On Functors Preserving Coproducts and Algebras with Iterativity

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Abstract

An algebra for a functor H is called completely iterative (cia, for short) if every flat recursive equation in it has a unique solution. Every cia is corecursive, i.e. it admits a unique coalgebra-to-algebra morphism from every coalgebra. If the converse also holds, H is called a cia functor. We prove that whenever the base category is hyper-extensive (i.e. countable coproducts are 'well-behaved') and H preserves countable coproducts, then H is a cia functor. Surprisingly few cia functors exists among standard finitary set functors: in fact, the only ones are those preserving coproducts; they are given by $X \mapsto W \times (-) + Y$ for some sets W and Y.

Keywords: terminal coalgebra, free algebra, corecursive algebra, hyper-extensive category

1. Introduction

Iteration and (co)recursion are of central importance in computer science. Iterative algebraic theories were proposed by Elgot [12] as a formalism for iteration. Later Nelson [16] and Tiuryn [17] introduced iterative algebras for finitary signatures which yield an easier approach to iterative theories. For endofunctors H there are two related notions of algebras: *Corecursive algebras* introduced by Capretta et al. [10] are those H-algebras A such that every recursive equation expressed as a coalgebra for H has a unique solution

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(i.e., a coalgebra-to-algebra morphism into A). The other notion, *completely iterative algebras* (or *cia*, for short), introduced by the second author [15], are H-algebras A with the stronger property that every recursive equation with parameters in A has a unique solution (Definition 2.7). Corecursive algebras often fail to be cias. In the present paper we study endofunctors such that every corecursive algebra is a cia – we call them *cia functors*.

It was already observed by Arbib and Manes [9] that every endofunctor preserving countable coproducts has the free algebra on an object Y (equivalently, the initial algebra for $H(-) + Y$) carried by

$$FY = \mu(H(-) + Y) = \coprod_{n<\omega} H^n Y,$$

(similarly to the formula $\Sigma^* = \coprod_{n<\omega} \Sigma^n$ for the free monoid on the set Σ).

Our first result is that every endofunctor preserving countable coproducts and having a terminal coalgebra is a cia functor (Corollary 4.4). This is based on a description of the free cia on an object Y (equivalently, the terminal coalgebra for $H(-) + Y$) as a coproduct

$$\nu(H(-) + Y) = \nu H + FY = \nu H + \coprod_{n<\omega} H^n Y$$

of the terminal coalgebra and the free algebra on Y (Theorem 3.5). We deduce that, for H preserving countable coproducts and having a terminal coalgebra, we obtain cia functors $H(-) + Y$ for all objects Y (Corollary 4.6). All this holds in every *hyper-extensive* base category (Definition 2.1), e.g., in sets, posets, graphs and all presheaf categories.

In particular, if the base category is also cartesian closed, then $X \mapsto W \times X + Y$ is a cia functor for every pair of objects W and Y. For standard finitary set functors we prove a surprising converse: the only cia functors are those of the above form $X \mapsto W \times X + Y$. Standard means that the given functor preserves inclusions and finite intersections. For non-standard finitary functors this "almost" holds in the sense that the above formula holds for all non-empty sets X.

Finally, we investigate Eilenberg-Moore algebras for the monad T of free cias for H. In general, these are characterized as the complete Elgot algebras for H [6]. In the setting of the present paper the monad T is also the monad of free corecursive algebras. The Eilenberg-Moore algebras for the latter monad were characterized as Bloom algebras for H provided that H is an accessible functor on a locally presentable category [3, Theorem 4.15]. We prove that under our assumptions on H complete Elgot algebras and Bloom algebras for H are the same (Theorem 6.5).

The present paper is a revised version of the conference paper [5]. Here we present full proofs of all our results, and besides, Section 6 is new.

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2. Preliminaries

Throughout the paper H denotes an endofunctor on a hyper-extensive category (recalled below) having a terminal coalgebra

$$t : \nu H \to H(\nu H).$$

By the famous Lambek Lemma [13], the coalgebra structure t is invertible and its inverse makes νH an H-algebra.

We denote by $\mathsf{Alg}\, H$ the category of H-algebras and their morphisms.

Definition 2.1 (Adámek, Bloom, Milius, Velebil [2])**.** A category is called *hyper-extensive* if it has countable coproducts which are

(1) *universal*, i.e., preserved by pullbacks along any morphism,

(2) *disjoint*, i.e., coproduct injections are monomorphic and have pairwise intersection 0 (the initial object), and

(3) *coherent*, i.e., given pairwise disjoint morphisms $a_n : A_n \to A$, $n \in \mathbb{N}$, each of which is a coproduct injection, then their copairing $[a_n]_{n\in\mathbb{N}} : \coprod_{n\in\mathbb{N}} A_n \to A$ is also a coproduct injection.

Example 2.2. The categories of sets, posets, graphs, unary algebras, and presheaf categories are hyper-extensive.

Remark 2.3. (1) We write $A + B$ for the coproduct of the objects A and B and denote coproduct injections by $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$. Furthermore we write $\mathsf{can} = [H\mathsf{inl}, H\mathsf{inr}] : HA + HB \to H(A + B)$ for the canonical morphism for any objects A and B.

(2) Recall that a category with finite coproducts is *extensive* if it has pullbacks along coproduct injections and conditions (1) and (2) are satisfied [11]. Equivalently, in a diagram of the following form

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & Z \arrow[d, "h"'] & Y \arrow[l, "y"'] \arrow[d, "g"] \\ A \arrow[r, "\mathsf{inl}"'] & A+B & B \arrow[l, "\mathsf{inr}"] \end{tikzcd}$$

the top row is a coproduct if and only if the squares are pullbacks. Another, more compact, equivalent characterization of extensivity states that the canonical functor $\mathcal{C}/A \times \mathcal{C}/B \to \mathcal{C}/(A+B)$ is an equivalence of categories for any pair of objects A and B.

(3) The somewhat technical condition (3) in Definition 2.1 is not a consequence of the other two. In fact, let $\mathcal{C}$ be the category of Jónsson-Tarski algebras, i.e., binary algebras A whose operation $A \times A \to A$ is a bijection. Then $\mathcal{C}$ has disjoint and universal countable (in fact, all) coproducts but is not hyperextensive [2].

Definition 2.4 (Capretta, Uustalu, Vene [10])**.** An algebra $a : HA \to A$ is called *corecursive* if for every coalgebra $e : X \to HX$ there exists a unique algebra-to-coalgebra morphism $e^\dagger : X \to A$:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX \arrow[r, "He^\dagger"'] & HA \arrow[u, "a"'] \end{tikzcd} \tag{2.1}$$

Examples 2.5. (1) The terminal coalgebra νH (considered as an algebra) is obviously corecursive. This is the initial corecursive algebra [10].

Furthermore, let Y be an object of $\mathcal{C}$ and assume that the functor $H(-) + Y$ has a terminal coalgebra TY. Then its structure

$$TY \xrightarrow{\alpha_Y} HTY + Y$$

has an inverse which is the copairing of two morphisms denoted by

$$HTY \xrightarrow{\tau_Y} TY \xleftarrow{\eta_Y} Y.$$

It follows that TY is a coproduct of HTY and Y with the above coproduct injections. It is easy to show that (TY, τ_Y) is a corecursive algebra.

(2) The trivial terminal algebra $H1 \to 1$ is corecursive, and if (A, a) is a corecursive algebra so is (HA, Ha) [10, Prop. 21]. Furthermore, if $\mathcal{C}$ has limits, then corecursive algebras are closed under limits in the category of algebras for H [3, Prop. 2.4]. It follows that all members of the *terminal-coalgebra* chain

$$1 \longleftarrow H1 \longleftarrow HH1 \longleftarrow \cdots$$

are corecursive algebras.

(3) A particular instance of point (1) is given by a signature $\Sigma = (\Sigma_n)_{n<\omega}$ of operation symbols with prescribed arity and considering the corresponding polynomial endofunctor H_Σ on Set defined by

$$H_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n.$$

For an operation symbol $\sigma \in \Sigma_n$ we write

$$\sigma(x_1, \dots, x_n)$$

in lieu of $(\sigma, (x_1, \dots, x_n))$ for elements in the summand of $H_\Sigma X$ corresponding to $n < \omega$. The terminal coalgebra νH_Σ is carried by the set of all Σ-*trees*, i.e., rooted and ordered trees with nodes labeled in Σ such that every node with n children is labeled by an n-ary operation symbol. The coalgebra structure t on νH_Σ is given by *tree-tupling*: t^{-1} assigns to $\sigma(s_1, \dots, s_n)$ with $\sigma \in \Sigma_n$ and $s_i \in \nu H_\Sigma$, $i = 1 \dots, n$, the Σ-tree obtained by joining the Σ-trees $s_1, \dots, s_n$ by a root node labeled by σ.

For every set Y we denote by

$$T_\Sigma Y$$

the algebra of all Σ-*trees over* Y, i.e., Σ-trees whose leaves are labeled by constant symbols in Σ_0 or elements of Y. This is the terminal coalgebra for $H_\Sigma(-) + Y$, and therefore it is a corecursive algebra.

Remark 2.6. For a polynomial endofunctor H_Σ on Set we can view a coalgebra $e : X \to H_\Sigma X$ as a system of *recursive equations* over the set X of (recursion) variables. For every variable $x \in X$, $e(x) = \sigma(x_1, \dots, x_n)$ is understood as a formal equation

$$x \approx \sigma(x_1, \dots, x_n).$$

The map $e^\dagger$ in Definition 2.4 is then a *solution* of the system of equations in the Σ-algebra A: the commutative square (2.1) states that $e^\dagger$ turns the above formal equations into actual identities in A:

$$e^\dagger(x) = \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_n)).$$

Definition 2.7 (Milius [15])**.** An algebra $a : HA \to A$ is called *completely iterative* (or *cia*, for short) if the algebra $[a, A] : HA + A \to A$ is corecursive for the endofunctor $H(-) + A$. That means that for every *(flat) equation morphism* $e : X \to HX + A$ there exists a unique *solution*, i.e., a unique morphism $e^\dagger$ such that square below commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[a,A]}"'] \end{tikzcd} \tag{2.2}$$

Examples 2.8. (1) If $H(-) + Y$ has a terminal coalgebra TY (cf. Example 2.5(1)), then (TY, τ_Y) is a cia. In fact, (TY, τ_Y) is a free cia on Y with the universal morphism η_Y [15].

(2) For a polynomial functor H_Σ on Set the above example states that the algebra $T_\Sigma Y$ of all Σ-trees over Y is the free cia on the set Y. Let us denote by

$$C_\Sigma Y$$

the subalgebra of $T_\Sigma Y$ given by all Σ-trees over Y which have only a finite number of leaves labeled in Y (and the remaining, possibly infinitely many, leaves are labeled in Σ_0). This algebra is corecursive but, whenever Σ contains an operation symbol of arity at least 2, not a cia. Moreover, $C_\Sigma Y$ is the free corecursive algebra on Y [3].

As a concrete example, consider the signature Σ consisting of a single binary operation σ. Then the equation morphism $e : \{x_1, x_2\} \to H_\Sigma\{x_1, x_2\} + \{y\}$ given by the recursive equations

$$x_1 \approx \sigma(x_1, x_2) \qquad \text{and} \qquad x_2 \approx y$$

has the unique solution $e^\dagger : \{x_1, x_2\} \to T_\Sigma\{y\}$ given as follows

$$e^\dagger : x_1 \mapsto \begin{array}{ccccccc} &&&\sigma&& \\ &&\sigma&&y& \\ &\sigma&&y&& \\ \vdots&&y&&& \end{array} \qquad x_2 \mapsto y.$$

This demonstrates that $C_\Sigma\{y\}$ is not a cia because the above infinite Σ-tree is not contained in it.

Definition 2.9. A *cia functor* is an endofunctor such that every corecursive algebra for it is a cia. (It then follows that cias and corecursive algebras coincide).

Notation 2.10. (1) If a free H-algebra on Y exists, we denote it by FY and its structure and universal morphism by

$$\varphi_Y : HFY \to FY \qquad \text{and} \qquad \eta^F_Y : Y \to FY,$$

respectively.

In the case of a polynomial set functor H_Σ, the free Σ-algebra $F_\Sigma Y$ is the subalgebra of $T_\Sigma Y$ on all finite Σ-trees over Y.

(2) If a free corecursive H-algebra on Y exists, we denote it by CY and its structure and universal morphism by

$$\psi_Y : HCY \to CY \qquad \text{and} \qquad \eta^C_Y : Y \to CY,$$

respectively.

3. Functors Preserving Countable Coproducts

Assumption 3.1. In this and the subsequent section we assume that H is an endofunctor on a hyper-extensive category having a terminal coalgebra and preserving countable coproducts.

Fact 3.2 (Arbib and Manes [9])**.** *A free algebra on Y is*

$$FY = H^*Y = \coprod_{n<\omega} H^nY \qquad \textit{with coproduct injections } j_n : H^nY \to H^*Y.$$

Its algebra structure and universal morphism are given by

$$\varphi_Y \cdot Hj_n = j_{n+1} \quad (n > 0) \qquad \textit{and} \qquad \eta^F_Y = j_0 : Y \to H^*Y,$$

respectively, using that $HFY \cong \coprod_{n<\omega} H^{n+1}Y$.

Notation 3.3. We denote by

$$\sigma_Y : H^*Y = \coprod_{n<\omega} H^nY \to Y + H\left(\coprod_{n<\omega} H^nY\right) = Y + HH^*Y$$

the isomorphism inverse to $[\eta^F_Y, \varphi_Y] : Y + HH^*Y \to H^*Y$. It is defined by the following commutative diagrams:

$$\begin{tikzcd}
Y \arrow[d, "j_0"'] \arrow[dr, "\mathsf{inl}"] & \\
H^*Y \arrow[r, "\sigma_Y"'] & Y + HH^*Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
H^nY \arrow[r, "Hj_{n-1}"] \arrow[d, "j_n"'] & HH^*Y \arrow[d, "\mathsf{inr}"] \\
H^*Y \arrow[r, "\sigma_Y"'] & Y + HH^*Y
\end{tikzcd}
\quad \text{for } n > 0. \tag{3.1}$$

Lemma 3.4. *In a hyper-extensive category, given a coproduct $A = \coprod_{n<\omega} A_n$ with injections $a_n : A_n \to A$, the subobjects*

$$\bar{a}_k = [a_0, [a_n]_{n\geq k}] : A_0 + \coprod_{n\geq k} A_n \to A \qquad (k \geq 1)$$

have the intersection $a_0 : A_0 \to A$.

Proof. It is our task to prove that every morphism $f : B \to A$ factorizing through the morphisms $\bar{a}_k$, for every $k \geq 1$, factorizes through a_0. Due to hyper-extensivity, f has the form $f = \coprod_{n<\omega} f_n$ for morphisms $f_n : B_n \to A_n$ with $B = \coprod_{n<\omega} B_n$. We now prove that since f factorizes through $\bar{a}_k$ it follows that $B_n \cong 0$ for all $1 \leq n < k$. Indeed, for any $k > 2$, let

$$\overline{A}_k = A_0 + \coprod_{n\geq k} A_n, \qquad \overline{B}_k = B_0 + \coprod_{n\geq k} B_n \qquad \text{and} \qquad \overline{f}_k = f_0 + \coprod_{n\geq k} f_n,$$

and consider for $1 \leq n < k$ the pullback squares below:

$$\begin{tikzcd}
\overline{B}_k \arrow[r, "\bar{b}_k"] \arrow[d, "\overline{f}_k"'] & B \arrow[l, dashed, bend left, "h"] \arrow[dl, "f'"'] \arrow[d, "f"] & B_n \arrow[l, "b_n"'] \arrow[d, "f_n"] \\
\overline{A}_k \arrow[r, "\bar{a}_k"'] & A & A_n \arrow[l, "a_n"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & B_n \arrow[d, "b_n"] \\
\overline{B}_k \arrow[r, "\bar{b}_k"'] & B
\end{tikzcd}$$

Since f factorizes through $\bar{a}_k$ we have the diagonal morphism f' on the left such that the triangle below it commutes. Using the universal property of the left-hand pullback, we then obtain a unique $h : B \to \overline{B}_k$ such that $\overline{f}_k \cdot h = f'$ and $\overline{b}_k \cdot h = \mathsf{id}_B$. This shows that the coproduct injection $\overline{b}_k$ is a split epimorphism, and since it is also a monomorphism by extensivity, we see that $\overline{b}_k$ is an isomorphism. Now consider the pullback on the right above, which expresses that the coproduct injections b_n and $\overline{b}_k$ are disjoint. Since the morphism at the bottom is an isomorphism, so is the morphism at the top, whence $B_n \cong 0$ for all $1 \le n < k$.

Since this holds for every $k \ge 1$, we have shown that $B_n \cong 0$ for all $n \ge 1$. Thus, we obtain $B \cong B_0$ as desired. □

Theorem 3.5. *The terminal coalgebra for $Y + H(-)$ is $H^*Y + \nu H$ with the following coalgebra structure*

$$H^*Y + \nu H \xrightarrow{\sigma_Y + t} Y + HH^*Y + H(\nu H) \cong Y + H(H^*Y + \nu H).$$

Proof. We will prove that for a given coalgebra $e : X \to Y + HX$ there exists precisely one morphism $h : X \to H^*Y + \nu H$ such that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "e"'] & H^*Y + \nu H \arrow[d, "\sigma_Y + t"] \\ Y + HX \arrow[r, "Y + Hh"'] & Y + H(H^*Y + \nu H) \end{tikzcd} \tag{3.2}$$

(a) Uniqueness. We define pairwise disjoint subobjects $\overline{X}_n$, $n \in \mathbb{N} \setminus \{0\} \cup \{\infty\}$ of X and prove that h is uniquely determined by the given equation morphism e on each of them. That will conclude the proof of uniqueness since we will see that X is the coproduct of all of those subobjects. To start, we put

$$X_0 = X \qquad \text{and} \qquad e_0 = e,$$

and denote the coproduct injections of $Y + HX$ by

$$HX \xrightarrow{i_0} Y + HX \qquad \text{and} \qquad Y \xrightarrow{\bar{i}_0} Y + HX.$$

Next form the pullbacks of e along these injections:

$$\begin{tikzcd} X_1 \arrow[r, "i_1"] \arrow[d, "e_1"'] & X_0 \arrow[d, "e_0"] & \overline{X}_1 \arrow[l, "\bar{i}_1"'] \arrow[d, "\bar{e}_1"] \\ HX \arrow[r, "i_0"'] & Y + HX & Y \arrow[l, "\bar{i}_0"] \end{tikzcd} \tag{3.3}$$

By extensivity, $X = X_1 + \overline{X}_1$ with injections i_1 and $\bar{i}_1$. The component $\overline{h}_1 := h \cdot \bar{i}_1$ of h at $\overline{X}_1$ is determined by e as follows

$$h \cdot \bar{i}_1 = \left(\overline{X}_1 \xrightarrow{\bar{e}_1} Y \xrightarrow{j_0} H^*Y \xrightarrow{\mathsf{inl}} H^*Y + \nu H \right).$$

This follows from the commutative diagram below (note that from (3.1) we see that the right-hand and lower arrows compose to $Y \xrightarrow{j_0} H^*Y \xrightarrow{\mathsf{inl}} H^*Y + \nu H$):

$$\begin{tikzcd}
\overline{X}_1 \arrow[r, "\bar{i}_1"] \arrow[d, "\bar{e}_1"'] & X \arrow[r, "h"] \arrow[d, "e"] & H^*Y + \nu H \\
Y \arrow[r, "\bar{i}_0"'] & Y + HX \arrow[r, "Y+Hh"'] & Y + H(H^*Y + \nu H) \arrow[u, "(\sigma_Y+t)^{-1}=\sigma_Y^{-1}+t^{-1}"']
\end{tikzcd} \tag{3.4}$$

In order to analyze the complementary coproduct component $h \cdot i_1$, we form the pullbacks of e_1 along the coproduct injections of $HX_0 = HX_1 + H\overline{X}_1$:

$$\begin{tikzcd}
X_2 \arrow[r, "i_2"] \arrow[d, "e_2"'] & X_1 \arrow[d, "e_1"] & \overline{X}_2 \arrow[l, "\bar{i}_2"'] \arrow[d, "\bar{e}_2"] \\
HX_1 \arrow[r, "Hi_1"'] & HX_0 & H\overline{X}_1 \arrow[l, "H\bar{i}_1"]
\end{tikzcd}$$

Then $X_1 = X_2 + \overline{X}_2$ and the component $\overline{h}_2 = h \cdot i_1 \cdot \bar{i}_2$ of h at $\overline{X}_2$ is determined by e as follows:

$$h \cdot i_1 \cdot \bar{i}_2 = \left(\overline{X}_2 \xrightarrow{\bar{e}_2} H\overline{X}_1 \xrightarrow{H\bar{e}_1} HY \xrightarrow{j_1} H^*Y \xrightarrow{\mathsf{inl}} H^*Y + \nu H \right).$$

This follows from the commutative diagram below (from (3.1) we see that the right-hand and lower arrows compose to $HY \xrightarrow{j_1} H^*Y \xrightarrow{\mathsf{inl}} H^*Y + \nu H$):

$$\begin{tikzcd}
\overline{X}_2 \arrow[r, "\bar{i}_2"] \arrow[d, "\bar{e}_2"] & X_1 \arrow[r, "i_1"] \arrow[d, "e_1"'] & X \arrow[r, "h"] \arrow[d, "e"'] & H^*Y+\nu H \\
H\overline{X}_1 \arrow[r, "H\bar{i}_1"] \arrow[dd, "H\bar{e}_1"] & HX \arrow[r, "i_0"] \arrow[d, "He"'] & Y+HX \arrow[r, "Y+Hh"] \arrow[d, "Y+He"'] & Y+H(H^*Y+\nu H) \arrow[u, "(\sigma_Y+t)^{-1}"'] \\
 & H(Y+HX) \arrow[r, "\mathsf{inr}"] \arrow[d, equal] & Y+H(Y+HX) \arrow[r, "Y+H(Y+Hh)"] \arrow[d, equal] & Y+H(Y+H(H^*Y+\nu H)) \arrow[u, "Y+H(\sigma_Y+t)^{-1}"'] \arrow[d, equal] \\
HY \arrow[r, "\mathsf{inl}"'] \arrow[ur, "H\bar{i}_0"] & HY+HHX \arrow[r, "\mathsf{inr}"'] & Y+HY+HHX \arrow[r, "Y+HY+HHh"'] & Y+HY+HH(H^*Y+\nu H)
\end{tikzcd} \tag{3.5}$$

We continue this process recursively: given a coproduct $X_n \xrightarrow{i_n} X_{n-1} \xleftarrow{\bar{i}_n} \overline{X}_n$ and a morphism $e_n : X_n \to HX_{n-1}$ we form its pullbacks along the coproduct injection of $HX_{n-1} = HX_n + H\overline{X}_n$:

$$
\begin{tikzcd}
X_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "e_{n+1}"'] & X_n \arrow[d, "e_n"'] & \overline{X}_{n+1} \arrow[l, "\bar{i}_{n+1}"'] \arrow[d, "\overline{e}_{n+1}"] \\
HX_n \arrow[r, "Hi_n"'] & HX_{n-1} & H\overline{X}_n \arrow[l, "H\bar{i}_n"]
\end{tikzcd}
\tag{3.6}
$$

Since compositions of coproduct injections are always coproduct injections, we obtain coproduct injections

$$
\bar{i}^*_{n+1} = \left(\overline{X}_{n+1} \xrightarrow{\bar{i}_{n+1}} X_n \xrightarrow{i_n} X_{n+1} \xrightarrow{i_{n-1}} \cdots \xrightarrow{i_1} X \right) \tag{3.7}
$$

for $n < \omega$ and morphisms

$$
\widehat{e}_{n+1} = \left(\overline{X}_{n+1} \xrightarrow{\overline{e}_{n+1}} H\overline{X}_n \xrightarrow{H\overline{e}_n} H^2\overline{X}_{n-1} \xrightarrow{H^2\overline{e}_{n-1}} \cdots \xrightarrow{H^n\overline{e}_1} H^nY \right). \tag{3.8}
$$

The component $\overline{h}_{n+1} := (\overline{X}_{n+1} \xrightarrow{\bar{i}^*_{n+1}} X \xrightarrow{h} H^*Y + \nu H)$ of h at $\overline{X}_{n+1}$ is determined by e via the commutativity of the following square

$$
\begin{tikzcd}
\overline{X}_{n+1} \arrow[rr, "\bar{i}^*_{n+1}"] \arrow[d, "\widehat{e}_{n+1}"'] & & X \arrow[d, "h"] \\
H^nY \arrow[r, "j_n"'] & H^*Y \arrow[r, "\mathsf{inl}"'] & H^*Y + \nu H
\end{tikzcd}
\tag{3.9}
$$

The proof is by an obvious inductive continuation of the diagrams (3.4) and (3.5). Observe also that by composing pullback squares we obtain the following pullback of e along $i_0 \cdot H\bar{i}^*_{n-1}$:

$$
\begin{tikzcd}
\overline{X}_n \arrow[r, "\bar{i}_n"] \arrow[d, "\overline{e}_n"'] \arrow[rrrrrr, bend left, "\bar{i}^*_n"] & X_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "e_{n-1}"'] & X_{n-2} \arrow[r, "i_{n-2}"] \arrow[d, "e_{n-2}"'] & \cdots \arrow[r, "i_3"] \arrow[d, phantom, "\cdots"] & X_2 \arrow[r, "i_2"] \arrow[d, "e_2"'] & X_1 \arrow[r, "i_1"] \arrow[d, "e_1"'] & X_0{=}X \arrow[d, "e"] \\
H\overline{X}_{n-1} \arrow[r, "H\bar{i}_{n-1}"'] \arrow[rrrrr, bend right, "H\bar{i}^*_{n-1}"'] & HX_{n-2} \arrow[r, "Hi_{n-2}"'] & HX_{n-3} \arrow[r, "Hi_{n-3}"'] & \cdots \arrow[r, "Hi_2"'] & HX_1 \arrow[r, "Hi_1"'] & HX_0 \arrow[r, "i_0"'] & Y{+}HX
\end{tikzcd}
\tag{3.10}
$$

Now the coproduct injections in (3.7) are clearly pairwise disjoint for $n \in \mathbb{N}$. Therefore, by hyper-extensivity, we have a coproduct injection $[\bar{i}^*_{n+1}]_{n<\omega}$ which we denote by

$$\overline{X}_\infty \xrightarrow{\bar{i}_\infty} X \qquad \text{for} \qquad \overline{X}_\infty := \coprod_{n<\omega} \overline{X}_{n+1}.$$

Then $h \cdot \bar{i}_\infty$ is, as proved by (3.9), determined by e. Let $i_\infty : X_\infty \to X$ be the complementary coproduct component, i.e., we have the coproduct

$$\overline{X}_\infty \xrightarrow{\bar{i}_\infty} X \xleftarrow{i_\infty} X_\infty.$$

Since the pullbacks (3.10) have pairwise disjoint coproduct injections as their upper arrows, they form together the pullback on the left below:

$$\begin{tikzcd}
\overline{X}_\infty = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots \arrow[r, "\bar{i}_\infty"] \arrow[d, "\coprod \bar{e}_n"'] & X \arrow[d, "e"] & X_\infty \arrow[l, "i_\infty"'] \arrow[dd, "e_\infty"] \\
Y + H\overline{X}_1 + H\overline{X}_2 + \cdots \arrow[r, "Y + H\bar{i}_\infty"] \arrow[d, equal] & Y + HX \arrow[d, equal] & \\
Y + H\overline{X}_\infty \arrow[r, "\mathsf{inl}"'] & Y + H\overline{X}_\infty + HX_\infty & HX_\infty \arrow[l, "\mathsf{inr}"] \arrow[ul, "\mathsf{inr} \cdot Hi_\infty"']
\end{tikzcd} \tag{3.11}$$

By extensivity, we obtain a morphism $e_\infty : X_\infty \to HX_\infty$ complementary to $\coprod \bar{e}_n$, i.e. such that the outside of the above diagram commutes. This morphism is the structure of an H-coalgebra on X_∞. In order to finish the proof of unicity of $h : X \to H^*Y + \nu H$, we only have to verify that the remaining coproduct component $h \cdot i_\infty$ is determined by e. To this end it suffices to prove that $h \cdot i_\infty$ factorizes through the coproduct injection $\mathsf{inr} : \nu H \to H^*Y + \nu H$, which we show below. Indeed, given a factorization $k : X_\infty \to \nu H$ such that the following square commutes:

$$\begin{tikzcd}
X_\infty \arrow[r, "i_\infty"] \arrow[d, "k"'] & X \arrow[d, "h"] \\
\nu H \arrow[r, "\mathsf{inr}"'] & H^*Y + \nu H
\end{tikzcd} \tag{3.12}$$

it follows that k is the unique(!) H-coalgebra morphism from e_∞ to t, i.e., the square below commutes:

$$\begin{tikzcd}
X_\infty \arrow[r, "k"] \arrow[d, "e_\infty"'] & \nu H \arrow[d, "t"] \\
HX_\infty \arrow[r, "Hk"'] & H(\nu H)
\end{tikzcd} \tag{3.13}$$

To see this consider the diagram below:

$$\begin{tikzcd}
X \arrow[rrr, "h"] \arrow[ddd, "e"'] & & & H^*Y + \nu H \arrow[ddd, "\sigma_Y + t"] \\
 & X_\infty \arrow[ul, "i_\infty"'] \arrow[r, "k"] \arrow[d, "e_\infty"'] \arrow[ur, phantom, "(3.12)"] \arrow[l, phantom, "(3.11)"] & \nu H \arrow[ur, "\mathsf{inr}"'] \arrow[d, "t"] & \\
 & HX_\infty \arrow[r, "Hk"] \arrow[dl, "\mathsf{inr} \cdot Hi_\infty"'] \arrow[dr, phantom, "(3.12)"] & H(\nu H) \arrow[dr, "\mathsf{inr}"] & \\
Y + HX \arrow[rrr, "Y + Hh"'] & & & Y + HH^*Y + H(\nu H)
\end{tikzcd}$$

Its outside is the square (3.2), and all inner parts, except perhaps the inner square, commute. Thus, that square also commutes since the coproduct injection inr is monomorphic (see Definition 2.1).

The proof that $h \cdot i_\infty$ factorizes throught $\mathsf{inr} : \nu H \to H^*Y + \nu H$ is based on Lemma 3.4, which shows that $\mathsf{inr} : \nu H \to H^*Y + \nu H$ is the intersection of the following coproduct injections

$$b_k = \left(\coprod_{n \geq k} H^n Y + \nu H \xrightarrow{[j_n]_{n \geq k} + \nu H} H^*Y + \nu H \right) \qquad (k \geq 1).$$

Thus, we only need to verify that $h \cdot i_\infty$ factorizes through every b_k. For $k = 1$ consider the diagram below:

$$\begin{tikzcd}
X_\infty \arrow[r, "i_\infty"] \arrow[dd, "e_\infty"'] & X \arrow[r, "h"] \arrow[d, "e"'] & H^*Y + \nu H \\
 & Y + HX \arrow[r, "Y + Hh"] & Y + HH^*Y + H(\nu H) \arrow[u, "(\sigma_Y + t)^{-1} = \sigma_Y^{-1} + t^{-1}"'] \\
HX_\infty \arrow[r, "Hi_\infty"] & HX \arrow[r, "Hh"'] \arrow[u, "\mathsf{inr}"] & HH^*Y + H(\nu H) \arrow[u, "\mathsf{inr}"'] \arrow[uu, bend right=60, "b_1"']
\end{tikzcd}$$

The right-hand part commutes by (3.1), for the left-hand one see the upper right-hand part of (3.11), the upper middle part commutes by (3.2) and the remaining lower middle part trivially commutes.

Given a factorization of $h \cdot i_\infty$ through b_k via f, then $H(h \cdot i_\infty)$ factorizes through Hb_k via Hf. We conclude that $h \cdot i_\infty$ factorizes through b_{k+1} by using this and the diagram below:

$$
\begin{tikzcd}
X_\infty \arrow[r, "i_\infty"] \arrow[ddd, "e_\infty"'] & X \arrow[r, "h"] \arrow[d, "e"'] & H^*Y+\nu H & \\
& Y+HX \arrow[r, "Y+Hh"] & Y+HH^*Y+H(\nu H) \arrow[u, "(\sigma_Y+t)^{-1}=\sigma_Y^{-1}+t^{-1}"'] & \\
& HX \arrow[u, "\mathsf{inr}"] \arrow[r, "Hh"'] & HH^*Y+H(\nu H) \arrow[u, "\mathsf{inr}"'] & \coprod_{n\geq k+1} H^nY+\nu H \arrow[uul, bend right, "b_{k+1}"'] \\
HX_\infty \arrow[ur, "Hi_\infty"] \arrow[r, "Hf"'] & H\left(\coprod_{n\geq k} H^nY+\nu H\right) \arrow[ur, "Hb_k"] \arrow[r, "\cong"'] & \coprod_{n\geq k} H^{n+1}Y+H(\nu H) \arrow[ur, "\mathsf{id}+t^{-1}"] &
\end{tikzcd}
$$

All its inner parts, except perhaps the right-hand one clearly commute. For the remaining right-hand part, we consider the components of the coproduct in its lower left-hand corner separately: the right-hand component with domain $H(\nu H)$ yields t^{-1} on both paths. We further consider the components of $H(\coprod_{n\geq k} H^nY)$ with the help of the diagram below:

$$
\begin{tikzcd}
H(H^nY) \arrow[rr, equal] \arrow[d, "H\mathsf{in}_n"'] \arrow[dr, "Hj_n"] & & H^{n+1}Y \arrow[r, "\mathsf{in}_n"] \arrow[dd, "j_{n+1}"'] \arrow[dr, "\mathsf{in}_{n+1}"'] & \coprod_{n\geq k} H^{n+1}Y+\nu H \arrow[d, "\mathsf{id}+t^{-1}"] \\
H\left(\coprod_{n\geq k} H^nY+\nu H\right) \arrow[d, "Hb_k"'] & HH^*Y \arrow[dl, "\mathsf{inl}"] \arrow[d, "\mathsf{inr}"] & & \coprod_{n\geq k} H^nY+\nu H \arrow[dd, "b_{k+1}"] \\
HH^*Y+H(\nu H) \arrow[d, "\mathsf{inr}"'] & Y+HH^*Y \arrow[dl, "\mathsf{inl}"'] \arrow[r, shift right, "\sigma_Y^{-1}"'] & H^*Y \arrow[l, shift right, "\sigma_Y"'] \arrow[dr, "\mathsf{inl}"] & \\
Y+HH^*Y+H(\nu H) \arrow[rrr, "\sigma_Y^{-1}+t^{-1}"'] & & & H^*Y+\nu H
\end{tikzcd}
$$

Its upper central part commutes by (3.1), the left-hand triangle commutes by the definition of b_k and the right-hand rhombus by the definition of b_{k+1}; all other inner parts clearly commute.

We conclude that h is unique since it is equal to

$$X = \coprod_{n\geq 1} \overline{X}_n + X_\infty \xrightarrow{[\overline{h}_n]+k} H^*Y + \nu H.$$

(b) Existence: For the given coalgebra e we define i_n, $\bar{i}_n$, e_n and $\overline{e}_n$ by (3.3) and (3.6), and we also define $e_\infty : X_\infty \to HX_\infty$ by (3.11) where $X = \overline{X}_\infty + X_\infty$ with $\overline{X}_\infty = \coprod_{n\geq 1} \overline{X}_n$. We furthermore use notations (3.7) and (3.8).

Define $k : X_\infty \to \nu H$ by (3.12) and for all $n \geq 1$ put

$$\overline{h}_n = \left(\overline{X}_n \xrightarrow{\widehat{e}_n} H^nY \xrightarrow{j_n} H^*Y\right). \tag{3.14}$$

We prove that $[\overline{h}_n] + k : X \to H^*Y + \nu H$ is a coalgebra morphism for $Y + H(-)$, i.e., the square below commutes:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "e"'] & \coprod_{n\geq 1} \overline{X}_n + X_\infty \arrow[r, "{[\overline{h}_n]+k}"] \arrow[d, "\coprod_{n\geq 1} \overline{e}_n + e_\infty"'] & H^*Y + \nu H \arrow[d, "\sigma_Y + t"] \\
Y + HX \arrow[r, equal] & Y + \coprod_{n\geq 1} H\overline{X}_n + HX_\infty \arrow[r, "{Y+[H\overline{h}_n]+Hk}"'] & Y + HH^*Y + H(\nu H)
\end{tikzcd}$$

Its right-hand coproduct component with domain X_∞ is the square (3.12) defining k by the commutativity of the right-hand part of (3.11).

Let us verify that the coproduct components with domain $\overline{X}_n$ commute. We proceed by induction on n. For the base case we obtain the following commutative diagram (for the right-hand triangle see (3.1), and for the left-hand one see (3.3)):

$$\begin{tikzcd}
\overline{X}_1 \arrow[r, "\overline{e}_1"] \arrow[d, "\bar{i}_0"'] \arrow[dr, "\overline{e}_1"] \arrow[rr, bend left, "\overline{h}_1"] & Y \arrow[r, "j_0"] \arrow[ddr, "\mathsf{inl}"] & H^*Y \arrow[dd, "\sigma_Y"] \\
X = X_0 \arrow[d, "e=e_0"'] & Y \arrow[dl, "\bar{i}_0=\mathsf{inl}"] \arrow[dr, "\mathsf{inl}"] & \\
Y + HX \arrow[rr, "{Y+[H\overline{h}_n]}"'] & & Y + HH^*Y
\end{tikzcd}$$

For the induction step with $n > 1$ consider the diagram below:

$$\begin{tikzcd}
\overline{X}_n \arrow[rr, "\hat{e}_n"] \arrow[rrr, bend left, "\overline{h}_n"] \arrow[d, "\overline{i}^*_n"'] \arrow[dr, "\overline{e}_n"] & & H^nY \arrow[r, "j_n"] \arrow[d, "Hj_{n-1}"] & H^*Y \arrow[ddd, "\sigma_Y"] \\
X \arrow[ddd, "e"'] & H\overline{X}_{n-1} \arrow[ur, "H\hat{e}_{n-1}"] \arrow[r, "H\overline{h}_{n-1}"'] \arrow[d, "H\overline{i}^*_{n-1}"] & HH^*Y \arrow[ddr, "\mathsf{inr}"] & \\
& HX \arrow[ddl, "i_0 = \mathsf{inr}"'] \arrow[ddrr, "{[H\overline{h}_n]}"] & & \\
& & & \\
Y + HX \arrow[rrr, "{[Y + [H\overline{h}_n]]}"'] & & & Y + HH^*Y
\end{tikzcd}$$

The upper part and the middle triangle under it commute by (3.14), the upper left-hand triangle follows immediately from (3.8). The right-hand part commutes by (3.1), and the left-hand part is the outside of (3.10). The remaining parts clearly commute. □

Corollary 3.6. *The free cia on an object Y is*

$$CY = H^*Y + \nu H$$

with algebra structure $\varphi_Y + t^{-1} : H(H^*Y + \nu H) \cong HH^*Y + H(\nu H) \to H^*Y + \nu H$.

Indeed, this follows from Example 2.8(1).

Example 3.7. (a) It is well known that the identity functor on Set has the free cias (equivalently, final coalgebras for $(-) + Y$) $TY = \mathbb{N} \times Y + 1$ where $\mathbb{N}$ is the set of natural numbers. It follows from Theorem 3.5 that the same formula holds in every hyper-extensive category with a terminal object 1. To see this, one first shows that

$$N := \coprod_{n<\omega} 1 \quad \text{with} \quad 1 \xrightarrow{\mathsf{in}_0} N \xleftarrow{[\mathsf{in}_{n+1}]_{n<\omega}} N$$

forms a natural number object, i.e., an initial algebra for $1 + (-)$. Using distributivity we see that for any object Y the free algebra Id^*Y is

$$\mathsf{Id}^*Y = \coprod_{n<\omega} Y \cong \left(\coprod_{n<\omega} 1\right) \times Y = N \times Y. \tag{3.15}$$

Finally, we clearly have $\nu\mathsf{Id} = 1$. By Theorem 3.5, we thus obtain

$$TY \cong N \times Y + 1.$$

(b) For the above formula giving the free cia for Id on every Y it is *not* sufficient that $\mathcal{C}$ be an extensive category. As a counterexample consider the category $\mathcal{C} = \mathsf{CHaus}$ of compact Hausdorff spaces. Its limits and finite coproducts are created by the forgetful functor into Set, thus CHaus is extensive. However, it is not hyper-extensive since countable coproducts are not universal. For $Y = 1$ (the one point space) the formula (3.15) gives an uncountable space since $\coprod_{n<\omega} 1$ is the Stone-Čech compactification of an infinite discrete space. However, $T1 = \nu(\mathsf{Id} + 1)$ is a countable space; for the terminal ω^{op}-chain

$$1 \leftarrow 1 + 1 \leftarrow 1 + 1 + 1 \leftarrow \cdots$$

of the functor $\mathsf{Id} + 1$ on CHaus has the corresponding underlying chain in Set. The limit in Set is countable, giving the set $N + 1$. The limit in CHaus is then a compact space on this set, in fact, it is the one-point compactification of the discrete space on N. Since the functor $X \mapsto X + 1$ preserves this limit, this is its terminal coalgebra, which means that $T1$ is countable.

Example 3.8. Extending Example 3.7(a), recall that the functor $HX = \Sigma \times X$ on Set has the free cias $TY = \Sigma^* \times Y + \Sigma^\omega$, where Σ^* and Σ^ω are the usual sets of strings (words) and sequences (streams) on Σ.

It follows from Theorem 3.5 that the same formula holds in every hyper-extensive category $\mathcal{C}$ with finite products commuting with countable coproducts. Note that all categories in Example 2.2 fulfil this.

Examples of such categories are presheaf categories, posets, graphs and unary algebras.

Given an object Σ of $\mathcal{C}$, the functor $HX = \Sigma \times X$ has the terminal coalgebra

$$\Sigma^\omega = \lim_{n<\omega} \Sigma^n$$

which is the limit of the ω^{op}-chain of projections as follows:

$$1 \xleftarrow{!} \Sigma \xleftarrow{\Sigma \times !} \Sigma \times \Sigma \xleftarrow{\Sigma \times \Sigma \times !} \Sigma \times \Sigma \times \Sigma \leftarrow \cdots$$

The free algebras H^*Y are obtained as follows: put

$$\Sigma^* = \coprod_{n<\omega} \Sigma^n.$$

Then $H^*Y = \Sigma^* \times Y$. Thus, according to Theorem 3.5, the free cia for H on Y is given by

$$TY = \Sigma^* \times Y + \Sigma^\omega.$$

Similarly, given another object W of $\mathcal{C}$, the functor $H'X = W + \Sigma \times X$ has the free cias $T'Y = \Sigma^* \times (W + Y) + \Sigma^\omega$.

Example 3.9. In Theorem 3.5 it is not sufficient that H preserves finite coproducts. In fact, consider the ultrafilter functor $U : \mathsf{Set} \to \mathsf{Set}$ which assigns to every set X the set of all ultrafilters on X and to a map $f : X \to Y$ the map Uf sending an ultrafilter $\mathcal{A}$ on X to $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}\}$. It preserves finite coproducts and $\nu U = 1$. But for Y infinite, $Y + U(-)$ has no fixed points; for suppose that $TY \cong Y + UTY$, then TY must be infinite since Y is and therefore $|TY| < |UTY|$ contradicting the isomorphism.

4. Corecursiveness vs. Complete Iterativity

Under Assumption 3.1 we prove in this section that H is a cia functor, i.e., every corecursive algebra is a cia. Throughout this section we fix an algebra $a : HA \to A$.

Notation 4.1. (1) Define morphisms

$$a^n : H^n A \to A$$

by the following induction:

$$a^0 = \mathsf{id}_A \qquad \text{and} \qquad a^{n+1} = (H^{n+1}A = HH^nA \xrightarrow{Ha^n} HA \xrightarrow{a} A).$$

(2) For every equation morphism $e : X \to HX + A$ we use the notation of the proof of Theorem 3.5, except that Y is replaced by A everywhere (and the order of summands is swapped). Thus we use the morphisms

$$i_n, \bar{\bar{i}}_n, e_n, \overline{e}_n, e_\infty, i_\infty, \bar{\bar{i}}_\infty, \widehat{e}_n, \text{ and}, \bar{\bar{i}}_n^*$$

as in that proof. For example, $i_0 : HX \to HX + A$ is the coproduct injection.

Construction 4.2. Let $a : HA \to A$ be an algebra. Given an equation morphism $e : X \to HX + A$ and a coalgebra-to-algebra morphism $s : X_\infty \to A$:

$$\begin{tikzcd}
X_\infty \arrow[r, "s"] \arrow[d, "e_\infty"'] & A \\
HX_\infty \arrow[r, "Hs"'] & HA \arrow[u, "a"']
\end{tikzcd} \tag{4.1}$$

we define a morphism $e_s^\dagger : X \to A$ on the components of the coproduct $X = \left(\coprod_{n \geq 1} \overline{X}_n\right) + X_\infty$ (with injections $\bar{i}_n^*$, for every $n \geq 1$, and i_∞) separately as follows:

$$\begin{tikzcd}
\overline{X}_n \arrow[r, "\widehat{e}_n"] \arrow[d, "\bar{i}_n^*"'] & H^{n-1}A \arrow[d, "a^{n-1}"] \\
X \arrow[r, "e_s^\dagger"'] & A
\end{tikzcd} \quad \text{for } n \geq 1, \text{ and} \quad
\begin{tikzcd}
X_\infty \arrow[d, "i_\infty"'] \arrow[dr, "s"] & \\
X \arrow[r, "e_s^\dagger"'] & A
\end{tikzcd} \tag{4.2}$$

Proposition 4.3. *The morphism $e_s^\dagger$ is a solution of e. Moreover, every solution of e is of the form $e_s^\dagger$ for some coalgebra-to-algebra morphism s.*

Proof. (1) We verify the commutativity of (2.2) for $e_s^\dagger$ by considering the coproduct components of $X = \coprod_{n \geq 1} \overline{X}_n + X_\infty$ separately. For the components $\overline{X}_n$ we proceed by induction on n. For the base case $n = 1$ we have the diagram below:

$$\begin{tikzcd}
\overline{X}_1 \arrow[rrr, "\bar{e}_1 = \widehat{e}_1"] \arrow[dr, "\bar{i}_1^* = \bar{i}_1"] \arrow[ddd, "\bar{e}_1"'] & & & A \arrow[dl, equal] \arrow[ddd, equal] \\
& X \arrow[r, "e_s^\dagger"] \arrow[d, "e"'] & A & \\
& HX + A \arrow[r, "He_s^\dagger + A"'] & HA + A \arrow[u, "{[a,A]}"'] \arrow[dr, equal] & \\
A \arrow[ur, "\bar{i}_0 = \mathsf{inr}"] \arrow[rrr, equal] & & & A
\end{tikzcd} \tag{4.3}$$

Its left-hand part is the right-hand square of (3.3), its upper part commutes by (4.2) and the lower and right-hand parts are trivial; since the outside also trivially commutes, so does the inner square when precomposed by $\bar{i}_1^*$, as desired.

For the induction step with $n > 1$ we consider the following diagram:

$$\begin{tikzcd}
\overline{X}_n \arrow[rrrr, "\hat{e}_n"] \arrow[dr, "\overline{i}^*_n"] \arrow[dddd, "\overline{e}_n"'] & & & & H^{n-1}A \arrow[dll, "a^{n-1}"'] \arrow[dddd, equal] \\
& X \arrow[r, "e^\dagger_s"] \arrow[d, "e"'] & A & & \\
& HX + A \arrow[r, "He^\dagger_s + A"'] & HA + A \arrow[u, "{[a,A]}"] & & \\
& HX_0 = HX \arrow[u, "i_0 = \mathsf{inl}"] \arrow[rr, "He^\dagger_s"'] & & HA \arrow[ul, "\mathsf{inl}"] \arrow[uul, "a"'] & \\
H\overline{X}_{n-1} \arrow[ur, "H\overline{i}^*_{n-1}"'] \arrow[rrrr, "H\hat{e}_{n-1}"'] & & & & HH^{n-2}A \arrow[ul, "Ha^{n-2}"']
\end{tikzcd} \tag{4.4}$$

Its upper part commutes by (4.2), the left-hand part by (3.10), the right-hand part commutes by the definition of a^{n-1} (see Notation 4.1(1)), the lower part commutes by the induction hypothesis, and the remaining two inner parts trivially commute. That the outside commutes follows from (3.8) by an easy induction. Thus, the inner square commutes when precomposed with $\overline{i}^*_n$, as desired.

Finally, for the coproduct component X_∞ we consider the following diagram:

$$\begin{tikzcd}
X \arrow[rrr, "e^\dagger_s"] \arrow[dddd, "e"'] & & & A \\
& X_\infty \arrow[ul, "i_\infty"'] \arrow[urr, "s"'] \arrow[d, "e_\infty"'] & & \\
& HX_\infty \arrow[d, "Hi_\infty"'] \arrow[dr, "Hs"] & & \\
& HX \arrow[r, "He^\dagger_s"'] \arrow[dl, "\mathsf{inl}"'] & HA \arrow[uuur, "a"'] \arrow[dr, "\mathsf{inl}"] & \\
HX + A \arrow[rrr, "He^\dagger_s + A"'] & & & HA + A \arrow[uuuu, "{[a,A]}"']
\end{tikzcd} \tag{4.5}$$

Its upper part and the middle triangle commute by (4.2)[2], its left-hand part

[2]Note that HX is now the left-hand coproduct component while in the previous section it was the right-hand one in $Y + HX$.

is the right-hand part of (3.11), the lower and right-hand parts trivially commute, and the remaining inner part commutes by (4.1). Thus, the outside commutes when precomposed by i_∞, as desired.

(2) Suppose that $e^\dagger$ is any solution of e, and let $s = e^\dagger \cdot i_\infty : X_\infty \to A$. We will now prove that s is a coalgebra-to-algebra morphism from $e_\infty : X_\infty \to HX_\infty$ to $a : HA \to A$ and that $e^\dagger = e_s^\dagger$. To see the former, take Diagram (4.5) and replace $e_s^\dagger$ by $e^\dagger$. Now the outside commutes, and it follows that the part exhibiting s as coalgebra-to-algebra morphism commutes since so do all other inner parts.

To complete the proof we now show by induction on n that

$$e^\dagger \cdot \bar{i}_n^* = a^{n-1} \cdot \widehat{e}_n : \overline{X}_n \to A,$$

cf. (4.2). It then follows that $e^\dagger \cdot \bar{i}_n = e_s^\dagger \cdot \bar{i}_n$, and together with $e^\dagger \cdot i_\infty = s = e_s^\dagger \cdot i_\infty$ we can conclude that $e^\dagger = e_s^\dagger$.

For the base case $n = 1$ consider Diagram (4.3) with $e_s^\dagger$ replaced by $e^\dagger$. Then the inner square commutes, and therefore so does the desired upper part since all other inner parts commute, as explained in part (1) of our proof.

Similary, for the induction step with $n > 1$ consider Diagram (4.4) with $e_s^\dagger$ replaced by $e^\dagger$. Then the inner square commutes, and therefore so does the desired upper part since all other inner parts commute, as explained in part (1). This completes the proof. □

Corollary 4.4. *H is a cia functor.*

Indeed, if (A, a) is a corecursive H-algebra and $e : X \to HX + A$ is a given equation morphism, we have a unique s as in (4.1). Now note that Proposition 4.3 establishes a bijective correspondence between solutions of e and coalgebra-to-algebra morphisms from e_∞ to a, and therefore there exists a unique solution of e.

Finally, note that both cias and corecursive algebras form full subcategories of the category of all algebras for H. Thus Corollary 4.4 establishes an isomorphism between the categories of cias and corecursive algebras for H.

The following proposition needs no assumptions on H nor on the base category, except that binary coproducts exist.

Proposition 4.5. *If H is a cia functor, then so is $H(-)+Y$ for every object Y.*

Proof. Let $[a, y] : HA + Y \to A$ be a corecursive algebra for $H(-) + Y$.

(1) The algebra $a : HA \to A$ is corecursive for H. Indeed, for every coalgebra $e : X \to HX$ we form the following coalgebra for $H(-) + Y$:

$$f = (X \xrightarrow{e} HX \xrightarrow{\mathsf{inl}} HX + A).$$

Now consider the diagram below:

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "e"'] \arrow[dd, bend right=60, "f"'] & A \\
HX \arrow[r, "Hs"] \arrow[d, "\mathsf{inl}"'] & HA \arrow[u, "a"'] \arrow[d, "\mathsf{inl}"] \\
HX + Y \arrow[r, "Hs+Y"'] & HA + Y \arrow[uu, bend right=60, "{[a,y]}"']
\end{tikzcd}$$

This shows that there is a bijective correspondence between coalgebra-to-algebra morphisms from e to a (w.r.t. H) and those from f to $[a, y]$ (w.r.t. $H(-) + Y$). Since the former exists uniquely, so does the latter, hence A is corecursive for H.

(2) From (1) we have by assumption on H that (A, a) is a cia for H. It follows that $(A, [a, y])$ is a cia for $H(-)+Y$; indeed, to give a cia (A, a) for H together with a morphism $y : Y \to A$ is equivalent to giving a cia $(A, [a, y])$ for $H(-) + Y$, see the proof of [15, Theorem 2.10]. $\square$

Corollary 4.6. *Let H be a functor having a terminal coalgebra and preserving countable coproducts. Then $H(-) + Y$ is a cia functor for every object Y.*

5. Finitary Set Functors

We have seen in Corollary 4.6 that for every functor H on a hyper-extensive category preserving countable coproducts, $H(-) + Y$ are cia functors (i.e., every corecursive algebra is a cia). In particular, if $\mathcal{C}$ is cartesian closed, then the functor $X \mapsto W \times X + Y$ is a cia functor. For standard finitary functors on $\mathcal{C} = \mathsf{Set}$ we we now prove the converse: if H is a cia functor then it has the form $X \mapsto W \times X + Y$ for some sets W and Y. Recall *standard* means that H preserves

(1) inclusions, i.e., $X \subseteq Y$ implies $HX \subseteq HY$ and the H-image of the inclusion map $X \hookrightarrow Y$ is the inclusion map $HX \hookrightarrow HY$, and

(2) finite intersections.

Further recall from [7] that a standard set functor H is *finitary* iff for every set X we have $HX = \bigcup HY$ where the union ranges over finite subsets $Y \subseteq X$. An example of a finitary functor on Set is the polynomial functor H_Σ, see Example 2.5(3).

Assumption 5.1. Throughout this section H denotes a standard, finitary set functor.

Definition 5.2. (1) By a *presentation* of H is meant a finitary signature Σ and natural epitransformation $\varepsilon : H_\Sigma \to H$, i.e., every component ε_X is a surjective map.

(2) An ε-*equation* (over a given set X) is an expression $\sigma(x_1, \dots x_n) = \tau(z_1, \dots, z_m)$ where $\sigma(x_1, \dots x_n)$ and $\tau(z_1, \dots, z_m)$ are elements of $H_\Sigma X$ that are merged by ε_X.

Remark 5.3. All ε-equations form an equivalence relation. More precisely, for any set X all ε-equations over X form precisely the kernel equivalence of ε_X. Moreover, the elements of HX may be regarded as equivalence classes of the elements $\sigma(x_1, \dots, x_n)$ of $H_\Sigma X$ modulo this equivalence.

Example 5.4. The finite power-set functor $\mathcal{P}_\mathsf{f}$ has a presentation with Σ having a single n-ary operation for every n, and ε sending $\sigma(x_1, \dots, x_n)$ to $\{x_1, \dots, x_n\}$.

The following lemma was proved in [8]. We present a (short) proof since we refer to it later.

Lemma 5.5. *Every finitary set functor has a presentation $\varepsilon : H_\Sigma \to H$, and the category $\mathsf{Alg}\, H$ is isomorphic to the variety of all Σ-algebras satisfying all ε-equations.*

Proof. Define a signature $\Sigma = (\Sigma_n)_{n<\omega}$ by $\Sigma_n = Hn$ where we regard n as the finite ordinal $\{0, \dots, n-1\}$ for all n. By the Yoneda lemma we have a natural transformation $\varepsilon_X : H_\Sigma X \to HX$ assigning to every $\sigma(x_1, \dots, x_n)$ represented as a function $x : n \to X$ the element $Hx(\sigma)$. Since H is finitary, ε_X is surjective.

Every H-algebra $a : HA \to A$ defines the corresponding Σ-algebra $a \cdot \varepsilon_A : H_\Sigma A \to A$ which clearly satisfies all ε-equations. This defines a full embedding of $\mathsf{Alg}\, H$ into $\mathsf{Alg}\, H_\Sigma$ (which is identity on morphisms). We now easily prove that every Σ-algebra satisfying all ε-equations has the above form $(A, a \cdot \varepsilon_A)$. Indeed, given $a^\Sigma : H_\Sigma A \to A$ satisfying all ε-equations, define $a : HA \to A$ by $a([\sigma(a_1, \dots, a_n]) = a^\Sigma(\sigma(a_1, \dots, a_n))$. Since we know from Remark 5.3 that a^Σ merges all pairs in the kernel of ε_A, this is well-defined and we clearly have $a^\Sigma = a \cdot \varepsilon_A$. Thus, our full embedding defines the desired isomorphism between H-algebras and Σ-algebras satisfing all ε-equations. $\square$

Remark 5.6. (1) Denote by C_1 the constant functor with value $1 = \{c\}$, and by $C_{0.1}$ its subfunctor with $C_{0,1}\emptyset = \emptyset$ and $C_{0,1}X = 1$ else. For every natural transformation $\alpha : C_{0,1} \to H$ there exists a unique extension to $\alpha' : C_1 \to H$.

Indeed, since H is standard, it preserves the (empty) intersection of the coproduct injections $\mathsf{inl}, \mathsf{inr} : 1 \to 1 + 1$. Since $H\mathsf{inl}(\alpha_1(c)) = \alpha_{1+1}(c) = H\mathsf{inr}(\alpha_1(c))$, there exists a unique element t of $H\emptyset$ such that the inclusion map $v : \emptyset \to 1$ fulfils $\alpha_1(c) = Hv(t)$. We put $\alpha'_\emptyset(c) = t$.

(2) All constants in our presentation of H are *explicit*. This means that whenever some n-ary symbol σ has the property that some ε-equation has the form $\sigma(x_1, \dots, x_n) = \sigma(z_1, \dots, z_n)$, where the variables x_i are pairwise distinct and none of them equals some z_j, then there exists a constant symbol τ in Σ for which we have the following ε-equation: $\sigma(x_1, \dots, x_n) = \tau$. Indeed, for every set $X \neq \emptyset$ we have an element

$$\alpha_X = \varepsilon_X(\sigma(a_1, \dots, a_n)) \in HX$$

independent of the choice of $a_1, \dots, a_n$ in X. This defines a natural transformation $\alpha : C_{0,1} \to H$. Let $\alpha' : C_1 \to H$ be its extension according to item (1). The element $\alpha'_\emptyset(c)$ of $H\emptyset$ has, since ε is an epitransformation, the form $\varepsilon_\emptyset(\tau)$ for some nullary symbol τ. Then the desired ε-equation holds because for $X = \{x_1, \dots, x_n\}$ and the unique empty map $u : \emptyset \to X$ we have

$$\begin{aligned} \varepsilon_X(\sigma(x_1, \dots, x_n)) &= \alpha_X(c) = \alpha'_X(c) = Hu \cdot \alpha'_\emptyset(c) \\ &= Hu \cdot \varepsilon_\emptyset(\tau) = \varepsilon_X \cdot Hu(\tau) = \varepsilon_X(\tau). \end{aligned}$$

Definition 5.7. A presentation $\varepsilon : H_\Sigma \to H$ is *reduced* provided that for every ε-equation

$$\sigma(x_1, \dots, x_n) = \tau(z_1, \dots, z_m)$$

the following hold:

(1) if $x_1, \dots, x_n$ are pairwise distinct, then they all lie in $\{z_1, \dots, z_n\}$, and

(2) if, moreover, $z_1, \dots, z_n$ are also pairwise distinct, then $\sigma = \tau$.

Proposition 5.8. *Every finitary set functor has a reduced presentation.*

Proof. (a) Assume that the above condition (1) holds for all ε-equations. Then we can restrict ε so that also (2) becomes true. Indeed, denote by $\sim$ the following equivalence on Σ: $\sigma \sim \tau$ iff there exists an ε-equation $\sigma(x_1, \dots, x_n) = \tau(z_1, \dots, z_m)$ with pairwise distinct variables on both sides. Condition (1) implies that $n = m$ and there exists a permutation $(i_1, \dots, i_n)$ with $x_1 = z_{i_1}, \dots, x_n = z_{i_n}$. This implies that the image of the summand $\{\sigma\} \times X^n$ under ε_X is equal to the image of $\{\tau\} \times X^m$. Consequently, if Σ' is a choice class of $\sim$, then the restriction ε' of ε to H'_Σ, as a subfunctor of H_Σ, is still an epi-transformation. And the presentation ε' fulfils (1) and (2) for all ε'-equations in Definition 5.7.

(b) It remains to prove that every presentation ε can be modified to one satisfying (1) in Definition 5.7 for all equations. Let σ be an n-ary symbol of Σ. For $i = 1, \dots, n$ we say that the coordinate i is *inessential* for σ if we have an ε-equation of the following form:

$$\sigma(x_1, \dots, x_n) = \sigma(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n)$$

all of whose $n + 1$ variables are pairwise distinct. The remaining coordinates will be called *essential*. Without loss of generality we can assume that the essential coordinates are precisely $1, \dots, n'$ for some $n' \leq n$. From Remark 5.3 it follows easily that the following is also an ε-equation:

$$\sigma(x_1, \dots, x_n) = \sigma(x_1, \dots, x_{n'}, z, \dots, z).$$

Form the signature Σ' with the same symbols as Σ but with arities n' in lieu of n. We define a presentation $\varepsilon' : H_{\Sigma'} \to H$ as follows: for each nonempty set X it sends every element $\sigma(x_1, \dots, x_{n'})$ to $\varepsilon_X(\sigma(x_1, \dots, x_{n'}, z, \dots, z))$, where z is arbitrary. And to define $\varepsilon_\emptyset$, use Remark 5.6(2): whenever a symbol σ has no essential coordinate (and hence σ becomes a constant symbol in Σ'), there exists a constant symbol τ in Σ and an ε-equation $\sigma(x_1, \dots, x_n) = \tau$. Define $\varepsilon'_\emptyset(\sigma) = \varepsilon_\emptyset(\tau)$. This presentation ε' clearly satisfies condition (1) of Definition 5.7 for all equations. $\square$

Notation 5.9. From now on we assume that a reduced presentation of H is given.

Recall the notation TY, FY and CY from Examples 2.8 and Notation 2.10. All these objects exist since H is finitary (and therefore so are all $H(-)+Y$). The corresponding notation for H_Σ will be $T_\Sigma Y$, $F_\Sigma Y$ and $C_\Sigma Y$. The monad units of T and C are denoted by η and η^C, respectively.

As mentioned in Example 2.8, $T_\Sigma Y$ can be described as the algebra of all Σ-trees over Y. And $C_\Sigma Y$ and $F_\Sigma Y$ are its subalgebras on all trees with finitely many leaves labeled in Y, or all finite trees, respectively.

Since TY is a corecursive algebra, there exists a unique homomorphism of H-algebras

$$m_Y : CY \to TY$$

with $m_Y \cdot \eta_Y^C = \eta_Y$. The corresponding H_Σ-algebra morphism is denoted by

$$m_Y^\Sigma : C_\Sigma Y \to T_\Sigma Y.$$

Remark 5.10. In [4] we described FY and TY as the following quotients of the Σ-algebras $F_\Sigma Y$ and $T_\Sigma Y$, respectively. Recall from Lemma 5.5 that every H-algebra $a : HA \to A$ may be regarded as the H_Σ-algebra with structure $a \cdot \varepsilon_A : H_\Sigma A \to A$.

(1) $FY = F_\Sigma Y/{\sim_Y}$, where $\sim_Y$ is the congruence of *finite application of ε-equations*. That is, the smallest congruence with $\sigma(x_1, \dots, x_n) \sim_Y \tau(z_1, \dots, z_m)$ for every ε-equation

$$\sigma(x_1, \dots, x_n) = \tau(z_1, \dots, z_m)$$

over Y. The universal map $\eta_Y^F : Y \to FY$ is the composite of the one of $F_\Sigma Y$ with the canonical quotient map $F_\Sigma Y \twoheadrightarrow F_\Sigma Y/{\sim_Y}$.

(2) $TY = T_\Sigma Y/{\sim_Y^*}$, where $\sim_Y^*$ is the congruence of (possibly infinitely many) *applications of ε-equations* as explained below. The universal map is $\widehat{\eta}_Y = \widehat{\varepsilon}_Y \cdot \eta_Y^\Sigma$, where $\eta_Y^\Sigma : Y \to T_\Sigma Y$ is the universal map of the free cia for H_Σ on Y and $\widehat{\varepsilon}_Y : T_\Sigma Y \twoheadrightarrow T_\Sigma Y/{\sim_Y^*}$ is the canonical quotient map.

The definition of a *application of ε-equations* is based on the concept of *cutting* a Σ-tree at level k: the resulting finite Σ-tree $\partial_k t$ is obtained from t by deleting all nodes of depth larger than k and relabeling all nodes at level

k by a symbol $\perp \notin Y$. Then using $\sim_{Y \cup \{\perp\}}$ we define, for Σ-trees t and s in $T_\Sigma Y$,

$$t \sim_Y^* s \qquad \text{iff} \qquad \partial_k t \sim_{Y \cup \{\perp\}} \partial_k s \quad \text{for every } k < \omega.$$

Not surprisingly, CY can be described analogously:

Proposition 5.11. *The free corecursive H-algebra CY is the quotient of the Σ-algebra $C_\Sigma Y$ modulo the application of ε-equations: $CY = C_\Sigma Y/{\sim_Y^*}$.*

Proof. This is based on the following description of CY presented in [3]: denote by $\oplus$ the binary coproduct of H-algebras in $\mathsf{Alg}\, H$. By Lemma 5.5, this is, equivalently, the coproduct in the variety of all Σ-algebras satisfying all ε-equations. Then we have

$$CY = \nu H \oplus FY.$$

Analogously, if $\boxplus$ denotes the binary coproduct of Σ-algebras, we of course have

$$C_\Sigma Y = \nu H_\Sigma \boxplus F_\Sigma Y.$$

For arbitrary H-algebras A and B we know that $A \oplus B$ is the quotient of $A \boxplus B$ modulo the application of ε-equations. Moreover, we have $T = T_\Sigma/{\sim^*}$ and $FY = F_\Sigma Y/{\sim}$. It follows immediately that $T \oplus FY = (T_\Sigma \boxplus F_\Sigma Y)/{\sim^*}$, as claimed. $\square$

Lemma 5.12. *Suppose that CY is a cia for H. For every equation morphism $e : X \to H_\Sigma X + Y$ with the unique solution $e^\ddagger : X \to T_\Sigma Y$ we can form an equation morphism*

$$\overline{e} = (X \xrightarrow{e} H_\Sigma X + Y \xrightarrow{\varepsilon_X + \eta_Y^C} HX + CY).$$

Then the square below commutes:

$$\begin{tikzcd} X \arrow[r, "\overline{e}^\dagger"] \arrow[d, "e^\ddagger"'] & CY \arrow[d, "m_Y"] \\ T_\Sigma Y \arrow[r, "\hat{\varepsilon}_Y"'] & TY \end{tikzcd} \tag{5.1}$$

Proof. Put

$$\widetilde{e} = (X \xrightarrow{e} H_\sigma X + Y \xrightarrow{\varepsilon_X + \eta_Y} HX + TY).$$

We prove that both sides of the square (5.1) are solutions of $\widetilde{e}$ in the cia TY for H.

(1) That $\widehat{\varepsilon}_Y \cdot \overline{e}^{\ddagger}$ solves $\widetilde{e}$ is due to the following diagram:

$$\begin{tikzcd}
X \arrow[r, "e^{\ddagger}"] \arrow[d, "e"'] \arrow[dd, bend right=60, "\tilde{e}"'] & T_\Sigma Y \arrow[r, "\hat{\varepsilon}_Y"] & TY \\
H_\Sigma X + Y \arrow[r, "H_\Sigma e^{\ddagger}+Y"'] \arrow[d, "\varepsilon_X+\eta_Y"'] & H_\Sigma T_\Sigma Y + Y \arrow[r, "H_\Sigma \hat{\varepsilon}_Y+Y"'] \arrow[u, "{[\tau_Y^\Sigma,\eta_Y^\Sigma]}"'] \arrow[d, "\varepsilon_{TY}+\eta_Y"] & H_\Sigma TY + Y \arrow[u, "{[\tau_Y\cdot\varepsilon_{TY},\eta_Y]}"] \arrow[d, "\varepsilon_{TY}+\eta_Y"'] \\
HX + TY \arrow[r, "He^{\ddagger}+Y"'] & HT_\Sigma Y + Y \arrow[r, "H\hat{\varepsilon}_Y+TY"'] & HTY + TY \arrow[uu, bend right=60, "{[\tau_Y,TY]}"']
\end{tikzcd}$$

The left-hand part commutes by the definition of $\widetilde{e}$, and the right-hand one does so trivially. The upper left-hand square commutes by the definition of $e^{\ddagger}$. For the lower two consider the coproduct components separately: the left-hand one commutes since ε is natural, and the right-hand one trivially commutes. And for the remaining upper right-hand part one considers the coproduct components separately once more: the right-hand one states that $\widehat{\varepsilon}_Y \cdot \eta^{\Sigma}_X = \eta_Y$, and for the left-hand one we use that TY considered as an H_Σ-algebra (with the structure $\tau_Y \cdot \varepsilon_{TY}$) is a quotient of the free H_Σ-algebra $(T_\Sigma Y, \tau^{\Sigma}_Y)$ via the quotient algebra morphism $\widehat{\varepsilon}_Y$ as explained in Remark 5.10(2).

(2) That $m_Y \cdot \overline{e}^{\dagger}$ solves $\widetilde{e}$ is due to the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "\overline{e}^{\dagger}"] \arrow[d, "e"'] \arrow[ddd, bend right=60, "\tilde{e}"'] & & CY \arrow[r, "m_Y"] & TY \\
H_\Sigma X+Y \arrow[dd, "\varepsilon_X+\eta_Y"] \arrow[dr, "\varepsilon_X+\eta_Y^C"] & & & \\
& HX+CY \arrow[r, "H\overline{e}^{\dagger}+CY"'] \arrow[dl, "HX+m_Y"] & HCY+CY \arrow[uu, "{[\psi_Y,CY]}"'] \arrow[d, "HCY+m_Y"] & \\
HX+TY \arrow[rr, "H\overline{e}^{\dagger}+TY"'] & & HCY+TY \arrow[r, "Hm_Y+TY"'] & HTY+TY \arrow[uuu, "{[\tau_Y,TY]}"']
\end{tikzcd}$$

The left-hand part commutes by the definition of $\widetilde{e}$, and the upper left-hand inner part commutes since $\overline{e}^{\dagger}$ is a solution of $\overline{e}$. For the triangle on the left consider the coproduct components separately: the right-hand one commutes since $m_Y \cdot \eta^C_Y = \eta_Y$ (see Notation 5.9), and the left-hand component trivially commutes. The middle lower part obviously commutes. Finally, for the right-hand part consider the coproduct components separately once more: the left-hand component commutes since m_Y is an H-algebra morphism from (CY, ψ_Y) to (TY, τ_Y), and the right-hand component trivially commutes. $\square$

Theorem 5.13. *For a standard finitary set functor H the following conditions are equivalent:*

(1) H is a cia functor,

(2) $H = H_0(-) + Y$ where H_0 preserves countable coproducts and Y is a set, and

(3) $H = W \times (-) + Y$ for some sets W and Y.

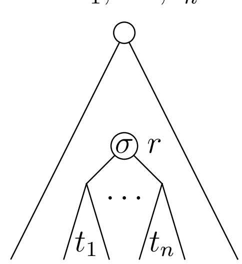
Proof. (2) $\Rightarrow$ (3). Since H is finitary, so is H_0, by the description of finitarity following Assumptions 5.1. Therefore, H_0 preserves all coproducts. Trnková proved [18, Theorem IX.8], that every coproduct-preserving set functor preserves colimits, thus it is a left adjoint. It is well known that the only right adjoint set functors R are the representable ones: for given $L \dashv R$, put $W = L1$, then the elements $1 \to RY$ bijectively correspond to the maps $W \to Y$, thus, R is naturally isomorphic to $\mathsf{Set}(W, -)$. Consequently, H_0 is left adjoint to $\mathsf{Set}(W, -)$, hence it is naturaly isomorphic to $W \times (-)$.

(3) $\Rightarrow$ (1). This follows from Corollary 4.6.

(1) $\Rightarrow$ (2). Let $\varepsilon : H_\Sigma \to H$ be a reduced presentation.

(a) We prove below that all arities in Σ are 1 or 0. Let W be the set of all unary symbols and Y that of all constants. Then $H_\Sigma X = W \times X + Y$. Furthermore, it follows that ε is a natural isomorphism. Indeed, each ε_X is, besides being surjective, also injective: it cannot merge distinct elements (w, x) and (w', x') of $W \times X$ because this would yield an ε-equation $w(x) = w'(x')$. Since the presentation is reduced, this implies $w = w'$ and $x = x'$. Analogously for all other pairs of elements of $H_\Sigma X$.

(b) Assume that some symbol α of Σ has arity at least 2. Then we derive a contradiction to H being a cia functor. Given a Σ-tree t we call a node r *pure* if the trees $t_1, \dots, t_n$ rooted at the children of r are pairwise distinct:



(σ an n-ary operation symbol).

Observe that an ε-equation applicable to a pure node r must have the form

$$\sigma(x_1, \ldots, x_n) = \tau(y_1, \ldots, y_m)$$

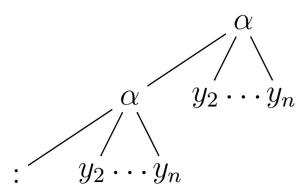
for some $\tau \in \Sigma_m$, where $x_1, \ldots, x_n$ are pairwise distinct.

Consider the following equation morphism $e : X \to H_\Sigma X + Y$ with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_2, \ldots, y_n\}$:

$$e(x_1) = \alpha(x_1, y_2, \ldots, y_n) \qquad \text{and} \qquad e(x_i) = y_i \quad \text{for } i = 2, \ldots, n.$$

Then the unique solution $e^\ddagger : X \to T_\Sigma Y$ assigns to x_1 the Σ-tree below:

$$e^\ddagger(x_1) =$$



Next consider the equation morphism

$$\overline{e} = (X \xrightarrow{e} H_\Sigma X + Y \xrightarrow{\varepsilon_X + \eta_Y^C} HX + CY).$$

Since CY is a cia, this has a unique solution $e^\dagger : X \to CY$. It assigns to x_1 an element of CY which by Proposition 5.11 has the form

$$\overline{e}^\dagger(x_1) = \overline{\varepsilon}_Y(s) \qquad \text{for some } s \in C_\Sigma Y,$$

where $\overline{\varepsilon}_Y : C_\Sigma Y \twoheadrightarrow C_\Sigma Y/{\sim^*} \cong CY$ denotes the canonical quotient map. From Lemma 5.12 we know that

$$\widehat{\varepsilon}_Y(t) = \widehat{\varepsilon}_Y \cdot e^\ddagger(x_1) = m_Y \cdot \overline{e}^\dagger(x_1) = m_Y \cdot \overline{\varepsilon}_Y(s).$$

Therefore, we obtain $t \sim_Y^* s$.

We derive the desired contradiction by proving that every tree obtained from t by a finite application of ε-equations has a leaf labeled by y_2 at every positive level. From this we conclude immediately that the same holds for all trees obtained from t by an infinite application of ε-equations. However, $t \sim_Y^* s$ where s has only finitely many leaves labeled by y_2.

(b1) Assume first that a single ε-equation is applied to t and let t' be the resulting tree. Let r be the node of t at which the application takes place.

Then r is not a leaf labeled in Y; for recall that all ε-equations have operation symbols on both sides, thus, they are not applicable to leaves labeled in Y. Therefore, r is a pure node labeled by α. The ε-equation in question thus has the form

$$\alpha(u_1, \dots, u_n) = \tau(z_1, \dots, z_m)$$

for some $\tau \in \Sigma_m$ and with the u_i pairwise distinct.

If r has depth k, then the tree t' has label y_2 at all levels $1, \dots, k$, since those leaves of t are unchanged. Furthermore, we have $u_2 = z_p$ for some $p = 1, \dots, m$ since ε is a reduced presentation. Therefore, y_2 occurs at level $k+1$ because the p-th child of r in t' is a leaf labeled by y_2. For the levels greater than $k+1$ we use that $u_1 = z_q$ holds for some $q = 1, \dots, m$, again because ε is a reduced presentation. Since the first subtree of r in t is t itself, it follows that the q-th child of r in t' is t itself. Thus, a label y_2 of depth n in t yields a label y_2 of depth $k+1+n$ of t'.

(b2) Assume next that two ε-equations are applied to t. The resulting tree t'' can be obtained from t' in (b1) by a single application of an ε-equation. Let r' be the node of t' at which the application takes place. We can assume $r \neq r'$ (for if $r = r'$ we can obtain t'' from t by a single application on an ε-equation; this follows from Remark 5.3). If r' does not lie in the subtree of t' with root r, then r' is a pure node labeled by α and we argue as in (b1).

Suppose therefore that r' lies in the subtree rooted at r. If this is the q-th subtree from (b1) above (the one with $u_1 = z_q$), then we also argue as in (b1) using that the q-th subtree is t itself. Otherwise, if r' lies in any other subtree of r, then the labels y_2 of the q-th subtree are unchanged.

The remaining cases of three and more applications of ε-equations are completely analogous. This yields the desired contradiction: if $t \sim^* \bar{t}$, then $\bar{t}$ has label y_2 at every level $1, 2, 3, \dots$, thus $t \sim_Y^* s$ cannot be true. $\square$

Remark 5.14. (1) The implication (1) $\implies$ (2) does not hold for non-standard finitary set functors. A counterexample is the functor C_1^2 which maps the empty set to 2 and all non-empty sets to 1. This functor is easily seen to be a cia functor, but it is not of the form stated in (2) and (3).

(2) For finitary, but not necessarily standard, set functors, Theorem 5.13 can also be formulated. In this formulation one replaces (3) by

$$H \cong W \times (-) \times C_y$$

for a function $y : Y_0 \to Y$, and where C_y is the functor which is constant of value Y on the full subcategory Set' of all non-empty sets and maps, whereas $C_y\emptyset = Y_0$ and $C_y f = y$ for every $f : \emptyset \to X$ with $X \neq \emptyset$. The result that every finitary cia functor has the above form then follows from Theorem 5.13 and the following facts:

(a) If two set functors agree on Set', then they have the same corecursive algebras and the same cias.

(b) For every set functor H there exists a standard set functor $\hat{H}$ (called the *Trnková hull* of H) which agrees with H on Set' [8, Theorem III.4.5]. In fact, $\hat{H}\emptyset$ can be defined as the equalizer

$$\hat{H}\emptyset \xrightarrow{\ e\ } H1 \underset{Hf}{\overset{Ht}{\rightrightarrows}} H2,$$

where $t, f : 1 \hookrightarrow 2$ are the two obvious inclusions.

(c) If H is a finitary set functor whose Trnková hull has the form $W \times (-) + Y$, then $H \cong W \times (-) + C_y$ where $Y_0 = H\emptyset$, $Y = \hat{H}\emptyset$ and $y : H\emptyset \to \hat{H}\emptyset$ is the unique morphism with $e \cdot y = Hu$ where $u : \emptyset \to 1$ is the empty map.

Example 5.15. The implication (2) $\implies$ (3) in Theorem 5.13 does not generalize to non-finitary functors; at least, not if we assume that a measurable cardinal exists. Recall that this means that an ultrafilter $\mathcal{U}$ on a set Z exists which

(a) is ω-complete, i.e., for every disjoint decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$ we have $X_n \in \mathcal{U}$ for some n, and

(b) nontrivial, i.e., $\mathcal{U}$ contains no finite subset of X.

The subfunctor U_0 of the ultrafilter functor of Example 3.8 of all ω-complete ultrafilters preserves countable coproducts; indeed, every element of $U_0(\coprod_{n\in\mathbb{N}} X_n)$ is an ultrafilter containing X_n for some $n \in \mathbb{N}$ and then it corresponds to an element of $U_0 X_n$. Since $U_0 1 = 1$, we have that $\nu U_0 = 1$.

By Theorem 4.6, U_0 is a cia functor. However, U_0 is not a finitary functor since the above ultrafilter $\mathcal{U} \in U_0 Z$ does not correspond to a (trivial) ultrafilter on a finite subset of Z. Hence, U_0 is not of the form $W \times (-) + Y$ as in Theorem 5.13.

6. Elgot Algebras and Bloom Algebras

Throughout this section H denotes an endofunctor on a hyper-extensive category preserving countable coproducts and having a terminal coalgebra νH. We are going to prove that complete Elgot algebras for H [6] coincide with Bloom algebras [3] and are precisely the algebras for the completely iterative monad [1]. We know that H is then *iteratable*, i.e., for every Y the terminal coalgebra TY for $H(-) + Y$ exists, viz.

$$TY = \coprod_{n<\omega} H^n Y + \nu H.$$

This is the free cia on Y, by Corollary 3.6. According to Corollary 4.4, TY is also the free corecursive algebra on Y.

The assignment of a free cia TY to the given object Y is well-known to yield a monad $\mathbb{T}$; in fact, this monad is the *free completely iterative monad* on H, see [1, 15]. We will not recall the notion of a completely iterative monad here, as it is not needed in the present paper. However, let us recall that the unit of the monad $\mathbb{T}$ is given by $\eta_Y : Y \to TY$ from Example 2.8(1), and the multiplication is given by freeness: $\mu_Y : TTY \to TY$ is the unique algebra morphism extending id_{TY} from the free cia TTY on TY to the cia TY.

The present section concerns the Eilenberg-Moore algebras for the monad $\mathbb{T}$. In previous joint work with J. Velebil [6] we called them complete Elgot algebras and described them as algebras for H equipped with an operation $(-)^\dagger$ that assigns to every equation morphism $e : X \to HX + A$ a solution $e^\dagger : X \to A$ satisfying two easy and well-motivated axioms that we now recall.

Notation 6.1. Given morphisms $e : X \to HX + Y$ and $h : Y \to Z$ we write

$$h \bullet e = (X \xrightarrow{e} HX + Y \xrightarrow{HX+h} HX + Z).$$

Definition 6.2. A *complete Elgot algebra* for H is a triple $(A, a, \dagger)$ where $a : HA \to A$ is an algebra and $\dagger$ is an operation that assigns to every equation morphism $e : X \to HX + A$ a solution $e^\dagger : X \to A$ (i.e., the square (2.2) commutes) such that the following two properties hold:

(1) *Functoriality*: given two equation morphisms $e : X \to HX + A$ and $f : Y \to HY + A$ and a coalgebra morphism $h : X \to Y$ we have that

$f^\dagger \cdot h = e^\dagger$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "h"'] & HX + A \arrow[d, "Hh+A"] \\ Y \arrow[r, "f"'] & HY + A \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} X \arrow[dr, "e^\dagger"] \arrow[dd, "h"'] & \\ & A \\ Y \arrow[ur, "f^\dagger"'] & \end{tikzcd}$$

(2) *Compositionality*: Given $e : X \to HX + Y$ and $f : Y \to HY + A$ we form the following equation morphism

$$e \blacksquare f = (X{+}Y \xrightarrow{[e,\mathsf{inr}]} HX{+}Y \xrightarrow{HX+f} HX{+}HY{+}A \xrightarrow{\mathsf{can}+A} H(X{+}Y){+}A);$$

compositionality states that

$$(e \blacksquare f)^\dagger \cdot \mathsf{inl} = (f^\dagger \bullet e)^\dagger : X \to A.$$

A morphism of complete Elgot algebras from $(A, a, \dagger)$ to $(B, b, \ddagger)$ is a morhism $h : A \to B$ *preserving solutions*, i.e., for every $e : X \to HX + A$ the following triangle commutes:

$$\begin{tikzcd} & X \arrow[dl, "e^\dagger"'] \arrow[dr, "(h \bullet e)^\ddagger"] & \\ A \arrow[rr, "h"'] & & B \end{tikzcd}$$

Recall that every morphism of complete Elgot algebras is an H-algebra morphism from (A, a) to (B, b) [6, Lemma 5.2]. Further recall from *loc. cit.* that every cia for H is a complete Elgot algebra; in fact, one readily proves that the operation assigning to a given equation morphism its unique solution satisfies functoriality and compositionality. Further examples of complete Elgot algebras are algebras on cpos with continuous algebra structure and algebras on non-empty complete metric spaces with contracting algebra structure [6].

Theorem 6.3 (Adámek, Milius, Velebil [6])**.** *The category of Eilenberg-Moore algebras for the monad $\mathbb{T}$ is isomorphic to the category of complete Elgot algebras.*

Note that this result does not require that H preserves coproducts; it holds for any iteratable endofunctor on a category with binary coproducts. Of course, in the light of Corollary 4.4, the monad $\mathbb{T}$ is also the monad of free corecursive algebras. For an accessible endofunctor on a locally presentable category we described the Eilenberg-Moore algebras for that monad as follows:

Definition 6.4 (Adámek, Haddadi, Milius [3]). A *Bloom algebra* is a triple $(A, a, \dagger)$ where $a : HA \to A$ is an H-algebra and $\dagger$ is an operation assigning to every coalgebra $e : X \to HX$ a coalgebra-to-algebra morphism $e^\dagger : X \to A$ so that $\dagger$ is functorial. This means that we obtain a functor

$$\dagger : \mathsf{Coalg}\, H \to \mathcal{C}/A.$$

More explicitly, given a coalgebra morphism h from (X, e) to (Y, f) we have $f^\dagger \cdot h = e^\dagger$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "h"'] & HX \arrow[d, "Hh"] \\ Y \arrow[r, "f"'] & HY \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} X \arrow[dr, "e^\dagger"] \arrow[dd, "h"'] & \\ & A \\ Y \arrow[ur, "f^\dagger"'] & \end{tikzcd}$$

Bloom algebras form a category together with solution preserving algebra morphisms (defined completely analogously as for complete Elgot algebras).

We will now prove that under our current assumption Bloom algebras and complete Elgot algebras are the same concept. Recall that the terminal coalgebra νH is considered as an algebra for H.

Theorem 6.5. *If H preserves countable coproduts and has a terminal coalgebra, then the following categories are isomorphic:*

(1) the Eilenberg-Moore category $\mathcal{C}^{\mathbb{T}}$ of the free completely iterative monad on H,

(2) the slice category $\nu H/\mathsf{Alg}\, H$

(3) the category of Bloom algebras for H, and

(4) the category of complete Elgot algebras for H.

Proof. The isomorphism (1) $\cong$ (4) was proved in [6, Theorem 5.8] for every iteratable endofunctor H.

The rest follows from various results in [3]. In that paper we assumed that H is accessible and $\mathcal{C}$ is locally presentable. However, for our purposes we only apply those result of *loc. cit.* that do not depend on those assumptions, as we now explain. First, the isomorphism (2) $\cong$ (3) was proved in [3, Proposition 3.4] for every endofunctor H having a terminal coalgebra νH.

The other results of *loc. cit.* we apply now make use of coproducts in $\mathsf{Alg}\, H$. But since H preserves countable coproducts, we know that the forgetful functor from $\mathsf{Alg}\, H$ to $\mathcal{C}$ creates countable coproducts. Hence, for example $TY = \coprod_{n<\omega} H^n Y + \nu H$ is a coproduct in $\mathsf{Alg}\, H$ of the free algebra $H^*Y = \coprod_{n<\omega} H^n Y$ on Y and the algebra $(\nu H, t^{-1})$. By [3, Theorem 3.16], TY is then a free Bloom algebra on Y. That is, the forgetful functor U_B of the category of Bloom algebra has the left adjoint T. It is now easy to prove that U_B is monadic, i.e., the isomorphism (1) $\cong$ (3) holds. The argument is given in the proof of [3, Theorem 4.15]; we repeat it here for the convenience of the reader (and to make clear that no extra assumptions are needed).

Before we proceed, let us recall [3, Lemma 3.7] that if $(A, a, \dagger)$ is a Bloom algebra and $h : (A, a) \to (B, b)$ is an algebra morphism, then there is a unique structure of a Bloom algebra on (B, b) such that h is a solution preserving algebra morphism.

We now prove that U_B is monadic. By Beck's Theorem [14, 4.4.4], it suffices to prove that U_B creates coequalizers of U_B-split pairs. This means that given a parallel pair of solution preserving algebra morphisms

$$f, g : (A, a, \dagger) \to (B, b, \ddagger)$$

and given morphisms in $\mathcal{C}$ as follows

$$\begin{array}{ll} k : B \to C & \text{with } k \cdot f = k \cdot g, \\ s : C \to B & \text{with } k \cdot s = id_C, \text{ and} \\ t : B \to A & \text{with } s \cdot k = f \cdot t \text{ and } id_B = g \cdot t, \end{array}$$

there exists a unique structure $(C, c, *)$ of a Bloom algebra such that k is a solution preserving algebra morphism; moreover, k is then a coequalizer in the category of Bloom algebras for H. Indeed, firstly, C carries a unique structure of an H-algebra such that k is an algebra morphism, namely:

$$c = (HC \xrightarrow{Hs} HB \xrightarrow{b} B \xrightarrow{k} C)$$

Secondly, by the above lemma there exists a unique structure $(C, c, *)$ of a Bloom algebra for which k is a solution preserving algebra morphism. It only remains to verify that k is a coequalizer in the category of Bloom algebras for H. To this end, let $h : (B, b, \ddagger) \to (D, d, +)$ be a solution preserving algebra morphism with $h \cdot f = h \cdot g$. There exists a unique algebra morphism $h' : (C, c) \to (D, d)$ with $h = h' \cdot k$. In order to see that h' preserves solutions (i.e., for every $e : X \to HX$ we have $h' \cdot e^* = e^+$) we use that both k and h do:

$$h' \cdot e^* = h' \cdot k \cdot e^\ddagger = h \cdot e^\ddagger = e^+. \qquad \square$$

7. Conclusions and Open Problems

For endofunctors H preserving countable coproducts and having a terminal coalgebra we have described the free corecursive algebra on an object Y as $\nu H + \coprod_{n<\omega} H^n Y$. In addition, we have shown that H is a cia functor, i.e., every corecursive algebra for H is a cia. For this we assumed that the base category has well-behaved countable coproducts, i.e., the category is hyper-extensive. It is an open problem whether our results hold in more general categories, e.g., in all extensive locally presentable ones.

For accessible functors H on locally presentable categories, the free corecursive algebra on Y was described in previous work [3] as the coproduct of FY (the free algebra on Y) and νH (considered as an algebra) in the category $\mathsf{Alg}\, H$. If H preserves countable coproducts, this is quite similar to the above desciption of the free cia, since coproducts of algebras are then formed on the level of the underlying category and therefore $FY = \coprod_{n<\omega} H^n Y$. But the proof techniques are completely different, and a common generalization of the two results is open.

We have also characterized all cia functors among standard finitary set functors: they are precisely the functors $X \mapsto W \times X + Y$ for some sets W and Y. Moreover, for arbitrary finitary set functors that formula holds for all non-empty sets. In Example 5.15 we have seen that the same result does not hold for all, not necessarily finitary, set functors. But that example required an assumption about set theory. It is an open problem whether that assumption was really necessary.

Our results can be stated in terms of corecursive monads [3] and completely iterative ones [1] as follows: a functor H having a terminal coalgebra νH and preserving countable coproducts has a free corecursive monad of the form $\coprod_{n<\omega} H^n(-) + \nu H$, and this is also the free completely iterative monad on H.

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