Terminal Coalgebras and Free Iterative Theories

Jiří Adámek [1] and Stefan Milius

Institute of Theoretical Computer Science,
Technical University of Braunschweig,
Germany

Abstract

Every finitary endofunctor H of **Set** can be represented via a finitary signature Σ and a collection of equations called "basic". We describe a terminal coalgebra of H as the terminal Σ-coalgebra (of all Σ-trees) modulo the congruence of applying the basic equations potentially infinitely often. As an application we describe a free iterative theory on H (in the sense of Calvin Elgot) as the theory of all rational Σ-trees modulo the analogous congruence. This yields a number of new examples of iterative theories, e.g., the theory of all strongly extensional, rational, finitely branching trees, free on the finite power-set functor, or the theory of all binary, rational unordered trees, free on one commutative binary operation.

Key words: terminal coalgebra, rational tree, iterative theory, basic equation

1 Introduction

It is well-known that for any finitary signature Σ an initial Σ-algebra I_Σ is the algebra of all finite Σ-trees, and a terminal Σ-coalgebra T_Σ is the algebra of all (finite and infinite) Σ-trees. We now prove the analogous statement for every finitary endofunctor H of **Set**. Firstly, we express H as a quotient of the polynomial functor H_Σ, given by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

Email addresses: `J.Adamek@tu-bs.de` (Jiří Adámek), `milius@iti.cs.tu-bs.de` (Stefan Milius).

[1] Support of the Ministry of Education of the Czech Republic MSM 6840770014 is acknowledged.

for some finitary signature Σ. In fact, being finitary (i.e., preserving directed colimits) is, for set functors, equivalent to being a quotient of some H_Σ. Moreover, the quotient is expressed by a collection of *basic equations*, i.e., equations of the form

$$\sigma(x_1, \dots, x_n) = \varrho(y_1, \dots, y_k)$$

where σ and ϱ are operation symbols and x_i and y_i are variables (not necessarily distinct).

Example: the finite-power-set functor $\mathscr{P}_f$ is a quotient of the polynomial functor

$$H_\Sigma X = 1 + X + X^2 + \cdots$$

(of the signature Σ which has one n-ary operation σ_n for every $n \in \mathbb{N}$) via the basic equations

$$\sigma_n(x_1, \dots, x_n) = \sigma_k(y_1, \dots, y_k)$$

where n and k are arbitrary numbers and the variables are such that the set $\{x_1, \dots, x_n\}$ is equal to $\{y_1, \dots, y_k\}$.

Now given such a presentation of H, it is well known that an initial H-algebra I has the form

$$I = I_\Sigma/_\sim$$

where $\sim$ is the congruence generated by the basic equations. That is, two finite Σ-trees t and s are congruent iff t can be obtained from s by a finite application of the basic equations. We prove below that a terminal H-coalgebra has the form

$$T = T_\Sigma/_{\sim^*}$$

where $\sim^*$ is the congruence of finite and infinite applications of the basic equations. The infinite application has a simple definition, inspired by the description of the terminal $\mathscr{P}_f$-coalgebra provided by M. Barr [14]: Given infinite Σ-trees t and s denote by $\partial_k t$ and $\partial_k s$ the trees we obtain from them by cutting them at level k. Then we define $\sim^*$ as follows:

$$t \sim^* s \qquad \text{iff} \qquad \partial_k t \sim \partial_k s \qquad \text{for all } k = 0, 1, 2, \dots$$

Example: a terminal $\mathscr{P}_f$-coalgebra is the coalgebra of all finitely branching strongly extensional trees, i.e., finitely branching unordered trees such that distinct children of every node define non-bisimilar subtrees, see [27]. The reason is that they form a choice class of the above congruence $\sim^*$: every unordered tree is congruent to a unique strongly extensional tree.

The main result of our paper is the above description of a terminal coalgebra of any finitary set functor H. From this we (easily) derive a concrete description of a free iterative theory $\mathscr{R}_H$ on H. Iterative theories were introduced by C. Elgot [17] as a means of an algebraic description of infinite computations. He presented two main examples: the theory Pfn of timed terminal behaviors, or partial functions, see [17], and the theory $\mathscr{R}_\Sigma$ of rational Σ-trees, which is a free iterative theory on Σ, see [18]. Recall that a *Σ-tree on X* (a set of variables) is a tree [2] whose inner nodes are labeled in Σ_n where n is the number of children, and whose leaves are labeled in $\Sigma_0 + X$. Such a tree is *rational*, see [20], if it has, up to isomorphism, only finitely many subtrees. The theory $\mathscr{R}_\Sigma$ assigns to every X the Σ-algebra $R_\Sigma X$ of all rational Σ-trees on X.

We now describe all free iterative theories on finitary endofunctors H of **Set**. Represent H as a quotient of H_Σ modulo basic equations. For every set X of variables denote by $\approx^*$ the congruence on the rational-tree algebra $R_\Sigma X$ obtained by potentially infinite applications of the basic equations. Then the free iterative theory $\mathscr{R}_H$ assigns to every set X the quotient algebra $R_\Sigma X/_{\approx^*}$ of all rational Σ-trees modulo $\approx^*$. This extends considerably the known concrete examples of iterative theories; e.g., in the compendium [16] one finds, besides the mentioned theories Pfn and $\mathscr{R}_\Sigma$, and the theory of synchronization trees, only examples based on complete metric spaces.

Example: one commutative binary operation. This corresponds to algebras on the endofunctor H assigning to every set X the set HX of all unordered pairs in X. This is represented via Σ consisting of one binary operation $*$ and the basic equation $x * y = y * x$. Here $\mathscr{R}_\Sigma$ is the theory of ordered rational binary trees, and $\mathscr{R}_H$ is the theory of unordered rational binary trees.

Related Work. An extended abstract of the present paper was presented at the workshop Coalgebraic Methods in Computer Science 2003, see [5].

Several constructions of terminal coalgebras T for finitary set functors H have been studied in the literature. For example M. Barr shows, in case H is also ω-continuous, the terminal coalgebra as a Cauchy completion of a natural metric on the initial algebra I of H, see [14], and the first current author provided in [6] a natural ordering on I for which T is a free (ideal) completion of I. For general finitary endofunctors J. Worrell [27] proved that the dual of the transfinite initial-algebra construction introduced in [4] stops after $\omega+\omega$ steps and yields a terminal coalgebra. The construction presented below is new and independent of the above mentioned results.

Free iterative theories over polynomial functors were concretely described by C. Elgot and his colaborators as the theories of rational trees, see [18]. The authors proved in [9] and [10] that, more generally, every finitary endofunctor of **Set** generates a free iterative monad. And we described this monad coalgebraically as a certain colimit. The description presented in the current paper is much more concrete. For endofunctors of base categories other than **Set**

[2] Trees are considered to be rooted, ordered, labeled trees, unless stated otherwise, and they are always taken up to isomorphism.

such a description is not known.

Acknowledgement. The authors are grateful to the referees for a number of suggestions which improved the presentation.

2 Initial Algebras

2.1 Assumption. We assume throughout the present section, whose aim is to prepare ground for Section 3, that a *finitary* endofunctor H of **Set** is given. This means as proved in [13] one of the following equivalent properties:

(i) H preserves directed colimits,
(ii) every element of HX, where X is an arbitrary set, lies in the image of Hm for some finite subset $m: M \hookrightarrow X$

and

(iii) H is a quotient of some polynomial functor, i.e., there exists a natural transformation $\varepsilon: H_\Sigma \to H$ with epimorphic components where H_Σ is a polynomial functor, see Example 2.2 below.

For convenience we also assume that H preserves monomorphisms, however, all the results hold without this assumption. In fact, for every endofunctor H there exists a monomorphism preserving endofunctor H' such that

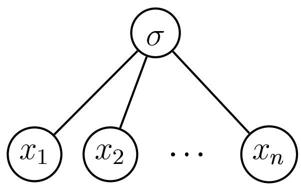
(a) for all $X \neq \emptyset$ we have $HX = HX'$ (and analogously on morphisms), and
(b) $H\emptyset = \emptyset$ if and only if $H'\emptyset = \emptyset$.

Consequently, both the categories of algebras and the categories of coalgebras for H and H', respectively, are isomorphic.

2.2 Example. For every (finitary) signature $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ the corresponding *polynomial endofunctor* H_Σ given on objects X by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

is finitary. The elements of $H_\Sigma X$ are written in the form $\sigma(x_1, \ldots, x_n)$ for $\sigma \in \Sigma_n$ and $(x_1, \ldots, x_n) \in X^n$ and they are called *flat terms*. They correspond to *flat trees*



of height 1 (for $n > 0$) or 0 (for $n = 0$). Pairs of flat terms are called *basic equations*.

2.3 Remark. The equivalence of the condition (i)–(iii) was proved in [13], let us make this explicit here:

(i) → (ii) Express X as a directed colimit of finite subsets.

(ii) → (iii) Put

$$\Sigma_n = H(n) \qquad \text{for all } n = 0, 1, 2, \ldots$$

and use the Yoneda Lemma: the component $\varepsilon_X\colon \coprod H(n) \times X^n \to HX$ is given by

$$\varepsilon_X(\sigma, f) = Hf(\sigma) \qquad \text{for all } f\colon n \to X \text{ and all } \sigma \in H(n).$$

(iii) → (i) Polynomial functors preserve directed colimits because coproducts and finite products commute with directed colimits in **Set**. The proof of the statement that all quotients of finitary functors are finitary is only a bit more technical, see V.3.9 in [13] or a simpler proof in [11], 5.2.

2.4 Remark. The condition (iii) in 2.1 presents H via a finitary signature Σ and a natural transformation $\varepsilon\colon H_\Sigma \to H$ having epimorphic components. Therefore each component

$$\varepsilon_X\colon \coprod_{n<\omega} \Sigma_n \times X^n \to HX$$

is fully described by its kernel equivalence, which we can present in the form of basic equations

$$\sigma(x_1, \ldots, x_n) = \varrho(y_1, \ldots, x_k)$$

for $\sigma \in \Sigma_n$, $\varrho \in \Sigma_k$ and for tuples $\sigma(\vec{x}), \varrho(\vec{y})$ in $H_\Sigma X$ (where X is a set of variables including all x_i and y_j) satisfying

$$\varepsilon_X\big(\sigma(\vec{x})\big) = \varepsilon_X\big(\varrho(\vec{y})\big).$$

We shall call these basic equations the *ε-equations.*

2.5 Examples. (i) The functor $\mathscr{P}_2$ assigning to a set A the set $\mathscr{P}_2 A$ of all subsets of power at most 2 is a quotient of H_Σ where Σ consists of a binary operation β and a constant b. Here

$$\varepsilon_X\colon X \times X + 1 \to \mathscr{P}_2 X$$

sends a pair (x, y) to $\{x, y\}$ and the unique element of 1 to $\emptyset$. The ε-equations are all consequences of the commutativity of β:

$$\beta(x, y) = \beta(y, x).$$

(ii) Consider the finite-power-set functor $\mathscr{P}_f$ assigning to a set X the set $\mathscr{P}_f X = \{A \subseteq X; A \text{ finite}\}$. Here we can use the signature Σ where Σ_n contains a unique n-ary operation for any $n = 0, 1, 2, \dots$ and obtain a natural epitransformation

$$\varepsilon_X \colon 1 + X + X^2 + X^3 + \cdots \to \mathscr{P}_f X$$

sending an n-tuple to the set of its members. The ε-equations equate two flat terms iff the sets of variables appearing in the terms are equal.

(iii) P. Aczel and N. Mendler use in [3] the following subfunctor $(-)^3_2$ of the polynomial functor $X \mapsto X^3$:

$$X^3_2 = \big\{(x_1, x_2, x_3) \in X^3; x_i = x_j \text{ for some } i \neq j\big\}.$$

This can be represented as a quotient of H_Σ where $\Sigma_2 = \{\sigma, \tau, \varrho\}$ and $\Sigma_n = \emptyset$ else. The corresponding basic equations are

$$\sigma(x, x) = \tau(x, x) = \varrho(x, x).$$

2.6 Notation. We denote by $\mathbf{Alg}\, H$ the category of *H-algebras*, i.e., sets A equipped with a structure morphism $\alpha \colon HA \to A$, and *homomorphisms* f between H-algebras, defined by the commutativity of the following square

$$\begin{array}{ccc} HA & \xrightarrow{\ \alpha\ } & A \\ {\scriptstyle Hf}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ HA' & \xrightarrow[\ \alpha'\ ]{} & A' \end{array}$$

2.7 Examples. (i) If $H = H_\Sigma$, then $\mathbf{Alg}\, H_\Sigma$ is the usual category of Σ-algebras and homomorphisms.

(ii) For every presentation $\varepsilon \colon H_\Sigma \to H$ we can consider $\mathbf{Alg}\, H$ as the variety of Σ-algebras presented by all ε-equations.

More precisely, every H-algebra $\alpha \colon HA \to A$ defines a Σ-algebra

$$H_\Sigma A \xrightarrow{\ \varepsilon_A\ } HA \xrightarrow{\ \alpha\ } A$$

which, since ε_A is an epimorphism, determines α completely. The full subcategory of $\mathbf{Alg}\, H_\Sigma$ on all these algebras is presented by ε-equations. In fact:

(a) The above algebra satisfies every ε-equation $u = v$ in $H_\Sigma X$ because given an interpretation $f \colon X \to A$ of the variables, then the interpretation of the two (flat) terms is $Hf(u)$ and $Hf(v)$, respectively, and $\alpha \cdot \varepsilon_A$ merges these two elements because ε_X merges u and v and $\alpha \cdot \varepsilon_A \cdot Hf = \alpha \cdot H_\Sigma f \cdot \varepsilon_X$.

(b) Whenever a Σ-algebra $\bar{\alpha}\colon H_\Sigma A \to A$ satisfies all ε-equation, then given $u, v \in H_\Sigma A$ with $\varepsilon_A(u) = \varepsilon_A(v)$, it follows that $\bar{\alpha}(u) = \bar{\alpha}(v)$, thus, $\bar{\alpha}$ factorizes through ε_A—in other words, A lies in $\mathbf{Alg}\, H$.

2.8 Remark. (i) As we just observed, every category $\mathbf{Alg}\, H$, where H is finitary, is a variety presented by basic equations. Conversely, every variety presented by basic equations is equivalent to $\mathbf{Alg}\, H$ for a finitary set functor, see [13].

(ii) As with every variety, $\mathbf{Alg}\, H$ is a reflective subcategory of $\mathbf{Alg}\, H_\Sigma$: for every Σ-algebra A the congruence $\sim$ generated by ε-equations in A yields a quotient-algebra

$$q\colon A \to A/_\sim \qquad \text{with} \qquad A/_\sim \text{ in } \mathbf{Alg}\, H.$$

This is a reflection, i.e., for every homomorphism $f\colon A \to B$ in $\mathbf{Alg}\, H_\Sigma$ with B in $\mathbf{Alg}\, H$ there exists a unique homomorphism $\bar{f}\colon A/_\sim \to B$ in $\mathbf{Alg}\, H$ with $f = \bar{f} \cdot q$.

2.9 Initial-Algebra Construction. Recall from [4] that every finitary endofunctor H has an initial algebra

$$I = \operatorname*{colim}_{i<\omega} H^i\emptyset.$$

More precisely, we consider the unique chain $\omega \to \mathbf{Set}$ with objects $H^i\emptyset$ and connecting morphisms w_{ij} such that

$$H^0\emptyset = \emptyset,\ H^{i+1}\emptyset = H(H^i\emptyset), \text{ and } w_{i+1,j+1} = Hw_{ij}.$$

Then a colimit $I = \operatorname{colim}_{n<\omega} W_n$ is an initial algebra whose structure map φ is given by the isomorphism

$$\varphi\colon HI \xrightarrow{\simeq} \operatorname*{colim}_{i<\omega} H(H^i\emptyset) = \operatorname*{colim}_{0<j<\omega} H^j\emptyset = I.$$

Observe that since H *preserves monomorphisms*, each $w_{i,i+1}$ is a monomorphism. Consequently, the colimit maps of $I = \operatorname{colim} W_i$ are all monomorphisms.

2.10 Example. For a polynomial functor H_Σ we can describe an initial algebra

$$I_\Sigma$$

as the algebra of all finite Σ-*trees*. Here a tree labeled in Σ is called a Σ-tree iff every node with n children is labelled in Σ_n. In particular, all leaves are labeled in Σ_0. The initial algebra construction yields

$$\begin{aligned} H_\Sigma\emptyset &= \Sigma_0 = \text{all } \Sigma\text{-trees of depth } 0, \\ H_\Sigma H_\Sigma\emptyset &= \coprod_{n<\omega} \Sigma_n \times \Sigma_0^n = \text{all } \Sigma\text{-trees of depth} \leq 1, \\ &\vdots \\ H_\Sigma^{i+1}\emptyset &= \text{all } \Sigma\text{-trees of depth} \leq i \end{aligned}$$

etc.

2.11 Example. Given a finitary functor H and a set X (of generators), the functor $H(-) + X$ is also finitary. A free H-algebra on X is easily seen to be the initial algebra of $H(-) + X$ (and vice versa). It is given as a colimit of the unique ω-chain with objects W_i and connecting morphisms w_{ij} $(i \leq j < \omega)$ such that

$$W_0 = \emptyset,\ W_{i+1} = HW_i + X, \text{ and } w_{i+1,j+1} = Hw_{i,j} + \mathrm{id}_X.$$

The corresponding right-hand injections injections $X \hookrightarrow W_{i+1}$ yield the universal map of the free algebra.

We call this chain a *free-algebra construction* of H on X.

2.12 Example. An initial $\mathscr{P}_f$-algebra is the set of all hereditarily finite sets. Recall the hierarchy W_i ($i \in \mathbf{Ord}$) of constructive sets given by $W_0 = \emptyset$, $W_{i+1} = \mathscr{P}W_i$ and $W_j = \bigcup_{i<j} W_i$ for limit ordinals j. All the sets W_i with $i < \omega$ are finite, and coincide with the above initial-algebra construction. Thus, the set

$$W_\omega$$

of all hereditarily finite sets is an initial $\mathscr{P}_f$-algebra w.r.t the identity function

$$\mathscr{P}_f W_\omega \xrightarrow{\mathrm{id}} W_\omega$$

since $\mathscr{P}_f W_\omega = W_\omega$.

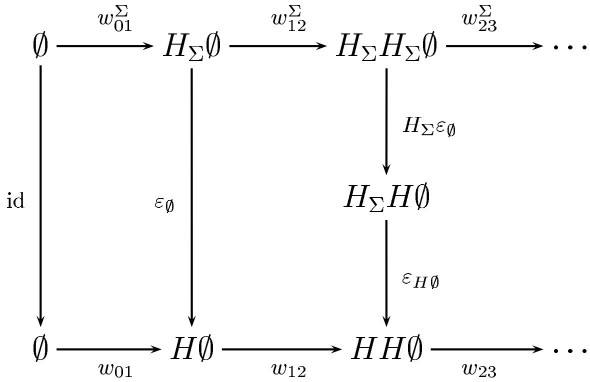
2.13 Remark. The initial-algebra constructions W_i^Σ of H_Σ and W_i of H are connected by the unique natural transformation

$$\widetilde{w}_i \colon W_i^\Sigma \to W_i \qquad (i < \omega)$$

for which the formation of next step is given by

$$\widetilde{w}_{i+1} \equiv W_{i+1}^\Sigma = H_\Sigma W_i^\Sigma \xrightarrow{H_\Sigma \widetilde{w}_i} H_\Sigma W_i \xrightarrow{\varepsilon_{W_i}} HW_i = W_{i+1} \tag{2.1}$$

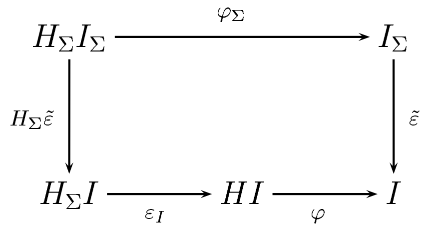
The first steps are as follows:



2.14 Notation. We denote by

$$\widetilde{\varepsilon}\colon I_\Sigma \to I$$

the reflection of the initial Σ-algebra I_Σ in **Alg** H, see Remark 2.8(ii), i.e., $\widetilde{\varepsilon}$ is the unique homomorphism of Σ-algebras from I_Σ to I:



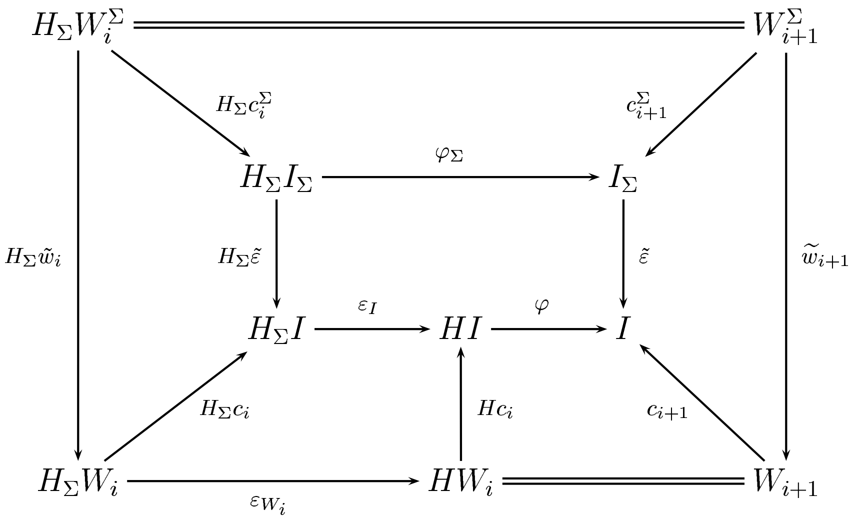
(2.2)

where φ_Σ and φ are the structures of the initial Σ-algebra, and the initial H-algebra, respectively.

2.15 Lemma. *A reflection $\widetilde{\varepsilon}$ of the initial Σ-algebra is given by the colimit*

$$\widetilde{\varepsilon} = \operatorname*{colim}_{i<\omega} \widetilde{w}_i\colon I_\Sigma \to I.$$

PROOF. For the sake of proof let us denote by $\widetilde{\varepsilon}$ the above colimit. It is our only task to show that for this morphism the above square (2.2) commutes. The colimit cocone $c_i\colon W_i \to I$ yields a colimit cocone $Hc_i\colon HW_i \to HC$ and the algebra structure $\varphi\colon HI \to I$ is defined by $\varphi{\cdot}Hc_i = c_{i+1}$, see [4]. Analogously we have $c_i^\Sigma\colon W_i^\Sigma \to I_\Sigma$ and $\varphi_\Sigma{\cdot}H_\Sigma c_i^\Sigma = c_{i+1}^\Sigma$. The desired square commutes when precomposed with $H_\Sigma c_i$ for every i. In fact, the upper part of the diagram (2.3) below commutes by the definition of φ_Σ, for the left-hand part remove H_Σ and use the definition of $\widetilde{\varepsilon}$, and for the right-hand part use the definition of $\widetilde{\varepsilon}$ once again. The lower left-hand part commutes by naturality of ε, and the lower right-hand part commutes by the definition of φ. Finally, the outer shape commutes due to the definition of $\widetilde{w}_{i+1}$, see (2.1). Consequently, since $(H_\Sigma c_i^\Sigma)_{i<\omega}$ is collectively epimorphic, the desired inner square commutes, whence the lemma follows.



(2.3)

□

2.16 Corollary. *For every set X a free H-algebra FX is a reflection of the free Σ-algebra $F_\Sigma X$ with the reflection map*

$$\widetilde{\varepsilon} = \operatorname*{colim}_{i<\omega} \widetilde{w}_i \colon F_\Sigma X \to FX$$

where $\widetilde{w}_i$ is the unique natural transformation between the free-algebra constructions (*see Example* 2.11) *with*

$$\widetilde{w}_{i+1} = \varepsilon_{W_i} \cdot H_\Sigma \widetilde{w}_i + \mathrm{id}_X \qquad (i < \omega).$$

In fact, this is Lemma 2.15 applied to $H_\Sigma(-) + X$ (which is the polynomial functor of the signature Σ expanded by nullary operation symbols from X) and $H(-) + X$.

2.17 Notation. (i) We denote by

$$1 = \{\bot\}$$

a terminal object of **Set**.

(ii) We shall write

$$F_\Sigma \quad \text{and} \quad F$$

for a free H_Σ-algebra and a free H-algebra on 1, respectively.

(iii) We also denote by

$$W_i^\Sigma \quad \text{and} \quad W_i \qquad (i < \omega)$$

the initial-algebra constructions of $H_\Sigma(-)+1$ and $H(-)+1$, respectively with connecting morphisms

$$w_{ij}^\Sigma \quad \text{and} \quad w_{ij} \qquad (i \leq j < \omega)$$

so that

$$F_\Sigma = \operatorname*{colim}_{i<\omega} W_i^\Sigma \qquad \text{and} \qquad F = \operatorname*{colim}_{i<\omega} W_i.$$

(iv) We finally denote by

$$\widetilde{w}_i \colon W_i^\Sigma \to W_i \qquad (i < \omega)$$

the natural transformation of Corollary 2.16 where $X = 1$.

Observe that

$$W_i^\Sigma = \text{all } \big(\Sigma + \{\bot\}\big)\text{-trees of depth} < i$$

and

$$F_\Sigma = \text{all finite } \big(\Sigma + \{\bot\}\big)\text{-trees.}$$

2.18 Corollary. *The kernel equivalence of* $\widetilde{\varepsilon} = \operatorname{colim}_{i<\omega} \widetilde{w}_i$ *is the congruence* $\sim$ *of application of* ε*-equations:*

$$t \sim s \quad \textit{iff} \quad t \textit{ can be obtained from } s \textit{ by applying } \varepsilon\textit{-equations (finitely many times)}$$

for all trees $t, s \in F_\Sigma$.

In fact, due to Example 2.7(ii) a reflection of F_Σ in $\mathbf{Alg}\, H$ is the quotient algebra $F_\Sigma/_\sim$ with the canonical quotient homomorphism $F_\Sigma \to F_\Sigma/_\sim$. Applying Corollary 2.16 to $X = 1$, we see that $\widetilde{\varepsilon}$ is this canonical map.

2.19 Corollary. *For every* $i < \omega$ *we have*

$$\widetilde{w}_i(t) = \widetilde{w}_i(s) \quad \textit{iff} \quad s \sim t \qquad (s, t \in W_i^\Sigma).$$

In fact, the colimit cocone of $F_\Sigma = \operatorname{colim} W_i^\Sigma$ is formed by the inclusion maps $c_i^\Sigma \colon W_i^\Sigma \hookrightarrow F_\Sigma$. And the colimit cocone $c_i \colon W_i \to F$ of $F = \operatorname{colim} W_i$ is formed by monomorphisms, see 2.9. Thus the present corollary follows from the preceding one due to the commutative square

$$
\begin{array}{ccc}
W_i^{\Sigma} & \xrightarrow{c_i^{\Sigma}} & F_{\Sigma} \\
{\scriptstyle \tilde{w}_i}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}} \\
W_i & \xrightarrow[c_i]{} & F
\end{array}
$$

3 Terminal Coalgebras

3.1 Assumption. Throughout this section H denotes a finitary endofunctor of **Set**. We still assume without loss of generality that H preserves monomorphisms. Furthermore, we assume that a fixed presentation

$$\varepsilon\colon H_{\Sigma} \to H$$

is given.

3.2 Notation. We denote by **Coalg** H the category of H-*coalgebras*, i.e., sets A equipped with a structure map $\alpha\colon A \to HA$, and *homomorphisms* f between H-coalgebras defined by the commutativity of the following square

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & HA \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Hf} \\
A' & \xrightarrow[\alpha']{} & HA'
\end{array}
$$

A *terminal coalgebra*, i.e., a terminal object of **Coalg** H, exists due to the finitarity of H, see [14], and we denote it by T with the structure morphism

$$T \xrightarrow{\psi} HT.$$

Recall that by Lambek's Lemma [23], ψ is an isomorphism; thus T can also be viewed as an H-algebra.

3.3 Examples. (i) For the polynomial functor

$$H_{\Sigma}X = X \times X + 1$$

we can consider coalgebras as deterministic systems with binary input and with halting states: given a map

$$\alpha\colon A \to A \times A + 1$$

then A is the set of all states, the halting states are mapped to $\perp$ in the right-hand summand, and non-halting states are mapped to the pair of next states. Homomorphisms are the usual functional bisimulations of systems. A terminal coalgebra T_Σ can be described as the coalgebra of all binary trees.

(ii) $\mathscr{P}_f$-coalgebras can be viewed as finitely branching graphs: A is the set of all nodes, and $\alpha\colon A \to \mathscr{P}_f A$ assigns to every node the set of all neighbors. Beware! The homomorphisms of $\mathscr{P}_f$-coalgebras are stronger than the usual graph morphisms; in fact, a $\mathscr{P}_f$-coalgebra homomorphism $h\colon A \to B$ is a graph morphism reflecting edges, i.e., for each edge $h(a) \to b$ in B there exists an edge $a \to a'$ in A with $h(a') = b$. We mention a description of the terminal $\mathscr{P}_f$-coalgebra in Example 3.18 below.

3.4 Terminal-Coalgebra Construction. The initial-algebra construction of [4] recalled in 2.9 above was restricted to ω because we work with finitary functors; in [4] it was defined for all ordinals. In case of the dual terminal-coalgebra construction, we work at the beginning with $\mathbf{Ord}^{\mathrm{op}}$ (the class of all ordinals with ordering opposite to the usual one), but we then show that all ordinals up to $\omega + \omega$ are sufficient.

Let

$$V\colon \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Set}$$

be the essentially unique chain of objects V_i and morphisms $v_{ij}\colon V_i \to V_j$ $(i \geq j)$ such that

$$V_0 = 1,\ V_{i+1} = HV_i, \text{ and } v_{i+1,j+1} = Hv_{ij},$$

and for all limit ordinals j

$$V_j = \lim_{i<j} V_i \quad \text{with the limit cone } \left(v_{ji}\right)_{i<j}.$$

We say that this construction *converges in λ steps* if $v_{\lambda+1,\lambda}\colon HV_\lambda \to V_\lambda$ is an isomorphism. Or, equivalently, all v_{ji} $(j > i \geq \lambda)$ are isomorphisms. It then follows that

$$T = V_\lambda$$

is a terminal coalgebra w.r.t $v_{\lambda+1,\lambda}^{-1}\colon T \to HT$.

3.5 Example. The terminal coalgebra construction of every polynomial functor H_Σ, which we denote by

$$V_i^\Sigma$$

converges in ω steps because H_Σ (being a coproduct of right adjoint functors) preserves ω^{op}-limits. We identify

$$V_0^\Sigma = \{\perp\}$$

with the singleton tree labelled by $\perp$ and

$$V_i^\Sigma = \coprod_{n\in\mathbb{N}} \Sigma_n \times \left(V_{i-1}^\Sigma\right)^n$$

with the set of all trees of depth $\leq i$ such that

all leaves at level i are labeled by $\perp$,
all leaves at levels $< i$ are labeled in Σ_0, and
all inner nodes with n children are labelled in Σ_n $(n > 0)$.

The connecting maps

$$v_{ij} \colon V_i^\Sigma \to V_j^\Sigma \qquad (j < i < \omega)$$

cut every tree at level j and label every leaf at level j by $\perp$. A limit of the chain

$$\{\perp\} \xleftarrow{v_{1,0}} H_\Sigma\{\perp\} \xleftarrow{v_{2,1}} H_\Sigma H_\Sigma\{\perp\} \xleftarrow{v_{3,2}} \cdots .$$

is the set T_Σ of all (finite and infinite) Σ-trees. The limit cone takes a tree $t \in T_\Sigma$ and assigns to it the sequence of cuttings at level $i = 0, 1, 2, \ldots$ We use the following

3.6 Notation. We denote by V_i^Σ the terminal-coalgebra construction of H_Σ and by

$$\partial_i \colon T_\Sigma \to V_i^\Sigma \qquad (i < \omega)$$

the limit cone of the corresponding limit $T_\Sigma = \lim_{i<\omega} V_i^\Sigma$. These are the functions assigning to every Σ-tree t the tree $\partial_i t$ obtained by cutting t at level i and labelling all leaves of level i by $\perp$.

That is, the nodes of $\partial_i t$ are precisely all nodes of t of depth at most i. All leaves of depth i are labelled by $\perp$, all other nodes are labelled as they were before.

3.7 Remark. For polynomial functors we see that $V_0^\Sigma = \{\perp\}$ is contained in $W_1^\Sigma = H_\Sigma\emptyset + 1$ and, more generally,

$$V_i^\Sigma \subseteq W_{i+1}^\Sigma \qquad \text{for all } i < \omega$$

since we described V_i^Σ as some of the trees forming W_{i+1}^Σ (namely those where the label $\perp$ is only used at the deepest level). This is no coincidence, as the following notation indicates.

3.8 Notation. (i) We denote by

$$V_i \qquad (i \in \mathbf{Ord}^{\mathrm{op}})$$

the terminal-coalgebra construction of H and define monomorphisms

$$m_i \colon V_i \to W_{i+1} \qquad (i < \omega)$$

by induction as follows [3]

$$m_0 = \mathsf{inl} \colon \{\bot\} \to H\emptyset + \{\bot\}$$

and

$$m_{i+1} \equiv V_{i+1} = HV_i \xrightarrow{Hm_i} HW_{i+1} \xrightarrow{\mathsf{inl}} HW_{i+1} + \{\bot\} = W_{i+2}.$$

If $H = H_\Sigma$, we denote these monomorphisms $m_i^\Sigma \colon V_i^\Sigma \to W_{i+1}^\Sigma$. They are just the inclusion maps of Remark 3.7.

3.9 Theorem (James Worrell [27]). *For every finitary functor the terminal-coalgebra construction converges in $\omega + \omega$ steps, and the connecting maps after ω*

$$v_{\omega+i,\omega} \colon V_{\omega+i} \to V_\omega \qquad (i < \omega)$$

are all monomorphisms. Shortly:

$$T = \lim_{i<\omega} V_{\omega+i} = \bigcap_{i<\omega} V_{\omega+i}.$$

The following result is a well-known fact proved in [21]. We include here a full proof for the convenience of the reader.

3.10 Lemma. *The canonical homomorphism*

$$\hat{\varepsilon} \colon T_\Sigma \to T,$$

i.e., the unique homomorphism of the H-coalgebra

$$T_\Sigma \xrightarrow{\psi_\Sigma} H_\Sigma T_\Sigma \xrightarrow{\varepsilon_{T_\Sigma}} HT_\Sigma,$$

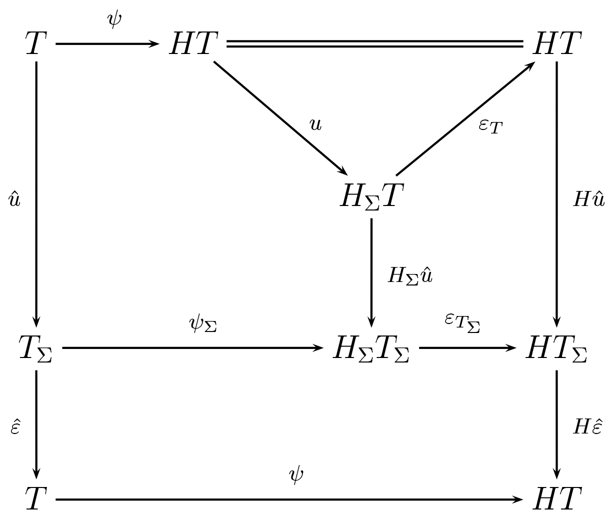
is an epimorphism.

PROOF. Let $u \colon HT \to H_\Sigma T$ split the epimorphism $\varepsilon_T \colon H_\Sigma T \to HT$, i.e., we have $\varepsilon_T u = \mathrm{id}$. Take the unique homomorphism $\hat{u} \colon T \to T_\Sigma$ of the H_Σ-coalgebra

$$T \xrightarrow{\psi} HT \xrightarrow{u} H_\Sigma T.$$

Then $\hat{\varepsilon}{\cdot}\hat{u} \colon T \to T$ is an H-coalgebra homomorphism:

[3] We denote the coproduct injections of $A + B$ by $\mathsf{inl} \colon A \to A + B$ and $\mathsf{inr} \colon B \to A + B$, respectively.



Thus, $\hat{\varepsilon} \cdot \hat{u} = \mathrm{id}$, which completes the proof. $\square$

3.11 Remark. Analogously to Remark 2.13 let us denote by

$$\hat{v}_i \colon V_i^\Sigma \to V_i \qquad (i < \omega + \omega)$$

the unique natural transformation between the terminal-coalgebra constructions of H_Σ and H such that for all i we have

$$\hat{v}_{i+1} \equiv H_\Sigma V_i^\Sigma \xrightarrow{H_\Sigma \hat{v}_i} H_\Sigma V_i \xrightarrow{\varepsilon_{V_i}} HV_i.$$

Observe that $\hat{v}_i$ is an epimorphism for every $i < \omega$ (easy proof by induction). However, $\hat{v}_\omega \colon T_\Sigma \to V_\omega$, which is (necessarily, due to naturality) the limit

$$\hat{v}_\omega = \lim_{i<\omega} \hat{v}_i$$

is, in general, not an epimorphism: a counterexample is $\mathscr{P}_f$ as we demonstrate below. Surprisingly, $\hat{v}_{\omega+\omega}$ is an epimorphism. This follows from Lemma 3.10 and the following

3.12 Lemma. $\hat{\varepsilon} = \lim\limits_{i<\omega+\omega} \hat{v}_i$.

The proof is simply a dual of Lemma 2.15 except that there we only had $i < \omega$, whereas here we have to also consider $\hat{v}_\omega = \lim_{i<\omega} \hat{v}_i$.

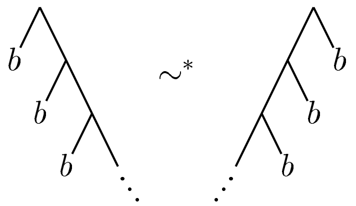
3.13 Definition. *Given Σ-trees t and s, we say that t can be obtained from s by (possibly infinitely many)* ***applications of ε-equations****, notation*

$$t \sim^* s,$$

provided that for every natural number k the cutting $\partial_k t$ can be obtained from the cutting $\partial_k s$ by (finitely many) applications of ε-equations. In symbols:

$$t \sim^* s \quad \textit{iff} \quad \partial_k t \sim \partial_k s \qquad (k < \omega).$$

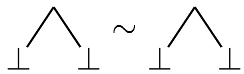
3.14 Example. For $\varepsilon\colon H_\Sigma \to \mathscr{P}_2$ of Example 2.5(i) we have



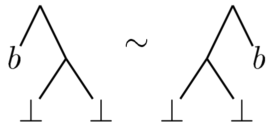
because in F_Σ we clearly have

$(k = 0)$ $\qquad \perp \sim \perp$

$(k = 1)$



$(k = 2)$



etc.

3.15 Theorem. *A terminal H-coalgebra T is the quotient of the terminal H_Σ-coalgebra T_Σ modulo the congruence of applications of ε-equations,*

$$T = T_\Sigma/_{\sim^*}.$$

Remark. We already denoted the canonical homomorphism by $\hat{\varepsilon}\colon T_\Sigma \to T$ and we know from Lemma 3.10 that it is an epimorphism. Thus, all we need to prove is that $\sim^*$ is the kernel equivalence of $\hat{\varepsilon}$. This makes T canonically isomorphic to $T_\Sigma/_{\sim^*}$.

PROOF. (1) We prove first that for every natural number i the kernel equivalence of $\hat{v}_i\colon V_i^\Sigma \to V_i$ (see Remark 3.11) is the congruence $\sim$ of Corollary 2.18. More precisely, we have $V_i^\Sigma \subseteq F_\Sigma$ and we prove

$$\hat{v}_i(t) = \hat{v}_i(s) \quad \text{iff} \quad t \sim s \qquad (i < \omega).$$

For this it is sufficient that the square

$$\begin{array}{ccc} V_i^\Sigma & \xrightarrow{\;m_i^\Sigma\;} & W_{i+1}^\Sigma \\ {\scriptstyle \hat{v}_i}\downarrow & & \downarrow{\scriptstyle \tilde{w}_{i+1}} \\ V_i & \xrightarrow[\;m_i\;]{} & W_{i+1} \end{array} \tag{3.1}$$

commutes: recall from 3.8 that m_i is a monomorphism and m_i^Σ an inclusion map, and use the fact that $\sim$ is the kernel equivalence of $\tilde{w}_{i+1}$, see Corollary 2.19.

The commutativity of the squares (3.1) follows by easy induction. For $i = 0$ both sides compose to $\mathsf{inr}\colon \{\bot\} \to H\emptyset + \{\bot\}$. In the induction step use the following diagram (based on the recursive definitions in 2.13, 3.8 and 3.11):

$$\begin{array}{ccccc}
V^\Sigma_{i+1} = H_\Sigma V^\Sigma_i & \xrightarrow{H_\Sigma m^\Sigma_i} & H_\Sigma W^\Sigma_{i+1} & \xrightarrow{\mathsf{inl}} & H_\Sigma W^\Sigma_{i+1} + 1 = W^\Sigma_{i+2} \\
\downarrow{\scriptstyle H_\Sigma \hat{v}_i} & & \downarrow{\scriptstyle H_\Sigma \tilde{w}_{i+1}} & & \downarrow{\scriptstyle H_\Sigma \tilde{w}_{i+1} + \mathrm{id}} \\
H_\Sigma V_i & \xrightarrow{H_\Sigma m_i} & H_\Sigma W_{i+1} & \xrightarrow{\mathsf{inl}} & H_\Sigma W_{i+1} + 1 \\
\downarrow{\scriptstyle \varepsilon_{V_i}} & & \downarrow{\scriptstyle \varepsilon_{W_{i+1}}} & & \downarrow{\scriptstyle \varepsilon_{W_{i+1}} + \mathrm{id}} \\
V_{i+1} = HV_i & \xrightarrow[Hm_i]{} & HW_{i+1} & \xrightarrow[\mathsf{inl}]{} & HW_{i+1} + 1 = W_{i+2}
\end{array}$$

In fact, this diagram commutes. The two right-hand squares are obvious, the lower left-hand one commutes due to the naturality of ε, and the remaining upper left-hand one by the induction hypothesis.

(2) The limit cone $l_i\colon V_{\omega+\omega} \to V_i$ $(i < \omega + \omega)$ of the terminal H-coalgebra $T = V_{\omega+\omega} = \operatorname{colim} V_i$ is collectively monomorphic. Since all the connecting morphisms $v_{\omega+i,\omega} = V_{\omega+i} \to V_\omega$ are monomorphic (see Theorem 3.9), it follows that the first ω projections l_i, $i < \omega$, are also collectively monomorphic. Therefore the commutative squares

$$\begin{array}{ccc}
T_\Sigma & \xrightarrow{\partial_i} & V^\Sigma_i \\
\downarrow{\scriptstyle \hat{\varepsilon}} & & \downarrow{\scriptstyle \hat{v}_i} \\
T & \xrightarrow[l_i]{} & V_i
\end{array} \qquad\qquad (i < \omega)$$

where ∂_i is the limit cone of T_Σ, see Notation 3.6, prove that

$$\hat{\varepsilon}(t) = \hat{\varepsilon}(s) \quad \text{iff} \quad \hat{v}_i(\partial_i t) = \hat{v}_i(\partial_i s) \quad \text{for all } i < \omega.$$

By part (1) this concludes the proof:

$$\hat{\varepsilon}(t) = \hat{\varepsilon}(s) \quad \text{iff} \quad \partial_i t \sim \partial_i s \quad \text{for all } i < \omega,$$

in other words

$$\hat{\varepsilon}(t) = \hat{\varepsilon}(s) \quad \text{iff} \quad t \sim^* s. \qquad \square$$

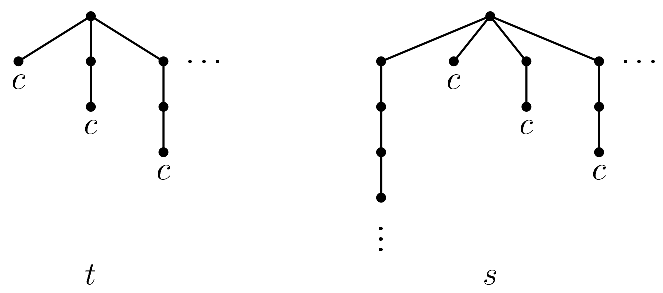
3.16 Example. A terminal $\mathscr{P}_2$-coalgebra can be described as the coalgebra of all non-ordered binary trees. In fact, T_Σ (in Example 2.5(i)) is the algebra of all ordered binary trees—we can simply ignore the labeling by b and β. And two ordered trees are congruent under $\sim^*$ iff they yield the same non-ordered tree (by forgetting the ordering of children).

3.17 Example. For infinitary functors the corresponding description of a terminal coalgebra does not work. We illustrate this on the countable-power-set functor $\mathscr{P}_c$, assigning to every set X the collection of all countable subsets of X. This functor is a quotient of the infinitary polynomial functor H_Σ where $\Sigma = \{c, \sigma\}$ with c nullary and σ ω-ary. The corresponding natural transformation $\varepsilon \colon H_\Sigma \to \mathscr{P}_c$ has components

$$\varepsilon_X \colon 1 + X^\omega \to \mathscr{P}_c X$$

sending the first summand to $\emptyset$ and given, on the second summand, by $(x_n)_{n<\omega} \mapsto \{x_n; n < \omega\}$.

A terminal coalgebra of $\mathscr{P}_c$ can be described as $T_\Sigma/\simeq$ where T_Σ is the algebra of all Σ-trees and $\simeq$ is the bisimilarity equivalence. It is clear that the following trees



are not bisimilar. However, $t \sim^* s$ because for every k we clearly have $\partial_k t \sim \partial_t s$.

3.18 Example. The terminal coalgebra of $\mathscr{P}_f$ has been described by James Worrell in [27]. It is the coalgebra formed by all (non-ordered) finitely branching strongly extensional trees, i.e., those non-ordered and finitely branching trees where subtrees defined by distinct children of a node are never bisimilar. We obtain this description from Theorem 3.15 as follows. Recall first the presentation of $\mathscr{P}_f$ in Example 2.5(ii).

Then clearly the set V_i^Σ consists of all finitely branching trees of depth less than i with all leaves at level i labelled by $\perp$, and $T_\Sigma = V_\omega^\Sigma$ is the coalgebra of all finitely branching trees.

It is not difficult to see that the sets V_i, $(i < \omega)$, consist of all finitely branching strongly extensional trees of depth less then i. And the maps $\hat{v}_i \colon V_i^\Sigma \to V_i$

compute the strongly extensional quotient obtained by forgetting the order of children, and then taking the quotient modulo the greatest bisimulation (which is always an equivalence). It follows that for two trees t and s in T_Σ we have $t \sim^* s$ iff for each natural number k the cuttings $\partial_k t$ and $\partial_k s$ have the same strongly extensional quotient.

Finally, one readily shows that finitely branching strongly extensional trees are in a one-to-one correspondence with equivalence classes of $\sim^*$.

Notice that in the present case the set V_ω consists of all equivalence classes of all countably branching trees modulo the relation defined analogously as $\sim^*$. This is *not* the terminal coalgebra T: we need the next ω steps! The subset $v_{\omega+1,\omega} \colon \mathscr{P}_f V_\omega \to V_\omega$ consists of all equivalence classes of trees in V_ω which are finitely branching at the root; in general, $V_{\omega+i}$ are the classes of all trees finitely branching up to level i. So $T = V_{\omega+\omega}$ is the intersection of all $V_{\omega+i}$, $(i < \omega)$, i.e., it consists of those classes in V_ω given by finitely branching trees.

4 Free Completely Iterative Theories

In the present section we describe for every finitary endofunctor H of **Set** a free completely iterative theory $\mathscr{T}_H$ on H in the sense of C. Elgot et al [18]. The description is analogous to that of a terminal coalgebra in the preceding section: we use a presentation of H as a quotient

$$\varepsilon \colon H_\Sigma \to H$$

for some signature Σ. Then H_Σ generates a free completely iterative theory $\mathscr{T}_\Sigma$ which, as proved in [18], assigns to every set X of variables the Σ-algebra $T_\Sigma X$ of all Σ-trees on X, i.e., trees where every node with $n > 0$ children is labeled in Σ_n and every leaf is labeled in $\Sigma_0 + X$. And we prove that the free completely iterative theory $\mathscr{T}_H$ on H assigns to every set X the quotient of $T_\Sigma X$ obtained by applying basic equations finitely or infinitely many times.

4.1 Recursive Tree-Equations. Given a signature Σ and a set X of variables, we denote by

$$\Sigma(X)$$

the signature obtained from Σ by adding new constant operation symbols labeled by elements of X. Then the initial algebra

$$I_{\Sigma(X)}$$

of the (polynomial) functor $H_{\Sigma(X)}$ is precisely a free Σ-algebra on X; it can be described as the algebra of all finite Σ-trees on X.

We also form a terminal coalgebra of $H_{\Sigma(X)}$ and denote it by

$$\psi_X^\Sigma \colon T_\Sigma X \to H_\Sigma T_\Sigma X + X.$$

By Lambek's Lemma [23] the coalgebra structure ψ_X^Σ is an isomorphism; therefore T_Σ is a coproduct of the set X of all variables (considered as singleton trees) and the set $H_\Sigma T_\Sigma X$ of all trees with root labeled in Σ. More precisely: $T_\Sigma X$ is a coproduct with injections

$$X \xrightarrow{\eta_X} T_\Sigma X \xleftarrow{\tau_X} H_\Sigma T_\Sigma X, \tag{4.1}$$

where η_X assigns to every variable the corresponding singleton tree, and τ_X expresses the Σ-algebra structure (of tree tupling) on $T_\Sigma X$.

We also denote, for every function $s \colon X \to T_\Sigma Y$ (which substitutes every variable x in X by a tree $s(x)$ on Y) by s^* the corresponding Σ-homomorphism

$$s^* \colon T_\Sigma X \to T_\Sigma Y \tag{4.2}$$

which carries out the substitution s in every leaf labeled by a variable.

An important property of the (co-)algebra $T_\Sigma X$ is the unique solvability of *recursive equation systems* of the following form

$$x_i = t_i(x_0, x_1, \dots, y_0, y_1, \dots) \tag{4.3}$$

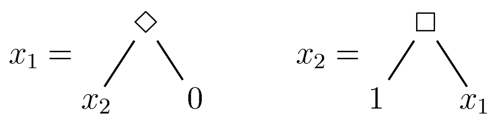
where $X = \{x_0, x_1, \dots\}$ is an arbitrary set of variables, $Y = \{y_0, y_1, \dots\}$ is an arbitrary set of parameters, and t_i is a Σ-tree on $X + Y$. By a *solution* we mean trees

$$x_i^\dagger \in T_\Sigma Y$$

(one for every variable $x_i \in X$) such that $x_i^\dagger$ is equal to t_i with x_0 substituted by $x_0^\dagger$, x_1 by $x_1^\dagger$ etc.:

$$x_i^\dagger = t_i\Big(x_0^\dagger/x_0, x_1^\dagger/x_1, \dots, y_0, y_1, \dots\Big)$$

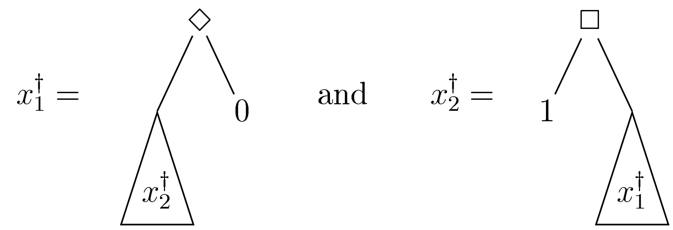
4.2 Example. For Σ consisting of binary operations $\diamond$ and $\square$, we solve the following equations



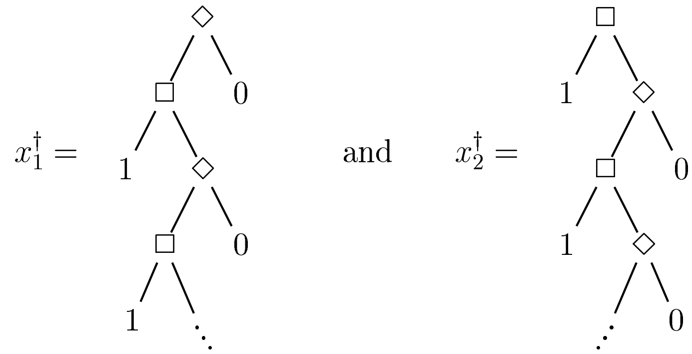
where $X = \{x_1, x_2\}$ and $Y = \{0, 1\}$. The unique solution is given by the trees

$$x_0^\dagger, x_1^\dagger \in T_\Sigma Y$$

(using parameters, but not variables) satisfying



Here they are:



4.3 Remark. Categorically, a system of equations $x_i = t_i$ as above is a morphism

$$e\colon X \to T_\Sigma(X+Y).$$

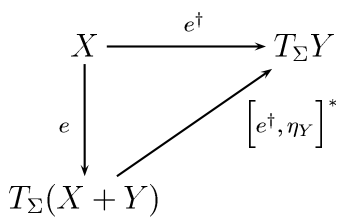
A solution of e is a morphism

$$e^\dagger\colon X \to T_\Sigma Y$$

having the property that $e^\dagger$ is equal to the composite of the morphism e with the "substitute $e^\dagger$" morphism from $T_\Sigma(X+Y)$ to $T_\Sigma Y$. The latter morphism is simply s^*, see 4.2, for the function

$$s = [e^\dagger, \eta_Y]\colon X+Y \to T_\Sigma Y$$

(substituting $e^\dagger(x_i)$ for every variable x_i, but leaving the parameters unchanged). Thus, the defining property of the solution morphism $e^\dagger$ is that the following triangle



commutes.

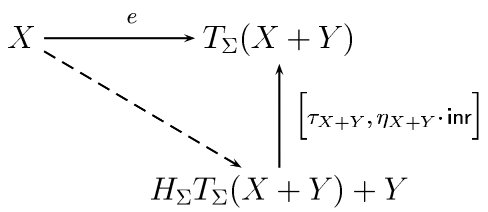
Almost all equation morphisms have a unique solution. Exceptions arise where on the right-hand sides of $x_i = t_i$ single variables are allowed—e.g., the equation $x_1 = x_1$ certainly does not have a unique solution. An equation morphism

$$e\colon X \to T_\Sigma(X+Y)$$

is called *guarded* provided that $e(x)$ is not a single variable for all $x \in X$. Observe that since by (3.1) we have

$$T_\Sigma(X+Y) = H_\Sigma T_\Sigma(X+Y) + X + Y,$$

e is guarded iff it factorizes through the coproduct injection of $H_\Sigma T_\Sigma(X + Y) + Y$ into $T_\Sigma(X+Y)$:



4.4 Observation. (1) Every guarded equation morphism $e\colon X \to T_\Sigma(X+Y)$ has a unique solution $e^\dagger\colon X \to T_\Sigma Y$.

(2) The above assignment of $T_\Sigma X$ and η_X to every set X is the object part of a monad

$$\mathscr{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$$

on **Set** whose unit $\eta^\Sigma\colon \mathrm{Id} \to T_\Sigma$ has the components η_X, and whose multiplication $\mu^\Sigma\colon T_\Sigma T_\Sigma \to T_\Sigma$ has components $\mu_X\colon T_\Sigma T_\Sigma X \to T_\Sigma X$ given by the flattening. Observe that $s^* = \mu_Y \cdot T_\Sigma s$ for all $s\colon X \to T_\Sigma Y$.

4.5 Remark. The above facts about the tree monad $\mathscr{T}_\Sigma$ generalize to monads called completely iterative in [18]. To formulate this, we first need the concept of an ideal theory of Calvin Elgot [17]. We formulate this in categorical language of monads instead of theories. This is equivalent as explained in [2]:

4.6 Definition. *A monad $\mathscr{S} = (S, \mu, \eta)$ on* **Set** *is called **ideal** provided that there is a subfunctor*

$$\sigma\colon S' \rightarrowtail S$$

such that

(i) $S = S' + \mathrm{Id}$ *with coproduct injections* σ *and* η

and

(ii) μ *can be restricted to a natural transformation* $\mu'\colon S'S \to S'$ *(with* $\sigma \cdot \mu' = \mu \cdot \sigma S$*).*

4.7 Remark. (1) The above subfunctor S', if it exists, is essentially unique, being the complement of the subfunctor $\eta\colon \mathrm{Id} \rightarrowtail S$.

(2) Given ideal monads $\mathscr{S}$ and $\overline{\mathscr{S}}$ (given by $\overline{S} = \overline{S}' + \mathrm{Id}$), a monad morphism $h\colon S \to \overline{S}$ is called *ideal* if h restricts to a natural transformation $h'\colon S' \to \overline{S}'$ with $h = h' + \mathrm{id}$.

4.8 Examples. (1) The tree monad $\mathscr{T}_\Sigma$ is ideal. We have, by (4.1)

$$T_\Sigma = H_\Sigma T_\Sigma + \mathrm{Id}$$

and the tree flattening $\mu_X\colon T_\Sigma T_\Sigma X \to T_\Sigma X$ restricts to

$$\mu'_X = H_\Sigma \mu_X\colon H_\Sigma T_\Sigma T_\Sigma X \to H_\Sigma T_\Sigma X,$$

which is the tree flattening of all nontrivial trees.

(2) Let H be a finitary endofunctor of **Set**. Then $H(-) + X$ is also finitary (for every set X), thus, it has a final coalgebra

$$\psi_X\colon TX \to HTX + X.$$

By Lambek's Lemma [23] the coalgebra structure ψ_X is an isomorphism $TX \cong H(TX) + X$ which makes TX a coproduct of HTX and X; we denote again by

$$\eta_X\colon X \to TX \quad \text{and} \quad \tau_X\colon HTX \to TX$$

the coproduct injections. We obtain an endofunctor T of **Set** with $T = HT + \mathrm{Id}$, and natural transformations $\psi\colon T \to HT + \mathrm{Id}$, $\eta\colon \mathrm{Id} \to T$ and $\tau\colon HT \to T$ such that ψ and $[\tau, \eta]$ are mutually inverse. We denote those by $\eta^\Sigma\colon \mathrm{Id} \to T_\Sigma$ and $\tau^\Sigma\colon H_\Sigma T_\Sigma \to T_\Sigma$ in case of a polynomial endofunctor H_Σ.

It has been proved in [2] that T is a part of a monad

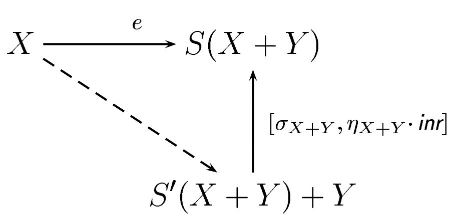
$$\mathscr{T}_H = (T, \eta, \mu)$$

which is ideal (with $T = HT + \mathrm{Id}$) since μ has the restriction

$$\mu' = H\mu\colon HTT \to HT.$$

We write $\mu^\Sigma\colon T_\Sigma T_\Sigma \to T_\Sigma$ in case of a polynomial endofunctor H_Σ.

4.9 Definition. *Let $\mathscr{S} = (S, \eta, \mu)$ be an ideal monad on* **Set**. *By an* ***equation morphism*** *is meant a morphism $e\colon X \to S(X+Y)$, and it is called* ***guarded*** *if it factorizes through $[\sigma_{X+Y}, \eta_{X+Y} \cdot \mathit{inr}]\colon S'(X+Y)+Y \to S(X+Y)$:*



By a ***solution*** *of e is meant a morphism $e^\dagger\colon X \to SY$ for which the following square*

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ S(X+Y) & \xrightarrow[S[e^\dagger,\eta_Y]]{} & SSY \end{array}$$

*commutes. The monad $\mathscr{S}$ is called **completely iterative** provided that every guarded equation morphism has a unique solution. And $\mathscr{S}$ is called **iterative** provided that every guarded equation morphism $e\colon X \to S(X+Y)$ with X and Y finite has a unique solution; such equation morphisms are called **finitary**.*

4.10 Example. The tree monad $\mathscr{T}_\Sigma$ is completely iterative for every signature Σ. More generally, given a finitary endofunctor H of **Set**, the above monad $\mathscr{T}_H$ is completely iterative see [26], [2], or [25] for a simple coalgebraic proof.

In fact, $\mathscr{T}_H$ can be characterized as a free completely iterative monad on H in the following sense:

4.11 Notation. We denote by $\mathsf{CIM}(\mathbf{Set})$ the category of all completely iterative monads on **Set** and ideal monad morphisms. We consider it as a concrete category over the endofunctor category $\mathbf{Set}^{\mathbf{Set}}$ via the functor

$$U\colon\ \mathsf{CIM}(\mathbf{Set}) \to \mathbf{Set}^{\mathbf{Set}}$$

assigning to every ideal monad $\mathscr{S}$ (carried by $S = S' + \mathrm{Id}$) the endofunctor S' and to every ideal monad morphism $h\colon \mathscr{S} \to \overline{\mathscr{S}}$ the natural transformation $h'\colon S' \to \overline{S}'$.

Example. For the above monad $\mathscr{T}_H$ we have $U\mathscr{T}_H = HT$.

4.12 Theorem. (see [2,25]) *For every finitary endofunctor H of* **Set** *the monad $\mathscr{T}_H$ is a free completely iterative monad on H. That is, given a completely iterative monad $\mathscr{S}$ and a natural transformation $f\colon H \to U\mathscr{S}$ there exists a unique ideal monad morphism $\overline{f}\colon \mathscr{T} \to \mathscr{S}$ with $f = U\overline{f}\cdot H\eta$.*

4.13 Corollary. *The tree monad $\mathscr{T}_\Sigma$ is a free completely iterative monad on the signature Σ.*

This has been proved already in [18], but the proof is much more involved than those of [2,25]. The monads $\mathscr{T}_\Sigma$ have been the only concretely described completely iterative monads so far. We are able to concretely describe the free completely iterative monad $\mathscr{T}_H$ on any finitary endofunctor H of **Set**:

4.14 Notation. Let H be a finitary endofunctor of **Set** represented as a quotient

$$\varepsilon \colon H_\Sigma \to H$$

of a polynomial endofunctor, see Remark 2.4. For every set X we thus have a quotient

$$\varepsilon + \mathrm{id}_X \colon H_\Sigma(-) + X \to H(-) + X.$$

(Observe that the ε-equations, as defined in 2.4, are precisely the same as the $(\varepsilon + \mathrm{id}_X)$-equations.) Denote by

$$\sim_X$$

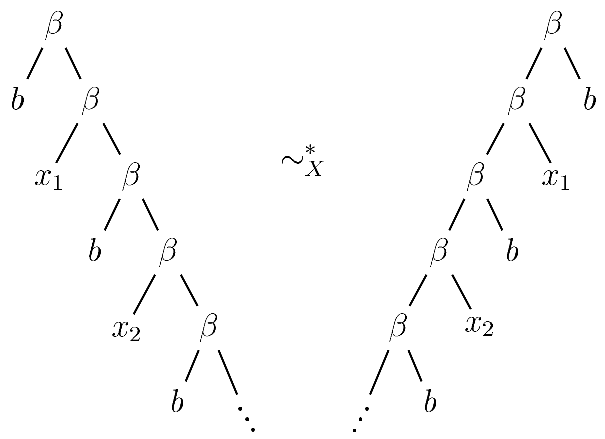
the corresponding congruence on the $\Sigma(X)$-algebra $F_{\Sigma(X)}$, see Definition 2.6.

Further, let

$$\sim_X^*$$

denote the congruence on $T_\Sigma X = T_{\Sigma(X)}$ of Definition 3.13 given by applying the ε-equations finitely or infinitely many times. That is, Σ-trees s and t over X are congruent iff $\partial_k s \sim_X \partial_k t$ holds for all $k < \omega$.

4.15 Example. For $H = \mathscr{P}_2$, see Example 2.5 (i), the congruence $\sim_X$ on the algebra $F_{\Sigma(X)}$ (of all finite binary trees with leaves labeled in $X + \{b, \bot\}$ and all inner nodes labeled by β) is just the commutativity of the operation β. And $\sim_X^*$ is the congruence on the algebra $T_{\Sigma(X)}$ (of all binary trees with leaves labeled in $X + \{b\}$) which uses the commutativity finitely or infinitely many times. Example:



4.16 Theorem. (Description of free completely iterative monads) *For every finitary endofunctor H of* **Set** *a free completely iterative monad $\mathscr{T}_H$ on H can be described as the quotient of the tree monad $\mathscr{T}_\Sigma$ modulo the monad congruence $\sim_X^*$ (X a set) of applying the basic equations finitely or infinitely many times.*

Remark. More detailed: given a presentation as a quotient $\varepsilon \colon H_\Sigma \to H$, then for the free completely iterative monad $\mathscr{T}_H$ on H we have the unique ideal monad morphism $h \colon \mathscr{T}_\Sigma \to \mathscr{T}_H$ such that the square

$$
\begin{array}{ccccc}
H_\Sigma & \xrightarrow{H\eta^\Sigma} & H_\Sigma T_\Sigma & \xrightarrow{\tau^\Sigma} & T_\Sigma \\
{\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\
H & \xrightarrow[H\eta]{} & HT & \xrightarrow[\tau]{} & T
\end{array}
\tag{4.4}
$$

commutes where h' is the restriction of h, see Remark 4.7(2). The theorem states that the components h_X are epimorphisms with the kernel equivalence $\sim_X^*$.

PROOF. Consider the following commutative diagram

$$
\begin{array}{ccccccc}
 & & & \overset{\tau^\Sigma}{\longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} & & & \\
T_\Sigma & \xleftarrow[\mu^\Sigma]{} & T_\Sigma T_\Sigma & \xleftarrow[\tau^\Sigma T_\Sigma]{} & H_\Sigma T_\Sigma T_\Sigma & \xleftarrow[H\eta^\Sigma T_\Sigma]{} & H_\Sigma T_\Sigma \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h*h} & & & & \downarrow{\scriptstyle \varepsilon*h} \\
T & \xleftarrow{\mu} & TT & \xleftarrow{\tau T} & HTT & \xleftarrow{H\eta T} & HT \\
 & & & \underset{\tau}{\longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} & & &
\end{array}
\tag{4.5}
$$

In fact, the left-hand square commutes since h is a monad morphism ($*$ denotes the parallel composition), and the commutativity of the right-hand part follows from naturality applied to the parallel components separately. The left-hand component with domain H_Σ commutes due to (4.4) and the right-hand one with domain T_Σ is trivial. For the lower part we have the commuting diagram

$$
\begin{array}{ccccc}
HT & \xrightarrow{H\eta T} & HTT & \xrightarrow{\tau T} & TT \\
 & \searrow\!\!\!\!\searrow & \downarrow{\scriptstyle H\mu} & & \downarrow{\scriptstyle \mu} \\
 & & HT & \xrightarrow[\tau]{} & T
\end{array}
$$

where the right-hand square commutes since T is an ideal monad (see Definition 4.6(ii)) and the left-hand triangle follows from the monad laws of T. Similarly, the upper part of (4.5) commutes. We conclude that, for any set X, the diagram

$$
\begin{array}{ccccc}
T_\Sigma X & \underset{[\tau_X^\Sigma,\eta_X^\Sigma]}{\overset{\psi_X^\Sigma}{\rightleftarrows}} & H_\Sigma T_\Sigma X + X & \xrightarrow{\varepsilon_{T_\Sigma X}+X} & HT_\Sigma X + X \\
{\scriptstyle h_X}\downarrow & & & \searrow{\scriptstyle (\varepsilon*h)_X+X} & \downarrow{\scriptstyle Hh+X} \\
TX & & \underset{\psi_X}{\overset{[\tau_X,\eta_X]}{\leftrightarrows}} & & HTX + X
\end{array}
\tag{4.6}
$$

commutes. In fact, recall that the coalgebra structure ψ_X is an isomorphism with the inverse $[\tau_X^\Sigma, \eta_X^\Sigma]$. Thus, in order to see that the outer shape of (4.6) commutes it suffices to check in the left-hand part the commutativity of the coproduct components of $H_\Sigma T_\Sigma X + X$. The left-hand component commutes by (4.5) and the right-hand one since $h{\cdot}\eta^\Sigma = \eta$ holds for the monad morphism h.

Therefore, $h_X \colon T_\Sigma X \to TX$ is the canonical coalgebra homomorphism, which is epimorphic by Lemma 3.10. Finally, by Theorem 3.15, the terminal coalgebra TX for $H(-) + X$ is the quotient of the terminal coalgebra $T_\Sigma X$ for $H_\Sigma(-) + X$ modulo $\sim_X^*$ and the kernel equivalence of h_X is $\sim_X^*$ as desired. □

4.17 Example. One commutative binary operation. Here algebras are precisely the H-algebras for the functor

$$HY = \text{all unordered pairs in } Y.$$

We have a canonical presentation of H as a quotient of the polynomial functor

$$H_\Sigma Y = Y \times Y.$$

Analogously, the polynomial functor $H_\Sigma(-) + X$ of one binary operation and constants labelled in X, has $H(-) + X$ as a canonical quotient. For H_Σ we get the completely iterative monad

$$T_\Sigma X = \text{all binary trees on } X.$$

And our H generates the free completely iterative monad

$$TX = T_\Sigma X/_{\sim_X^*},$$

where for trees $t, s \in T_\Sigma X$ we have

$$t \sim_X^* s \quad \text{iff} \quad s \text{ can be obtained from } t \text{ by swapping siblings (possibly infinitely often).}$$

This is completely analogous to Example 3.16:

$$TX = \text{all unordered binary trees on } X.$$

4.18 Example. The finite-power-set functor $\mathscr{P}_f$. We start with the signature Σ of Example 2.5(ii): the free completely iterative monad $\mathscr{T}_\Sigma$ assigns to every set X the algebra $T_\Sigma X$ of all finitely branching trees with leaves labeled in $X + 1$. We then obtain the free completely iterative monad on $\mathscr{P}_f$ as the quotient $\mathscr{T}_\Sigma/_{\sim^*}$. It is easy to see that for every tree $t \in T_\Sigma X$, given a node where two children are bisimilar subtrees, we can cut one of the subtrees away and obtain a tree $t' \sim_X^* t$. The bisimilarity here is related to labeled trees, of course: it is the biggest relation R on $T_\Sigma X$ such that given a pair $t_1 \mathrel{R} t_2$ of trees in $T_\Sigma X$, then for every child s_1 of t_1 there is a child s_2 of t_2 with $s_1 \mathrel{R} s_2$, and vice versa. By repeating this process (infinitely often) we obtain, for every tree $t \in T_\Sigma X$, a *strongly extensional* tree congruent to t under $\sim_X^*$; strong extensionality is defined, analogously to Example 3.18, by the property that distinct children of any parent are non-bisimilar.

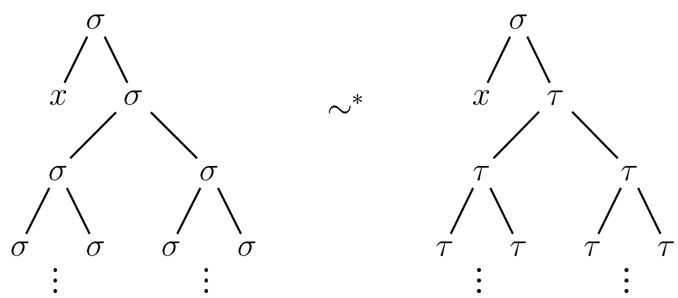
We conclude that a free completely iterative monad $\mathscr{T}_{\mathscr{P}_f}$ on $\mathscr{P}_f$ is described analogously to Example 3.18:

$$TX = \text{the algebra of all non-ordered strongly extensional finitely branching trees on } X+1.$$

4.19 Example. A free completely iterative monad on $(-)^3_2$, see Example 2.5(iii), is obtained from the monad $\mathscr{T}_\Sigma$ (of all binary trees with inner nodes labeled by σ, τ or ϱ and leaves labeled by variables) as a quotient

$$\mathscr{T}_H = \mathscr{T}_\Sigma / {\sim^*}.$$

Here $t \sim^* s$ means that t can be obtained from s by relabeling (finitely or infinitely many) inner nodes whose children are isomorphic subtrees. Example:



5 Free Iterative Theories

Iterative theories were introduced by C. Elgot in his fundamental paper [17], where he also proved the existence of free iterative theories. Later, these free theories were described as the theories of all rational trees, see [18,20]. The basic notion of Elgot is a Lawvere theory in which certain iterative equation have unique solutions. We use here, in lieu of Lawvere theories, the equivalent concept of a finitary monad. The concept of ideal and iterative theory of [17] then precisely correspond to ideal and iterative finitary monads, as explained in [9] and [10]. This section presents a description of a free iterative monad $\mathscr{R}_H$ on an arbitrary finitary endofunctor H of **Set**. Again, we present H as a quotient of a polynomial functor H_Σ. The free iterative monad $\mathscr{R}_\Sigma$ on H_Σ is the subtheory of the Σ-tree theory

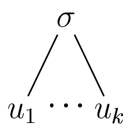
$$\mathscr{R}_\Sigma \subseteq \mathscr{T}_\Sigma$$

of all *rational* Σ*-trees* on X, where a tree is called rational iff it has, up to isomorphism, only finitely many subtrees. We describe $\mathscr{R}_H$ as the quotient of $\mathscr{R}_\Sigma$ obtained by (possibly infinite) applications of the basic equations.

5.1 Remark. (i) Rational Σ-trees in T_Σ can all be obtained as solutions of finite equation systems (4.3), i.e., such that $X = \{x_1, \dots, x_n\}$. We can, in fact, without loss of generality restrict to finite, *flat* equation systems, i.e., finite systems of equations

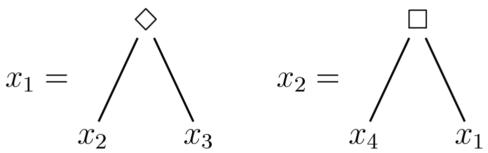
$$x_1 = t_1$$
$$\vdots$$
$$x_n = t_n$$

where each t_i is either a flat Σ-tree



$\sigma \in \Sigma_k,\ u_1, \dots, u_k \in X$ variables

or a single parameter from Y. For example, the rational tree $x_1^\dagger$ of Example 4.2 is obtained by solving the following flat system



$$x_3 = 0, \quad x_4 = 1.$$

Flat equation morphisms have the form

$$e\colon X \to H_\Sigma X + Y$$

and they are considered as (always guarded) equation morphisms by composition with the canonical embedding

$$H_\Sigma X + Y \hookrightarrow T_\Sigma(X + Y).$$

On the other hand, e is simply a coalgebra for the functor $H_\Sigma(-) + Y$.

(ii) More generally, for every endofunctor H a coalgebra

$$e\colon X \to HX + Y$$

of $H(-)+Y$ is called a *flat equation morphism*. It is considered to be an (always guarded) equation morphism by composition with the canonical morphism $HX + Y \to T(X + Y)$ whose components are (see Example 4.8)

$$HX \xrightarrow{H\eta_X} HTX \xrightarrow{\tau_X} TX \xrightarrow{T\mathsf{inl}} T(X+Y) \quad \text{and} \quad Y \xrightarrow{\eta_Y} TY \xrightarrow{T\mathsf{inr}} T(X+Y).$$

The solution of the corresponding equation morphism is denoted by $e^\dagger\colon X \to TY$ (by a slight abuse of notion).

5.2 Lemma. (see [2]) *For flat equation morphism e solution is corecursion. That is, $e^\dagger$ is the unique homomorphism from the coalgebra e of $H(-)+Y$ into the terminal coalgebra TY.*

5.3 Definition. (see [8]) *Given a finitary endofunctor H of* **Set**, *we define a subfunctor R of the above free completely iterative monad T on H (see Example* 4.8(2)) *by*

$$RY = \bigcup e^\dagger[X]$$

where the union ranges over all flat equation morphisms $e\colon X \to HX + Y$ with X finite.

5.4 Remark. R is a monad and it has a universal property analogous to T (see Notation 4.11 and Theorem 4.12): here we form the category $\mathsf{IM}(\mathbf{Set})$ of iterative monads (see Definition 4.9) and the forgetful functor $U^*\colon \mathsf{IM}(\mathbf{Set}) \to \mathbf{Set}^{\mathbf{Set}}$ given by $\mathscr{S} \mapsto S'$. The universal morphism $\lambda\colon H \to HR = U^*\mathscr{R}$ is here the codomain restriction of $H\eta\colon H \to HT$:

5.5 Theorem. (see [9]) *Every finitary endofunctor H of* **Set** *generates a free iterative monad, viz, the submonad $\mathscr{R}_H$ of $\mathscr{T}_H$ carried by the above subfunctor R. That is, given an iterative monad $\mathscr{S}$ and a natural transformation $f\colon H \to U^*\mathscr{S}$, there exists a unique ideal monad morphism $\overline{f}\colon \mathscr{R}_H \to \mathscr{S}$ with $f = U^*\overline{f}\cdot\lambda$.*

5.6 Notation. Let H be finitary endofunctor of **Set** represented as a quotient

$$\varepsilon\colon H_\Sigma \to H.$$

For every set X we denote by $\approx^*_X$ the congruence on the rational-tree algebra $R_\Sigma X$ which is the restriction of the congruence $\sim^*_X$ of Notation 4.14. That is, two rational Σ-trees s and t on X are congruent iff t can be obtained from s by (potentially) infinite applications of the basic equations. More precisely, iff $\partial_k s \sim_X \partial_k t$ for all $k < \omega$.

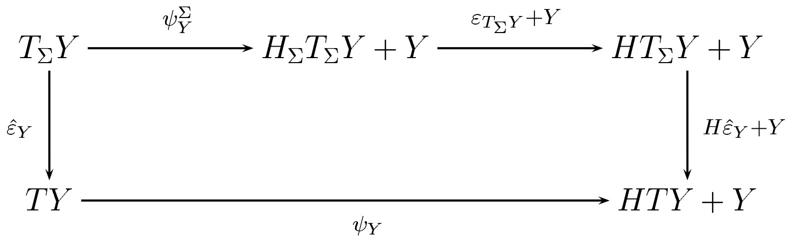
5.7 Theorem (Description of free iterative monads). *For every finitary endofunctor H on* **Set** *a free iterative monad $\mathscr{R}_H$ on H can be described as the quotient of the rational-tree monad $\mathscr{R}_\Sigma$ modulo the monad congruence $\approx^*_X$ (X a set) of applying the basic equations finitely or infinitely many times.*

Remark. We thus exhibit, for every presentation of H as a quotient $\varepsilon\colon H_\Sigma \to H$, a monad homomorphism $h\colon \mathscr{R}_\Sigma \to \mathscr{R}$ whose components h_X are epimorphisms with the kernel equivalence $\approx^*_X$.

PROOF. (1) Recall that by

$$T_\Sigma Y \xrightarrow{\psi^\Sigma_Y} H_\Sigma T_\Sigma Y + Y \qquad \text{and} \qquad TY \xrightarrow{\psi_Y} HTY + Y$$

we denote the terminal coalgebras of $H_\Sigma(-)+Y$ and $H(-)+Y$, respectively. By applying Lemma 3.10 to $H(-)+Y$, we obtain a quotient map $\hat{\varepsilon}_Y : T_\Sigma Y \to TY$ such that the square



commutes.

Observe that $\hat{\varepsilon}\colon T_\Sigma \to T$ is a natural transformation (in fact, a monad morphism: we denoted it by h in Theorem 4.16).

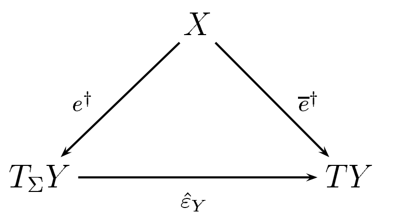
(2) Given a coalgebra of $H_\Sigma(-)+Y$, say

$$e\colon X \to H_\Sigma X + Y \qquad (X \text{ finite}),$$

we obtain a coalgebra of $H(-)+Y$:

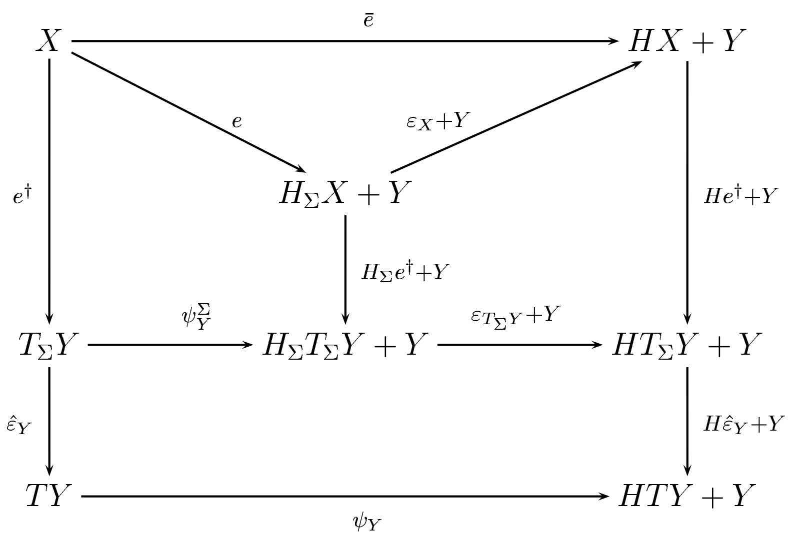
$$\overline{e} \equiv X \xrightarrow{e} H_\Sigma X + Y \xrightarrow{\varepsilon_X + Y} HX + Y$$

such that the following triangle



(5.1)

commutes. In fact, by Lemma 5.2, $e^\dagger$ is a coalgebra homomorphism w.r.t $H_\Sigma(-)+Y$ into a terminal coalgebra $T_\Sigma Y$. This clearly implies that $\hat{\varepsilon}_Y e^\dagger$ is a coalgebra homomorphism w.r.t $H(-)+Y$:



Since TY is terminal, we conclude $\overline{e}^\dagger = \hat{\varepsilon}_Y e^\dagger$.

This implies that $\hat{\varepsilon}_Y$ has a domain-codomain restriction $h_Y : R_\Sigma Y \to RY$. In fact, every element r of $R_\Sigma Y$ is a solution of some flat system $e\colon X \to H_\Sigma X + Y$ with X finite, more precisely, $r = e^\dagger(x)$ for some $x \in X$, see Remark 5.1(i). Then, by Definition 5.3 we have

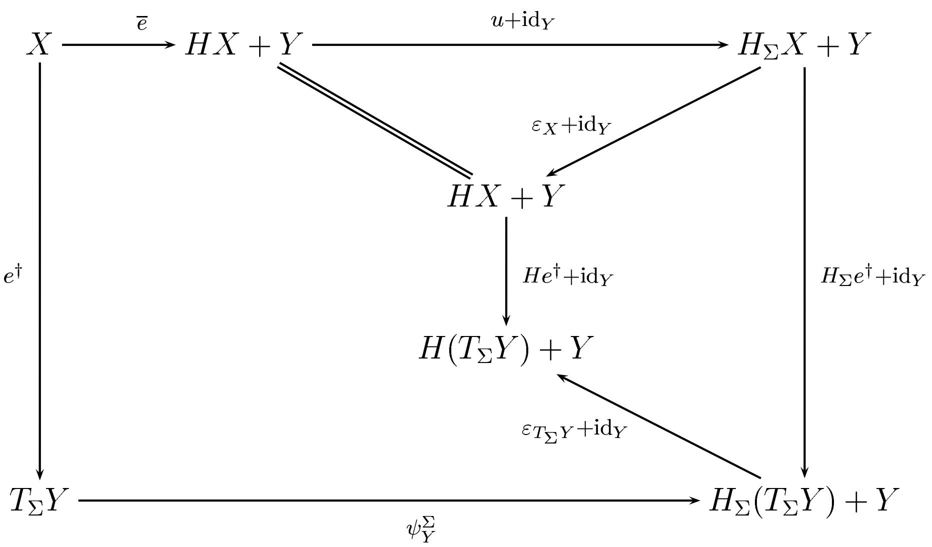
$$\hat{\varepsilon}_Y(r) = \hat{\varepsilon}_Y e^\dagger(x) = \overline{e}^\dagger(x) \in RY.$$

Since $\hat{\varepsilon}\colon T_\Sigma \to T$ is a monad morphism, it follows that the maps h_Y form a natural transformation $h\colon R_\Sigma \to R$, in fact, a monad morphism, $h\colon \mathscr{R}_\Sigma \to \mathscr{R}$.

It remains to prove that h_Y is surjective. To this end, for every flat equation $\overline{e}\colon X \to HX + Y$ choose a splitting of ε_X:

$$u\colon HX \to H_\Sigma X, \qquad \varepsilon_X \cdot u = \mathrm{id}_{HX}$$

and consider the flat equation $e = (u + \mathrm{id}_Y)\cdot\overline{e}\colon X \to H_\Sigma X + Y$ w.r.t. H_Σ. Then the above triangle (5.1) commutes. To verify this, we only need to prove that $e^\dagger$ is a coalgebra homomorphism w.r.t. $H(-)+Y$ from $\overline{e}$ to the coalgebra $T_\Sigma Y \cong H_\Sigma T_\Sigma Y + Y \xrightarrow{\varepsilon_{T_\Sigma Y} + \mathrm{id}_Y} H(T_\Sigma Y) + Y$. In fact, from Lemma 5.2 we know that $e^\dagger$ is a coalgebra homomorphism from e to $T_\Sigma Y$. That is, in the following diagram



the outward square commutes. Since the right-hand part commutes by naturality of ε, it follows that $e^\dagger$ is a homomorphism from $\overline{e}$ w.r.t $H(-)+Y$, as requested. This shows that h_Y is surjective: every element $\overline{r} \in RY$ has the form $\overline{r} = \overline{e}^\dagger(x)$ for some flat equation $\overline{e}\colon X \to HX + Y$ with X finite and some $x \in X$, see Remark 5.1(i), and then we have

$$\overline{r} = h_Y\big(e^\dagger(x)\big) \qquad \text{with} \qquad e^\dagger(x) \in R_\Sigma Y. \qquad \square$$

5.8 Examples. (i) One commutative binary operation, i.e.,

$$HY = \text{all unordered pairs in } Y.$$

The free iterative theory $\mathscr{R}_H$ assigns to every set X the algebra RX of all rational, binary unordered trees. This follows from Example 4.17.

(ii) The free iterative theory on $\mathscr{P}_f$ (the finite-power-set functor) is the theory of all non-ordered strongly extensional, rational, finitely branching trees. See Example 4.18.

(iii) The functor $(-)^3_2$, see Example 4.19, generates the free iterative theory $\mathscr{R}_H$ assigning to every set X the algebra $R_\Sigma X/_{\approx^*}$ where $R_\Sigma X$ are all binary, rational (ordered) trees with inner nodes labeled by $\{\sigma, \tau, \varrho\}$ and leaves labeled in X. And $\approx^*$ is the congruence allowing arbitrary changes of labels of nodes where two children define isomorphic subtrees.

6 Conclusions and Generalizations

The main result of the present paper is a description of a terminal coalgebra for every finitary endofunctor H of the category of sets: present H as a quotient functor of a polynomial functor (of some finitary signature Σ) modulo basic equations and then describe a terminal H-coalgebra T as a quotient

$$T = T_\Sigma/_{\sim^*}$$

of the terminal Σ-coalgebra T_Σ of all Σ-trees modulo the congruence $\sim^*$ of finite and infinite application of those basic equations. This is completely analogous to the well-known fact that an initial H-algebra I, i.e., an initial algebra of the variety of Σ-algebras presented by our basic equations, is a quotient $I = I_\Sigma/_\sim$ of the initial Σ-algebra I_Σ of all finite Σ-trees modulo the congruence $\sim$ of (finite) application of the basic equations. As a consequence of our description of terminal coalgebras we were able to describe all free iterative monads in **Set** in the sense of C. Elgot.

The reader may wonder why we restricted ourselves to finitary functors: in Example 3.17 we show, however, that the corresponding result does not hold for the countable-power-set functor. Next the reader may wonder why we restricted ourselves to the category of sets. In fact, the two main ingredients of our description of terminal coalgebras of finitary endofunctors seem to be that

(a) every finitary endofunctor is a quotient of a polynomial functor, and

(b) the initial-algebra construction converges after ω steps and the terminal-coalgebra construction converges after $\omega + \omega$ steps and the steps between ω and $\omega + \omega$ are monomorphisms.

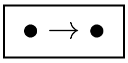
Both of these facts are true in every strongly locally finitely presentable category, whenever the finitary endofunctor preserves strong monomorphisms and epimorphisms as proved in [7]. For example, the category **Gra** of graphs (= sets with a binary relation) is strongly finitely presentable. The concept of a signature Σ in this category, following G. M. Kelly and A. J. Power [22], assigns to every finite graph n (up to isomorphism) a graph Σ_n. The corresponding polynomial functor H_Σ is defined on objects X by

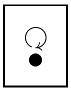
$$H_\Sigma X = \coprod_n \hom(n, X) \bullet \Sigma_n$$

where $M \bullet -$ denotes a coproduct indexed by the set M. There is an important third ingredient, besides (a) and (b) above, which plays a rôle in our description of terminal coalgebra above, namely:

(c) for every presentation $\varepsilon \colon H_\Sigma \to H$ the canonical homomorphism $\hat{\varepsilon} \colon T_\Sigma \to T$ between the terminal coalgebras of H_Σ and H, respectively, is a quotient.

Unfortunately, this feature seems to request that all quotients are split epimorphisms (see the proof of Lemma 3.10 above). In fact, (c) fails in **Gra** consider the following signature Σ:

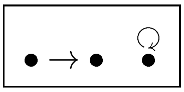
$\Sigma_0 =$  (0 = initial, empty, graph)

$\Sigma_1 = 1 =$  (1 = terminal graph)

with $\Sigma_n = \emptyset$ for all $n \neq 0, 1$. The corresponding polynomial functor is

$$H_\Sigma X = \Sigma_0 + \coprod_{\text{loops of } X} 1$$

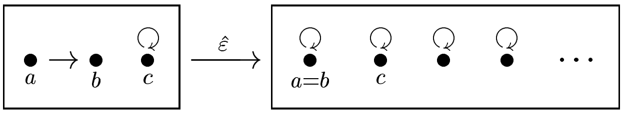
and its terminal coalgebra is easily computed: the terminal-coalgebra construction converges after 1 step to the following graph

$T_\Sigma \equiv$ 

(isomorphic to $H_\Sigma T_\Sigma$). Now let

$$\varepsilon \colon H_\Sigma \to H, \qquad HX = 1 + \coprod_{\text{loops of } X} 1$$

be the regular quotient obtained by merging Σ_0 to a single-node graph. A terminal coalgebra T of H is obtained from the terminal coalgebra of the set functor $X \mapsto X + 1$ by putting loops on all elements: T is thus a countable set of loops. Therefore $\hat{\varepsilon}$ has the following form



Consequently, the method used in the present paper in **Set** does not seem to have any analogy in **Gra**.

References

[1] P. Aczel, *Non-well-founded Set*, CSLI Lecture Notes 14, Center for the Study of Languages and Information, Stanford 1988.

[2] P. Aczel, J. Adámek, S. Milius and J. Velebil, Infinite trees and completely iterative theories, *Theoret. Comput. Sci.* **300** (2003), 1–45.

[3] P. Aczel and N. Mendler, A final coalgebra theorem, *Lecture Notes in Comput. Sci.* **389**, Springer Verlag 1989, 357–365.

[4] J. Adámek, Free algebras and automata realizations in the language of categories, *Comment. Math. Univ. Carolinæ* **15** (1974), 589–602.

[5] J. Adámek, On a description of terminal coalgebras and iterative theories, *Electron. Notes Theor. Comput. Sci.* **82** (2003).

[6] J. Adámek, Final coalgebras are ideal completions of initial algebras, *J. Logic Comput.* **12** (2002), 217–242.

[7] J. Adámek, On final coalgebras of continuous functors, *Theoret. Comput. Sci.* **294** (2003), 3–29.

[8] J. Adámek, S. Milius and J. Velebil, On rational monads and free iterative theories, *Electron. Notes Theor. Comput. Sci.* **69** (2002), 24 pp.

[9] J. Adámek, S. Milius and J. Velebil, Free iterative theories: A coalgebraic view, *Math. Structures Comput. Sci.* **13** (2003), 259–320.

[10] J. Adámek, S. Milius and J. Velebil, From iterative algebras to iterative theories, submitted, available via `http://www.iti.cs.tu-bs.de/~milius` extended abstract appeared in *Electron. Notes Theor. Comput. Sci.* **106** (2004), 3–24.

[11] J. Adámek and H.-E. Porst, On tree coalgebras and coalgebra presentations, *Theoret. Comput. Sci.* **311** (2004), 257–283.

[12] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge Univ. Press 1994.

[13] J. Adámek and V. Trnková, *Automata and Algebras in Categories*, Academic Publishers, Dordrecht (1990).

[14] M. Barr, Terminal coalgebras in well-founded set theory, *Theoret. Comput. Sci.* **124** (1994), 182–192.

[15] S. L. Bloom and C. C. Elgot, The existence and construction of free iterative theories, *J. Comput. System Sci.* **12** (1974), 305–318.

[16] S. L. Bloom and Z. Ésik, *Iteration Theories: The Equational Logic of Iterative Processes*, EATCS Monograph Series on Theoretical Computer Science, Springer-Verlag (1993).

[17] C. C. Elgot, Monadic computation and iterative algebraic theories, in: *Logic Colloquium '73* (eds: H. E. Rose and J. C. Shepherdson), North-Holland Publishers, Amsterdam (1975), 175–230.

[18] C. C. Elgot, S. L. Bloom and R. Tindell, On the algebraic structure of rooted trees, *J. Comput. System. Sci.* **16** (1978), 361–399.

[19] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math. 221, Springer-Verlag (1971).

[20] S. Ginali, Regular trees and the free iterative theory, *J. Comput. System Sci.* **18** (1979), 228–242.

[21] H.-P. Gumm and T. Schröder, Coalgebras of bounded type, *Math. Structures Comput. Sci.* **260** (2001), 71–86.

[22] G. M. Kelly and A. J. Power, Adjunction whose counits are coequalizers, and presentations of finitary enriched monads, *J. Pure Appl. Algebra* **89** (1993), 163–179.

[23] J. Lambek, A fixpoint theorem for complete categories, *Math. Z.* **103** (1968), 151–161.

[24] E. G. Manes, *Algebraic Theories*, Springer-Verlag, Berlin (1976).

[25] S. Milius, Completely iterative algebras and completely iterative monads, *Inform. and Comput.* **196** (2005), 1–41.

[26] L. S. Moss, Parametric corecursion, *Theoret. Comput. Sci.* **260** (2001), 139–163.

[27] J. Worrell, On the final sequence of a finitary set functor, *Theoret. Comput. Sci.* **338** (2005), 184–199.