelian groups is an example of an lfp category which is not DCC. We leave as an open problem the question whether every finitary endofunctor on Ab preserving nonempty binary intersections has a terminal coalgebra obtained in $\omega + \omega$ steps.

We have also seen that finitary endofunctors on the category of (extended) metric spaces have terminal coalgebras obtained in $\omega + \omega$ steps, provided that they preserve nonempty binary intersections and isometric embeddings. We also leave as an open problem the question whether every finitary endofunctor on Met preserving nonempty intersections (but not necessarily isometric embeddings) has a terminal coalgebra obtained in $\omega + \omega$ steps.

We have seen that that for finitary endofunctors on Set or K-Vec no extra assumption is needed: νF is always obtained in $\omega + \omega$ steps. It is interesting to ask about other DCC categories with this property; we currently have no example of a finitary endofunctor on a DCC category that does not have a terminal coalgebra obtained in $\omega + \omega$ steps.

Worrell's result holds, more generally, for λ-accessible set functors (i.e. those preserving λ-directed colimits): they have a terminal coalgebra obtained in $\lambda + \lambda$ steps. An appropriate generalization of DCC categories in which this result holds is also an item for future work.

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