On Algebras with Iteration

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Abstract. Several concepts of algebras with solutions of recursive equation systems are compared: CPO-enrichable algebras are proved to be iteration algebras of Z. Ésik, and iteration algebras are a special case of the recently introduced Elgot algebras (which are the monadic algebras for the free iterative monad). Another special case of iteration algebras are the iterative algebras of E. Nelson and J. Tiuryn, which are algebras with unique solutions of all guarded systems. For each of the above classes of algebras an example is provided showing that the inclusion in a wider class is proper.

1 Introduction

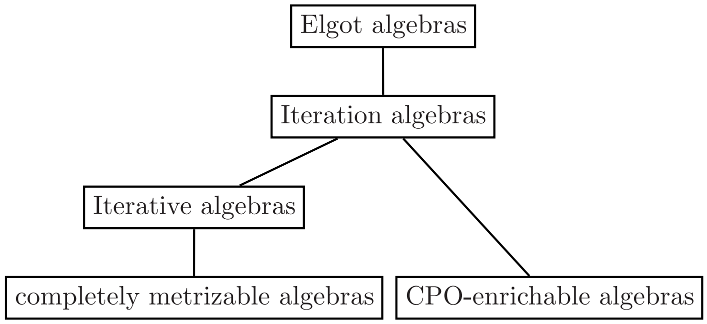
In program semantics we often need to consider models which are algebras such that all recursive specifications have a "clear" meaning. A classical case are CPO-enriched algebras where the least solution of a recursive specification is the canonical choice. Another approach, studied by Evelyn Nelson [24] and Jerzy Tiuryn [25] (based on iterative theories of Calvin Elgot [17]) are iterative algebras which we recall below: there all guarded recursive specifications have a unique solution. Zoltan Ésik introduced in [19] iteration algebras as algebras A in which every recursive specification has a solution and for which a choice of solutions can be performed so that certain axioms hold. These axioms are derived from those that all CPO-enriched algebras fulfil. The implication

$$\text{CPO-enriched} \implies \text{iteration algebra}$$

was proved in [15] for classical Σ-algebras, and we prove it for general endofunctors of **Set** as the main result of our paper.

We also compare the above models of recursion with two others: the first one are the completely metrizable algebras; more precisely, algebras enriched in **CMS**, the category of complete metric spaces, with contracting operations. We prove that these algebras are iterative, but not conversely: there are iterative algebras which are not completely metrizable. The other model are Elgot algebras which are the monadic algebras of the free iterative monad, see [8]: they are, like iteration algebras, algebras with a choice of solution for every recursive specification satisfying some axioms. In the case of algebras for an endofuntor, as considered in the present paper, the Elgot algebra axioms are a reduction of the axioms of iteration algebras. (The original concept of Ésik concerned iteration algebras for an algebraic theory; this is not subsumed in the concept of Elgot algebra). We present an example of an Elgot algebra that is not an iteration algebra.

The results of our paper are summarized in the following diagram of inclusions



We also provide examples demonstrating that each of the above inclusions is proper.

In section 2 we recall the concept of iterative algebra and provide examples demonstrating that an iterative algebra need not have a CPO-enrichment nor a complete metrization.

Elgot algebras and iteration algebras are introduced in Section 3, where we show that every iterative algebra is an iteration algebra but not conversely, and every iteration algebra is an Elgot algebra but not conversely.

The main result is presented in Section 4. We introduce the concept of canonical solution of an equation morphism in Elgot algebras, and then prove that for continuous algebras the least solution is always canonical. From that we conclude that every CPO-enrichable algebra is an iteration algebra. This generalizes the corresponding result for Σ-algebras presented in [15].

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2 Iterative Algebras and CIA's

Assumption 2.1. Throughout the paper H denotes an endofunctor of a category $\mathscr{A}$. Usually we take $\mathscr{A} = \mathbf{Set}$, but occasionally other examples (e.g. the category of complete metric spaces or CPOs) are considered. We assume that $\mathscr{A}$ has finite coproducts and we denote by inl and inr the coproduct injections of $X + Y$. The *canonical morphism* can: $HX + HY \to H(X + Y)$ has components $H\,\mathsf{inl}$ and $H\,\mathsf{inr}$.

For some results we will need the assumption that $\mathscr{A}$ is *locally finitely presentable* that is $\mathscr{A}$ has

(a) colimits and

(b) a set of *finitely presentable objects* X (which means that $\mathscr{A}(X,-)\colon \mathscr{A} \to \mathbf{Set}$ preserves filtered colimits) whose closure under filtered colimits is all of $\mathscr{A}$.

Notation 2.2. $\mathbf{Alg}\, H$ denotes the category of *algebras* for the endofunctor H, i.e., objects A of $\mathscr{A}$ together with a morphism $\alpha\colon HA \to A$ (the algebra structure). Morphisms, called *homomorphisms*, from (A,a) to (B,b) are the morphisms $f\colon A \to B$ of $\mathscr{A}$ for which $f{\cdot}a = b{\cdot}Hf$.

We also work with the dual concept of a *coalgebra* $a\colon A \to HA$. Coalgebra homomorphisms $f\colon (A,a) \to (B,b)$ are those morphisms $f\colon A \to B$ with $b{\cdot}f = Hf{\cdot}a$.

Example 2.3. (i) The classical Σ-algebras for a signature Σ are represented by $\mathscr{A} = \mathbf{Set}$ and $H = H_\Sigma$, the *polynomial functor* of the signature, which is defined on objects by

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^n \qquad n = \text{arity of } \sigma.$$

(ii) *Continuous algebras* are defined analogously, but instead of **Set** here we work with the category **CPO** whose objects (CPO's) are posets with joins of ω-chains and whose morphisms, the *continuous functions*, preserve joins of ω-chains. Since **CPO** has products and coproducts built up on those in **Set**, we can consider H_Σ above as an endofunctor of **CPO**. Then an H_Σ-algebra is simply a *continuous algebra* which means that its underlying set carries the structure of a CPO such that all operations are continuous functions.

(iii) Another important category is the category **CMS**. Its objects are complete metric spaces (i.e., such that every Cauchy sequence has a limit) with distances in the interval $[0,1]$. The morphisms, called nonexpanding maps, are the functions $f\colon (X,d_X) \to (Y,d_Y)$ for which the inequality $d_Y(f(x),f(x')) \le d_X(x,x')$ holds for all x,x' in X. Here also finite products and coproducts are built up on those in **Set**. Thus, for finitary signatures Σ we can consider H_Σ as an endofunctor of **CMS**. Then an H_Σ-algebra is a Σ-algebra whose underlying set carries the structure of a complete metric such that all operations are contracting.

Remark 2.4. The aim of our paper is to study solutions of recursive equations in a given algebra. For example, consider the endofunctor

$$H_\Sigma X = X \times X$$

of **Set** corresponding to algebras on one binary operation $*$. The recursive equation

$$x = x * x$$

has a solution in precisely those algebras which have an idempotent element. This is true, as we will see below, whenever the algebra can be enriched to a continuous algebra with a least element $\perp$. In fact, each such algebra has the least idempotent given by the join of the ω-chain

$$\perp \sqsubseteq \perp * \perp \sqsubseteq (\perp * \perp) * (\perp * \perp) \sqsubseteq \cdots$$

The concepts of iteration algebra and Elgot algebra studied below aim at a formalization of "canonical" solutions of recursive equational systems, where "canonical" can mean "the least one" in a continuous algebra. It can also mean "the unique one": as we will see below, completely metrizable algebras have unique solutions of all recursive equations.

We begin with the concept of algebras where this unique solvability plays a role:

Definition 2.5 (E. Nelson [24], J. Tiurin [25]). *Given a finitary (one-sorted) signature Σ, then a Σ-algebra A is called* ***iterative*** *provided that every finite system of guarded equations*

$$\begin{aligned} x_1 &= t_1(x_1, \ldots, x_n, a_1, \ldots, a_k) \\ &\vdots \\ x_n &= t_n(x_1, \ldots, x_n, a_1, \ldots, a_k) \end{aligned} \qquad \textit{with } a_1, \ldots, a_k \in A \tag{2.1}$$

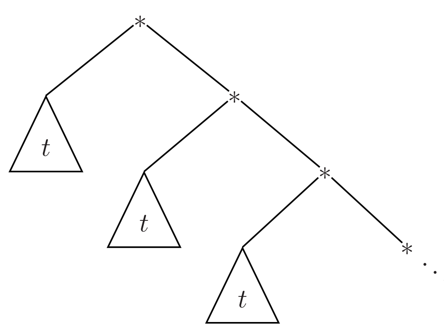
has a unique solution. Guardedness means that each right-hand side t_i is a term which is not equal to a single variable x_i.

Example 2.6. (i) An important example of an iterative Σ-algebra is the algebra

$$T_\Sigma Z \qquad \text{all } \Sigma\text{-trees on a set } Z.$$

Its elements are (rooted and ordered) labelled trees whose leaves are labelled in $\Sigma_0 + Z$ and nodes with out-degree p are labelled in Σ_p $(p > 0)$. Its operations are given by tree tupling. For example, let $\Sigma = \{*\}$ with $*$ binary. The unique solution of $x = x * x$ is the complete binary tree. The unique solution of $x = t * x$ is, for any tree t, the tree

$x^\dagger =$



(2.2)

(ii) The subalgebra

$$R_\Sigma Z$$

of $T_\Sigma Z$ on all *rational* trees, i.e., trees having up-to isomorphism only finitely many subtrees, is also iterative. Observe that e.g. the tree (2.2) is rational whenever t is.

Remark 2.7. (i) For Σ-algebras the concept of iterative algebra can be reformulated by means of *flat* systems of recursive equations: these are systems (2.1) where the right-hand side terms $t_j(x_1, \ldots, x_n, a_1, \ldots, a_k)$ have the simple form either $t_j = \sigma(x_{i_1}, \ldots, x_{i_p})$ for some p-ary symbol σ or $t_j = a_k$ (an element of A). It is easy to see that every system (2.1) can be substituted by a flat system (using additional variables) having the same solutions. Example: the equation $x = t * x$ can be substituted by the flat system

$$\begin{aligned} x &= y * x \\ y &= t \end{aligned}$$

Therefore, a Σ-algebra is iterative iff every flat system of equations (2.1) has a unique solution.

A flat equation system assigns to every variable in the set $X = \{x_1, \ldots, x_n\}$ an element of the set $H_\Sigma X + A$. Thus, it is a morphism $e\colon X \to H_\Sigma X + A$ with X a finite set. This generalizes to arbitrary endofunctors, see Definition 2.8.

For a flat equation morphism

$$e\colon X \to H_\Sigma X + A$$

a *solution* $e^\dagger$ assigns to every variable in X an element in A. Thus, $e^\dagger$ is a morphism

$$e^\dagger\colon X \to A.$$

To say that $e^\dagger$ is a solution of e means that the formal equations in (2.1) become identities under the substitution $x/e^\dagger(x)$ for all $x \in X$. This is equivalent to stating that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle [a,A]} \\ H_\Sigma X + A & \xrightarrow[H_\Sigma e^\dagger + A]{} & H_\Sigma A + A \end{array} \tag{2.3}$$

commutes.

(ii) We can also consider equation systems with infinitely many variables in place of just $x_1, \ldots, x_n$. Again, it is sufficient to consider flat systems, and again, these are morphisms of the form $e\colon X \to H_\Sigma X + A$, but now X can be arbitrary. The concept of iterative algebra is, however, not stable under this type of generalization. For example, the iterative algebra $R_\Sigma Z$ of rational trees does not have solutions of non-finitary flat systems of equations in general. We thus need the concept of a (completely) iterative algebra. Recall the concept of finitely presentable object from 2.1. In **Set** these are precisely the finite sets.

Definition 2.8. *Let A be an object of the category $\mathscr{A}$.*

(i) *By a **flat equation morphism** in A is meant a morphism $e\colon X \to HX + A$. The morphism e is called **finitary** provided that X is a finitely presentable object.*

(ii) *A* ***completely iterative algebra*** *(or CIA, for short) is an algebra* $a\colon HA \to A$ *such that every flat equation morphism* $e\colon X \to HX + A$ *has a unique* ***solution****, i.e., a unique morphism* $e^\dagger\colon X \to A$ *for which the square*

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle [a,A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array} \tag{2.4}$$

commutes.

(iii) *The algebra A is called* ***iterative*** *provided that every finitary flat equation morphism has a unique solution.*

Remark 2.9. We use the concept of iterative algebra only in case H is a *finitary functor*, in other words, H preserves filtered colimits.

Examples 2.10. Here we consider the category of sets.

(1) Unary algebras are iterative (for $H = \mathrm{Id}$) iff the operation $a\colon A \to A$ has a unique fixed point x and no cycles of length > 1. And they are CIAs iff, moreover, for every infinite sequence x_n ($n \in \mathbb{N}$) in A with $ax_{n+1} = x_n$ we have $x_n = x_0$ for all n. See [7] and [21].

(2) For general signatures Σ no simple description of iterative algebras is known. An example of a CIA is the algebra $T_\Sigma Z$. The algebra $R_\Sigma Z$ is iterative, but it is not a CIA; this is a free iterative Σ-algebra on Z, see [24].

Examples 2.11. Given complete metric spaces X and Y, then every hom-set $\mathbf{CMS}(X, Y)$ carries the pointwise metric $d_{X,Y}$ defined as follows: Given nonexpanding maps $f, g\colon X \to Y$ then

$$d_{X,Y}(f, g) = \sup_{x \in X} d_Y\big(f(x), g(x)\big).$$

America and Rutten [13] call a functor $H\colon \mathbf{CMS} \to \mathbf{CMS}$ *contracting* if there exists a constant $\varepsilon < 1$ such that for arbitrary morphisms $f, g\colon X \to Y$ we have

$$d_{HX,HY}(Hf, Hg) \leq \varepsilon \cdot d_{X,Y}(f, g).$$

For every contracting endofunctor of **CMS** all nonempty algebras $a\colon HA \to A$ were proved in [21] to be CIAs: the unique solution of an equation morphism $e\colon X \to HX + A$ is obtained as a limit

$$e^\dagger = \lim_{n \in \mathbb{N}} e_n^\dagger$$

of the Cauchy sequence of approximations. Here $e_0^\dagger\colon X \to A$ is an arbitrary nonexpanding map, and given $e_n^\dagger$ we define $e_{n+1}^\dagger$ by "approximating" the square of Definition 2.8:

$$
\begin{array}{ccc}
X & \xrightarrow{\;e_{n+1}^{\dagger}\;} & A \\
{\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\
HX + A & \xrightarrow[\;He_n^{\dagger}+A\;]{} & HA + A
\end{array}
\tag{2.5}
$$

Many set functors H have a lifting to contracting endofunctors H' of **CMS**. That is, for the forgetful functor $U\colon$ **CMS** $\to$ **Set** the following square

$$
\begin{array}{ccc}
\mathbf{CMS} & \xrightarrow{\;H'\;} & \mathbf{CMS} \\
{\scriptstyle U}\downarrow & \cong & \downarrow{\scriptstyle U} \\
\mathbf{Set} & \xrightarrow[\;H\;]{} & \mathbf{Set}
\end{array}
\tag{2.6}
$$

commutes up to natural isomorphism. For example, if $HX = X^n$, define

$$H'(X,d) = \big(X^n, \tfrac{1}{2} \cdot d'\big)$$

(where d' is the maximum metric) which is a contracting functor with $\varepsilon = \frac{1}{2}$. Since coproducts of $\frac{1}{2}$-contracting liftings are $\frac{1}{2}$-contracting liftings of coproducts, we conclude that every polynomial endofunctor has a contracting lifting to **CMS**.

Definition 2.12. *Let H be a set functor with a lifting* (2.6) *to* **CMS**. *We call an H-algebra $\alpha\colon HA \to A$ completely metrizable if there exists a complete metric, d, on A such that α is a nonexpanding map from $H'(A,d)$ to (A,d).*

Every completely metrizable algebra A is a CIA see [21]—but not conversely:

Example 2.13. A completely iterative Σ-algebra which is not completely metrizable. Let Σ be a unary signature of two operation symbols f and g. Let A be the algebra on the set

$$\big\{a_n\big\}_{n\in\mathbb{N}} \cup \{b, c\},$$

where the operations f, g are defined by:

$$
\begin{aligned}
f(a_i) = g(a_i) &= a_{i+1}, \qquad i = 0, 1, \ldots \\
f(b) = f(c) &= b \\
g(b) = g(c) &= c.
\end{aligned}
$$

The algebra A is a CIA since each of the operations f, g has a unique fixed point in $\{b, c\}$; in fact, both f, g are constant on $\{b, c\}$. Thus, each nonempty composite of these operations has a unique fixed point in $\{b, c\}$, and none of the a_i's is fixed under a nonempty composite of these operations.

If A were completely metrizable, the sequence

$$f(a_0), f^2(a_0), f^3(a_0), \ldots$$

would converge to the fixed point of f; thus

$$\lim_{n\to\infty} a_n = b.$$

But for the analogous reason, using g in lieu of f,

$$\lim_{n\to\infty} a_n = c.$$

Examples 2.14. Here we work in the category CPO, see Example 2.3. A CPO is called *strict* if it has a least element $\bot$. Notice that each hom-set $\mathbf{CPO}(X,Y)$ is a CPO under the pointwise ordering. An endofunctor H of $\mathbf{CPO}$ is called *locally continuous* if each induced map $\mathbf{CPO}(X,Y) \to \mathbf{CPO}(HX,HY)$ is continuous, see [26]. Given a locally continuous endofunctor H of $\mathbf{CPO}$, the following holds for all strict algebras $\alpha\colon HA \to A$: every flat equation morphism $e\colon X \to HX+A$ has the least solution given by the join

$$e^\dagger = \bigsqcup_{n\in\mathbb{N}} e_n^\dagger \colon X \to A$$

where $e_0^\dagger\colon X \to A$ is the least element of $\mathbf{CPO}(X,A)$ (the constant map of value $\bot$) and given $e_n^\dagger$ we define $e_{n+1}^\dagger$ by the same approximation as in (2.5).

However, solutions are, in general, not unique. For example, the identity functor is locally continuous, and the algebra $A = \{\bot, 0\}$ in which $\alpha\colon A \to A$ is the identity map has the smallest solution of every system of equations. But for the equation $\mathsf{inl}\colon 1 \to 1 + A$ both $\bot$ and 0 are solutions.

Many set functors H have a lifting to locally continuous endofunctors H' of $\mathbf{CPO}$. That is, for the forgetful functor $U\colon \mathbf{CPO} \to \mathbf{Set}$ the following square

$$\begin{array}{ccc} \mathbf{CPO} & \xrightarrow{H'} & \mathbf{CPO} \\ {\scriptstyle U}\downarrow & \cong & \downarrow{\scriptstyle U} \\ \mathbf{Set} & \xrightarrow[H]{} & \mathbf{Set} \end{array} \tag{2.7}$$

commutes up to natural isomorphism. For example, if $HX = X^n$ we can define $H'(X,\sqsubseteq)$ to have the componentwise ordering on X^n, this is a locally continuous functor. Since coproducts of locally continuous functors are locally continuous, we conclude that every polynomial endofunctor has a locally continuous lifting to $\mathbf{CPO}$.

Definition 2.15. *Let H be a set functors with a lifting* (2.7) *to* $\mathbf{CPO}$*. We call an H-algebra $a\colon HA \to A$* CPO*-enrichable if there exists a* CPO *ordering $\sqsubseteq$ with $\bot$ on A such that a is a continuous function from $H'(A,\sqsubseteq)$ to $(A,\sqsubseteq)$.*

Example 2.16. Not every iterative algebra is CPO-enrichable. We demonstrate this on unary algebras: consider the algebra on the unit interval $I = [0,1]$ whose operations are all endofunctions f which are contracting with $\varepsilon = \frac{1}{2}$:

$$\big|f(x) - f(y)\big| \le \tfrac{1}{2}|x-y|.$$

This algebra is **CMS**-enrichable, thus completely iterative as explained in Example 2.11: consider the usual metric on I. We show there is no CPO structure $\leq$ on I, with least element b, say, such that for every contracting self map $f\colon I \to I$, the unique fixed point of f is the least upper bound of the sequence

$$b, f(b), f^2(b), \ldots$$

Indeed, for each $b \in I$, we will define two contractions f, g such that

$$\begin{aligned}
&f(b) = g\big(g(b)\big)\\
&g(b) = f\big(f(b)\big)\\
&f(b) \neq g(b).
\end{aligned}$$

Thus, for any CPO-structure on A,

$$\sup_n f^n(b) = \sup_n g^n(b).$$

But

$$b \leq f(b) \leq f\big(f(b)\big) = g(b),$$

so that $f(b) \leq g(b)$. But similarly,

$$b \leq g(b) \leq g\big(g(b)\big) = f(b),$$

so $g(b) \leq f(b)$, shoving $f(b) = g(b)$, contradicting the assumptions.

There are two cases. If $b = 0$, define

$$\begin{aligned}
f(x) &:= \frac{x+1}{2}\\
g(x) &:= \frac{9-4x}{12}
\end{aligned}$$

Then f and g are contractions with constant $1/2$. We have

$$f(0) = 1/2 \neq 3/4 = g(0),$$

and

$$\begin{aligned}
g(3/4) &= 1/2\\
f(1/2) &= 3/4,
\end{aligned}$$

completing the proof.

If $b \neq 0$, let

$$f(x) := x/2$$

and let $g(x)$ be any contraction with $g(b) = f(f(b)) = b/4$ and $g(g(b)) = g(b/4) = f(b) = b/2$. For example, let $g(x)$ be the piece-wise linear function defined by

$$g(x) := \begin{cases} b/2 & 0 \leq x \leq b/4\\ -x/3 + 7b/12 & b/4 \leq x \leq b\\ b/4 & b \leq x \leq 1. \end{cases}$$

Notation 2.17 (see [1]). Given an object Z of the base category $\mathscr{A}$ such that the endofunctor $H(-) + Z$ has a terminal coalgebra

$$TZ$$

then TZ is a CIA, in fact, a free CIA on Z. More detailed: by Lambek's Lemma the structure morphism

$$\alpha_Z \colon TZ \to HTZ + Z$$

of the terminal coalgebra is invertible. We denote the components of α_Z^{-1} by

$$\tau_Z \colon HTZ \to TZ \qquad \text{(an algebra structure)}$$

and

$$\eta_Z \colon Z \to TZ.$$

Then (TZ, τ_Z) is a free CIA on Z where η_Z is the corresponding universal morphism, see [21].

Definition 2.18 (see [1]). *An endofunctor H is called **iteratable** provided that for every object Z a terminal coalgebra TZ for $H(-) + Z$ exists. The monad $\mathbb{T}$ given objectwise by $Z \mapsto TZ$ (more precisely: the free-*CIA *monad) is called the **free completely iterative** monad on H.*

In fact, the concept of a completely iterative monad was introduced in [18], and T is indeed a free completely iterative monad on H, see [1].

Remark 2.19. The finitary variant of 2.17 and 2.18 can be performed in every locally finitely presentable category of Peter Gabriel and Friedrich Ulmer, see [12] or [20], which is a cocomplete category $\mathscr{A}$ with a set of finitely presentable objects whose closure under filtered colimits is all of $\mathscr{A}$. (Examples: **Set**, **Pos**, or **Alg** K for every finitary endofunctor K of **Set**.) Let H be a finitary endofunctor of $\mathscr{A}$. For every object Z of $\mathscr{A}$ there exists a free iterative algebra on Z, see [7], and we denote its algebra structure by

$$\hat{\tau}_Z \colon HRZ \to RZ$$

and its universal arrow by

$$\hat{\eta}_Z \colon Z \to RZ.$$

Analogously to 2.17 we have

$$RZ = HRZ + Z$$

with coproduct injections $\hat{\tau}_Z$ and $\hat{\eta}_Z$, see [7].

Remark 2.20. (a) It was proved in [7] that the monad $\mathbb{R} = (R, \hat{\mu}, \hat{\eta})$ of free iterative algebras for H, where

$$\hat{\mu}_Z \colon RRZ \to RZ$$

denotes the unique homomorphism of (iterative) algebras with $\hat{\mu}_Z \cdot \hat{\eta}_{RZ} = \mathrm{id}$, is a free iterative monad on H. It is called the *rational monad* of H.

(b) Example: the rational monad of the polynomial functor H_Σ of **Set** is the monad

$$\mathbb{R}_\Sigma$$

of rational Σ-trees, see Example 2.10(2).

Notation 2.21 (see [8]). (i) For every CIA $a\colon HA \to A$ where H is iteratable we have the unique homomorphism

$$\tilde{a}\colon TA \to A \qquad \text{with} \quad \tilde{a}\cdot\eta_A = \mathrm{id}_A\,.$$

This represents the CIA as an Eilenberg-Moore algebra $(A, \tilde{a})$ for the monad $\mathbb{T}$.

(ii) For every iterative algebra

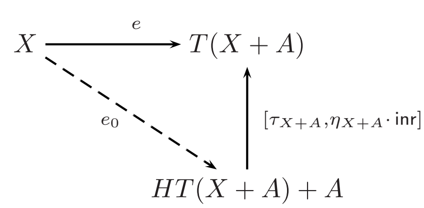
$$a\colon HA \to A$$

where H is finitary we have the unique homomorphism

$$\hat{a}\colon RA \to A \qquad \text{with} \quad \hat{a}\cdot\hat{\eta}_A = \mathrm{id}_A\,.$$

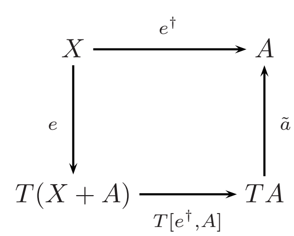
This represents A as an Eilenberg-Moore algebra for the monad $\mathbb{R}$.

Definition 2.22. *By an* ***equation morphism*** *in an object A is meant a morphism $e\colon X \to T(X + A)$. It is called* ***guarded*** *if e factorizes through the coproduct injection of $HT(X + A) + A$ in $T(X + A) = HT(X + A) + X + A$:*



(2.8)

If A is a CIA*, then a* ***solution*** *of e is a morphism $e^\dagger\colon X \to A$ such that the square*

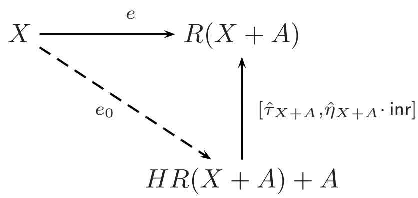


commutes.

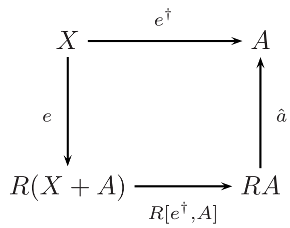
Remark 2.23. Guardedness generalizes here the concept mentioned in Definition 2.5: no right-hand side of the system (2.1) is a single variable, i.e., e is disjoint from the coproduct injection $\eta_{X+Y}\cdot\mathsf{inl}\colon X \to T(X + A)$. The following result demonstrates that solving flat equations suffices for solving guarded ones.

Theorem 2.24 ([21]). *If H is an iteratable endofunctor, then in every* CIA*, A, every guarded equation morphism has a unique solution $e^\dagger\colon X \to A$.*

Definition 2.25. *By a* ***rational equation morphism*** *in an object A is meant a morphism $e\colon X \to R(X+A)$ where X is finitely presentable. The morphism e is called* ***guarded*** *if it factorizes through the coproduct injection of $HR(X+A)+A$ into $R(X+A) = HR(X+A) + X + A$:*



If A is an iterative algebra, then a ***solution*** *of e is a morphism $e^\dagger\colon X \to A$ such that the square*



commutes.

Theorem 2.26 ([7]). *Let H be a finitary endofunctor of a locally fintely presentable category. Then in every iterative algebra A every guarded rational equation morphism has a unique solution.*

Remark 2.27. (i) In order to be able to compare iterative algebras with iteration algebras in Section 3 below, we need to provide solutions of *all*, not necessarily guarded, rational equation morphisms in every iterative algebra A. However, there are types of equations in systems (2.1) such as

$$x = x$$

(cycle of length 1)

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_1 \end{aligned}$$

(cycle of length 2) etc. that in non-trivial algebras do not have unique solutions. We call variables x_i which occur in a cycle of any length *ungrounded*. If the system (2.1) has no ungrounded variables, then it has a unique solution in every algebra, see [23] or [10].

(ii) The concept of an ungrounded variable can be introduced categorically: Let $e\colon X \to T(X+A)$ be an equation morphism and let

$$i_0 = X \xrightarrow{\mathsf{inl}} X + A \xrightarrow{\eta_{X+A}} T(X+A)$$

be the inclusion of variables. Form the pullback

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & X \\ {\scriptstyle e_1}\big\downarrow & & \big\downarrow{\scriptstyle e} \\ X_0 = X & \xrightarrow[i_0]{} & T(X+A) \end{array}$$

We conclude that

$$i_1\colon X_1 \to X$$

is the subobject of all variables for which the right-hand side of the equation system e is also a variable. Thus, the complementary subobject

$$\bar{i}_1\colon \overline{X}_1 \to X$$

(given by $X = X_1 + \overline{X}_1$ with injections i_1 and $\bar{i}_1$) is the subobject of all "guarded" variables. All ungrounded variables live in X_1. Moreover

$$e_1\colon X_1 \to X$$

is the domain-codomain restriction of e. Let us form the pullback

$$\begin{array}{ccc} X_2 & \xrightarrow{i_2} & X_1 \\ {\scriptstyle e_2}\big\downarrow & & \big\downarrow{\scriptstyle e_1} \\ X_1 & \xrightarrow[i_1]{} & X_0 \end{array}$$

Here

$$i_2 \cdot i_1\colon X_2 \to X$$

is the inclusion of variables in X_1 where the right-hand of the equation system is a variable from X_1. All ungrounded variables live in X_2. We then proceed by forming the analogous pullback

$$\begin{array}{ccc} X_3 & \xrightarrow{i_3} & X_2 \\ {\scriptstyle e_3}\big\downarrow & & \big\downarrow{\scriptstyle e_2} \\ X_2 & \xrightarrow[i_2]{} & X_1 \end{array}$$

etc., and conclude that the intersection of all $X_n \hookrightarrow X$:

$$i_\infty\colon X_\infty = \bigcap_{n\in\mathbb{N}} X_n \hookrightarrow X$$

is the set of all ungrounded variables.

Finally, let A be an algebra with a chosen element $\bot$. Then a solution $e^\dagger\colon X \to A$ is *strict* if $e^\dagger \cdot i_\infty$ is the constant function with value $\bot$.

Remark 2.28. Strict solution can also be defined for H-algebras A in abstract categories provided that

(a) a global element $\bot\colon 1 \to A$ has been chosen in A

and

(b) the abstract category has "well-behaved" coproducts.

Recall that a category is called *extensive* if it has finite coproducts which are

(i) universal, i.e., pullbacks along coproduct injections exist and form coproducts

and

(ii) disjoint, i.e., the pullback of two distinct coproduct injections has 0 (the initial object) as the domain.

It is easy to verify that every extensive category has the following property:

(iii) given coproduct injections $a_1\colon A_1 \to B$ and $a_2\colon A_2 \to B$ (of two coproducts) which are disjoint, then also $[a_1, a_2]\colon A_1 + A_2 \to B$ is a coproduct injection.

What we need in order to work easily with strict solutions is the following generalization of (i)–(iii) from finite to countable coproducts:

Definition 2.29. *A category is called* ***hyper-extensive*** *if it has countable coproducts which*

(a) *are disjoint and universal, and*
(b) *given pairwise disjoint coproduct injections $A_n \to B$ $(n \in \mathbb{N})$, then the induced morphism $\coprod A_n \to B$ is also a coproduct injection.*

Example 2.30. Sets, posets, graphs and presheaves form hyper-extensive categories. Also **CPO** and **CMS** are hyper-extensive. The (extensive) category of compact Hausdorff spaces is not hyper-extensive.

Definition 2.31 ([10]). *Let H be an iteratable endofunctor of a hyper-extensive category. For every equation morphism $e\colon X \to T(X + A)$ form pullbacks as follows:*

$$\begin{array}{ccccccc}
\cdots\ X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X \\
\downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} \\
\cdots\ X_2 & \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X = X_0 & \xrightarrow[i_0=\eta_{X+A}\cdot \mathsf{inl}]{} & T(X+A)
\end{array}$$

Then the subobjects

$$i_n^* = i_1 \cdot i_2 \cdot \cdots \cdot i_n\colon X_n \to X \qquad (n \geq 1)$$

are called the ***derived subobjects*** *of e.*

Remark 2.32. (i) Due to extensivity each i_n is a coproduct injection. This implies that the above pullbacks exist.

(ii) We thus have, for every n, a coproduct $X_n = X_{n+1} + \overline{X}_{n+1}$ where $i_n \colon X_{n+1} \to X_n$ is the above coproduct injection; the "complementary" coproduct injection is denoted by

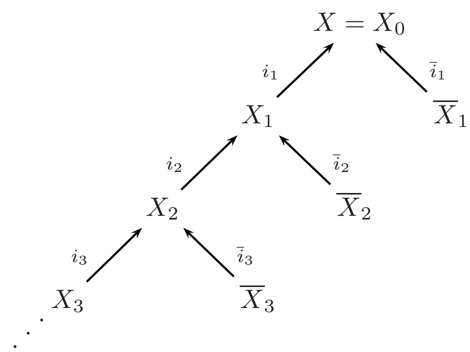
$$\bar{i}_n \colon \overline{X}_{n+1} \to X_n.$$

Consequently, $X_n = X_{n+1} + \overline{X}_{n+1}$ with coproduct injections i_{n+1} and $\bar{i}_{n+1}$.

(iii) The composites i_n^* are thus also coproduct injections, and so are

$$i_n^* \cdot \bar{i}_{n+1} \colon \overline{X}_{n+1} \to X.$$

The following diagram illustrates the situation:



(iv) The above coproduct injections $i_n * \bar{i}_{n+1}$ are pairwise disjoint. Due to hyper-extensivity we deduce that the induced morphism

$$[i_n^* \cdot \bar{i}_{n+1}] \colon \coprod_{n \in \mathbb{N}} \overline{X}_{n+1} \to X \tag{2.9}$$

is also a coproduct injection. This injection, then, represents the subobject of X "of all grounded variables". Let

$$i_\infty \colon X_\infty \to X \tag{2.10}$$

denote the "complementary" coproduct injection. That is, we have

$$X = X_\infty + \coprod_{n \in \mathbb{N}} \overline{X}_{n+1}$$

with injections i_∞ and (2.9). Then $i_\infty \colon X_\infty \to X$ is the subject "of all ungrounded variables".

Definition 2.33. *An equation morphism $e \colon X \to T(X + A)$ is called* ***preguarded*** *if it has no ungrounded variables, that is, $X_\infty = 0$, or equivalently*

$$X = \coprod \overline{X}_{n+1}.$$

Example 2.34. Consider for $H = H_\Sigma$ of **Set** a system (2.1) of equations as an equation morphism $e\colon X \to T(X + A)$. Then X_1 is the subobject of all variables x_i that are not guarded, i.e., where the equation in (2.1) has the form $x_i = x_j$. Consequently

$$\overline{X}_1 = \text{all “guarded” variables.}$$

The subobject X_2 consists of all variables $x_i \in X$ such that in (2.1) we have equations forming a 2-cycle, that is, of the form

$$\begin{aligned} x_i &= x_j \\ x_j &= x_k. \end{aligned}$$

Consequently, $\overline{X}_2$ (which is the complement of X_2 in X_1) consists of all variables x_i which are “guarded in two steps”: in (2.1) we have two equations

$$\begin{aligned} x_i &= x_j \\ x_j &= t \qquad\qquad \text{for } t \in H_\Sigma T(X + A) + A. \end{aligned}$$

And so on. We see that the subobject $\coprod_{n\in\mathbb{N}} \overline{X}_{n+1}$ consists of all grounded variables (i.e., those “guarded in finitely many steps”), thus

$$X_\infty = \text{all ungrounded variables.}$$

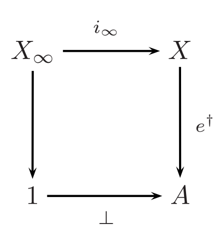
Remark 2.35. In the semantics based on CPOs, where one considers the least solution of equations, ungrounded variables are always assigned the value $\bot$ (the least element). It turns out that for providing canonical solutions for algebras in **Set** (where no order is considered), all one needs is to choose an element $\bot$ in the algebra—and then, when $\bot$ is assigned to all ungrounded variables, we get unique solutions. We refer to the result proved in [10] for CIA’s.

Definition 2.36 ([10]). *Let $\mathscr{A}$ be a hyper-extensive category with a terminal object* 1*.*

(1) *By a **strict algebra** is meant an algebra $a\colon HA \to A$ together with a chosen global element*

$$\bot\colon 1 \to A.$$

(2) *Let A be a strict* CIA *for an iteratable functor H. A solution $e^\dagger$ of an equation morphism $e\colon X \to T(X+A)$ is called **strict** provided that its restriction to X_∞ “is constantly $\bot$”, that is, the square below commutes:*



Theorem 2.37 ([10]). *Let H be an iteratable endofunctor of a hyper-extensive category. Then in every strict* CIA *every equation morphism has a unique strict solution.*

Remark 2.38. For finitary set functors we have the analogous result concerning all strict iterative algebras A: let

$$e\colon X \to R(X+A), \qquad X \text{ finite},$$

be a rational equation morphism. The derived subobjects $i_n^*\colon X_n \to X$ are defined precisely as in Definition 2.31, we only substitute $i_0 = \eta_{X+A} \cdot \mathsf{inl}$ by

$$X \xrightarrow{\mathsf{inl}} X + A \xrightarrow{\hat{\eta}_{X+A}} R(X+A).$$

Since X is now finite and the subobjects X_n form a decreasing chain, there is a least one—which plays the role that X_∞ plays for CIAs:

Definition 2.39. *Let A be a strict iterative algebra for a finitary endofunctor of a hyper-extensive, locally finitely presentable category. For every rational equation morphism $e\colon X \to R(X+A)$ let, $n \in \mathbb{N}$ have the property that $i_{n+1}\colon X_{n+1} \to X_n$ is an isomorphism. Then a solution $e^\dagger\colon X \to A$ of e is called **strict** provided that its restriction to X_n "is constantly $\bot$", that is, we have a commutative square*

$$\begin{array}{ccc} X_n & \xrightarrow{i_n} & X \\ \downarrow & & \downarrow{\scriptstyle e^\dagger} \\ 1 & \xrightarrow[\bot]{} & A \end{array}$$

Theorem 2.40 ([10]). *Let H be a finitary endofunctor of a hyper-extensive, locally finitely presentable category. Then in every strict iterative algebra every rational equation morphism has a unique strict solution.*

Remark 2.41. Hyper-extensivity is essential in the above theorem. In fact, we showed in [10] that there is a locally finitely presentable category $\mathscr{A}$ which, although it is extensive (in fact, a topos) does not have the above property: We constructed a flat equation morphism with infinitely many solutions in a given CIA.

Remark 2.42. The endofunctor H is called *strict* if a morphism $\bot\colon 1 \to H0$ has been chosen. In that case every algebra $a\colon HA \to A$ is considered strict via the morphism

$$1 \xrightarrow{\bot} H0 \xrightarrow{Hu} HA \xrightarrow{a} A$$

where $u\colon 0 \to A$ is the unique morphism.

3 Elgot Algebras and Iteration Algebras

Throughout this section H denotes an iteratable endofunctor of a category with binary coproducts and a terminal object 1. We compare two models of "algebras with iteration"; by this we mean algebras A equipped with a function $e \mapsto e^\dagger$ which assigns to each equation morphism e in A a solution $e^\dagger$ in such a way that certain axioms are satisfied. The first model, the Elgot algebras, was introduced in [8], where it was proved that the category of Elgot algebras is the Eilenberg-Moore category of the rational monad (see 2.20). The second model, iteration algebras, was introduced in [15] in a slight variation of the concept defined by Zoltan Ésik in [19].

Notation 3.1. Let $h\colon A \to B$ be a morphism. Given a flat equation morphism $e\colon X \to HX + A$, the function h "transports" e to an equation morphism in B:

$$h \bullet e \equiv X \xrightarrow{e} HX + A \xrightarrow{\mathrm{id}+h} HX + B.$$

Given flat equation morphisms

$$e\colon X \to HX + Y \qquad \text{and} \qquad f\colon Y \to HY + A,$$

we define an equation morphism

$$f \blacksquare e\colon X + Y \to H(X+Y) + A$$

by its components as follows:

$$(f \blacksquare e)\cdot \mathsf{inl} \equiv X \xrightarrow{e} HX + Y \xrightarrow{HX+f} HX + HY + A \xrightarrow{\mathrm{can}+A} H(X+Y) + A$$

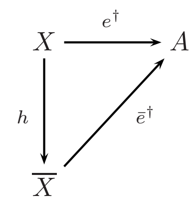
and

$$(f \blacksquare e)\cdot \mathsf{inr} \equiv Y \xrightarrow{f} HY + A \xrightarrow{\mathsf{inr}} HX + HY + A \xrightarrow{\mathrm{can}+A} H(X+Y) + A.$$

Definition 3.2. *By a **complete Elgot algebra** is meant a triple* $(A, a, (-)^\dagger)$ *where* $a\colon HA \to A$ *is an algebra and* $(-)^\dagger$ *is a function assigning to every flat equation morphism* $e\colon X \to HX + A$ *a solution* $e^\dagger\colon X \to A$ (cf. (2.4)) *so that the following two conditions are satisfied:*
Functoriality. *For every "morphism of equations", that is, coalgebra homomorphism of* $H(-) + A$*:*

$$\begin{array}{ccc} X & \xrightarrow{e} & HX + A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh+A} \\ \overline{X} & \xrightarrow[\bar{e}]{} & H\overline{X} + A \end{array}$$

the triangle

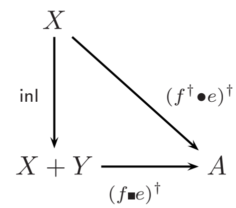


commutes.

Compositionality. *Given flat equation morphisms*

$$e\colon X \to HX + Y \qquad \text{and} \qquad f\colon Y \to HY + A$$

the triangle



commutes.

Remark 3.3. We sometimes denote the complete Elgot algebra by A only. For two complete Elgot algebras we denote, whenever there is no danger of confusion, by $e^\dagger$ the chosen solution of a flat equation morphism e for both algebras.

Example 3.4. (i) Every CIA fulfils Functoriality and Compositionality (of the unique solutions), see [8]. Thus, CIAs are complete Elgot algebras.

(ii) Every CPO-enrichable algebra is a complete Elgot algebra under the choice

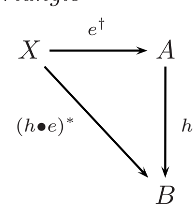
$$e^\dagger = \text{the least solution of } e.$$

For the classical case $H = H_\Sigma$ this was proved in [15]. The generalization to an arbitrary H was proved in [8], 3.8.

We now recall the concept of Elgot algebra from [8] which differs from complete Elgot algebra in our restriction to finitary equation morphisms, see Definition 2.8.

Definition 3.5. *By an* ***Elgot algebra*** *is meant a triple* $(A, a, (-)^\dagger)$ *where* (A, a) *is an algebra and* $(-)^\dagger$ *is a function assigning to every finitary flat equation morphism* $e\colon X \to HX + A$ *a solution* $e^\dagger\colon X \to A$ *satisfying Functoriality and Compositionality.*

A ***homomorphism of Elgot algebras*** $(A, a, (-)^\dagger)$ *and* $(B, b, (-)^*)$ *is a morphism* $h\colon A \to B$ *preserving solutions of finitary flat equations* $e\colon X \to HX + A$ *in the sense that the triangle*



commutes. Analogously for complete Elgot algebras (drop "finitary" from the last sentence).

Lemma 3.6 ([8]). *Every Elgot algebra homomorphism h is an algebra homomorphism, i.e., $h \cdot a = b \cdot Hh$. For iterative algebras (considered as Elgot algebras) the reverse implication also holds.*

Theorem 3.7 ([8]). *The category of (complete) Elgot algebras and homomorphisms is isomorphic to the Eilenberg-Moore category of algebras for the rational monad $\mathbb{R}$, see* 2.20 *(or the free completely iterative monad $\mathbb{T}$, see* 2.18*, respectively).*

Notation 3.8. For every Elgot algebra $(A, a, (-)^\dagger)$ the corresponding Eilenberg-Moore algebra for the monad $\mathbb{R}$ is denoted by

$$\hat{a} \colon RA \to A.$$

In fact, as proved in [8], RA is a free Elgot algebra with the universal morphism $\hat{\eta}_A \colon A \to RA$, and then $\hat{a}$ is the unique homomorphism of Elgot algebras with

$$\hat{a} \cdot \hat{\eta}_A = \mathrm{id}_A \,.$$

Analogously for complete Elgot algebras $(A, a, (-)^\dagger)$: we denote by

$$\tilde{a} \colon TA \to A$$

the Eilenberg-Moore algebra which is the unique homomorphism of complete Elgot algebras with

$$\tilde{a} \cdot \eta = \mathrm{id}_A \,.$$

Remark 3.9. (i) The notation $\bullet$ of 3.1 can be extended to equation morphisms $e \colon X \to T(X + A)$: given $h \colon A \to B$ we obtain an equation morphism

$$h \bullet e \equiv X \xrightarrow{e} T(X + A) \xrightarrow{T(X+h)} T(X + B).$$

If e is guarded (see Definition 2.25) so is $h \bullet e$. Analogously for rational equation morphisms.

(ii) Let A be a complete Elgot algebra. Although we only have a choice of solutions $e \mapsto e^\dagger$ for flat equation morphisms, we immediately obtain a "canonical" choice of solutions for all guarded equation morphisms: in the free CIA, TA, solve (uniquely) the equation morphism $\eta_A \bullet e \colon X \to T(X + TA)$, and compose the solution with $\tilde{a} \colon TA \to A$. We extend this now to all equation morphism by using Theorem 2.37:

Definition 3.10. (i) *A **strict complete Elgot algebra** is a complete Elgot algebra A together with a choice of an element $\bot \colon 1 \to A$. We then consider the* CIA *TA as strict via $\eta_A \cdot \bot \colon 1 \to TA$.*

(ii) *A homomorphism h between strict complete Elgot algebras A and B with chosen elements $\bot_A$ and $\bot_B$, respectively, is said to be **strict** if $h \cdot \bot_A = \bot_B$.*

(iii) *Every equation morphism* $e\colon X \to T(X+A)$ *in* A *defines an equation morphism* $\eta_A \bullet e\colon X \to T(X+TA)$ *in* TA*, and we denote by*

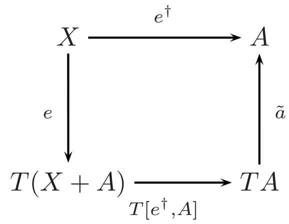
$$e^{\ddagger}\colon X \to TA$$

the unique strict solution in TA (*see* 2.37)*. The morphism*

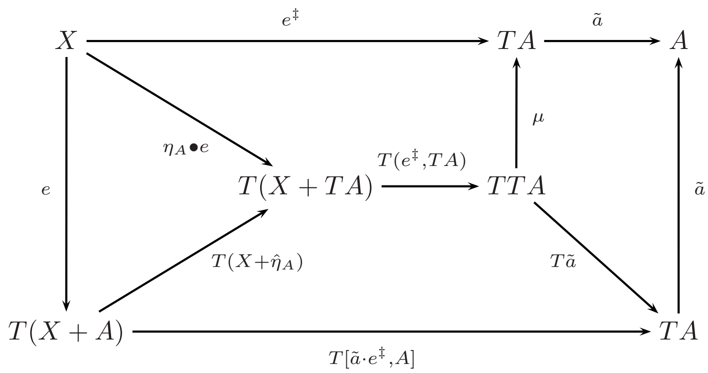
$$e^{\dagger} \equiv X \xrightarrow{e^{\ddagger}} TA \xrightarrow{\tilde{a}} A$$

is the called the ***canonical solution*** *of* e *in* A*.*

Remark 3.11. The canonical solution is indeed a solution of e, that is, the square



commutes. In fact, in the diagram



the upper part commutes by definition of $e^{\ddagger}$, the right-hand part does because $(A, \tilde{a})$ is an Eilenberg-Moore algebra, and the left-hand triangle is the definition of $\eta_A \bullet e$. The lower part is obvious: delete T and consider the components of $X + A$ separately. Thus, the outward square commutes, as requested.

Remark 3.12. The situation with Elgot algebras is analogous: let a *strict Elgot algebra* be an Elgot algebra A together with a chosen element $\bot\colon 1 \to A$. Then RA is considered as a strict iterative algebra via

$$1 \xrightarrow{\bot} A \xrightarrow{\hat{\eta}_A} RA.$$

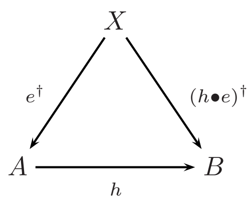
As in 3.10(ii), a strict homomorphism $h\colon A \to B$ is one that preserves the chosen elements; in symbols: $h \cdot \bot_A = \bot_B$. For every rational equation morphism $e\colon X \to R(X+A)$ we denote by

$$e^{\ddagger}\colon X \to RA$$

the unique strict solution (see Theorem 2.40) of $\hat{\eta}_A \bullet e\colon X \to R(X + RA)$ in the strict iterative algebra RA. Then the *canonical solution* of e in A is defined by

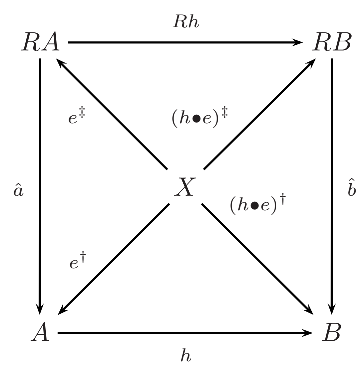
$$e^\dagger \equiv X \xrightarrow{e^\ddagger} RA \xrightarrow{\hat{a}} A.$$

Lemma 3.13. *Every strict homomorphism $h\colon A \to B$ of Elgot strict algebras preserves canonical solutions. That is, given a rational equation morphism $e\colon X \to R(X + A)$, then the triangle*



commutes.

Proof. (1) We only need to show that this statement holds in the free iterative algebras RA and RB: it then follows that it holds for A and B. To see this consider the diagram below:

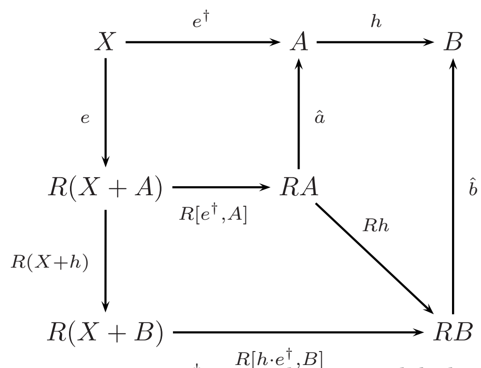


We are to prove that the lower triangle commutes. This follows from the fact that the other three triangles and the outward square commute: in fact, the upper triangle commutes by assumption, the left-hand and right-hand squares commute by the definition of the canonical solution and the outward square commutes since every homomorphism of Elgot algebras is, equivalently, a morphism of the corresponding algebras for the monad R.

(2) Assume that A and B in the lemma are iterative algebras. We know, due to Lemma 3.6 that h is an algebra homomorphism. Given an equation morphism $e\colon X \to R(X + A)$, we have the unique strict solution $e^\dagger$ in A. It is clear that

$$h\cdot e^\dagger\colon X \to B$$

is strict: if $e^\dagger \cdot i_n^*$ factors through $\perp_A$, then $(h\cdot e^\dagger)\cdot i_n^*$ factors through $h\cdot \perp_A = \perp_B$. It remains to prove that $h\cdot e^\dagger$ is a solution of $h \bullet e$. In fact, the diagram below commutes:



The upper parts commute since $e^\dagger$ is a solution and h, being a homomorphism of iterative algebras, is a homomorphism of the corresponding algebras of the monad R. The lower part commutes trivially: delete R and consider the components separately. $\square$

Remark 3.14. We next introduce the iteration algebras of Zoltan Ésik [19] also called strong iteration algebras in [15]. We use the terminology of Elgot algebras whereas the original definition used algebraic theories $\mathscr{T}$ and theory morphism in $\mathbf{Pow}_{\mathrm{A}}$. Let us first explain the translation between these terminologies:

(i) Recall that an *algebraic theory* $\mathscr{T}$ is a category whose objects are natural numbers and every number n is a coproduct of n copies of 1. Every set A defines an algebraic theory $\mathbf{Pow}_{\mathrm{A}}$ whose morphisms from n to p are the functions $A^p \to A^n$ (with the obvious composition and identity morphisms). Shortly: $\mathbf{Pow}_{\mathrm{A}}(n, p) = \mathbf{Set}(A^p, A^n)$.

(ii) An *algebra* for $\mathscr{T}$ is a functor $\mathscr{A} \colon \mathscr{T}^{\mathrm{op}} \to \mathbf{Set}$ preserving finite products. Put $A = \mathscr{A}(1)$. For every morphism $f \colon n \to p$ we obtain the function $\mathscr{A}(f) \colon A^p \to A^n$, and it is easy to verify that $f \mapsto A(f)$ is a *theory morphism* from $\mathscr{T}$ to $\mathbf{Pow}_{\mathrm{A}}$ (that is, a functor preserving finite coproducts which is the identity map on objects). We denote this theory morphism by $\overline{\mathscr{A}} \colon \mathscr{T} \to \mathbf{Pow}_{\mathrm{A}}$.

Conversely, every theory morphism from $\mathscr{T}$ to $\mathbf{Pow}_{\mathrm{A}}$ has the form $\overline{\mathscr{A}}$ for a unique algebra for $\mathscr{T}$.

Consequently, algebras for $\mathscr{T}$ can be identified with theory morphisms $\mathscr{T} \to \mathbf{Pow}_{\mathrm{A}}$. And they can also be identified with algebras for the corresponding finitary monad $\mathbb{T}$ given on objects $p \in \mathbb{N}$ by $Tp = \mathscr{T}(1, p)$. Recall that, conversely, given a finitary monad $\mathbb{T}$ in $\mathbf{Set}$, it is determined by the algebraic theory with $\mathscr{T}(n, p) = (Tp)^n$.

(iii) A *preiteration theory* is a pair $(\mathscr{T}, \dagger)$ where $\mathscr{T}$ is an iteration theory and $\dagger$ is a family of functions from $\mathscr{T}(n, n+p)$ to $\mathscr{T}(n, p)$. For example, let H be a strict (see Remark 2.42) endofunctor of $\mathbf{Set}$. Then the theory associated with the rational monad $\mathbb{R}$ (see 2.20) is a preiteration theory in the sense that given $e \in \mathscr{T}_{\mathbb{R}}(n, n+p)$ which is a function $e \colon n \to R(n+p)$ in $\mathbf{Set}$, we denote by $e^\dagger$ its unique strict solution in the (free) iterative algebra Rp, and obtain $e^\dagger \in \mathscr{T}_{\mathbb{R}}(n, p)$.

(iv) We see that Elgot algebras A for H, which are precisely the monadic algebras for $\mathbb{R}$, are in a bijective correspondence with theory morphisms $\overline{A} \colon \mathscr{T}_{\mathbb{R}} \to \mathbf{Pow}_{\mathrm{A}}$. Zoltan Ésik defined iteration algebras, for any preiteration theory $\mathscr{T}$, as those algebras A such that given two morphisms $e, f \colon n \to n+p$ in $\mathscr{T}$ then

$\overline{A}(f^\dagger) = \overline{A}(g^\dagger)$ holds whenever for every morphism $u\colon n \to p$ of $\mathscr{T}$ we have $\overline{A}[u, \mathrm{id}_p]\cdot e = \overline{A}[u, \mathrm{id}_p]\cdot f$. This specializes, for $\mathscr{T} = \mathscr{T}_{\mathbb{R}}$, to the following

Definition 3.15. *Let H be a strict endofunctor of a hyper-extensive, locally finitely presentable category. An **iteration algebra** is an Elgot algebra A such that for arbitrary rational equation morphisms $e, f\colon X \to R(X + A)$ the chosen solutions are equal whenever for each $u\colon X \to A$ the morphism*

$$u^\# \equiv R(X + A) \xrightarrow{R[u,A]} RA \xrightarrow{\hat{a}} A$$

merges e and f. Shortly:

$$u^\#\cdot e = u^\#\cdot f \quad (\forall u) \qquad \textit{implies} \quad e^\dagger = f^\dagger.$$

Remark 3.16. (a) Iteration algebras were first introduced by Zoltan Ésik in [19] for iteration theories in **Set**. Our concept above is, for **Set**, a special case of the free iteration theory $\mathbb{R}$ on a strict endofunctor $H\colon$ **Set** $\to$ **Set**. In the more general case the functoriality (see Definition 3.2) need not hold.

(b) Ésik's iteration algebras are called strong iteration algebras in [15]. The reason is that "iteration algebra" is reserved for the following wider concept: for each map $v\colon Y \to A$ put

$$v^\flat \equiv RY \xrightarrow{Rv} RA \xrightarrow{\hat{a}} A.$$

Given a rational equation morphism $e\colon X \to R(X + Y)$, then the unique strict solution of $\hat{\eta}_Y \bullet e\colon X \to R(X+RY)$ (see Notation 3.1) in the iterative algebra RY is denoted by $e^\ddagger\colon X \to RY$. An Elgot algebra A is called an iteration algebra in [15] provided that given rational equation morphisms $e, f\colon X \to R(X + Y)$ we have that

$$u^\flat\cdot e = u^\flat\cdot f \quad (\forall u\colon X + Y \to A) \qquad \text{implies} \qquad v^\flat\cdot e^\ddagger = v^\flat\cdot f^\ddagger \quad (\forall v\colon Y \to A).$$

(c) We will use Definition 3.15 in the remainder of the paper, but the results remain valid for the weaker version.

Example 3.17. (i) Every CPO-enrichable Σ-algebra is an iteration algebra, see [15], 7.1.5. We generalize this to H-algebras in Section 4.

(ii) Given a finitary endofunctor H of **Set**, then every iterative algebra for H is an iteration algebra. This follows from Example 7.1.6 in [15] applied to the free iteration theory of H. We present a full proof here because this provides some details left out in [15].

Proposition 3.18. *Given a finitary endofunctor H of* **Set***, then every strict iterative algebra for H is an iteration algebra.*

Proof. Let A be an iterative algebra for H. For the trivial one-point algebra $A = 1$ the statement clearly holds. So assume A has at least two elements. Given rational equation morphisms

$$e, f\colon X \to R(X + A)$$

with

$$u^{\#}\cdot e = u^{\#}\cdot f \colon X \to A,$$

for all $u\colon X \to A$, we show that $e^{\dagger} = f^{\dagger}$.

(1) If $e(x) \in \hat{\eta}_{X+A}[X]$, then $e(x) = f(x)$. In fact, it is sufficient to show that $f(x) \in \eta_{X+A}[X]$. Then, for every $u\colon X \to A$, we have

$$u\bigl(e(x)\bigr) = u\bigl(f(x)\bigr),$$

since $u^{\#}\cdot\hat{\eta}_X = u$ and $u^{\#}\cdot e = u^{\#}\cdot f$. Thus, since A has at least two elements, $e(x) = f(x)$.

In order to obtain a contradiction, assume $f(x) \notin \hat{\eta}_{X+A}[X]$. The equation morphism

$$g \equiv 1 \xrightarrow{f(x)} R(X+A) \xrightarrow{R(t+\mathrm{id})} R(1+A) \qquad (t\colon X \to 1 \text{ unique})$$

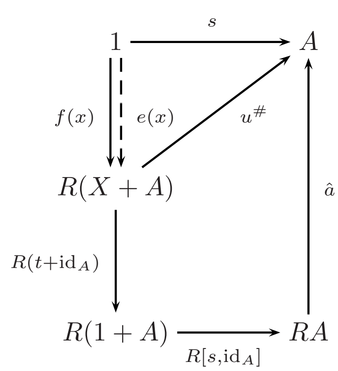
is clearly guarded, and we prove that every function $s\colon 1 \to A$ is a solution of g. The function $u = s\cdot t\colon X \to A$ fulfils

$$u\cdot y = s \qquad \text{for all } y\colon 1 \to X$$

and consequently

$$u^{\#} = \hat{a}\cdot R[s, \mathrm{id}_A]\cdot R(t + \mathrm{id}_A)\colon R(X+A) \to A.$$

Therefore, $u^{\#}\cdot f(x) = u^{\#}\cdot e(x)$ implies that the diagram



commutes: to see that the upper triangle commutes notice that since $e(x) \in \hat{\eta}_{X+A}[X]$ we have $u^{\#}\cdot(e(x)) = u\cdot y$ for some element $y\colon 1 \to X$. We conclude that an (arbitrary) morphism s is a solution of g. Since A has more than one element, this contradicts the uniqueness of $g^{\dagger}$.

(2) By symmetry, the above statement (1) proves that e and f have the same non-guarded variables and they are identical when restricted to them, shortly: $e_1 = f_1$ in the notation of Definition 2.31. This implies that e and f have the same derived subobjects, and that

$$e_i = f_i \qquad \text{for all } i = 1, 2, 3, \ldots$$

(3) We prove that $f^\dagger$ (the unique strict solution of f, see Theorem 2.40) is a solution of e. Then by (2) $f^\dagger$ is a strict solution of e, that is, $f^\dagger = e^\dagger$. In fact, for $u = f^\dagger\colon X \to A$ we have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\;f^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \hat{a}} \\ R(1+A) & \xrightarrow[R[f^\dagger, A]]{} & RA \end{array}$$

because the lower passage from X to A is $u^\# \cdot e = u^\# \cdot f$ and we thus can change the left-hand arrow from e to f. □

Example 3.19. An iteration algebra which is not iterative. Let Σ consist of a unary operation s and a constant $\perp$. The algebra $A = \{0, \perp\}$ with $s = \mathrm{id}$ is not iterative since the equation $x \approx sx$ has two solutions. But A is CPO-enrichable, and thus an iteration algebra.

Example 3.20. An Elgot algebra which is not an iteration algebra. Consider the polynomial functor

$$H_\Sigma X = X + X,$$

corresponding to the signature Σ having two unary operations s and t. The corresponding rational monad R_Σ assigns to every set X the set

$$R_\Sigma X = \{s, t\}^* \times X + \{s, t\}^\infty,$$

where $\{s, t\}^*$ is the set of all finite strings on s, t, and $\{s, t\}^\infty$ is the set of all ultimately periodic sequences on s, t; that means infinite sequences of the form $w = uv^\omega = uvvv\ldots$, where u, v are finite sequences on s, t and v is nonempty.

Let $C = \{0, 1\}$ be equipped with the algebraic structure

$$\gamma\colon R_\Sigma C \to C$$

defined as follows:

$$\gamma(w, i) = i \quad \text{for all } w \in \{s, t\}^*, \quad i = 0, 1 \tag{3.1}$$

and

$$\gamma(v) = \begin{cases} 1 & \text{for } v = ut^\omega, \quad u \in \{s, t\}^*, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We shall prove that γ is the structure maps of an Eilenberg-Moore algebra for R_Σ. For the unit law, note that

$$\hat{\eta}_C(i) = (\epsilon, i) \qquad i = 0, 1,$$

where ϵ is the empty sequence in $\{s,t\}^*$. Then, by (3.1), we see that

$$\gamma\big(\hat{\eta}_C(i)\big) = i.$$

In order to show that the composition law

$$\begin{array}{ccc} R_\Sigma R_\Sigma C & \xrightarrow{\hat{\mu}_C} & R_\Sigma C \\ {\scriptstyle R_\Sigma \gamma}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ R_\Sigma C & \xrightarrow[\gamma]{} & C \end{array} \tag{3.3}$$

holds, we first observe that the elements of $R_\Sigma R_\Sigma C$ have one of the following forms:

1. $(w', (w, i))$, with $w, w' \in \{s,t\}^*$, $i = 0, 1$,
2. (w', v), with $w' \in \{s,t\}^*$, $v \in \{s,t\}^\infty$, or
3. v, with $v \in \{s,t\}^\infty$.

We show that the equation (3.3) holds by case analysis. In the first case, we compute

$$\begin{aligned} \gamma\cdot\hat{\mu}_C\big(w', (w,i)\big) &= \gamma(ww', i) \\ &= i \\ &= \gamma(w', i) \\ &= \gamma\big(w', \gamma(w,i)\big) \\ &= \gamma\Big(R_\Sigma\gamma\big(w', (w,i)\big)\Big). \end{aligned}$$

In the second case, we have

$$\gamma\cdot\mu_C(w', v) = \gamma(w'v) = \begin{cases} 1 & \text{if } v = ut^\omega \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\gamma\cdot R_\Sigma\gamma(w', v) = \gamma\big(w', \gamma(v)\big) = \begin{cases} \gamma(w', 1) = 1 & \text{if } v = ut^\omega \\ \gamma(w', 0) = 0 & \text{otherwise.} \end{cases}$$

Finally, the third case holds since

$$\mu_C(v) = R_\Sigma\gamma(v) = v.$$

Thus, C is an Elgot algebra.

To see that C is not an iteration algebra, note that both unary operations s_C, t_C are the identity on C, see (3.1). Now consider the two formal equations:

$$x = s(x) \qquad \text{and} \qquad x = t(x), \tag{3.4}$$

which give rise to two rational equation morphisms

$$e, f\colon 1 \to R_\Sigma(1 + C).$$

We prove that for every element $u\colon 1 \to C$, we have

$$u^\# \cdot e = u^\# \cdot f.$$

In fact, $u^\# \cdot e$ is the element of C which is the second coordinate of the right-hand side of (3.4) for $x = u$ in C; in symbols,

$$u^\# \cdot e = s_C(u).$$

Similarly we have $u^\# \cdot f = t_C(u)$. The solutions $e^\dagger, f^\dagger\colon 1 \to C$ are obtained by unfolding the above equations (3.4), and applying γ to the resulting sequences (see Remark 3.11):

$$e^\dagger = \gamma(sss\dots) = 0 \neq 1 = \gamma(ttt\dots) = f^\dagger.$$

Thus, C is not an iteration algebra.

4 Continuous Algebras

In the present section we prove that every algebra enrichable over CPO is an iteration algebra. This holds for every iteratable endofunctor H of **Set** possessing a locally continuous lifting to CPO, see 2.14. Unfortunately, this does not imply that the lifted endofunctor has a terminal coalgebra (for that we would need the strict variant of CPO). However, we prove that whenever H has a locally continuous lifting $\overline{H}$ and is iteratable, then $\overline{H}$ is also iteratable.

Assumption 4.1. Throughout this section H denotes an iteratable set functor with a locally continuous lifting to CPO.

Theorem 4.2. *If an iteratable functor $H\colon$ **Set** $\to$ **Set** has a locally continuous lifting $\overline{H}\colon$ CPO $\to$ CPO, then $\overline{H}$ is also iteratable. Moreover, the free completely iterative monad $\overline{T}$ of $\overline{H}$ is locally continuous, and it is a lifting of the free completely iterative monad of H.*

Remark. Except for the fact that $\overline{T}$ is locally continuous, Theorem 4.2 was proved in [22], Example 2.8. We decided to include a full proof for the convenience of the reader.

Proof. By definition, for every set Z the functor $H(-) + Z$ has a terminal coalgebra TZ. As proved in [4], this implies that the "classical" terminal-coalgebra construction (dualizing the initial-algebra construction of [2]) stops: we denote by $W^Z\colon \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Set}$ the ordinal-indexed chain with objects W_i^Z and morphisms $w_{i,j}^Z$ determined up to natural isomorphism uniquely by

$$W^Z = 1$$

$$W_{i+1}^Z = HW_i^Z + Z \qquad \text{and} \qquad w_{i+1,j+1}^Z = Hw_{i,j}^Z + \mathrm{id}_Z \quad (i, j \in \mathbf{Ord},\ i \geq j)$$

and given on limit ordinals by forming a limit. Then for every set Z there exists i such that $w^Z_{i+1,i}$ is an isomorphism, and it follows that

$$TZ = W^Z_i$$

is a terminal coalgebra for $H(-) + Z$ with the structure map

$$\left(w^Z_{i+1,i}\right)^{-1} \colon TZ \to HTZ + Z,$$

see [4].

To prove that $\overline{H}$ is iteratable, let Z be an arbitrary CPO and let

$$\overline{W}^Z \colon \mathbf{Ord}^{\mathrm{op}} \to \mathbf{CPO}$$

be the corresponding chain: $\overline{W}^Z_0 = 1$ and $\overline{W}^Z_{i+1} = \overline{H}^Z_i \overline{W}^Z_i + Z$. Since the forgetful functor $U \colon \mathbf{CPO} \to \mathbf{Set}$ preserves coproducts and limits, we conclude that for the terminal-coalgebra construction of the set UZ we have a natural isomorphism

$$W^{UZ} \cong U \cdot \overline{W}^Z \colon \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Set}\,.$$

Consequently, $Uw^Z_{i+1,i}$ is an isomorphism in **Set** for some i, which implies that all $Uw^Z_{j,i}$ with $j \geq i$ are isomorphisms. Thus, $w^Z_{j,i}$ with $j \geq i$ are monomorphisms in **CPO**, and since **CPO** is wellpowered, there exists $j \geq i$ such that $w^Z_{j+1,j}$ is an isomorphism. This proves that $\overline{H}(-) + Z$ has a terminal coalgebra

$$\overline{T}Z = \overline{W}^Z_j\,.$$

Moreover, for H we also have $T(UZ) = W^{UZ}_j$, which shows that

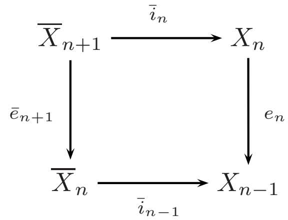
$$U\overline{T}Z \cong T(UZ)$$

and this gives a natural isomorphism

$$U\overline{T} \cong TU$$

(since the above ordinal j can be chosen arbitrarily large). Thus, $\overline{T}$ is a lifting of T. Finally, $\overline{T}$ is locally continuous because it is a large limit of the endofunctors $\overline{W}_i \colon$ CPO $\to$ CPO where $\overline{W}_0$ is the constant functor of value 1, $\overline{W}_{i+1} = H\overline{W}_i +$ Id, and on limit ordinals one forms a limit. In fact, since H is locally continuous, it is easy to verify that $\overline{W}_i$ is locally continuous by transfinite induction on i. Thus, $\overline{T} = \lim \overline{W}_i$ is also locally continuous. □

Remark 4.3. The category **CPO** is clearly hyper-extensive (see 2.29). Recall the concept of a derived subobject $i_{n+1} \colon X_{n+1} \to X_n$ of an equation morphism and the "complement" $\bar{i}_{n+1} \colon \overline{X}_{n+1} \to X_n$ (see Remark 2.32). The extensivity implies that for every $n \geq 1$ there is a unique morphism $\bar{e}_{n+1} \colon \overline{X}_{n+1} \to X_n$ such that the square



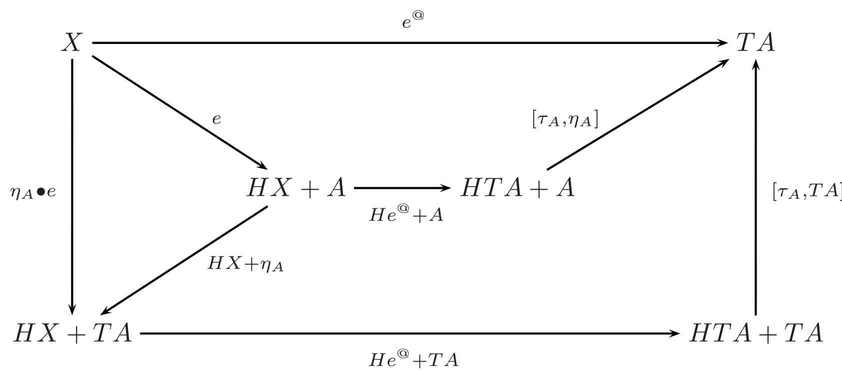
is a pullback. The morphism with components $\bar{e}_2 \cdot \bar{e}_3 \cdot \dots \cdot \bar{e}_{n+1} \colon \overline{X}_{n+1} \to \overline{X}_1$ is denoted by

$$u \colon \coprod_{n \in \mathbb{N}} \overline{X}_{n+1} \to \overline{X}_1. \tag{4.1}$$

Lemma 4.4. *For every flat equation morphism $e \colon X \to HX + A$ the corresponding coalgebra homomorphism $e^{@} \colon X \to TA$ is the unique solution of $\eta_A \bullet e$ (see Notation 3.1) in TA:*

$$e^{@} = (\eta_A \bullet e)^{\dagger}. \tag{4.2}$$

Proof. The following diagram



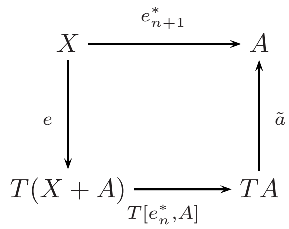
commutes: the upper square expresses the fact that $e^{@}$ is a coalgebra homomorphism for $H(-) + A$ (cf. 2.17); the left-hand triangle commutes by the definition of $\eta_A \bullet e$, and the lower part obviously commutes. Therefore, $e^{@}$ is a solution of $\eta_A \bullet e$ in TA. □

Theorem 4.5. *Let H be an iteratable, locally continuous endofunctor of* **CPO** *whose completely iterative monad T is locally continuous. Then every strict H-algebra A becomes a strict complete Elgot algebra by assigning to every flat equation morphism the least solution. Moreover, for every equation morphism $e \colon X \to T(X + A)$ the canonical solution (see* (3.10)*) is the least solution of e.*

Remark 4.6. (i) The least solution of $e \colon X \to T(X + A)$ is obtained as the join

$$e^* = \bigsqcup_{n \in \mathbb{N}} e_n^* \qquad \text{in } \mathbf{CPO}(X, A) \tag{4.3}$$

of approximations, where e_0^* is the least element of $\mathbf{CPO}(X, A)$ (the constant function on $\perp$), and given e_n^*, then e_{n+1}^* is defined by the commutative square



This follows from the fact that, since T is locally continuous, the self-map of $\mathbf{CPO}(X, A)$ given by $\tilde{a}{\cdot}T[-, A]{\cdot}e$ is continuous.

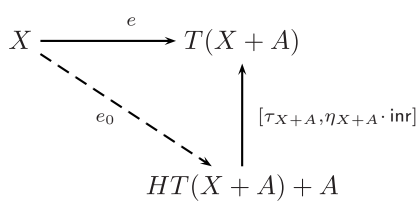
The statement of Theorem 4.5 makes use of the canonical solution $e^\dagger$ for an equation morphism $e\colon X \to T(X + A)$, see Definition 3.10. Its existence follows by applying Theorem 2.37 in the hyperextensive category **CPO**.

Proof. The fact that $(A, a, (-)^*, \bot)$ is a strict complete Elgot algebra has been proved in [8], Proposition 3.5. Recall that $(-)^*$ assigns to every equation morphism $e\colon X \to HX + A$ the least solution $e^*\colon X \to A$. We now prove that for every equation morphism $e\colon X \to T(X + A)$ the canonical solution

$$e^\dagger = \tilde{a}{\cdot}e^\ddagger\colon X \to A \tag{4.4}$$

of 3.10 is the least solution of e. Recall that $e^\ddagger\colon X \to TA$ is the unique strict solution of $\eta_A \bullet e$ in TA.

(1) Suppose e is guarded (see 2.22):



(4.5)

Let us form a new flat equation morphism (see 2.17)

$$\hat{e} \equiv T(X + A) \xrightarrow{\alpha_{X+A}} HT(X + A) + X + A \xrightarrow{[\mathsf{inl}, e_0, \mathsf{inr}]} HT(X + A) + A. \tag{4.6}$$

Since $\hat{e}$ is a coalgebra for $H(-) + A$, we have the unique coalgebra homomorphism

$$\hat{e}^@\colon T(X + A) \to TA$$

from which $e^\ddagger$ (the unique strict solution of $\eta_A \bullet e$) can be computed as follows:

$$e^\ddagger \equiv X \xrightarrow{\mathsf{inl}} X + A \xrightarrow{\eta_{X+A}} T(X + A) \xrightarrow{\hat{e}^@} TA.$$

(See the proof of 3.9 in [21] with $f := \eta_A$ and $s := \hat{e}^@$.)

In order to prove the equation (4.4), we use the fact that $\tilde{a}\colon TA \to A$, which is a homomorphism of complete Elgot algebras, preserves the solution of $\eta_A \bullet \hat{e}$

which by Lemma 4.4 yields

$$\begin{aligned}\tilde{a}\cdot e^{\ddagger} &= \tilde{a}\cdot\hat{e}^{@}\cdot\eta_{X+A}\cdot\mathsf{inl}\\ &= \tilde{a}\cdot\big(\eta_A \bullet \hat{e}\big)^{\dagger}\cdot\eta_{X+A}\cdot\mathsf{inl}\\ &= \big(\tilde{a}\bullet(\eta_A\bullet\hat{e})\big)^{*}\cdot\eta_{X+A}\cdot\mathsf{inl}\,.\end{aligned}$$

It is easy to see that $\tilde{a}\bullet(\eta_A\bullet\hat{e}) = (\tilde{a}\cdot\eta_A)\bullet\hat{e}$ and since $\tilde{a}\cdot\eta_A = \mathrm{id}$, we obtain

$$\tilde{a}\cdot e^{\ddagger} = \hat{e}^{*}\cdot\eta_{X+A}\cdot\mathsf{inl}\,.$$

It remains to prove that the right-hand side is e^*:

$$e^{*} = \hat{e}^{*}\cdot\eta_{X+A}\cdot\mathsf{inl}\,.$$

Since both sides are solutions of e, it follows that $e^* \sqsubseteq \hat{e}^*\cdot\eta_{X+A}\cdot\mathsf{inl}$, and we will prove the opposite inequality. For that we only need to verify that

$$\tilde{a}\cdot T[e^*, A] \text{ is a solution of } \hat{e}. \tag{4.7}$$

In fact, this implies

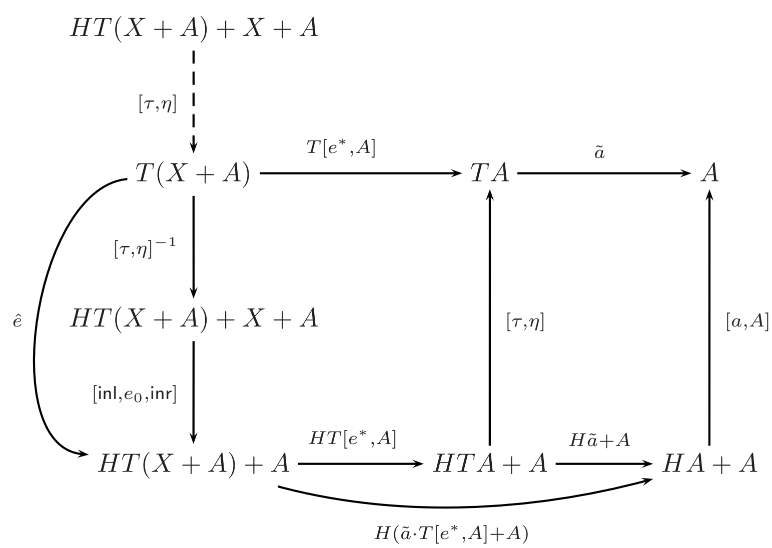
$$\hat{e}^{*} \sqsubseteq \tilde{a}\cdot T[e^*, A]$$

therefore

$$\hat{e}^{*}\cdot\eta_{X+A}\cdot\mathsf{inl} \sqsubseteq \tilde{a}\cdot T[e^*, A]\cdot\eta_{X+A}\cdot\mathsf{inl} = e^{*} \tag{4.8}$$

where the last equality follows from $T[e^*, A]\cdot\eta_{X+A} = \eta_A\cdot[e^*, A]$ as desired.

Since by (4.6) and Notation 2.17 we have $\hat{e} = [\mathsf{inl}, e_0, \mathsf{inr}]\cdot[\tau_{X+A}, \eta_{X+A}]^{-1}$, the proof of (4.7) is performed by verifying the commutativity of the following diagram (where we drop the subscripts of τ and η):



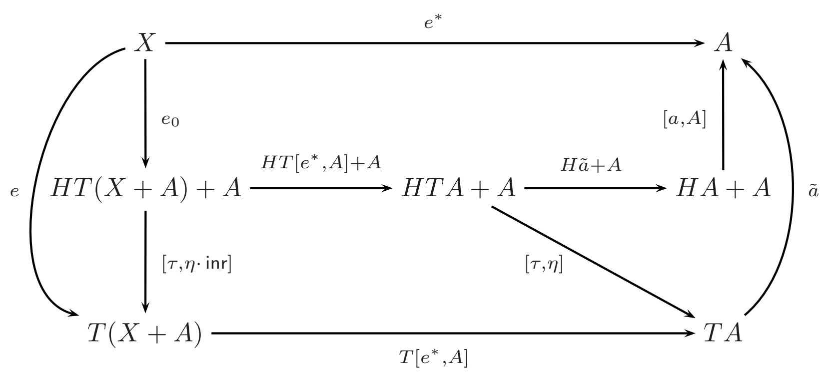
The right-hand square commutes since $\tilde{a}$ is a homomorphism of complete Elgot algebras (thus an algebra homomorphism by Lemma 3.6) with $\tilde{a}\cdot\eta_A = \mathrm{id}$. Let us verify that the left-hand square commutes when post-composed with $\tilde{a}$. To this end we precompose it with the isomorphism $[\tau_{X+A}, \eta_{X+A}]$ and consider the components of $HT(X+A)+X+A$ separately. The right-hand component yields $\tilde{a}\cdot\eta_A = \mathrm{id}_A$ on both possible paths, the left-hand component yields the result

$$\tilde{a}\cdot T[e^*, A]\cdot\tau_{X+A} = \tilde{a}\cdot\tau_A\cdot HT[e^*, A]$$

due to the naturality of τ, thus it remains to check the middle component. The upper passage is the morphism e^*, see (4.8). The lower passage yields

$$[a, A]\cdot(H\tilde{a} + A)\cdot HT[e^*, A]\cdot e_0 \colon X \to A,$$

and we verify that this is e^*, too. To this end consider the diagram below:



The outward square commutes since e^* is a solution of e; the left-hand part commutes by (4.5); the lower part commutes by the naturality of τ; finally the right-hand square commutes as before in the last diagram. This concludes the proof of part (1) showing the equation

$$e^\dagger = \tilde{a}\cdot e^\ddagger = e^*$$

for all guarded equation morphisms e.

(2) Let e be *preguarded*, i.e., let X be a coproduct of the complementary subobjects of Remark 2.32

$$i_n^*\cdot\bar{i}_{n+1} \colon \overline{X}_{n+1} \to X \qquad (n \in \mathbb{N}),$$

in symbols: $X = \coprod_{n\in\mathbb{N}} \overline{X}_{n+1}$. By using the coproduct injection $\bar{i} \colon \overline{X}_1 \to X$ and the morphism $u \colon X \to \overline{X}_1$ of (4.1) we can define the "guarded modification" of e as the equation morphism

$$f \equiv \overline{X}_1 \xrightarrow{\bar{i}_1} X \xrightarrow{e} T(X+A) \xrightarrow{T(u+A)} T(\overline{X}_1 + A). \tag{4.9}$$

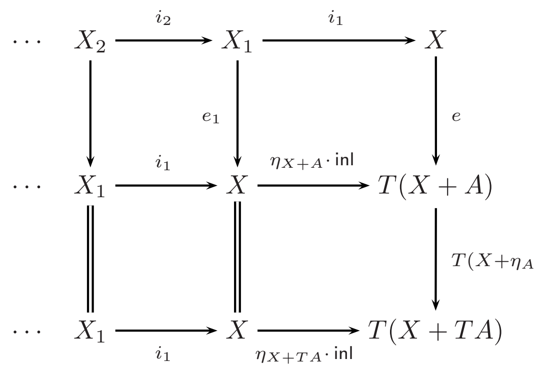
It was proved in [10] that f is guarded and in every CIA the unique solution of f provides a unique solution of e because

(a) every solution $e^\dagger\colon X \to A$ of e yields a solution $e^\dagger \cdot \bar{i}_1$ of f

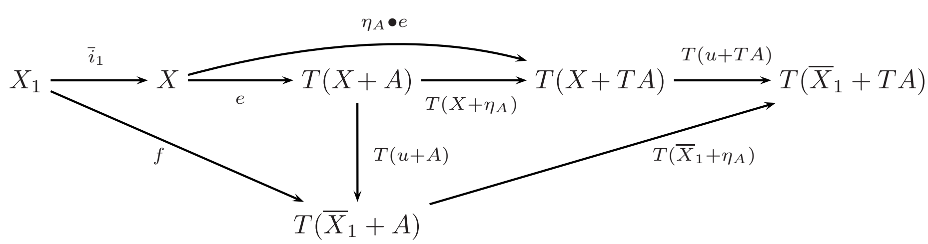
and

(b) every solution $f^\dagger\colon \overline{X}_1 \to A$ of f yields a solution $f^\dagger \cdot u$ of e.

We now apply this result to $\eta_A \bullet e$ and TA as the CIA. Observe first that e and $\eta_A \bullet e$ have the same derived subobjects:



Consequently, $\eta_A \bullet e$ is preguarded (since e is). The guarded modification of $\eta_A \bullet e$ is simply $\eta_A \bullet f$; to see this consider the diagram below:



The upper path is the desired guarded modification of $\eta_A \bullet e$. The left-hand triangle commutes by the definition (4.9) of f, and the right-hand one commutes trivially. Therefore, by (b) above we conclude that the unique strict solution $e^\ddagger$ of $\eta_A \bullet e$ is equal to $f^\ddagger \cdot u$. Thus

$$e^\dagger = \tilde{a} \cdot e^\ddagger = \tilde{a} \cdot f^\ddagger \cdot u = f^\dagger \cdot u.$$

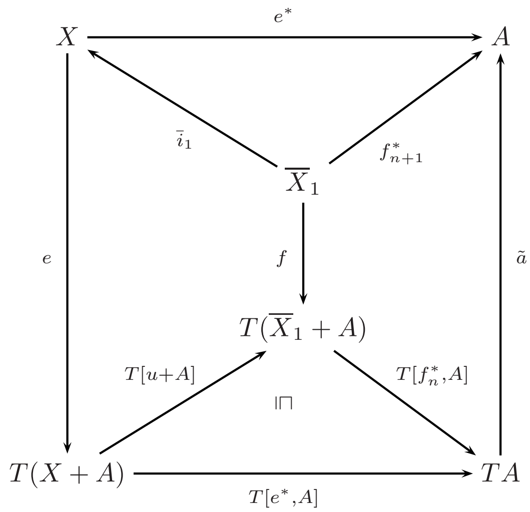
Since f is guarded, we thus get from case (1)

$$e^\dagger = f^* \cdot u.$$

It remains to prove that the right-hand side in the last equation is the least solution $e^* = \bigsqcup_{n \in \mathbb{N}} e_n^*$ (see Remark 4.6). For that we only need to verify that

$$f_n^* \cdot u \sqsubseteq e^* \qquad \text{for all } n.$$

The case $n = 0$ is clear from $\bot \cdot u = \bot$. (Here $\bot$ denotes the least element of $\mathrm{CPO}(X, A)$.) For the induction step consider the following diagram:



The inequality in the lower-triangle follows from the induction hypothesis since T is locally continuous, the right-hand square is the definition of f^*_{n+1}, and the left-hand square is the definition of f. Since the outward square commutes, we conclude

$$f^*_{n+1} \sqsubseteq e^* \cdot \bar{i}_1.$$

Now e^* fulfils

$$e^* \cdot \bar{i}_1 \cdot u = e^*.$$

In fact, the morphism $\bar{i}_1 \cdot u$ has the property that every solution of a preguarded equation equalizes it with id_X: see Remark 4.13 in [10]. Thus,

$$f^*_{n+1} \cdot u \sqsubseteq e^*.$$

(3) Let e be arbitrary. Recall from Remark 2.32 that the coproduct injections $i^*_n \cdot \bar{i}_{n+1} \colon \overline{X}_{n+1} \to X$ together with $i_\infty \colon X_\infty \to X$ form a coproduct

$$X = X_\infty + \coprod_{n \in \mathbb{N}} \overline{X}_{n+1}.$$

We proved in [10] that by modifying the first coproduct component of e (the component $e \cdot i_\infty$) we obtain a "preguarded modification" of e, that is, a preguarded equation morphism $f \colon X \to T(X + A)$ with the same strict solutions. More precisely, we define f to have the same components as e on each $\overline{X}_{n+1}$, and its first component $f \cdot i_\infty \colon X_\infty \to T(X + A)$ is

$$X_\infty \to 1 \xrightarrow{\perp} A \xrightarrow{\mathsf{inr}} X + A \xrightarrow{\eta_{X+A}} T(X + A).$$

Moreover, given a CIA, then the unique solution of f provides a unique strict solution of e because

(a) every strict solution of e is a solution of f,

and

(b) the unique solution of f is a strict solution of e.

We now apply this result to $\eta_A \bullet e$ and TA as the CIA. Observe that the pre-guarded modification of $\eta_A \bullet e$ is simply $\eta_A \bullet f$ (the verification of the individual components is trivial). Therefore, by (b) above the unique strict solution $e^\ddagger$ of $\eta_A \bullet e$ is equal to $f^\ddagger$ (the unique solution $\eta_A \bullet f$). Consequently, by case (2),

$$e^\dagger = \tilde{a}\cdot e^\ddagger = \tilde{a}\cdot f^\ddagger = f^\dagger = f^*.$$

It remains to prove

$$f^* \sqsubseteq e^*$$

to conclude $e^\dagger = e^*$. In fact, we verify that

$$f_k^* \sqsubseteq e_k^* \qquad (k \in \mathbb{N})$$

holds in $\mathbf{CPO}(X, A)$ by induction: $f_0^* = \bot = e_0^*$ and for the induction step use the inequality $f \sqsubseteq e$ in $\mathbf{CPO}(X, T(X + A))$: from $f_k^* \sqsubseteq e_k^*$ we easily obtain $f_{k+1}^* \sqsubseteq e_{k+1}^*$. $\square$

Remark 4.7. For every iteratable functor $H\colon \mathbf{Set} \to \mathbf{Set}$ we proved in [6] that the rational monad $\mathbb{R}$ is a submonad of the monad $\mathbb{T}$. More detailed: for every set Z the free CIA, TZ, is of course iterative, thus, there exists a unique homomorphism $\varepsilon_Z\colon RZ \to TZ$ of H-algebras satisfying $\varepsilon_Z\cdot\hat{\eta}_Z = \eta_Z$ (see 2.17 and 2.19). It is easy to verify that this defines a monad morphism

$$\varepsilon\colon \mathbb{R} \to \mathbb{T}.$$

In [6] the objects RZ and morphisms $\varepsilon_Z\colon RZ \to TZ$ are constructed coalgebraically and each ε_Z is shown to be a monomorphism. From this we easily derive the following

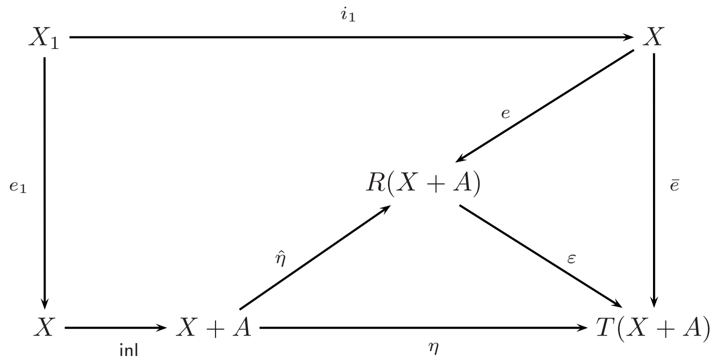
Lemma 4.8. *For every iteratable functor $H\colon$ **Set** $\to$ **Set** and every strict* CIA*, A, strict solutions of rational equation morphisms w.r.t. $\mathbb{R}$ and $\mathbb{T}$ are the same.*

That is, given $e\colon X \to R(X + A)$ with X finite, then the strict solution of e w.r.t. $\mathbb{R}$ is the strict solution of

$$\bar{e} \equiv X \xrightarrow{e} R(X + A) \xrightarrow{\varepsilon_{X+A}} T(X + A)$$

w.r.t. $\mathbb{T}$.

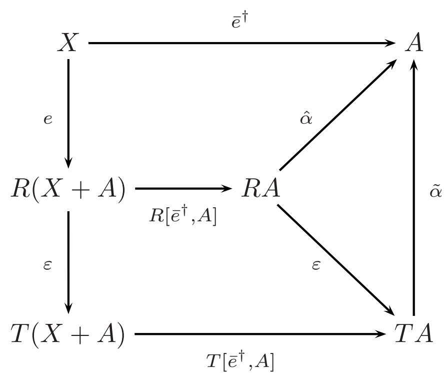
Proof. (1) We first verify that the strictness of solutions has the same meaning for e and $\bar{e}$: in fact, the derived subobjects of $\bar{e}$ (see 2.31) are equal to the derived subobjects of e (see 2.38). To prove this for the first one, let $i_1\colon X_1 \to X$ be the derived subobject of e:



In the above diagram all inner parts commute, the left-hand square is a pullback and ε_{X+A} is a monomorphism. Consequently the outward square is a pullback, too. Thus, i_1 is the derived subject of $\bar{e}$.

It now follows that $i_2\colon X_2 \to X_1$ has the same meaning for e and $\bar{e}$, etc.

(2) We prove that if $\bar{e}^\dagger$ is the unique strict solution of $\bar{e}$ w.r.t. $\mathbb{T}$, then it solves e (and is thus the unique strict solution w.r.t. $\mathbb{R}$). In the diagram



the outward square commutes since $\bar{e}^\dagger$ is a solution, the lower part commutes due to the naturality of ε, and the right-hand part does because A is an iterative algebra and the homomorphism $\hat{\alpha}$, $\tilde{\alpha}\cdot\varepsilon_A\colon RA \to A$ are merged by the universal arrow $\hat{\eta}\colon A \to RA$:

$$\hat{\alpha}\cdot\hat{\eta}_A = \mathrm{id}_A = \tilde{\alpha}\cdot\eta_A = \tilde{\alpha}\cdot\varepsilon_A\cdot\hat{\eta}_A.$$

Consequently, the upper square commutes, as required. □

Theorem 4.9. *Let $H\colon$ **Set** $\to$ **Set** be an iteratable functor with a locally continuous lifting to* CPO*. Then every continuous algebra is an iteration algebra.*

Proof. Let A be a continuous algebra, that is, an H-algebra with a lifting to an $\overline{H}$-algebra $\overline{A}$ (where $\overline{H}\colon$ **CPO** $\to$ **CPO** is a locally continuous lifting of H). From Theorem 4.5 we know that the canonical solutions in A, which, clearly are the canonical solutions for $\overline{A}$ w.r.t. $\overline{H}$, are the least solutions. Let $e_1, e_2\colon X \to R(X+A)$ fulfil $u^\#\cdot e_1 = u^\#\cdot e_2$ for all $u\colon X \to A$. Denote by $\overline{X}$ the discrete CPO on X. Then the two endomorphisms of $\mathbf{CPO}(\overline{X}, \overline{A})$ defined by

$$u \mapsto \hat{a}\cdot R[u, \mathrm{id}_A]\cdot e_i \qquad (\text{for } u\colon X \to A) \qquad i = 1, 2,$$

are equal. Thus, they have the same least fixed point. By 4.8 these are the canonical solutions of e_1 and e_2, respectively, in the Elgot algebra A. □

5 Conclusions

Our paper is devoted to a comparison of various types of algebras with solutions of systems of recursive equations.

One type are iteration algebras of Zoltan Ésik [19] which he introduced as the semantics counter-part of iteration theories studied in the monograph [15]. In the present paper iterative algebras are studied for the free iteration theories (called rational monads) on endofunctors. They turn out to be a bit more restrictive than the Eilenberg-Moore algebras of free iterative monads: the latter are studied in [8] under the name Elgot algebras, and we provide an example of an Elgot algebra that is not an iteration algebra in 3.20. (Iteration algebras were introduced in [19] for more general iteration theories; they are not Elgot algebras in general.)

The most wide-spread type of algebras with iteration are the algebras enrichable over **CPO** (with continuous operations) where the least solutions are taken. We prove that every such algebra is an iteration algebra; for algebras over a signature this has already been proved in the monograph [15], our Theorem 4.9 extends this to algebras for iteratable set functors. However, the converse does not hold: an iteration algebra which is not **CPO** enrichable is presented in Example 2.16.

Evelyn Nelson [24] and Jerzy Tiuryn [25] studied iterative algebras, where guarded systems of recursive equations have unique solutions. Such algebras are iteration algebras, see Proposition 3.18, but not vice versa, as Example 3.19 demonstrates. Every algebra enrichable over **CMS**, the category of complete metric spaces, with contracting operations is iterative: see Example 2.11. An iterative algebra need not be enrichable either over **CMS** or over **CPO**, see Examples 2.13 and 2.16.

There is an obvious parallel to the topic of the present paper: a comparison of iterative algebraic theories of Calvin Elgot and iteration theories of Stephen Bloom and Zoltan Ésik. This is a topic of our future research, the first step is [9].

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