Initial Algebras, Terminal Coalgebras, and the Theory of Fixed Points of Functors

Draft: July 28, 2020

Jiří Adámek    Stefan Milius    Lawrence S. Moss
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>7</td>
</tr>
<tr>
<td>1 Introduction</td>
<td>11</td>
</tr>
<tr>
<td>1.1 Why are initial algebras interesting?</td>
<td>11</td>
</tr>
<tr>
<td>1.2 Why are terminal coalgebras interesting?</td>
<td>13</td>
</tr>
<tr>
<td>1.3 Induction and Coinduction</td>
<td>14</td>
</tr>
<tr>
<td>1.4 Other Fixed Points</td>
<td>16</td>
</tr>
<tr>
<td>1.5 Algebraic versus coalgebraic concepts</td>
<td>16</td>
</tr>
<tr>
<td>1.6 The aim of this book</td>
<td>17</td>
</tr>
<tr>
<td>2 Algebras and Coalgebras</td>
<td>19</td>
</tr>
<tr>
<td>2.1 Algebras</td>
<td>19</td>
</tr>
<tr>
<td>2.2 Initial algebras</td>
<td>26</td>
</tr>
<tr>
<td>2.3 Recursion and induction</td>
<td>35</td>
</tr>
<tr>
<td>2.4 Coalgebras</td>
<td>39</td>
</tr>
<tr>
<td>2.5 Terminal coalgebras</td>
<td>47</td>
</tr>
<tr>
<td>2.6 Corecursion and bisimulation</td>
<td>55</td>
</tr>
<tr>
<td>2.7 Summary of this chapter</td>
<td>64</td>
</tr>
<tr>
<td>3 Finitary Iteration</td>
<td>65</td>
</tr>
<tr>
<td>3.1 Initial-algebra chain</td>
<td>65</td>
</tr>
<tr>
<td>3.2 Examples of initial algebras</td>
<td>68</td>
</tr>
<tr>
<td>3.3 Terminal-coalgebra chain</td>
<td>77</td>
</tr>
<tr>
<td>3.4 Summary of this chapter</td>
<td>84</td>
</tr>
<tr>
<td>4 Finitary Set Functors</td>
<td>85</td>
</tr>
<tr>
<td>4.1 Limits and colimits of algebras and coalgebras</td>
<td>85</td>
</tr>
<tr>
<td>4.2 Weakly terminal coalgebras</td>
<td>88</td>
</tr>
<tr>
<td>4.3 Presentation of finitary set functors</td>
<td>92</td>
</tr>
<tr>
<td>4.4 Iterating the terminal-coalgebra chain to $\omega + \omega$</td>
<td>102</td>
</tr>
<tr>
<td>4.5 Finite Power-Set Functor: the Terminal Coalgebra vs. the $\omega^{\text{op}}$-Limit</td>
<td>109</td>
</tr>
<tr>
<td>4.6 Summary of this Chapter</td>
<td>115</td>
</tr>
<tr>
<td>5 Finitary Iteration in Enriched Settings</td>
<td>117</td>
</tr>
<tr>
<td>5.1 Canonical fixed points in CPO-enriched categories</td>
<td>118</td>
</tr>
<tr>
<td>5.2 CMS-enriched categories</td>
<td>131</td>
</tr>
<tr>
<td>5.3 Solving domain equations</td>
<td>140</td>
</tr>
<tr>
<td>5.4 Summary of this chapter</td>
<td>144</td>
</tr>
</tbody>
</table>
## Contents

### 6 Transfinite Iteration 145
- 6.1 The initial-algebra chain ........................................... 147
- 6.2 The terminal-coalgebra chain ..................................... 160
- 6.3 Subfunctors and quotient functors ................................ 167
- 6.4 Canonical fixed points in CPO-enriched categories .......... 173
- 6.5 Summary of this chapter ........................................... 175

### 7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras 177
- 7.1 Corecursive algebras ............................................... 179
- 7.2 Completely Iterative Algebras ................................... 185
- 7.3 Recursive coalgebras ............................................... 195
- 7.4 Summary of this Chapter .......................................... 203

### 8 Well-Founded Coalgebras 205
- 8.1 Well-Founded Coalgebras and Well-Founded Graphs .......... 206
- 8.2 Factorization of Coalgebra Homomorphisms .................... 213
- 8.3 The Next Time Operator on Coalgebras ......................... 215
- 8.4 The Well-Founded Part of a Coalgebra ......................... 221
- 8.5 Closure Properties of Well-Founded Coalgebras ............... 224
- 8.6 The General Recursion Theorem ................................ 228
- 8.7 The Converse of the General Recursion Theorem ............... 233
- 8.8 Summary of this Chapter .......................................... 239

### 9 State Minimality and Well-Pointed Coalgebras 241
- 9.1 Simple Coalgebras ................................................ 241
- 9.2 Pointed and Reachable Coalgebras ............................... 245
- 9.3 Well-pointed Coalgebras ........................................... 254
- 9.4 Summary of this chapter ........................................... 261

### 10 Fixed Points Determined by Finite Behaviour 263
- 10.1 Locally Finitely Presentable Categories ....................... 263
- 10.2 The Rational Fixed Point ........................................ 267
- 10.3 Iterative Algebras ............................................... 275
- 10.4 The Rational Fixed Point of a Set Functor .................... 285
- 10.5 Full Abstractness and Finitely Generated Objects .......... 287
- 10.6 Beyond the Rational Fixed Point ............................... 297
- 10.7 Summary of this chapter .......................................... 298

### 11 Sufficient Conditions for Terminal Coalgebras 301
- 11.1 Using the Adjoint Functor Theorems ........................... 301
- 11.2 Subfunctors and Quotients ..................................... 302
- 11.3 Finitary and accessible functors ............................... 304
- 11.4 Adjunctions and duality ........................................ 308
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.5</td>
<td>Liftings of terminal coalgebras</td>
<td>311</td>
</tr>
<tr>
<td>11.5.1</td>
<td>Relations</td>
<td>311</td>
</tr>
<tr>
<td>11.5.2</td>
<td>Preorders</td>
<td>312</td>
</tr>
<tr>
<td>11.5.3</td>
<td>Complete Partial Orders</td>
<td>312</td>
</tr>
<tr>
<td>11.5.4</td>
<td>Complete Metric Spaces</td>
<td>313</td>
</tr>
<tr>
<td>11.5.5</td>
<td>Kleisli Categories</td>
<td>315</td>
</tr>
<tr>
<td>11.5.6</td>
<td>Eilenberg-Moore Algebras</td>
<td>318</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Interaction between Initial Algebras and Terminal Coalgebras</td>
<td>323</td>
</tr>
<tr>
<td>12.1</td>
<td>Canonical morphisms</td>
<td>324</td>
</tr>
<tr>
<td>12.2</td>
<td>Cauchy completion</td>
<td>327</td>
</tr>
<tr>
<td>12.3</td>
<td>Ideal completion</td>
<td>330</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>Derived Functors</td>
<td>335</td>
</tr>
<tr>
<td>13.1</td>
<td>Free algebras</td>
<td>335</td>
</tr>
<tr>
<td>13.2</td>
<td>Composite functors</td>
<td>336</td>
</tr>
<tr>
<td>13.3</td>
<td>Relatively terminal coalgebras</td>
<td>340</td>
</tr>
<tr>
<td>13.4</td>
<td>Mutual recursion</td>
<td>341</td>
</tr>
<tr>
<td>13.5</td>
<td>Parametric Fixed Points</td>
<td>343</td>
</tr>
<tr>
<td>13.6</td>
<td>Initial double-algebras</td>
<td>348</td>
</tr>
<tr>
<td>13.7</td>
<td>Coproducts of monads</td>
<td>351</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>Recursion and corecursion</td>
<td>357</td>
</tr>
<tr>
<td>14.1</td>
<td>Recursion and induction on $\mathbb{N}$, a review</td>
<td>357</td>
</tr>
<tr>
<td>14.2</td>
<td>Primitive recursion</td>
<td>360</td>
</tr>
<tr>
<td>14.3</td>
<td>Dual results</td>
<td>362</td>
</tr>
<tr>
<td>14.4</td>
<td>Completely iterative algebras</td>
<td>365</td>
</tr>
<tr>
<td>14.5</td>
<td>The operation $T$</td>
<td>367</td>
</tr>
<tr>
<td>14.6</td>
<td>The Substitution Theorem</td>
<td>368</td>
</tr>
<tr>
<td>14.7</td>
<td>The Solution Theorem</td>
<td>371</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>Special Topics</td>
<td>375</td>
</tr>
<tr>
<td>15.1</td>
<td>Collections of endofunctors with terminal coalgebras</td>
<td>375</td>
</tr>
<tr>
<td>15.2</td>
<td>Variations on Cantor's Theorem</td>
<td>375</td>
</tr>
<tr>
<td>15.3</td>
<td>The interval $[0,1]$ as a terminal coalgebra</td>
<td>378</td>
</tr>
<tr>
<td>15.4</td>
<td>Terminal coalgebras and corecursive algebras related to subsets of the reals</td>
<td>381</td>
</tr>
<tr>
<td>15.5</td>
<td>The Vietoris functor $\mathcal{V}$ on compact Hausdorff spaces</td>
<td>387</td>
</tr>
<tr>
<td>15.6</td>
<td>Measurable spaces</td>
<td>389</td>
</tr>
<tr>
<td>15.7</td>
<td>Posets and related categories</td>
<td>391</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A Primer on Fixed Points in Ordered and Other Structures</td>
<td>403</td>
</tr>
<tr>
<td>A.1</td>
<td>Fixed Points in Posets</td>
<td>403</td>
</tr>
<tr>
<td>A.2</td>
<td>Pataraia’s Theorem</td>
<td>406</td>
</tr>
<tr>
<td>A.3</td>
<td>Fixed Points in Complete Metric Spaces</td>
<td>408</td>
</tr>
<tr>
<td>Contents</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------------------------------------</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>B Appendix: Set Functors</td>
<td>409</td>
<td></td>
</tr>
<tr>
<td>B.1 Examples</td>
<td>409</td>
<td></td>
</tr>
<tr>
<td>B.2 Basic facts and preservation properties</td>
<td>410</td>
<td></td>
</tr>
<tr>
<td>B.3 Finitary set functors</td>
<td>414</td>
<td></td>
</tr>
<tr>
<td>B.4 Trnková hull</td>
<td>417</td>
<td></td>
</tr>
<tr>
<td>B.5 Accessible Functors</td>
<td>418</td>
<td></td>
</tr>
<tr>
<td>B.6 Standard Functors</td>
<td>419</td>
<td></td>
</tr>
<tr>
<td>B.7 Accessible coreflection</td>
<td>420</td>
<td></td>
</tr>
<tr>
<td>B.8 Weak pullbacks</td>
<td>422</td>
<td></td>
</tr>
<tr>
<td>B.9 Preservation properties of particular set functors</td>
<td>422</td>
<td></td>
</tr>
<tr>
<td>Index</td>
<td>427</td>
<td></td>
</tr>
<tr>
<td>Bibliography</td>
<td>431</td>
<td></td>
</tr>
</tbody>
</table>
Preface

Initial algebras for endofunctors on a category have been used since the 1970s in algebraic specification and for the semantics of inductive data types definitions. They provide a generic framework to study notions such as recursive function definitions and proofs by structural induction. This has been developed for example in the work of the ADJ group on initial-algebra semantics of abstract data types. Domain theory is another example. Dana Scott’s model of the lambda-calculus [219] works with initial algebras for endofunctors on the category of domains or complete partial orders [110]. This usage developed into Michael Smyth and Gordon Plotkin’s treatment of the solution of recursive domain equations [224]. The 1980s and 90s saw the further development of such topics and their treatment in textbooks such as Ernest Manes and Michael Arbib’s book [170] on algebraic semantics of programming, and Samson Abramsky and Achim Jung’s survey of domain theory [2].

At the turn of the new millenium, the dual concept, coalgebras for endofunctors, came within the focus of increased attention by theoretical computer scientists. While the study of coalgebras had its roots in earlier work parallel to algebras, it was sparked in earnest by Peter Aczel’s book [3] on non-well founded sets, where coalgebras were mentioned at the end. His observation was that the infinite processes that we see in theoretical computer science may be profitably studied as elements of the terminal coalgebra, with specifications of them coming from other coalgebras. Aczel also exhibited, together with Nax Mendler, Robin Milner and David Park’s notion of bisimulation from process algebra as a coalgebraic notion. This then led to the breakthrough for coalgebras, which came with Jan Rutten’s seminal paper [208] on coalgebras as a theory of systems. It demonstrated that many types of state-based systems studied by theoretical computer scientists in fields like automata theory, concurrency, and verification arise as examples of coalgebras for endofunctors. Moreover, the terminal coalgebra yields a fully abstract domain for the behaviour of states of systems. This laid the basis for the new subject of universal coalgebra. In the 2000s the subject then rapidly developed and has unified a host of topics that looked similar but were not always understood that way. These were topics from theoretical computer science and logic like automata theory, process calculi, streams, non-wellfoundeded sets, and modal logic, and also areas of mathematics such as power series. Meanwhile universal coalgebra has become a diverse research field offering a generic framework for the semantics of state-based systems, specification and proof principles such as corecursion and coinduction, and the development of coalgebraic logics. In the last few years, generic methods and algorithms for reasoning, model checking, minimization and learning of coalgebras have become a focus of research. Coalgebra continues to be a lively and actively developed area of research, and we hope to offer our readers a source that will help them to enter the field.
Throughout this development it was the pull of category theory that has provided the language and conceptual apparatus that is needed to unify topics and pose new questions. Therefore, the aim of our book is to give a category-theoretic account of initial algebras, terminal coalgebras, and, as the title of our book suggests, to pursue the topic of fixed points of functors more generally. We also put a focus on the interplay of algebras and coalgebras, a feature that most other texts on those subjects miss, treating one or the other only. Many of the results in this book are stated and proved for categories that go beyond the category of sets. Furthermore, we use special features of the category of sets such as presentations of finitary functors and transfinite recursion. We frequently call on facts about set functors, so much so that we provide an appendix on this topic. A number of these facts are based on work by the Prague mathematical community, especially Věra Trnková and her colleagues and students Václav Koubek and Jan Reiterman. We know that all of this extra background will make our work less “accessible” to some readers, and we therefore provide some background on them as needed.

The central topics of our book are:

1. The iterative constructions of initial algebras and terminal algebras as colimits of chains and limits of op-chains, respectively. We discuss this at great length, providing hosts of concrete examples and developing generalizations to the transfinite setting. We also develop the enriched setting, in particular, enrichment over the categories of complete partial orders and of complete metric spaces, that has been so useful in topics like domain theory.

2. Fixed points of finitary functors, and accessible functors more generally. For example, we develop and extend the work of James Worrell [244, 245] on the terminal coalgebra of accessible set functors.

3. Categorical recursion theory, especially corecursion, completely iterative algebras and well-founded coalgebras. This not only connects our subject to the past but also highlights topics that we believe will be ever more important in the future.

4. The relations between various fixed points. In particular, we treat the rational fixed point of a functor, which is a fully abstract domain of ‘finite-state’ behaviour with instances such as the regular languages from automata theory. Here again we go beyond sets for most results.

It has been said that there never is a good time to write a book about anything, and this is especially true of a book coming from an active research field. We are not only trying to summarize algebra, coalgebra and related fields, we are also contributing to them. Some of the results in this book appear here for the first time, and in many other cases we have revised our text in order to develop a subject that only came into existence a few years ago. We hope that an up-to-the-moment book itself will prove useful to the reader.

We have tried hard to be scholarly about the history of results, and we hope that those whose work was not mentioned correctly, or at all, will forgive us.

We have neglected several topics that are near and dear to the hearts of many in the coalgebra community. We did this partly to keep the book of reasonable length, and
partly because those topics are already treated in books. In particular, we are thinking
of bisimulation and of coalgebraic modal logic. Bisimulation is treated in Bart Jacob’s
2016 book [136]; that book also has a lot of material on another topics which we do
not treat: predicate liftings. Coalgebraic modal logic is the topic of Dirk Pattinson and
Lutz Schröder’s book. In addition, one also should read Jan Rutten’s The Method of
Coalgebra: exercises in coinduction [206]. These are all excellent resources, and they will
be especially useful for newcomers to the subject.

But now we hope our readers will have as much fun with this book as we have had in
writing it.

Jiří Adámek
Stefan Milius
Larry Moss
1 Introduction

1.1 Why are initial algebras interesting?

Recursion and induction are important tools in mathematics and computer science. In functional programming, for example, recursion is a definition principle for functions over the (inductive) structure of data types such as natural numbers, lists or trees. And induction is the corresponding proof principle used to prove properties of programs. An important question of theoretical computer science concerns the semantics of such definitions. Initial Algebra Semantics, studied since the 1970’s, uses the tools of category theory to unify recursion and induction at the appropriate abstract conceptual level. In this approach, the type of data on which one wants to define functions recursively and to prove properties inductively is captured by an endofunctor \( F \) on the category of sets (or another appropriate base category). This functor describes the signature of the data type constructors. An \( F \)-algebra is a set \( A \) together with a map \( \alpha : FA \to A \), and an initial algebra for the functor \( F \) provides a canonical minimal model of a data type with the desired constructors.

Let us illustrate this by a concrete example: Consider the endofunctor on sets given by \( FX = X + 1 \), i.e. the set construction adding a fresh element to the set \( X \). An algebra \( \alpha : A \to A + 1 \) for \( F \) is just a set equipped with a unary operation and a constant, and the initial algebra is the algebra of natural numbers \( \mathbb{N} \) with the successor function and the constant 0. The abstract property of initiality of that algebra is precisely the usual principle of recursion on natural numbers: given an \( F \)-algebra \( X \), i.e. a unary operation on \( u : X \to X \) and a constant \( x \in X \), there exists precisely one function \( f \) from the natural numbers to \( X \) with \( f(0) = x \) and \( f(n + 1) = u(f(n)) \). In other words, functions from \( \mathbb{N} \) to \( X \) can be defined by recursion, and the uniqueness yields the proof principle of induction.

As a second example, consider the set functor \( FX = X \times X + 1 \). An algebra \( \alpha : A \times A + 1 \to A \) consists of a set equipped with a binary operation and a constant. Then the initial algebra is the algebra of finite binary trees, and initiality yields a tree-recursion principle.

In the present book initial algebras are studied for all categories \( \mathcal{A} \) and endofunctors \( F : \mathcal{A} \to \mathcal{A} \). It was J. Lambek [163] who first studied algebras for \( F \) as pairs consisting of an object \( A \) of \( \mathcal{A} \) and a morphism \( a : FA \to A \); the corresponding \( F \)-algebra homomorphisms (shortly homomorphisms) from \((A, \alpha)\) to \((A', \alpha')\) are those morphisms \( h : A \to A' \).
1 Introduction

in $\mathcal{A}$ for which the square below commutes:

\[
\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
\downarrow Fh & & \downarrow h \\
FA' & \xrightarrow{\alpha'} & A'
\end{array}
\]

If $\alpha$ is invertible (thus $FA \cong A$) we call the algebra a fixed point of $F$.

The category of algebras is denoted by $\text{Alg } F$. By an initial algebra $\mu F$ of $F$ is meant its initial object: this is an algebra such that for every algebra there exists a unique homomorphism from $\mu F$.

**Lambek’s Lemma.** If $F$ has an initial algebra, then it is a fixed point.

We conclude immediately that even for $\mathcal{A} = \text{Set}$, there are important endofunctors that do not have an initial algebra: the power-set functor $P$. A fundamental result of set theory known as Cantor’s Theorem says that no set $A$ is in bijective correspondence with $P A$. (The short proof may be found in Example 2.2.7(1).) So no set is a fixed point of the power-set functor.

In the present book, we study the existence and construction of initial algebras. There is a general procedure for constructing the initial algebra for $F$ starting from the initial object $0$ of $\mathcal{A}$, first used by Adámek [8]. Denoting by $!: 0 \to F0$ the unique morphism, we form the corresponding $\omega$-chain:

\[
0 \xrightarrow{!} F0 \xrightarrow{F!} FF0 \xrightarrow{F^2!} F^30 \xrightarrow{F^3!} \cdots
\]

If the colimit exists and is preserved by $F$, then that colimit carries the initial algebra

$$\mu F = \text{colim}_{n \in \omega} F^n0.$$

If $F$ does not preserve that colimit, we iterate further and obtain a transfinite chain $F^n0$, where $i$ ranges over all ordinal numbers. At successor ordinals $i + 1$ we apply $F$, so that $F^{i+1}0 = FFi0$, and at limit ordinals $\lambda$ we take colimits: $F^\lambda0 = \text{colim}_{i<\lambda} F^i0$. If $F$ preserves one of these colimits, we obtain an initial algebra. We go into detail on this construction in Chapter 3 for finite ordinals and in Chapter 6 for general ones.

We illustrate the behaviour of this construction and other methods for obtaining initial algebras by numerous examples. For example the functor $FX = X \times X + 1$ (of binary algebras with a constant) yields the chain with $F^10 = 1$ and $F^{n+1}0 = F^n0 \times F^n0 + 1$. This recursion allows us to represent $F^n0$ by the set all binary trees of height less than $n$, up to isomorphism. The initial algebra $\text{colim}_{n<\omega} F^n0 = \bigcup_{n<\omega} F^n0$ is the algebra of all finite binary trees. The constant is the trivial single-node tree, and the binary operation is tree tupling, i.e. the operation that assigns to a pair $(t_1, t_2)$ of binary trees the binary tree having $t_1$ and $t_2$ rooted at the children of the root:

\[
(t_1, t_2) \mapsto \begin{array}{c} t_1 \\
| \downarrow \searrow \nearrow \\
t_2 \end{array}
\]

Shortly: $\mu F = \text{algebra of finite binary trees}$. 

12
1.2 Why are terminal coalgebras interesting?

A coalgebra for a functor $F$ is the dual concept of an $F$-algebra: it consists of an object $A$ and a morphism $\alpha: A \to FA$. Jan Rutten [208] presented a persuasive survey of applications of this idea to the theory of discrete dynamical systems which are ubiquitous in computer science. The idea is that in a coalgebra $\alpha: A \to FA$ for a set functor $F$, $A$ is the set of states in a systems, and $\alpha$ assigns to a given state the set of possible successor states. The point of working with a functor is that it leads to a generic description of maps between such systems, as we shall see shortly. For example, a deterministic automaton with input alphabet $\Sigma$ can be described by the set $A$ of its states together with a function

$$\alpha: A \to \{0, 1\} \times A^\Sigma$$

whose first component $A \to \{0, 1\}$ describes the predicate “accepting state”, and the second one

$$A \to A^\Sigma$$

$$A \times \Sigma \to A$$

describes the next-state function. This is a coalgebra for the set functor $F$ given by

$$FX = \{0, 1\} \times X^\Sigma.$$

A non-deterministic automaton is given by a function from $A$ to $\{0, 1\} \times (\mathcal{P}A)^\Sigma$ and is thus a coalgebra for the endofunctor $F\mathcal{P}$ composed of the power-set functor and the above functor $F$.

For another example, consider a dynamic system with states accepting binary input and having also deadlock states (not reacting to inputs). This is given by a set $A$ of states and a function

$$\alpha: A \to A \times A + 1$$

assigning to every deadlock state the element of $1$ and to every other state the pair of the possible next states. This is a coalgebra for the set functor $FX = X \times X + 1$.

The category $\mathbf{Coalg} F$ of coalgebras has as morphisms from $(A, \alpha)$ to $(A', \alpha')$ the $F$-coalgebra homomorphisms (shortly: homomorphisms) which are the morphisms $h: A \to A'$ such that the square below commutes:

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
h \downarrow & & \downarrow \text{Fh} \\
A' & \xrightarrow{\alpha'} & FA'
\end{array}$$

The terminal coalgebra $\nu F$ is (if it exists) the terminal object of this category: For every coalgebra there exists a unique homomorphism into $\nu F$. The intuition is that $\nu F$ is that the terminal coalgebra has as elements all types of behavior realized by states in coalgebras. If a system is represented by a coalgebra $(A, \alpha)$, then the unique homomorphism $f: A \to \nu F$ assigns to a state its “abstract behaviour of the states in $A$”, as we demonstrate on numerous examples in Section 2.3 and later.
1 Introduction

Example 1.2.1. The functor $FX = \{0,1\} \times X^\Sigma$ has the terminal coalgebra $\mathcal{P}(\Sigma)^*$, the set of all formal languages over the alphabet $\Sigma$:

$$\nu F = \mathcal{P}(\Sigma)^*.$$ 

Given an automaton $(A,\alpha)$, the unique homomorphism $f : A \to \mathcal{P}(\Sigma)^*$ assigns to every state the language this state accepts. For details, see Example 2.5.5.

Example 1.2.2. The functor $FX = X \times X + 1$, used for the dynamical systems with deadlocks leads to the concept of behaviour as a binary tree: the root represents the starting state. Every node that is not a deadlock has two children given by the possible next states. This leads to the coalgebra of all (finite and infinite) binary trees up to isomorphism as the terminal coalgebra, shortly:

$$\nu F = \text{binary trees}.$$ 

Given a dynamic system $(A,\alpha)$ the unique homomorphism $f : A \to \nu F$ assigns to every state the binary tree of all possible future developments in which the names of states are “abstracted away” and only the distinction deadlock/non-deadlock remains, see Example 2.5.11(4).

By dualizing Lambek’s Lemma, we see that $\nu F$ is always a fixed point of $F$. Thus, we can consider $\nu F$ also to be an algebra for $F$. (For example, binary trees form an algebra for $FX = X \times X + 1$, with tree tupling and the single-node tree, analogously to $\mu F$.) This algebra $\nu F$ has a strong property of solvability of recursive equations. Let us illustrate this with the functor $FX = X \times X + 1$ of one binary and one nullary operation.

Given a system of recursive equations

$$x_1 = t_1$$
$$x_2 = t_2$$
$$\vdots$$

where each $t_i$ is a $\Sigma$-term in the variables $x_1,x_2,\ldots$ for the above signature of a binary operation and constant, there exists a solution in $\nu F$. This means that to every $x_i$ we can assign a binary tree $x_i^\dagger$ such that the formal equations above become identities when the simultaneous substitution $x_i^\dagger/x_i$ is performed on the left- and right-hand sides of each equation in the system. Moreover, the solution of the system is unique, provided none of the right-hand sides is a bare variable. Algebras with this recursion property are called completely iterative. Milius [174] proved that whenever $\nu F$ exists, it is a completely iterative algebra – in fact, it can be characterized as the initial completely iterative algebra. We provide a theory of completely iterativity in Section 7.2.

1.3 Induction and Coinduction

One of the important roles of initial algebras is to enable a very general formulation of induction and recursion; this is discussed in Section 2.3. Jan Rutten also discusses in [208]
1.3 Induction and Coinduction

corecursion as an important construction principle dual to recursion, and coinduction as an important proof principle dual to induction; we come to this in Section 2.6.

In fact, induction can be formulated abstractly as follows: for a parallel pair \( f_1, f_2: \mu F \rightarrow A \) of morphisms in the base category \( \mathcal{A} \) with domain \( \mu F \), in order to prove \( f_1 = f_2 \) it is sufficient to present a morphism \( \alpha: FA \rightarrow A \) for which \( f_1 \) and \( f_2 \) are algebra homomorphisms from \( (\mu F, \iota) \) to \( (A, \alpha) \). Coinduction is the dual principle which for the terminal coalgebra \( \nu F \) allows us to prove equality of morphisms of the form \( f_1, f_2: A \rightarrow \nu F \).

It goes without saying that not every functor possesses a terminal coalgebra. This follows from the dual of Lambek’s Lemma: the power-set functor does not have a terminal coalgebra. We study the dual of the above initial algebra construction (explicitly used by Michael Barr [58] for the first time): Start with the terminal object 1 of \( \mathcal{A} \) and the unique morphism \( !: F1 \rightarrow 1 \) and form the \( \omega \text{op} \)-chain

\[
1 \xleftarrow{!} F1 \xleftarrow{F!} FF1 \xleftarrow{F^2!} F^3 \cdots .
\]

If the limit exists and is preserved by \( F \), this is the terminal coalgebra for \( F \):

\[
\nu F = \lim_{n<\omega} F^n1.
\]

Example 1.3.1. The above functors \( FX = \{0, 1\} \times X^\Sigma \) and \( FX = X \times X + 1 \) preserve all limits of \( \omega \text{op} \)-chains (called \( \omega \text{op} \)-limits, for short), and in particular their terminal coalgebras may be obtained by taking the limit of the \( \omega \text{op} \)-chain in (1.1).

If \( F \) does not preserve \( \omega \text{op} \)-limits, one may continue to iterate \( F \), obtaining a transfinite chain. We discuss this in Chapter 6.

Example 1.3.2. Graphs are nothing else than coalgebras for the power-set functor \( \mathcal{P} \): given a graph on the set \( A \) of vertices, then consider, for every vertex \( x \in A \), the set \( \alpha(x) \subseteq A \) of all neighbours of \( x \). This defines a coalgebra \( \alpha: A \rightarrow \mathcal{P}A \), and conversely, every coalgebra stems from a graph. But be careful: there are fewer coalgebra homomorphisms than graph homomorphisms. Given graphs \( (A, \alpha) \) and \( (B, \beta) \) a coalgebra homomorphism \( f: A \rightarrow B \) does not only fulfill

\[
\text{if } x \rightarrow x' \text{ in } A \text{ then } f(x) \rightarrow f(x') \text{ in } B
\]

but also

\[
\text{if } f(x) \rightarrow y' \text{ in } B \text{ then } x \rightarrow y \text{ in } A \text{ for some } y \text{ with } f(y) = y'.
\]

In other words, when we consider the graphs \( A \) and \( B \) as (unlabelled) transition systems, a coalgebra homomorphism is precisely a function whose graph is a bisimulation in the sense of Milner [185, 186] and Park [196]; see also [212].

Lambek’s Lemma tells us that there exists no terminal graph. However, for \( \mathcal{P}_f \) the subfunctor of \( \mathcal{P} \) mapping a set to the set of all its finite subsets, coalgebras are precisely the finitely branching graphs, and a terminal coalgebra exists. We mention this example because the limit of the \( \omega \text{op} \)-chain in (1.1) is not preserved by the functor \( \mathcal{P}_f \), and so one needs a more sophisticated construction. Moreover, there are several different interesting descriptions of the terminal coalgebra that we present in Section 4.5.
1 Introduction

As we shall see in Section 4.5, there are several different constructions of the terminal coalgebra for $\mathcal{P}_f$. For other functors on other categories, there are yet other constructions.

1.4 Other Fixed Points

In the study of state-based systems one is mostly interested in finite systems, or ones whose state space has a finite representation (e.g. a finite dimensional vector space or an orbit finite nominal set). For example, in classical automata theory the regular language are those accepted by finite automata, and rational streams over a field are precisely those behaviours obtained from systems whose state space is a finite dimensional vector space over the field.

We present a general treatment of regularity of system behaviour and the semantics of `finite' coalgebras. For this we assume that the functor $F$ is finitary (see Definition 4.3.1 for the formal definition). For example, finitary set functors are those determined by their action on finite sets; for them the behaviour of all finite coalgebras yields a subcoalgebra of the terminal coalgebra $\nu F$ (consisting of the behaviour of all coalgebras). This subcoalgebra turns out to be a fixed point of $F$ which we call the rational fixed point and denote it by $\varrho F$. For example, for the functor $FX = \{0,1\} \times X^\Sigma$ this is the coalgebra of all regular languages over the alphabet $\Sigma$. For $FX = X \times X + 1$ the rational fixed point $\varrho F$ is the subcoalgebra of $\nu F$ (of all binary trees) consisting of precisely those binary trees that have only finitely many subtrees (up to isomorphism). Besides all finite binary trees, the complete (infinite) binary tree is an example of a tree contained in $\varrho F$. These are precisely the regular trees in the sense of Courcelle [84].

The concept of a rational fixed point makes sense for endofunctors on other categories that $\text{Set}$, too. For example, for the functor $F$ on the category of vector spaces whose coalgebras are linear weighted automata we obtain the rational streams as $\varrho F$. We study the rational fixed point in Chapter 10.

1.5 Algebraic versus coalgebraic concepts

Although algebra and coalgebra are dual terms, and although this duality persists to the level of initial algebra and terminal coalgebra, $\text{Alg} F$ is not dual to $\text{Coalg} F$. There are easy examples of this; here is a very simple one: consider the poset \{x, y, z\} with $x \leq y$, $x \leq z$ as a category (with morphisms given by the order relation $\leq$), and consider the identity map on this poset as a functor. This poset has an initial object $x$ but no terminal one. Thus the identity functor has an initial algebra but no terminal coalgebra.

What is true is the following: every functor $F: \mathcal{A} \to \mathcal{A}$ defines a functor $F^{\text{op}}: \mathcal{A}^{\text{op}} \to \mathcal{A}^{\text{op}}$ by the same rule as $F$. The category of algebras for $F$ (in $\mathcal{A}$) is dual to the category of coalgebras for $F^{\text{op}}$ (in $\mathcal{A}^{\text{op}}$). Shortly,

$$(\text{Alg} F)^{\text{op}} = \text{Coalg}(F^{\text{op}}).$$

We present in Figure 1.1 a comparison between concepts and ideas in algebra and coalgebra. As this indicates, one reason why the coalgebraic concepts on the right of
Figure 1.1 are interesting is that they are the structures used in the mathematics of \textit{transition} and \textit{observation}, as opposed to \textit{operations}. Terminal coalgebras in this sense are like the most abstract collections of “transitions” or “observations”. We know that this is very vague, and so we hope that the examples throughout this book will explain what we mean.

We also would like to mention other sources, such as Rutten [208], a highly recommendable general source on coalgebra (this is the source of Figure 1.1), Gumm’s survey in [133], or Moss [188], which includes much conceptual discussion related to the set-theoretic topics. Last but not least, Jacobs’ textbook [136] on coalgebra is a very valuable source.

\section*{1.6 The aim of this book}

We present general conditions which guarantee the existence of an initial algebra or a terminal coalgebra. We are also interested in \textit{representations} of terminal coalgebras. The reason for this is that the existence theorems themselves frequently are fairly abstract, and so concrete representations make the terminal coalgebras more intuitive. We also study a number of topics related to terminal coalgebras, e.g. interaction between initial algebras and terminal coalgebras (Chapter 12). At this point, we want to mention the main categories and functors of interest in our study.

We begin with \textbf{Set}, the category of sets and functions. We are interested in the \textit{polynomial functors} obtained from the identity functor and the constant functors by products and coproducts (including exponents $X \mapsto X^B$ for a fixed set $B$).

Another functor which we shall study is the \textit{discrete probability measure functor} $\mathcal{D}$, where $\mathcal{D}X$ is the set of functions from $X$ to $[0,1]$ which have the value $0$ except on finitely many points and which sum to $1$. $\mathcal{D}$ takes a function $f: X \to Y$ to the function
1 Introduction

\( \mathcal{D} f : \mathcal{D} X \to \mathcal{D} Y \). For each \( \mu \in \mathcal{D} X \), \( \mathcal{D} f(\mu) \) is given by

\[
\mathcal{D} f(\mu)(y) = \mu(f^{-1}(y)) = \sum_{x \in f^{-1}(y)} \mu(x).
\]

We already mentioned the power-set functor \( \mathcal{P} \); this gives the set of subsets of \( X \). There are also a few refinements of \( \mathcal{P} \), including \( \mathcal{P}_f \), the finite power-set functor.

**Other categories** include many-sorted sets, multi-sets, \( K\text{-Vec} \) (vector spaces over a fixed field), \( \text{Nom} \) (nominal sets), \( \text{Rel} \) (sets and relations), \( \text{POS} \) (posets and monotone maps), and \( \text{CPO} \) (complete partial orders and continuous maps). Further, we shall consider \( \text{MS} \), the category of metric spaces and non-expanding maps, and also the full subcategory \( \text{CMS} \) of complete metric spaces with distances bounded by 1. Both of these categories have a “power-set-like” operation, and these will be of special interest.

**The structure of this book.** We have tried to collect interesting results about terminal coalgebras (and initial algebras) scattered throughout the literature. We have found some results not quite complete and we completed them. For technically more involved proofs we usually indicate the idea of the proof, but otherwise we provide references to where proofs can be found.
2 Algebras and Coalgebras

As the title of our book suggests, we are mainly interested in initial algebras and terminal coalgebras. But to understand what these are, we must discuss the more general concepts of algebras and coalgebras first. This is the purpose of this chapter.

Contents  This chapter has two halves: one devoted to algebras and the other to coalgebras. For algebras, we first discuss them in general, primarily using examples from the categories Set of sets, Set$^3$ of sorted sets, and CPO$_\bot$ of complete partial orders with a least element (see Example 2.1.7(2)). Then we turn to initial algebras. The main results are detailed constructions of initial algebras, mainly for polynomial endofunctors on these categories. Following this, we examine two concepts closely related to initial algebras: recursion and induction. Then we turn to the parallel topics for coalgebras: examples from the same categories, terminal coalgebras for polynomial functors, and finally corecursion and bisimulation.

In addition to the results which we have mentioned, the point of the chapter is also to present examples and intuitions. But the overall treatment in this chapter is in a sense preliminary: at numerous points we mention discussions later in the book which amplify what we do here.

Background needed for this chapter  To read this chapter, you need to know a few of the most basic definitions and concepts from category theory. These include initial and terminal objects (denoted 0 and 1, respectively), products and coproducts, monomorphisms and epimorphisms. Very little else is required. Later chapters in the book require more, of course. And throughout the book, we are mainly writing to readers well-versed in the basics of category theory. The main categories in this chapter are Set and CPO$_\bot$. It would be useful to have seen signatures and Σ-algebras, say as they appear in theoretical computer science or general algebra.

In general, in this book when we say “Recall X”, we assume that you have a passing knowledge of X.

2.1 Algebras

To specify algebras, we must have an underlying or base category, say $\mathcal{A}$, and an endofunctor $F$, i.e. a functor $F : \mathcal{A} \to \mathcal{A}$. With this in mind, we introduce the concepts of algebra and homomorphism. They are due to Lambek [163]. Often the action of $F$ on morphisms is quite obvious, so one surpresses it and states only its action on objects. For
example, \( FX = X \times X + 1 \) is given for objects \( X \), and it is understood that for every morphism \( f: X \to Y \) we have \( Ff = f \times f + \text{id}_A \).

**Definition 2.1.1.** An *algebra* for an endofunctor \( F \) (or an \( F \)-algebra) consists of an object \( A \) and a morphism \( \alpha: FA \to A \). We sometimes call \( A \) the *carrier* and \( \alpha \) the *structure*. So technically the algebra is a pair \((A, \alpha)\). But as usual we shorten this to \( A \) whenever we can. Usually we use upper-case letters for algebras and Greek letters for structures.

A homomorphism of \( F \)-algebras from \((A, \alpha)\) to \((B, \beta)\) (or a morphism of \( F \)-algebras) is a morphism \( h: A \to B \) of \( \mathcal{A} \) such that the square below commutes:

\[
\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
\downarrow{Fh} & & \downarrow{h} \\
FB & \xrightarrow{\beta} & B
\end{array}
\]  

It should be clear that if \( A, B, \) and \( C \) are algebras, and \( g: A \to B \) and \( h: B \to C \) are homomorphisms, then \( h \cdot g: A \to C \) is an algebra homomorphism. That is, composition of homomorphisms works as in the base category \( \mathcal{A} \). Similarly, the identity morphism from the base category is always an algebra homomorphism. This endows the collection of algebras for a fixed endofunctor with a category structure.

**Notation 2.1.2.** The category of \( F \)-algebras and homomorphisms is denoted by \( \text{Alg} \ F \). The base category \( \mathcal{A} \) is suppressed in our notation.

**Examples 2.1.3.** Algebras in \( \mathcal{A} = \text{Set} \) for various endofunctors.

1. Algebras for \( FX = X + 1 \). Recall that 1 denotes a terminal object, here a singleton set. The coproduct operation + is disjoint union. An algebra consists of a set \( A \) and a function \( \alpha: A + 1 \to A \). That is, on a set \( A \) (of data) a unary operation \( \alpha_1: A \to A \) is given together with an “initial datum”: an element expressed by \( \alpha_0: 1 \to A \). Indeed, to specify \( \alpha: A + 1 \to A \) is the same as to specify \( \alpha_1 \) and \( \alpha_0 \), the two components of \( \alpha \).

   A homomorphism from \((A, \alpha)\) to \((B, \beta)\) is a function \( h: A \to B \) between the data sets which preserves the unary operations, i.e. \( h \cdot \alpha_1 = \beta_1 \cdot h \), and the initial data, i.e. \( h \cdot \alpha_0 = \beta_0 \). Indeed, this is equivalent to the commutativity of the following square

\[
\begin{array}{ccc}
A + 1 & \xrightarrow{[\alpha_1, \alpha_0]} & A \\
\downarrow{h + \text{id}} & & \downarrow{h} \\
B + 1 & \xrightarrow{[\beta_1, \beta_0]} & B
\end{array}
\]

2. Analogously, algebras for \( FX = X \times X + 1 \) are sets (of data) with a binary operation and a constant. And homomorphisms are functions commuting with the operation and preserving the constant.

3. For a given set \( \Sigma \), the algebras for \( FX = \Sigma \times X \) are sets \( A \) endowed with unary operations \( \alpha_s: A \to A \) where \( s \) ranges through \( \Sigma \). Indeed, this is the same as specifying
2.1 Algebras

a function \( \alpha : A \times \Sigma \to A \). Homomorphisms from \( (A, \alpha) \) and \( (B, \beta) \) are the functions \( h : A \to B \) preserving all the unary operations: \( h \cdot \alpha_s = \beta_s \cdot h \) for every \( s \in \Sigma \).

(4) The power-set functor is defined by \( \mathcal{P}X = \{ M : M \subseteq X \} \), and for \( f : X \to Y \) and \( M \subseteq X \), \( \mathcal{P}f(M) \) is the image set \( f[M] \). For this functor, an algebra consists of a set \( A \) and a function \( \alpha : \mathcal{P}A \to A \). For example, joins in a complete lattice \( A \) form an algebra for \( \mathcal{P} \).

A homomorphism from \( (A, \alpha) \) to \( (B, \beta) \) is a function \( h : A \to B \) such that for every set \( M \subseteq A \) we have

\[ h(\alpha(M)) = \beta(h[M]). \]

If \( \alpha \) and \( \beta \) are both joins (of complete lattices), this states that \( h \) is join-preserving.

Next, we generalize Example 2.1.3, parts (1)–(3).

**Definition 2.1.4.** A signature is a collection \( \Sigma = (\Sigma_n)_{n \in \mathbb{N}} \) of sets \( \Sigma_n \), where \( \Sigma_n \) is called the set of \( n \)-ary operation symbols, in particular, the elements of \( \Sigma_0 \) are the nullary symbols, or constants. We associate to the signature \( \Sigma \) the polynomial functor

\[ H_\Sigma : \text{Set} \to \text{Set} \]

given by

\[ H_\Sigma X = \coprod_{n \in \mathbb{N}} \Sigma_n \times X^n. \]  \hfill (2.2)

The operation symbols from the signature are “inscribed in” the functor. This functor assigns to a function \( f : X \to Y \) the function

\[ H_\Sigma f = \coprod_{n \in \mathbb{N}} \text{id}_{\Sigma_n} \times f \times \cdots \times f \quad \text{n times} \]

**Example 2.1.5.** Let \( \Sigma \) be a signature. A \( \Sigma \)-algebra is a set \( A \) together with interpretations of the operation symbols; a symbol \( \sigma \) of arity \( n \) is interpreted as a function \( \sigma^A : A^n \to A \). Moreover, a homomorphism of \( \Sigma \)-algebras is a function \( h : A \to B \) such that for all \( \sigma \in \Sigma_n \) and all \( x_0, \ldots, x_{n-1} \in A \),

\[ h(\sigma^A(x_0, \ldots, x_{n-1})) = \sigma^B(h(x_0), \ldots, h(x_{n-1})). \]  \hfill (2.3)

This notion of a \( \Sigma \)-algebra is from general algebra.

We check that \( \Sigma \)-algebras are the same as algebras for the set functor \( H_\Sigma \). and the same for homomorphisms as we have defined them in (2.3) and homomorphisms of \( H_\Sigma \)-algebras.

Given a \( \Sigma \)-algebra \( A \), we define \( \alpha : H_\Sigma A \to A \) by coproduct components: on the summand \( \Sigma_n \times A^n \) of \( H_\Sigma A \) we let

\[ \alpha(\sigma, (x_0, \ldots, x_{n-1})) = \sigma^A(x_0, \ldots, x_{n-1}). \]  \hfill (2.4)

In the other direction, given \( \alpha \), we obtain a \( \Sigma \)-algebra \( A \) in the following way. For every \( \sigma \in \Sigma_n \) define \( \sigma^A \) using (2.4) in the right-to-left direction. This recovers algebras. And
as for homomorphisms, \( h: A \to B \) is a homomorphism of \( \Sigma \)-algebras iff the square below commutes:

\[
\begin{array}{ccc}
\coprod_{n \in \mathbb{N}} \Sigma_n \times A^n & \xrightarrow{\alpha} & A \\
\downarrow^{H \Sigma h} & & \downarrow^h \\
\coprod_{n \in \mathbb{N}} \Sigma_n \times B^n & \xrightarrow{\beta} & B
\end{array}
\]

Again, given a signature \( \Sigma \), the category of \( \Sigma \)-algebras is the same as the category of \( H \Sigma \)-algebras.

**Example 2.1.6.** Many-sorted algebras. Fix a set \( S \). An \( S \)-sorted set is just a family of sets \( X = (X_s)_{s \in S} \) indexed by \( S \). Given \( X = (X_s)_{s \in S} \) and \( Y = (Y_s)_{s \in S} \), an \( S \)-sorted function is a family \( f = (f_s)_{s \in S} \) of functions, where \( f_s: A_s \to B_s \). We thus work in the category \( \text{Set}^S \) of \( S \)-sorted sets and \( S \)-sorted functions. An \( S \)-sorted signature is a set \( \Sigma \) of operation symbols together with arities \( \text{ar}(\sigma) \) of members \( \sigma \) of \( \Sigma \). An arity has the form

\[
s_0 \times \ldots \times s_{n-1} \to s
\]

for an operation symbol whose \( n \) variables have sorts \( s_0, \ldots, s_{n-1} \), respectively, and whose result has sort \( s \).

Incidentally, in all of our discussions of many-sorted sets in this book, \( S \) can be arbitrary. Most of the time, taking \( S = 2 = \{0, 1\} \) illustrates our point: as categories, \( \text{Set} \) and \( \text{Set}^2 \) have enough differences to be interesting for us.

A many-sorted algebra is an \( S \)-sorted set \( (A_s)_{s \in S} \) together with an interpretation of the operation symbols; a symbol \( \sigma \) of arity \( s_0 \times \ldots \times s_{n-1} \to s \) is interpreted as a function \( \sigma^A: A_{s_1} \times \cdots \times A_{s_n} \to A_s \). Equivalently, a many-sorted algebra is an algebra for a polynomial endofunctor on \( \text{Set}^S \). If \( \Sigma \) consists of a single operation symbol of arity \( s_0 \times \ldots \times s_{n-1} \to s \), then the corresponding polynomial functor \( H \Sigma \) has empty sorts \( (H \Sigma X)_t \) for all \( t \neq s \) and its sort \( s \) is

\[
(H \Sigma X)_s = X_{s_0} \times \ldots \times X_{s_{n-1}}
\]

Its action on morphisms is analogous:

\[
(H \Sigma f)_s = f_{s_0} \times \ldots \times f_{s_{n-1}}
\]

In the case of general \( S \)-sorted signatures the polynomial functor \( H \Sigma: \text{Set}^S \to \text{Set}^S \) is defined by

\[
(H \Sigma X)_s = \coprod_{\sigma \in \Sigma} X_{s_0} \times \ldots \times X_{s_{n-1}},
\]

where the coproduct ranges over all \( \sigma \in \Sigma \) with the result sort \( s \) and the arity \( s_1 \times \ldots \times s_n \to s \). It is easy to verify that many-sorted \( \Sigma \)-algebras and homomorphisms in the above sense form precisely the category \( \text{Alg} H \Sigma \).
Example 2.1.7. Continuous algebras. These are algebras with a complete partial order on them and where operations are continuous. A **complete partial order**\(^1\), or *cpo* for short, is a poset \((A, \sqsubseteq)\) in which all \(\omega\)-chains \(a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots\) have a join. (The empty sequence is not an \(\omega\)-sequence, and so cpo’s need not have have a bottom element. Also the empty poset is a cpo.) A *continuous function* from a cpo \(A\) to a cpo \(B\) is a monotone function \(f: A \to B\) preserving joins of \(\omega\)-chains. An operation \(\sigma^A: A^n \to A\) is continuous if it is a continuous function on the cpo \(A^n\) ordered componentwise. Continuous algebras can be represented as \(F\)-algebras over one of the following categories:

1. The category \(\text{CPO}\) is the category of all cpos and all continuous maps between them. Products of cpos are formed as in \(\text{Set}\), i.e. they are cartesian products with the coordinatewise order and joins. Similarly, coproducts are as in \(\text{Set}\) the disjoint union with the order given by the order of the summands and with elements in different summands incomparable. Hence, a number of examples of \(F\)-algebras work analogously as for \(\text{Set}\):
   a. The algebras for \(FX = X + 1\) are cpos \(A\) together with a constant and a unary operation that is a continuous function on \(A\).
   b. The algebra for \(FX = X \times X + 1\) are cpos \(A\) with a constant and a binary operation \(A \times A \to A\) that is a continuous function.
   c. For every cpo \(X\) we denote by \(X_\bot\) its *lifting* obtained by adding a (new) least element \(\bot\) to \(X\). This defines an endofunctor \(FX = X_\bot\) assigning to every continuous function \(f\) its extension \(f_\bot\) preserving the least element \(\bot\). An algebra for this endofunctor is given by a cpo \(A\), a continuous unary operation \(\alpha: A \to A\) and a constant \(c_\bot \in A\) satisfying \(c_\bot \sqsubseteq \alpha(x)\) for all \(x \in A\).

2. The picture is more interesting for the (non full sub-)category \(\text{CPO}_\bot\) of all cpos with a least element \(\bot\), and all *strict* continuous maps (i.e. those preserving \(\bot\)). This subcategory is closed under products in \(\text{CPO}\). But its coproducts are not \(\text{Set}\)-based: a coproduct in \(\text{CPO}_\bot\) is obtained from that in \(\text{CPO}\) by indentifying in the disjoint union the least elements of all summands to one (least) element. Thus, here \(FX = X + 1\) is simply the identity functor on \(\text{CPO}_\bot\).
   a. Strict continuous algebras with one unary operation and one constant are the algebras for the functor \(FX = X_\bot + 1_\bot\) forming the coproduct of the lifting of \(X\) and the 2-chain \(1_\bot = \{\bot, *\}\). On morphisms \(F\) acts as expected: \(Ff = f_\bot + \text{id}_{1_\bot}\). To give an algebra \(\alpha: A_\bot + 1_\bot \to A\) in \(\text{CPO}_\bot\) is equivalent to giving a cpo \(A\) with \(\bot\), a constant \(\alpha(*) \in A\) and a strict continuous unary operation \(A \to A\).
   b. Analogously, algebras for the endofunctor on \(\text{CPO}_\bot\) defined by \(FX = (X \times X)_\bot + 1_\bot\) are the strict continuous algebras with one binary operation and one constant.

Example 2.1.8. Algebras over sets in context. The presence of variable binding operators, e.g. in the \(\lambda\)-calculus or name-abstraction in the \(\pi\)-calculus, turns structural induction principles into a more delicate issue. It was the idea of Fiore et al. [100] to deal with these issues by working in the category of sets in context. Denote by \(F\)

\(^1\)This is often called an \(\omega\)-complete partial order, in contrast to dcpos that we introduce in Example 5.1.2(3).
2 Algebras and Coalgebras

the category of finite sets and maps, and form the category $\mathbf{Set}^\mathcal{F}$ of presheaves having functors $\mathcal{F} \to \mathbf{Set}$ as objects and natural transformations as morphisms. One thinks of a finite set as a context $\Gamma$ of variables, and a presheaf $X : \mathcal{F} \to \mathbf{Set}$ assigns to every context $\Gamma$ a set $X(\Gamma)$ of ‘terms in context $\Gamma$’. For example, let $L$ be the presheaf of $\lambda$-terms, i.e. for a context $\Gamma = \{x_1, \ldots, x_n\}$, $L(\Gamma)$ is the set of $\alpha$-equivalence classes of $\lambda$-terms with free variables $x_1, \ldots, x_n$. The types of variable binding operations such as $\lambda$-abstraction may be encoded by an endofunctor on $\mathbf{Set}^\mathcal{F}$. For example, consider the functor $F : \mathbf{Set}^\mathcal{F} \to \mathbf{Set}^\mathcal{F}$ that maps a presheaf $X : \mathcal{F} \to \mathbf{Set}$ to $FX = V + X \times X + \delta X$, where $V : \mathcal{F} \hookrightarrow \mathbf{Set}$ is the inclusion functor and $\delta X(\Gamma) = X(\Gamma) + 1$. The shape of this functor $F$ reflects the constructors of $\lambda$-terms. In fact, the presheaf $L$ of $\alpha$-equivalence classes of $\lambda$-terms is an algebra for $F$: consider the algebra structure $\alpha : FL \to L$ whose three coproduct components are given by

1. the inclusion of variables $V(\Gamma) = \Gamma \hookrightarrow L(\Gamma)$ as $\lambda$-terms,
2. the operation $L(\Gamma) \times L(\Gamma) \to L(\Gamma)$ given, for two $\lambda$-terms $t_1$ and $t_2$ in context $\Gamma$, by $t_1 t_2$, the application of $t_1$ to $t_2$, and
3. the operation $\delta L(\Gamma) = L(\Gamma + \{x\}) \to L(\Gamma)$ of $\lambda$-abstraction, which takes a $\lambda$-term $t$ in context $\Gamma + \{x\}$ and assigns to it the $\lambda$-term $\lambda x. t$ in context $\Gamma$.

Similarly, one can deal with variable binding operators of higher arity: Fiore et al. [100] introduce binding signatures and their corresponding endofunctors on $\mathbf{Set}^\mathcal{F}$.

Example 2.1.9. Algebras over nominal sets. An alternative way to deal with the subtle issues of induction in the presence of variable binding are nominal sets (also known as sets with atoms). They go back to Fraenkel and Mostowski’s permutation models for set theory devised in the 1920s and 1930s. Gabbay and Pitts [106] rediscovered the ideas and show how to employ nominal sets to deal abstractly with syntax with variable binding operations. Let us recall the definition of the category $\mathbf{Nom}$ of nominal sets (see e.g. [201]). We fix a countably infinite set $\mathbb{A}$ of atomic names. Let $\mathbf{Perm}(\mathbb{A})$ denote the group of all finite permutations on $\mathbb{A}$ (generated by all transpositions $(a \ b)$ with $a, b \in \mathbb{A}$). Consider a set $X$ with an action of this group, denoted by $\pi \cdot x$ for a finite permutation $\pi$ and $x \in X$. A subset $A \subseteq \mathbb{A}$ is called a support of an element $x \in X$ provided that every permutation $\pi \in \mathbf{Perm}(\mathbb{A})$ that fixes all elements of $A$ also fixes $x$:

$$\pi(a) = a \text{ for all } a \in A \implies \pi \cdot x = x.$$  

A nominal set is a set with an action of the group $\mathbf{Perm}(\mathbb{A})$ where every element has a finite support. The category $\mathbf{Nom}$ is formed by nominal sets and equivariant maps, i.e. maps $f$ preserving the given group action: $f(\pi \cdot x) = \pi \cdot f(x)$.

Finite products and coproducts of nominal sets are given by cartesian products and disjoint union, respectively, with the evident ensuing nominal structure. It is a standard result that every element $x$ of a nominal set has the least support, denoted by $\text{supp}(x)$. The support plays the role of the set of free variables or names of the elements of the nominal set. For example, $\lambda$-terms (modulo $\alpha$-equivalence) form a nominal set $L$, where the support of a $\lambda$-term is the set of its free variables, and the group action is given by renaming free variables in a term.
For a nominal set $X$ we have the abstraction set $[X]$ which is the following nominal set

$$[A|X] = (A \times X) / \sim,$$

where the equivalence relation $\sim$ abstracts the notion of $\alpha$-equivalence known from the $\lambda$-calculus: $(a, x) \sim (b, y)$ holds iff $(ca) \cdot x = (cb) \cdot y$ for some fresh name $c$, i.e. $c$ is distinct from $a$ and $b$ and is not contained in the supports of $x$ or $y$. One writes $\langle a \rangle(x)$ for the equivalence class of $(a, x)$. The action of $[A|X]$ is given by $\pi \cdot \langle a \rangle(x) = \langle \pi(a) \rangle(\pi \cdot x)$.

The assignment $X \mapsto [A|X]$ is easily seen to be an endofunctor on $\text{Nom}$. Moreover, the operations of the $\lambda$-calculus are modelled by the functor $FX = A + X \times X + [A|X]$ on $\text{Nom}$. In fact, the nominal set $L$ of $\lambda$-terms modulo $\alpha$-equivalence is an algebra for $F$ in an obvious way.

Similarly, for every binding signature one obtains an endofunctor on $\text{Nom}$ such that terms over the signature modulo $\alpha$-equivalence form an algebra.

**General properties of the category $\text{Alg} F$** The foregoing part of this chapter was devoted to examples of functors and algebras. We close this section with a discussion of a few general properties of $\text{Alg} F$. These facts hold for all endofunctors on all categories.

**Proposition 2.1.10.** If the base category $\mathcal{A}$ has finite products, then a product of algebras $\alpha: FA \to A$ and $\beta: FB \to B$ is formed on the level of $\mathcal{A}$. That is, there exists a unique algebra structure on $A \times B$

$$\gamma: F(A \times B) \to A \times B$$

for which both projections (in $\mathcal{A}$) become homomorphisms. And $(A \times B, \gamma)$ is a product of the given algebras in $\text{Alg} F$.

**Proof.** Let $\pi_A$ and $\pi_B$ denote the projections in $\mathcal{A}$. Given $\gamma$ for which the following diagram

$$\begin{array}{c}
F(A \times B) \xrightarrow{\gamma} A \times B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
F \pi_A \quad \pi_A \quad \pi_B \\
\alpha \quad A \quad \beta \quad B
\end{array}$$

commutes, it follows that

$$\gamma = \langle \alpha \cdot F \pi_A, \beta \cdot F \pi_B \rangle.$$

So the uniqueness of $\gamma$ is clear. Now define $\gamma$ by the last equality. Then $\pi_A$ and $\pi_B$ are both homomorphisms. Consider an algebra $\delta: FD \to D$ and homomorphisms $h_A: D \to A$ and $h_B: D \to B$. Let the $\mathcal{A}$-morphism $h: D \to A \times B$ be $\langle h_A, h_B \rangle$; we check that $h$ is a morphism in $\text{Alg} F$ as well. That is, the square below commutes:

$$\begin{array}{ccc}
FD & \xrightarrow{\delta} & D \\
\downarrow Fh & & \downarrow h \\
F(A \times B) & \xrightarrow{\gamma} & A \times B
\end{array}$$
Indeed, $\pi_A$ and $\pi_B$ are collectively monic, thus, we only need to prove that the square above commutes when post-composed with $\pi_A$ (and $\pi_B$ – this follows by symmetry). We have

$$\pi_A \cdot (h \cdot d) = h_A \cdot d = a \cdot Fh_A \quad \text{and} \quad \pi_A \cdot (c \cdot Fh) = a \cdot F \pi_A \cdot Fh = a \cdot Fh_A.$$ 

To complete the proof, we show that $h$ is the unique algebra morphism. Suppose that in $\text{Alg} \, F$, a morphism $h': D \to A \times B$ satisfies $\pi_A \cdot h' = h_A$ and $\pi_B \cdot h' = h_B$. Then these all hold in the base category, so $h'$ must be $\langle h_A, h_B \rangle = h$. This completes the proof. \hfill $\square$

**Remark 2.1.11.** Similarly terminal objects and pullbacks – in fact, all limits in $\text{Alg} \, F$ – are formed on the level of $\mathcal{A}$. This is discussed in detail in Section 4.1 (see Remark 4.1.4).

**Corollary 2.1.12.** If $\mathcal{A}$ has a terminal object $1$, then the terminal algebra is $F1 \to 1$.

So there is not much to say about terminal algebras. Initial algebras, on the contrary, are really interesting.

**Corollary 2.1.13.** A homomorphism $h: (A, \alpha) \to (B, \beta)$ is monic in $\text{Alg} \, F$ iff $h$ is monic in $\mathcal{A}$.

Indeed, recall that $h$ is monic iff the following commutative square is a pullback

$$\begin{array}{ccc} A & \xrightarrow{id} & A \\ \downarrow{id} & & \downarrow{h} \\ A & \xrightarrow{h} & B \end{array}$$

and use that pullbacks of $F$-algebras are formed on the level of $\mathcal{A}$.

**Remark 2.1.14.** For epimorphisms, the situation is not as simple. However, every algebra homomorphism $h: (A, \alpha) \to (B, \beta)$ with $h$ epic in $\mathcal{A}$ is epic in $\text{Alg} \, F$.

**Remark 2.1.15.** We assume that the reader is familiar with the general notion of a subobject. For an object $A$, a subobject is represented by a monomorphism $m_B: B \to A$. If $m_B: B \to A$ and $m_C: C \to A$ are monomorphisms, we write

$$m_B \leq m_C$$

if $m_B$ factorizes through $m_C$. (That is, for some $k: B \to C$, $m_B = m_C \cdot k$.) Whenever $m_B \leq m_C \leq m_B$, then $k$ above is an isomorphism. In this case, we identify $m_B$ and $m_C$ as subobject representatives. Thus, we pass to the poset of equivalence classes. The category $\mathcal{A}$ is called well-powered if for every object $A$, only a set of subobjects exist.

Subalgebras of an algebra are understood to be subobjects in the category $\text{Alg} \, F$ of algebras for $F$.

## 2.2 Initial algebras

We have motivated initial algebras in Section 1.1. This section presents our first examples, mainly using endofunctors on $\text{Set}$ and $\text{CPO}_\perp$. In addition, we present some basic material on initial algebras.
2.2 Initial algebras

Definition 2.2.1. An algebra for $F$ is initial if it admits a unique homomorphism into every $F$-algebra. Shortly: it is the initial object of $\text{Alg} F$.

Example 2.2.2. If $F$ preserves the initial object $0$ of $\mathcal{A}$, then $(0, \text{id})$ is the initial algebra.

Notation 2.2.3. $\mu F$ or $\mu X F X$ is the usual notation for the (underlying object of) the initial algebra. The algebra structure is often denoted by

$$\nu: F(\mu F) \to \mu F.$$ 

Examples will follow soon. But even before presenting those, we would like to mention an important result.

Definition 2.2.4. By a fixed point of an endofunctor $F$ is meant an object $A$ together with an isomorphism $FA \cong A$.

Lambek’s Lemma 2.2.5 [163]. Every initial algebra of an endofunctor is a fixed point.

If an initial $F$-algebra exists, then it is a fixed point of $F$.

Proof. Let $(A, \alpha)$ be initial. Since $(FA, F\alpha)$ is an algebra, we have a homomorphism $h$:

$$\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
\downarrow{Fh} & & \downarrow{h} \\
FFA & \xrightarrow{F\alpha} & FA \\
\downarrow{F\alpha} & & \downarrow{\alpha} \\
FA & \xrightarrow{\alpha} & A
\end{array}$$

The square at the bottom obviously commutes. Consequently $\alpha \cdot h$ is an endomorphism of $(A, \alpha)$ in $\text{Alg} F$. Thus $\alpha \cdot h = \text{id}_A$ by initiality. Then $h \cdot \alpha = F\alpha \cdot Fh = \text{id}_{FA}$. Thus $h = \alpha^{-1}$. }

For endofunctors on $\text{Set}$ there is a converse:

Theorem 2.2.6 [231, Thm. II.4]. A set functor has an initial algebra iff it has a fixed point.

We shall discuss this result in more detail and greater generality in Chapter 6 (see Theorem 6.1.22).

Example 2.2.7. (1) The power-set functor $\mathcal{P}: \text{Set} \to \text{Set}$ does not have an initial algebra: Cantor’s Theorem tells us that for all sets $A$, there is no map of $A$ onto $\mathcal{P}A$. For the proof, suppose towards a contradiction that $f: A \to \mathcal{P}A$ is a surjection. Let $X = \{s \in S: s \notin f(s)\}$. Since $f$ is surjective, let $s_0$ be such that $f(s_0) = X$. Then we have a contradiction: $s_0 \in X$ iff $s_0 \in f(s_0)$ iff $s_0 \notin X$. Therefore, there exists no fixed point of $\mathcal{P}$.
(2) Let \((P, \leq)\) be a poset, considered as a category. An endofunctor \(f : P \to P\) is just an order-preserving function, and a fixed point is an element \(p \in P\) with \(F(p) = p\) as usual. An algebra is an element \(p \in P\) such that \(Fp \leq p\). Such an element \(p\) is a pre-fixed point, Lambek’s Lemma then says that if \(p\) is the least pre-fixed point, then \(p = Fp\).

(3) The initial algebra of the set functor \(FX = X + 1\) is the algebra of natural numbers:
\[
\mu X.X + 1 = \mathbb{N}.
\]
The algebra structure \(a = [a_1, a_0]\), compare Example 2.1.3(1), consists of the unary operation \(a_1\) which is the successor function \(s : \mathbb{N} \to \mathbb{N}\), and the constant \(a_0 = 0\). Shortly: \(\iota = [s, 0]\). Indeed, given an algebra \([\beta_1, \beta_0] : B + 1 \to B\) the unique homomorphism \(h : \mathbb{N} \to B\) is determined by
\[
h(0) = \beta_0 \\
h(sn) = \beta_1(h(n)) \quad \text{for all } n \in \mathbb{N}.
\]
See Example 2.2.9 for a generalization of this example to general base categories.

(4) Let \(\mathcal{P}_f\), the finite power-set functor, be the subfunctor of \(\mathcal{P}\) from Example 2.1.3(4) given by
\[
\mathcal{P}_f A = \{M \subseteq A : M \text{ is finite}\}.
\]
The initial algebra \(\mu \mathcal{P}_f\) can be described as the set of all hereditarily finite sets. These are finite sets all elements of which are hereditarily finite sets again.

In concrete terms,
\[
V_\omega = \emptyset \cup \mathcal{P}\emptyset \cup \mathcal{P}\mathcal{P}\emptyset \cup \cdots \cup \mathcal{P}^n\emptyset \cup \cdots.
\]
Each hereditarily finite set is a finite subset of \(V_\omega\), and conversely. Thus \(\mathcal{P}_f V_\omega = V_\omega\). The proof that \((V_\omega, \text{id})\) is initial for \(\mathcal{P}_f\) can be found in Example 3.2.9.

**Remark 2.2.8.** Although we use notation \(\mu F\) (and speak about the initial algebra), initial algebras are only unique up to isomorphism. Indeed, given an isomorphism \(i : A \to \mu F\) in the base category, then \(A\) with the algebra structure
\[
\alpha = (FA \xrightarrow{F i} F(\mu F) \xrightarrow{i} \mu F \xrightarrow{i^{-1}} A)
\]
is also initial. This follows from \(i : (A, \alpha) \to (\mu F, \iota)\) being an isomorphism in \(\text{Alg} \; F\).

Nevertheless, we shall always speak of the initial algebra for an endofunctor (whenever it exists).

Frequently, there are different ways to present an initial algebra; we shall see some examples in this chapter.

**Example 2.2.9.** We now present an example that applies to all of the base categories which we have seen so far in this chapter: \(\text{Set}, \text{Set}^S\), and \(\text{CPO}\). In fact, it applies to every base category that has countable copowers

\[
\mathbb{N} \bullet A = A + A + A \ldots
\]
2.2 Initial algebras

Then for every object \( A \) the endofunctor \( FX = X + A \) has the initial algebra

\[
\mu X.X + A = N \bullet A.
\]

If \( \text{in}_k : A \to N \bullet A \) are the coproduct injections for \( k \in \mathbb{N} \), then the algebra structure is

\[
t = [\alpha_1, \text{in}_0] : (N \bullet A) + A \to N \bullet A.
\]

where \( \alpha_1 \) is determined by the following commutative triangles

\[
\begin{array}{ccc}
N \bullet A & \xrightarrow{\alpha_1} & N \bullet A \\
\downarrow{\text{in}_k} & & \downarrow{\text{in}_{k+1}} \\
N & & (k \in \mathbb{N})
\end{array}
\]

Indeed, let \([\beta_1, \beta_0] : B + A \to B\) be an algebra. Then a homomorphism \( h \):

\[
\begin{array}{ccc}
N \bullet A + A & \xrightarrow{[\alpha_1, \alpha_0]} & N \bullet A \\
\downarrow{h + \text{id}} & & \downarrow{h} \\
B + A & \xrightarrow{[\beta_1, \beta_0]} & B
\end{array}
\]

is uniquely determined by \( h \cdot \text{in}_0 \) and \( h \cdot \text{in}_{k+1} = \beta_1 \cdot (h \cdot \text{in}_k) \): we have

\[
h = [\beta_0, \beta_1 \beta_0, \beta_1 \beta_1 \beta_0, \ldots].
\]

Remark 2.2.10. (1) In the next example, and at many later points, we use trees to describe algebras of special interest. Let us recall that a tree is a directed graph with a distinguished node called the root from which every other node can be reached by a unique directed path. A tree might well be infinite. We always identify isomorphic trees.

(2) We distinguish between unordered trees, defined as above, and ordered trees. An ordered tree comes with a linear order on the children of every node. In pictures, this linear order is the left-to-right order. In this chapter, all the trees will be ordered. But later in the book we shall also consider unordered trees.

A node is a leaf if it has no children. A tree is binary if every node which is not a leaf has precisely two children. The depth of a node of a tree is its distance from the root.

(3) A labelled tree (ordered or unordered) is a tree together with a function assigning to every node an element of a given set \( M \) (of labels). We consider also labelled trees up to (label-preserving) isomorphism.

(4) Let us also recall that the complete \( n \)-ary tree (in which every node has precisely \( n \) children) can be encoded as \( A^* \), the set of words over \( A = \{a_0, \ldots, a_{n-1}\} \), for any \( n \)-element set \( A \). The root is \( \varepsilon \), the empty word. And the \( n \) children of a word \( w \) are the words \( wa_0, \ldots, wa_{n-1} \).
Example 2.2.11. Initial \( \Sigma \)-algebra. It is well known from general algebra that \( \mu H_\Sigma \), the initial algebra of the signature \( \Sigma \), can be described as follows:

\[
\mu H_\Sigma = \text{all ground terms, i.e. terms without variables.}
\]

This is the smallest set such that

1. every constant symbol is a term, i.e. \( \Sigma_0 \subseteq \mu H_\Sigma \), and
2. given an \( n \)-ary symbol \( \sigma \in \Sigma_n \) and \( n \) terms \( t_0, \ldots, t_{n-1} \), then \( \sigma(t_0, \ldots, t_{n-1}) \) is a term.

The operations of the algebra \( \mu H_\Sigma \) are the obvious ones.

Given a \( \Sigma \)-algebra \( A \), every term computes in \( A \) to a (unique) element, and the resulting mapping from \( \mu H_\Sigma \) to \( A \) is the unique homomorphism from \( \mu H_\Sigma \) to \( A \).

We now provide an alternative description of \( \mu H_\Sigma \) using trees.

Definition 2.2.12. Given a signature \( \Sigma \), by a \( \Sigma \)-tree is meant an ordered tree labelled in \( \coprod_{n \in \mathbb{N}} \Sigma_n \) in such a way that every node of \( k \) children is labelled by a \( k \)-ary symbol. Every \( n \)-ary symbol \( \sigma \in \Sigma_n \) defines an \( n \)-ary operation of tree-tupling on the set of all \( \Sigma \)-trees: it sends an \( n \)-tuple \( t_0, \cdots, t_{n-1} \) to the following \( \Sigma \)-tree

\[
\sigma \begin{array}{c} \cdots \\ t_0 \\ t_{n-1} \end{array}
\]

Remark 2.2.13. In most of the classical literature (e.g., Courcelle’s paper [84]), \( \Sigma \)-trees are described in the following equivalent way: a \( \Sigma \)-tree is a parial function \( t: \mathbb{N}^* \rightarrow \coprod_{n \in \mathbb{N}} \Sigma_n \) such that

1. the domain of definition of \( t \) is nonempty and prefix-closed, i.e. whenever a word \( w_i \) lies in the domain, so does \( w \) (this entails that \( \varepsilon \) lies in the domain), and
2. whenever \( t(w) \in \Sigma_n \) then \( t(w_i) \) is defined if and only if \( i = 0, \ldots, n-1 \).

We will use this description of \( \Sigma \)-trees in the proof of Theorem 2.5.9. It is equivalent to the above one: given a partial function \( t \) as above, the nodes of the corresponding \( \Sigma \)-tree are the elements of the domain of definition of \( t \), \( \varepsilon \) is the root, the label of a node \( w \) is \( t(w) \), and the children of \( w \) with an \( n \)-ary label are \( w0, \ldots, w(n-1) \).

Proposition 2.2.14. \( \mu H_\Sigma \) is the algebra of all finite \( \Sigma \)-trees.

Proof. It is sufficient to find an isomorphism \( i \) between the above algebras of terms and of finite \( \Sigma \)-trees in \( \text{Alg } H_\Sigma \). It is defined by structural induction on terms as follows: for all \( \sigma \in \Sigma_0 \) let \( i(\sigma) \) be the single-node tree labelled by \( \sigma \):

\[
\sigma
\]
2.2 Initial algebras

and for all \( \sigma \in \Sigma_n, n > 0 \), use tree-tupling: the term \( \sigma(t_0, \ldots, t_{n-1}) \) is mapped to the following \( \Sigma \)-tree

\[
\begin{array}{c}
\sigma \\
\vdots \\
i(t_0) \\
i(t_{n-1})
\end{array}
\]

This is a bijection: the inverse function \( i^{-1} \) assigns to a tree \( t \) with root labelled by \( \sigma \in \Sigma_n \) the term \( \sigma(t_0, \ldots, t_{n-1}) \), where \( t_k \) is the term such that \( i(t_k) \) is the \( k \)’th maximum subtree of \( t \).

It is clear that \( i \) preserves the operations. \( \square \)

**Example 2.2.15.** (1) \( \mu X.X \times X + 1 \) is the algebra of all finite binary trees. (Labels are not needed since for every arity there is at most one operation symbol.)

(2) For a set \( B \), the set functor \( FX = B \times X + 1 \) is the polynomial functor of the signature \( \Sigma \) with \( \Sigma_1 = B \) and \( \Sigma_0 = \{\cdot\} \). That is, the elements of \( B \) appear as unary operation symbols, and we have a single nullary symbol, here written as a dot \( \cdot \) symbol.

Here are three examples of finite \( \Sigma \)-trees:

\[
\begin{array}{c}
\sigma \\
\vdots \\
i(t_0) \\
i(t_{n-1})
\end{array}
\]

We thus conclude that

\[ \mu X.B \times X + 1 = B^* \]

Indeed, the set of all \( \Sigma \)-trees for this signature is isomorphic to the set \( B^* \) of all words over \( B \).

**Example 2.2.16.** For many-sorted algebras, see Example 2.1.6, we also get

\[ \mu H = \text{all ground terms.} \]

The sorted set of all ground terms is defined as the set \( T = (T_s)_{s \in S} \) where each \( T_s \) (terms of output-sort \( s \)) is the smallest set such that

(1) every constant of sort \( s \) is a term in \( T_s \), and

(2) given a symbol \( \sigma \) of arity \( s_0 \times \ldots \times s_{k-1} \to s \) and \( k \) terms \( t_0 \in T_{s_0}, \ldots, t_{k-1} \in T_{s_{k-1}} \) then \( \sigma(t_0, \ldots, t_{k-1}) \) is a term in \( T_s \).

Again, we can instead work with \( \Sigma \)-trees which are ordered trees labelled in \( S \times \Sigma \), where the label \((s, \sigma)\) means that the node has sort \( s \) and corresponds to the operation \( \sigma \). This labelling is such that every node labelled by \((s, \sigma)\) with a symbol \( \sigma \) of arity \( s_1 \times \ldots \times s_n \to s \) has \( n \) children, and the \( i \)-th child has sort \( s_i \) for all \( i = 0, \ldots, n - 1 \).
The sort of a Σ-tree is defined to be the sort in its root label. We have the following description of \( \mu H_\Sigma \): for every sort \( s \),

\[
(\mu H_\Sigma)_s = \text{all finite } \Sigma\text{-trees of output sort } s.
\]

The proof is completely analogous to that of Proposition 2.2.14.

**Example 2.2.17.** Let us consider initial algebras in \( \mathsf{CPO}_\perp \) (see Example 2.1.7). Recall that coproducts are disjoint unions with bottom elements smashed to one.

1. The initial algebra for \( FX = X_\perp \) is the algebra \( \mu X.X_\perp = \mathbb{N}^\top \) of natural numbers extended by a largest element \( \infty \). The unary operation is the successor function \( s: \mathbb{N}^\top \to \mathbb{N}^\top \) where we put \( s(\infty) = \infty \).

   Indeed, given an algebra \( \alpha: A_\perp \to A \), the unique homomorphism \( h: \mathbb{N}^\top \to A_\perp \) fulfils \( h(0) = \perp \) (since \( h \) is strict) and \( h(sn) = \alpha(h(n)) \). This determines \( h \) on \( \mathbb{N} \), and continuity determines the value \( h(\infty) = h(\bigsqcup_{n \in \mathbb{N}} n) = \bigsqcup_{n \in \mathbb{N}} h(n) \).

2. For the functor \( FX = (X \times X)_\perp + 1_\perp \) of one binary continuous operation and one constant we have \( \mu X.(X \times X)_\perp + 1_\perp = \text{all binary trees} \) (finite and infinite!). The constant is the root-only tree. The binary operation is tree-tupling. And the ordering is the least one for which the root-only tree is the smallest element, and tree-tupling is continuous. Example:

\[
\begin{array}{cccccccc}
\bullet & \subseteq & \bullet & \subseteq & \bullet & \subseteq & \bullet & \subseteq \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

The above is an \( \omega \)-chain whose join is an (obvious) infinite tree. This explains the presence of infinite trees in \( \mu H_\Sigma \). Indeed, every infinite binary tree \( t \) has the form \( t = \bigsqcup_{n \in \mathbb{N}} t_n \), where \( t_n \) is the finite tree obtained by cutting \( t \) at the level \( n \). Clearly, \( t_0 \subseteq t_1 \subseteq t_2 \cdots \).

**Example 2.2.18.** (1) Let us come back to the category \( \mathsf{Nom} \) of nominal sets and the functor \( FX = A + X \times X + [A]X \) (see Example 2.1.9). We have seen that the nominal set \( L \) of all \( \lambda \)-terms modulo \( \alpha \)-equivalence is an algebra for \( F \). In fact, Gabbay and Pitts [106] have shown that it is the initial algebra for \( F \):

\[
\mu F = L = \text{all } \lambda \text{-terms modulo } \alpha \text{-equivalence}.
\]

The initiality of \( L \) provides a structural induction principle. For example, it allows to define the usual (capture avoiding) substitution of \( \lambda \)-terms seemlessly. Indeed, let

\[
t = [t_1, t_2, t_3]: A + L \times L + [A]L \to L
\]
2.2 Initial algebras

be the structure of the initial algebra, and suppose we would like to substitute a variable \( a \in A \) by the \( \lambda \)-term \( t \). Let \( s: A \to L \) be the unique equivariant map with \( s(a) = t_{[a]} \), where \( t_{[a]} \) denotes the \( \alpha \)-equivalence class of \( t \). Then one forms the algebra

\[
\sigma = [s, \iota_2, \iota_3]: A + L \times L \times [A]L \to L.
\]

It is now easy to verify that the unique algebra homomorphism \((L, \iota) \to (L, \sigma)\) maps every \( t'_{[a]} \in L \) to \( t'[t/a]_{[a]} \).

Similarly, terms over a binding signature \([100]\) form an initial algebra, and initiality can be used to define capture-avoiding substitution for such terms.

Note that \( \lambda \)-terms (but not modulo \( \alpha \)-equivalence) form the initial algebra for a different endofunctor on \( \text{Nom} \), \( F'X = A + X \times X + A \times X \). However, initiality then does not yield the desired capture-avoiding substitution of \( \lambda \)-terms.

(2) We also have seen that \( \lambda \)-terms modulo \( \alpha \)-equivalence form an algebra for the functor \( FX = V + X \times X + \delta X \) on the category \( \text{Set}^{\mathcal{F}} \) of sets in context (Example 2.1.8). In fact, Fiore et al. [100] have shown that this is the initial \( F \)-algebra. Again, initiality can be used to give a seamless definition of capture-avoiding substitution of \( \lambda \)-terms.

Initial algebras and free algebras

Remark 2.2.19. Initial algebras are closely related to free algebras. By a free \( F \)-algebra on an object \( A \) (of generators) in \( \mathcal{A} \) is meant an algebra

\[
\varphi_A: FA^\sharp \to A^\sharp
\]

together with a universal arrow \( \eta_A: A \to A^\sharp \). Universality means that for every algebra \( \beta: FB \to B \) and every morphism \( f: A \to B \) in \( \mathcal{A} \), there exists a unique homomorphism \( \bar{f}: A^\sharp \to B \) extending \( f \), i.e. a unique morphism of \( \mathcal{A} \) for which the diagram below commutes:

\[
\begin{array}{ccc}
FA^\sharp & \xrightarrow{\varphi_A} & A^\sharp \\
\downarrow Ff & & \downarrow f \\
FB & \xrightarrow{\beta} & B
\end{array}
\]

(2.5)

In the case where \( \mathcal{A} \) has binary coproducts, we can reduce (2.5) to a square as follows:

\[
\begin{array}{ccc}
FA^\sharp + A & \xrightarrow{[\varphi_A, \eta_A]} & A^\sharp \\
\downarrow Ff + \text{id}_A & & \downarrow \bar{f} \\
FB + A & \xrightarrow{[\beta, f]} & B
\end{array}
\]

(2.6)

Note that for an endofunctor \( F \) and fixed object \( A \), we get a new functor \( F(-) + A \) defined in the obvious way.
Proposition 2.2.20. Let $\mathcal{A}$ be a category with finite coproducts. For every endofunctor $F$, the free $F$-algebra on $A$ is precisely the initial algebra of $F(-) + A$.

That is, if $A^f$ is free, then $\mu X.FX + A = A^f$ with the algebra structure $[\varphi_A, \eta_A]$, and vice versa.

Proof. (1) Let $A^f$ together with $\varphi_A$ and $\eta_A$ be a free algebra. Given an algebra $B$ for $F(-) + A$, its algebra structure has the form $[\beta, f] : FB + A \to B$ for a unique pair of morphisms as in (2.6) above. And $\bar{f}$ above is the unique homomorphism of algebras for $F(-) + A$.

(2) Let $A^f$ be an initial algebra for $F(-) + A$, and let $\varphi_A : FA^f \to A^f$ and $\eta_A : A \to A^f$ denote the components of its algebra structure. This is a free $F$-algebra on $A$. Indeed, given an $F$-algebra $\beta : FB \to B$ and a morphism $f : B \to A$, we get a unique homomorphism $\bar{f}$ of algebras for $F(-) + A$ from $A^f$ to $B$. That is, a unique morphism for which the above square commutes. But this is equivalent to the commutativity of the diagram (2.5). Hence, $(A^f, \varphi_A)$ is a free $F$-algebra on $A$.

This concludes the proof.

Examples 2.2.21. (1) The identity functor on $\text{Set}$ has free algebras

$$A^f = \mathbb{N} \times A$$

with the algebra structure $\varphi_A : \mathbb{N} \times A \to \mathbb{N} \times A$ given by $\varphi(n, a) = (n + 1, a)$ and the universal morphism $\eta_A : A \to \mathbb{N} \times A$ taking $a$ to $(0, a)$.

More generally, if $\mathcal{A}$ is a category with countable coproducts, then a free algebra for $\text{Id}$ on an object $A$ is

$$A^f = \mathbb{N} \cdot A,$$

see Example 2.2.9.

(2) Let $\Sigma$ be a (one-sorted) signature. A free $\Sigma$-algebra (in $\text{Set}$) on a set $A$ is precisely an initial $\Sigma_A$-algebra, where $\Sigma_A$ is the signature $\Sigma$ expanded by nullary operations indexed by $A$. Indeed, from the formula $H_\Sigma X = \bigsqcup_{n \in \mathbb{N}} \Sigma_n \times X^n$, we see that $H_\Sigma(-) + A = H_{\Sigma_A}$.

We can describe $A^f$ as the algebra of all finite $\Sigma_A$-trees. That is, leaves are labelled either by nullary symbols of $\Sigma$ or by elements of $A$.

(3) Analogously for $S$-sorted signatures: the free $S$-sorted $\Sigma$-algebra on a set $A = (A_s)_{s \in S}$ is the algebra of all finite $\Sigma_A$-trees. Here $\Sigma_A$ is the signature $\Sigma$ with additional nullary operations of sorts $s$ indexed by $A_s$ for every sort $s$.

Definition 2.2.22. An endofunctor $F$ is called a varietor if every object generates a free algebra for $F$.

Thus for $\mathcal{A} = \text{Set}$ we have seen that $H_{\Sigma}$ is a varietor and $\mathcal{P}$ is not. A characterization of varietors on $\text{Set}$ will be presented in Corollary 6.1.34 on page 159.
Remark 2.2.23. A functor $F$ is a varietor if the forgetful functor from $\textbf{Alg} F$ to $\mathcal{A}$ has a left adjoint $A \mapsto A^\sharp$. This defines a monad $(M, \eta, \mu)$ on $\mathcal{A}$ by

$$MA = A^\sharp,$$

For a morphism $f: A \to B$, we have a unique homomorphism of $F$-algebras $Mf: A^\sharp \to B^\sharp$ with $Mf \cdot \eta_A = \eta_B \cdot f$. The unit $\eta: \text{Id} \to M$ has components the universal arrows $\eta_A: A \to A^\sharp$, and the monad multiplication $\mu: MM \to M$ has as its components the unique $F$-algebra homomorphism

$$\mu_A: (A^\sharp)^\sharp \to A^\sharp \quad \text{with} \quad \mu_A \cdot \eta_{A^\sharp} = \text{id}_{A^\sharp}.$$

This monad is free on $F$ with respect to the universal natural transformation $F \to M$ with components

$$FA \xrightarrow{F\eta_A} FA^\sharp \xrightarrow{\varphi_A} A^\sharp.$$

This was proved by Barr [56]. He also proved the converse which was later strengthened by Kelly [146]:

**Theorem 2.2.24.** Given a complete category, an endofunctor $F$ generates a free monad iff it is a varietor.

**Examples 2.2.25.** (1) If $\mathcal{A}$ has countable coproducts, then the free monad on $\text{Id}$ is $MA = \mathbb{N} \cdot A$; cf. see Example 2.2.9.

(2) The free monad on the set functor $FX = X \times X$ is given by

$$MA = \text{finite binary ordered trees with leaves labelled in } A.$$ 

Indeed, the functor $F(-) + A$ is the polynomial functor of the signature of one binary operation and constants indexed by $A$.

## 2.3 Recursion and induction

The most basic form of recursion concerns functions on the natural numbers: we specify a function $f$ by specifying its value at 0, and for all $n$, deriving the value $f(n + 1)$ from $f(n)$. As we have seen, the natural numbers form the initial algebra of $FX = X + 1$, see Example 2.2.7(2). This is a special case of the following

**Definition 2.3.1.** Let $F$ be an endofunctor with an initial algebra. Given an object $A$, we say that a morphism $f: \mu F \to A$ is recursively specified if there exists an algebra structure $\alpha: FA \to A$ (a specification of $f$) turning $f$ into a homomorphism.

**Example 2.3.2.** (1) The height of a finite tree. Here we work with finite binary trees as an initial algebra

$$T = \mu X \cdot X \times X + 1,$$

see Example 2.2.15. The function $\text{ht}: T_\Sigma \to \mathbb{N}$ assigns 0 to the root-only tree, and given a tree $t$ with maximum subtrees $t_1, t_2$, then

$$\text{ht}(t) = 1 + \max(\text{ht}(t_1), \text{ht}(t_2)).$$
In order to specify height recursively, we need an algebra structure on \( \mathbb{N} \) of the form 
\[
[\alpha_1, \alpha_0]: \mathbb{N} \times \mathbb{N} + 1 \to \mathbb{N}.
\]
The obvious candidate is 
\[
\alpha_1(n, m) = 1 + \max(n, m) \quad \text{and} \quad \alpha_0 = 0.
\]
Indeed, the square below (where \( \iota_1 \) is the tree-tupling and \( \iota_0 \) represents the root-only tree)

\[
\begin{array}{ccc}
T \times T + 1 & \xrightarrow{[\iota_1, \iota_0]} & T \\
\downarrow \text{ht} \times \text{ht} + \text{id} & & \downarrow \text{ht} \\
\mathbb{N} \times \mathbb{N} + 1 & \xrightarrow{[\alpha_1, \alpha_0]} & \mathbb{N}
\end{array}
\]
clearly commutes. It is easy to see by structural induction that \( \text{ht} \) is indeed the unique homomorphism from \( T \) to \( \mathbb{N} \).

(2) The same example now played with potentially infinite trees: put \( \mathcal{A} = \text{CPO}_\perp \) and 
\[ FX = (X \times X)_\perp + 1_\perp, \]
the functor with the initial algebra 
\[ T = (\text{finite and infinite}) \text{ binary trees}; \]
see Example 2.2.17(2). We want to recursively specify the function 
\[ \text{ht}: T \to \mathbb{N}_\perp \]
assigning \( \infty \) to every infinite tree. This requires an algebra structure for \( F \) (in \( \text{CPO}_\perp \)) on the set \( \mathbb{N}_\perp \). Analogously to the preceding example we put 
\[ [\overline{\alpha}_1, \overline{\alpha}_0]: \mathbb{N}_\perp \times \mathbb{N}_\perp + 1 \to \mathbb{N}_\perp \]
where \( \overline{\alpha}_1(n, m) = 1 + \max(n, m) \), which is understood as expected \( (1 + \infty = \infty \) and \( \max(n, \infty) = \infty \)) and \( \overline{\alpha}_0 = 0 \).

The leading idea in Example 2.3.2 and many similarly results is to use the initiality principle to establish a principle of definition by recursion. We now present an elegant categorical generalization of this idea, which generalizes recursion with parameters and is often called ‘primitive recursion’ in the literature:

**Theorem 2.3.3** (Primitive Recursion). Let \( F \) be a functor with an initial algebra on a category with finite products. Then for every morphism \( \alpha: F(A \times \mu F) \to A \) there exists a unique morphism \( h: \mu F \to A \) such that the square below commutes:

\[
\begin{array}{ccc}
F(\mu F) & \xrightarrow{\alpha} & \mu F \\
F(h, \text{id}_{\mu F}) & \downarrow \alpha & \downarrow h \\
F(A \times \mu F) & \xrightarrow{\alpha} & A
\end{array}
\]

(2.7)

**Proof.** Let \( \pi_1 \) and \( \pi_2 \) denote the projections of the product \( A \times \mu F \). Form the algebra 
\[ \overline{\alpha} = (F(A \times \mu F) \xrightarrow{\text{id}_{\mu F} \pi_2} F(A \times \mu F) \times F(\mu F) \xrightarrow{\alpha \times \text{id}} A \times \mu F). \]
2.3 Recursion and induction

Then \((A \times \mu F, \bar{\alpha})\) is an algebra. Let \(\bar{h} : \mu F \rightarrow A \times \mu F\) be the unique homomorphism. The right-hand component of \(\bar{h}\) is \(\text{id}_{\mu F}\). Indeed, the following diagram shows that \(\pi_2 \cdot \bar{h}\) is a homomorphism from \(\mu F\) to itself:

\[
\begin{array}{ccc}
F(\mu F) & \xrightarrow{\iota} & \mu F \\
F\bar{h} \downarrow & & \downarrow \bar{h} \\
F(A \times \mu F) & \xrightarrow{(\text{id}_F, \pi_2)} & F(A \times \mu F) \times F(\mu F) \xrightarrow{\alpha \times \iota} A \times \mu F \\
F\pi_2 \downarrow & & \downarrow \pi_2 \\
F(\mu F) & \xleftarrow{\pi_2} & \mu F
\end{array}
\tag{2.8}
\]

Thus, putting \(h = \pi_1 \cdot \bar{h}\) we have \(\bar{h} = \langle h, \text{id}_{\mu F} \rangle\), and extending the upper square above by \(\pi_1\) we obtain the desired square (2.7).

For the uniqueness, suppose that \(h\) makes (2.7) commutative. Then we see that for \(\bar{h} = \langle h, \text{id}_{\mu F} \rangle\) the upper square in diagram (2.8) commutes. We know that its left-hand component is commutative, and the right-hand one trivially commutes. \(\square\)

**Example 2.3.4.** Recursion with parameters and the classical primitive recursion for functions on natural numbers are special cases of Theorem 2.3.3: choose \(\mathcal{A} = \text{Set}\) and \(FX = X + 1\).

1. Recursion with parameters. Given \(a \in A\) and \(f : A \times \mathbb{N} \rightarrow A\) we form

\[
\alpha = (F(A \times \mu F) = A \times \mathbb{N} + 1) \xrightarrow{[\!a,f\!]} \mathbb{N} = \mu F).
\]

Spelling out the commutativity of (2.7), we see that this uniquely determines the function \(h : \mathbb{N} \rightarrow A\) such that

\[
h(0) = a \\
h(n+1) = f(h(n), n)
\]

For a concrete example, choose \(A = \mathbb{N}\), \(a = 0 \in \mathbb{N}\) and let \(f : \mathbb{N} \times \mathbb{N}\) be given by \(f(m, n) = m \cdot (n + 1)\). Then \(h : \mathbb{N} \rightarrow \mathbb{N}\) is the factorial function given by its usual recursive definition \(h(0) = 0\) and \(h(n+1) = h(n) \cdot (n + 1)\).

2. The classical formulation of primitive recursion defines a function \(h : \mathbb{N}^{k+1} \rightarrow \mathbb{N}\) by recursion on the first variable taking the remaining \(k\) variables as parameters. More precisely, given \(f : \mathbb{N}^k \rightarrow \mathbb{N}\) and \(g : \mathbb{N}^{k+2} \rightarrow \mathbb{N}\) the function \(h\) is uniquely determined by

\[
h(0, n_1, \ldots, n_k) = f(n_1, \ldots, n_k) \\
h(n+1, n_1, \ldots, n_k) = g(f(n, n_1, \ldots, n_k), n, n_1, \ldots, n_k).
\]

In order to obtain this as a special instance of (2.7) choose \(A = [\mathbb{N}^k, \mathbb{N}]\) the set of functions from \(\mathbb{N}^k\) to \(\mathbb{N}\) and define \(\alpha : \mathbb{N} \times A + 1 \rightarrow A\) componentwise using currying: the left-hand component is \(\text{curry} f : 1 \rightarrow [\mathbb{N}^k, \mathbb{N}]\) and the right-hand one is given by currying the function

\[
([\mathbb{N}^k, \mathbb{N}] \times \mathbb{N}) \times \mathbb{N}^k \cong [\mathbb{N}^k, \mathbb{N}] \times \mathbb{N}^k \xrightarrow{(\text{ev}, \pi_2)} \mathbb{N} \times \mathbb{N} \times \mathbb{N}^k \cong \mathbb{N}^{k+2} \xrightarrow{g} \mathbb{N},
\]

2 Algebras and Coalgebras

where $\pi_2$ and $\pi_3$ are the second and third product projections, respectively. This function clearly maps the triple given by the function $f(n, - , . . .)$ in $[\mathbb{N}^k, \mathbb{N}]$, $n_1, \ldots, n_k \in \mathbb{N}$ and $n \in \mathbb{N}$ to the right-hand side of the second equation above. It is therefore easy to see that the commutativity of (2.7) states precisely that $h$ is defined by primitive recursion from $f$ and $g$.

**Remark 2.3.5.** A classical induction proof takes a subset $A$ of $\mathbb{N} = \mu X.X + 1$, and if $0 \in A$ and $n \in A \Rightarrow sn \in A$, then it concludes $A = \mathbb{N}$. In other words, if $A$ is a subalgebra $(\mathbb{N}, \iota)$, where $\iota = [s, 0]$, then $A = \mathbb{N}$.

We now formulate the corresponding principle for $F$-algebras. Recall the concept of subalgebra from Remark 2.1.15.

**Lemma 2.3.6 (Induction Principle [166]).** If $F$ is an endofunctor with an initial algebra, then every subalgebra of $\mu F$ is an isomorphism.

**Proof.** Consider a subalgebra represented by a monomorphism $m$:

$$
\begin{array}{ccc}
FB & \xrightarrow{\beta} & B \\
Fm & \downarrow & \downarrow m \\
F(\mu F) & \xrightarrow{\iota} & \mu F
\end{array}
$$

We have a unique homomorphism $h : (\mu F, \iota) \rightarrow (B, \beta)$. Then $m \cdot h$ is an endomorphism of the initial object $\text{Alg} F$. But an initial object has only one endomorphism, the identity. Therefore $m \cdot h = \text{id}$. Since $m$ is monic, $m = h^{-1}$.

In $\text{Set}$, here is how this is used. To show that a given subset $A \subseteq \mu F$ is equal to $\mu F$, we need only show that $A$ is a subalgebra of $\mu F$.

**Examples 2.3.7.** (1) The case $\mathbb{N} = \mu X.X + 1$ is the most basic form of mathematical induction. To show that $A \subseteq \mathbb{N}$ is all of $\mathbb{N}$, one shows that $0 \in A$ and that $A$ is closed under the successor function.

(2) For the set $T = \mu X.X \times X + 1$ of all finite binary trees (cf. Example 2.2.15) the “tree-induction principle” states that to prove that a set $A \subseteq T$ contains all trees, one needs only to verify that

(a) $A$ contains the root-only tree, and

(b) with every pair $A$ contains its tree tupling.

(3) For $\Sigma^* = \mu X.X \times \Sigma + 1$ (cf. Example 2.2.15), a set $A \subseteq X^*$ contains all words provided that

(a) $A$ contains $\varepsilon$, and

(b) with every word $w$ it contains all $sw, s \in \Sigma$.
2.4 Coalgebras

The concept of a coalgebra for an endofunctor $F$ is formally just the dual of an algebra: it consists of an object $A$ and a morphism $\alpha: A \to FA$. A number of interesting types of systems can be formalized as coalgebras, thinking of the object $A$ as the collection of states and the structure $\alpha$ as the dynamics of the system.

**Definition 2.4.1.** A coalgebra for an endofunctor $F$ (or just $F$-coalgebra) consists of an object $A$ and a morphism $\alpha: A \to FA$.

Given another coalgebra $\beta: B \to FB$, a homomorphism (or a morphism of $F$-coalgebras) is a morphism $h: A \to B$ of $A$ such that the square below commutes:

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
h \downarrow & & \downarrow Fh \\
B & \xrightarrow{\beta} & FB
\end{array}$$

(2.9)

The category of coalgebras and homomorphisms for $F$ is denoted by $\text{Coalg}_F$.

**Examples 2.4.2.** The following examples of coalgebras over $\text{Set}$ stem essentially from Rutten’s fundamental paper [208].

1. **Systems with termination** are given by a set $A$ of states; for every state either a unique next state is specified, or the state is terminal. Such systems are precisely coalgebras for the endofunctor $FX = X + 1$

   Indeed, define $\alpha: A \to A + 1$ by assigning to a nonterminal state $x$ its next state $\alpha(x)$ (in the left-hand summand) and to a terminal state the unique element of the right-hand summand.

   Homomorphisms $h: (A, \alpha) \to (A', \alpha')$ between systems with termination are functions $h: A \to A'$ between their state sets preserving next states: if $\alpha(x) = x'$ in $A'$, then $h \cdot \alpha(x) = \alpha' \cdot h(x)$. Moreover, $x$ is terminal iff $h(x)$ is.

2. **Binary input and termination.** These are coalgebras for $FX = X \times X + 1$

   A coalgebra $\alpha: A \to A \times A + 1$ is given by a set $A$ of states $x$ which are either terminal ($\alpha(x)$ in the right-hand summand) or have, for every input $i = 0, 1$ precisely one next state $x_i$. Then $\alpha(x) = (x_0, x_1)$.

   Coalgebra homomorphisms preserve next states and preserve and reflect termination.

3. **Deterministic automata.** Let $\Sigma$ be a set of inputs. A deterministic automaton on a set $S$ of states is given by a state transition function $\delta_s: S \to S$ for every $s \in \Sigma$ and a set $A \subseteq S$ of *accepting states*. We can represent the state transition functions in the curried form by

   $$\delta: S \to S^\Sigma,$$
and the subset $A$ via the characteristic function

\[ \gamma : S \to \{0, 1\}, \quad A = \gamma^{-1}(1). \]

Thus, deterministic automata are precisely the coalgebras for the functor

\[ FX = \{0, 1\} \times X^\Sigma. \]

Indeed, to specify a coalgebra $\alpha : S \to \{0, 1\} \times S^\Sigma$ means precisely to specify a pair $\delta : S \to S^\Sigma$ and $\gamma : A \to \{0, 1\}$ and then to set $\alpha = (\gamma, \delta)$.

**Homomorphisms of coalgebras**

\[
\begin{array}{c}
S \xrightarrow{(\gamma, \delta)} \{0, 1\} \times S^\Sigma \\
h \downarrow \\
S' \xrightarrow{(\gamma', \delta')} \{0, 1\} \times (S')^\Sigma
\end{array}
\]

are precisely the functions preserving transitions:

\[ h \cdot \delta_s = \delta'_s \cdot h \quad \text{for all } s \in \Sigma \]

and preserving and reflecting accepting states:

\[ \gamma = \gamma' \cdot h \]

We have not mentioned initial states here, but we “recover” them in Example 2.5.5.

4. **Directed graphs.** A directed graph $(V, E)$ where $E \subseteq V \times V$ is, equivalently, a coalgebra for the power set functor $\mathcal{P}$. More precisely, given a coalgebra $\alpha : V \to \mathcal{P} V$, take $V$ as the set of vertices of the graph, and

\[ E = \{(u, v) \mid u \in V, v \in \alpha(u)\} \]

as its edges. Conversely, every graph $(V, E)$ defined a coalgebra $\alpha : V \to \mathcal{P} V$ with $\alpha(u) = \{v \mid (u, v) \in E\}$.

However, as pointed out in Example 1.3.2, coalgebra homomorphisms are more special than the usual multigraph morphisms viz. edge-preserving functions. The square

\[
\begin{array}{c}
V \xrightarrow{a} \mathcal{P} V \\
h \downarrow \\
V' \xrightarrow{a'} \mathcal{P} V'
\end{array}
\]

commutes iff

(a) $h$ preserves edges: $(u, v) \in E$ implies $(h(u), h(v)) \in E'$, and

(b) for every edge $(h(u), v') \in E'$ there exists $v \in V$ with $(u, v) \in E$ and $v' = h(v)$.
2.4 Coalgebras

(5) **Non-Deterministic automata.** Here transitions are given by relations on the state set $S$, or, equivalently, by functions

$$\delta_s : S \to \mathcal{P}S$$

where $\mathcal{P}$ is the power-set functor. These can be represented by a single function

$$\delta : S \to (\mathcal{P}S)^\Sigma.$$ 

We conclude that non-deterministic automata are coalgebras for the endofunctor

$$FX = \{0,1\} \times (\mathcal{P}X)^\Sigma.$$ 

Homomorphisms of coalgebras

$$\begin{array}{ccc}
S & \overset{\gamma,\delta}{\longrightarrow} & \{0,1\} \times (\mathcal{P}S)^\Sigma \\
h \downarrow & \ & \downarrow \id \times \mathcal{P}h \\
S' & \overset{\gamma',\delta'}{\longrightarrow} & \{0,1\} \times (\mathcal{P}S')^\Sigma
\end{array}$$

are precisely the functions that

(a) preserve transitions, i.e. for every $s \in \Sigma$, $y \in \delta_s(x)$ implies $h(y) \in \delta_s(h(x))$,

(b) reflect them, i.e. for every $y' \in \delta_s(h(x))$ there exists a $y \in \delta_s(x)$ with $h(y) = y'$, and

(c) preserve and reflect accepting states, i.e. $\gamma = \gamma' \cdot h$.

(6) **Moore and Mealy automata.** Here a set $\Sigma$ of inputs and a set $\Gamma$ of outputs are given. In a Moore automaton, every state emits an output, thus a function $\gamma : S \to \Gamma$ is given (generalizing the case $\Gamma = \{0,1\}$ in point (3)). We see that Moore automata are coalgebras for the endofunctor

$$FX = \Gamma \times X^\Sigma.$$ 

In a Mealy automaton the output depends not only on the state but also on the input. Thus, the output function has the form $\gamma : \Sigma \times S \to \Gamma$ that we consider curried as $S \to \Gamma^\Sigma$. Thus Mealy automata are coalgebras for the endofunctor

$$FX = (\Gamma \times X)^\Sigma.$$ 

(7) **Labelled transition systems (LTS).** Given a set $\Sigma$ of actions, an LTS consists of a set $S$ of states and a binary relation $\xrightarrow{s}$ on $S$ for every action $s$. We consider, again, relations as functions from $S$ to $\mathcal{P}S$, then the above $\Sigma$-tuple of relations forms a function

$$\alpha : S \to (\mathcal{P}S)^\Sigma.$$ 

Thus an LTS is precisely a coalgebra for the functor $FX = (\mathcal{P}X)^\Sigma$ which is composed of $\mathcal{P}$ and the polynomial functor $X \mapsto X^\Sigma$. An equivalent way to present a labelled transition system is by one relation $\xrightarrow{s} \subseteq S \times \Sigma \times S$ which is equivalent to giving a coalgebra $S \to \mathcal{P}(\Sigma \times S)$. Thus, an LTS is, equivalently, a coalgebra for the endofunctor $FX = \mathcal{P}(\Sigma \times X)$. 

41
Example 2.4.3. Given a signature \( \Sigma \) (see Example 2.1.5), \( \Sigma \)-coalgebras are coalgebras for the polynomial endofunctor \( H_\Sigma \) on \( \mathsf{Set} \). A coalgebra

\[
\alpha: A \to H_\Sigma A = \coprod_{n \in \mathbb{N}} \Sigma_n \times A^n.
\]

can be viewed as a system with the state set \( A \) and a dynamics consisting of two functions: one, denoted by \( \text{head}: A \to \coprod_n \Sigma_n \), states which of the operation symbols is assigned to the state \( x \in A \). The other function \( \text{body}(x, i) \) is defined for \( i = 0, \ldots, n - 1 \) where \( n \) is the arity of \( \text{head}(x) \). If \( \alpha(x) = (\sigma, (y_0, \ldots, y_{n-1})) \) and \( i < n \), then \( \text{body}(x, i) = y_i \).

This example subsumes

1. systems with termination: \( \Sigma \) has a unary operation and a constant,
2. systems with binary input and termination: \( \Sigma \) has a binary operation and a constant,
3. deterministic automata: consider two \( n \)-ary operations if the input set \( \Sigma \) has \( n \) elements (and observe that \( \{0, 1\} \times X^\Sigma \cong X^n + X^n \)),
4. Moore automata (\( n \)-ary operations indexed by \( \Gamma \)) and Mealy automata (\( n \)-ary operations indexed by \( \Gamma^n \)).

Example 2.4.4. Multigraphs. Let us now move from the base category \( \mathsf{Set} \) to the category \( \mathsf{Set} \times \mathsf{Set} \) of two-sorted sets. This is a special case of many-sorted sets; see Example 2.1.6. Recall that a multigraph, where multiple edges between a pair of vertices are allowed, can be represented by two sets \( V \) (vertices) and \( E \) (edges) and two functions \( s, t: E \to V \) (source and target of an edge). Define an endofunctor

\[
F: \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}
\]

by

\[
F(V, E) = (1, V \times V).
\]

on objects, and \( F(f, g) = (\text{id}, g \times g) \) on morphisms. Then a multigraph is precisely a coalgebra for \( F \): it consists of a two-sorted set \( A = \langle V, E \rangle \) and a two-sorted function

\[
\langle \alpha_v, \alpha_e \rangle : \langle V, E \rangle \to \langle 1, V \times V \rangle.
\]

Indeed, we can ignore \( \alpha_v \). And \( \alpha_e: E \to V \times V \) is precisely a pair of functions from \( E \) to \( V \), as desired.

This time, coalgebra homomorphisms are precisely the usual graph homomorphisms, that is, pairs \( h = (h_v, h_e) \) of functions \( h_v: V \to V' \) and \( h_e: E \to E' \) such that for every edge \( x \in E \), the edge \( h_e(x) \) has the corresponding source and target:

\[
h_v \cdot s(x) = s' \cdot h_e(x) \quad \text{and} \quad h_v \cdot t(x) = t' \cdot h_e(x). \quad (2.10)
\]

Indeed, this condition is equivalent to the commutativity of the following square:

\[
\begin{array}{ccc}
\langle V, E \rangle & \xrightarrow{\alpha} & \langle 1, V \times V \rangle \\
\downarrow_{(h_v, h_e)} & & \downarrow_{(\text{id}, h_v \times h_e)} \\
\langle V', E' \rangle & \xleftarrow{\alpha'} & \langle 1, V \times V \rangle
\end{array}
\]
We can clearly ignore the left-hand component, thus, this commutativity condition is equivalent to the commutativity of the square below:

\[
\begin{array}{ccc}
E \langle s, t \rangle & \rightarrow & V \times V \\
\downarrow h_v & & \downarrow h_v \times h_v \\
E' \langle s', t' \rangle & \rightarrow & V' \times V'.
\end{array}
\]

This is precisely (2.10).

**Example 2.4.5.** Weighted automata are non-deterministic automata with outputs and transition multiplicities in a semiring modelling e.g. costs or rewards (see e.g. [91]). A semiring \((S, +, 0, \cdot, 1)\) consists of two monoids \((S, +, 0)\) and \((S, \cdot, 1)\), the first of which is commutative, obeying the usual distributivity of multiplication over finite sums \(r \cdot 0 = 0 = 0 \cdot r\) and \(r \cdot (s + t) = r \cdot s + r \cdot t\) and \((r + s) \cdot t = r \cdot t + s \cdot t\). A weighted automaton with input alphabet \(\Sigma\) is a tuple \((i, (M_a)_{a \in \Sigma}, o)\), where \(i, o \in S^n\), for some \(n \in \mathbb{N}\) (viz. the number of states of the automaton), are the input and output vectors and each \(M_a\) is an \((n \times n)\)-matrix over \(S\). The \((i, j)\)-entry of the matrix \(M_a\) is the weight of the \(a\)-transition from state \(i\) to state \(j\), \(1 \leq i, j \leq n\).

Weighted automata are coalgebras in the category of (left) \(S\)-semimodules. An \(S\)-semimodule is a commutative monoid \((M, +, 0)\) equipped with a (left)-action \(S \times M \rightarrow M\) denoted by juxtaposition \(r \cdot m\) for \(r \in S\) and \(m \in M\), such that the following laws hold for all \(r, s \in S\) and \(m, n \in M\):

\[
\begin{align*}
(r + s)m &= rm + sm & r(m + n) &= rm + rn \\
0m &= 0 & r0 &= 0 \\
1m &= m & r(sm) &= (r \cdot s)m.
\end{align*}
\]

A morphism of \(S\)-semimodules is a monoid morphism \(h: M_1 \rightarrow M_2\) such that \(h(rm) = rh(m)\) for each \(r \in S\) and \(m \in M_1\). The category of \(S\)-semimodules and their morphisms is denoted by \(\text{S-Mod}\).

On this category one considers the functor \(FX = S \times X^\Sigma\), where \(\times\) denotes the (cartesian) product and \(X^\Sigma\) the \(\Sigma\)-fold power of the \(S\)-semimodule \(X\). Then a weighted automaton with \(n\) states (without the initial vector) is precisely a coalgebra for \(F\) on the (free) \(S\)-semimodule \(S^n\). Indeed, the output vector corresponds to a semimodule morphism \(o: S^n \rightarrow S\) and each matrix \(M_a\) represents a semimodule morphism \(\delta_a: S^n \rightarrow S^n\). Pairing the latter for \(a \in \Sigma\) yields, equivalently, a semimodule morphism \(\delta: S^n \rightarrow (S^n)^\Sigma\), and together with the output morphism an \(F\)-coalgebra \(\alpha = (o, \delta): S^n \rightarrow S \times (S^n)^\Sigma\).

Coalgebra homomorphisms between weighted automata considered as coalgebras are precisely the simulations in the sense of weighted automata theory [69, 96]. Indeed, a
coalgebra homomorphism

\[
\begin{array}{ccc}
S^n & \xrightarrow{(o,\delta)} & S \times (S^n)^\Sigma \\
\downarrow h & & \downarrow \id \times h^\Sigma \\
S^m & \xrightarrow{(o',\delta')} & S \times (S^m)^\Sigma
\end{array}
\]

corresponds to an \((n \times m)\)-matrix \(M_h\) such that \(oM_h = o'\) (i.e. \(o\) considered as a row vector of length \(n\) multiplied by the matrix \(M_h\) yields \(o'\) considered as a row vector of length \(m\)) and \(M_aM_h = M_hM'_a\) for every \(a \in \Sigma\), where \(M_a\) and \(M'_a\) are the matrices representing \(\delta_a: S^n \to S^n\) and \(\delta'_a: S^m \to S^m\), respectively.

**Example 2.4.6.** Here we consider coalgebras for endofunctors of \(\text{CPO}_\bot\), see Example 2.2.17. First, we need a definition. By an *ideal* of a cpo \(P\) is meant a nonempty subset closed both downwards and under \(\omega\)-joins. The ideals of \(P\) are exactly the sets of the form \(f^{-1}(\bot)\), where \(f: P \to 2\) ranges over morphisms of \(\text{CPO}_\bot\) with codomain the two-chain \(2 = \{\bot, \top\}\) where \(\bot \leq \top\).

1. A coalgebra for \(FX = X_\bot\) is a system with termination whose state space \(A\) has the structure of a cpo such that
   
   (a) all terminal states form an ideal, and
   
   (b) the next state function (on non-terminal states) is continuous.

   Indeed, this yields a strict and continuous function \(\alpha: A \to A_\bot\) sending all terminal states to the (new) \(\bot\) element.

2. A coalgebra for \(FX = X_\bot + 1_\bot\) can be viewed as a system with termination and one deadlock state \(\bot\). Every state is either a terminal state or \(\bot\), or it has a (unique) next state. In the cpo \(A\) of states the terminal states and the states with a next state are always incomparable. And conditions and (b) above hold.

   Indeed, this yields a strict and continuous function \(\alpha: A \to A_\bot + 1_\bot\) mapping deadlock states to the non-bottom element of the right-hand summand.

3. Continuous deterministic automata. These are automata with a cpo \(A\) as a state space and whose transition self-maps are strict and continuous. Moreover, all non-accepting states form an ideal.

   Such automata are precisely the coalgebras for the endofunctor of \(\text{CPO}_\bot\) defined (as in the case of \(\text{Set}\)) by

   \[
   FX = 2 \times X^\Sigma.
   \]

   (This is a product of the chain \(0 \leq 1\) and of \(n\) copies of \(X\) if \(|\Sigma| = n\).) Indeed, a coalgebra

   \[
   \alpha: A \to 2 \times A^\Sigma
   \]

   defines two strict and continuous functions

   \[
   \delta: A \to A^\Sigma \quad \text{and} \quad \gamma: A \to 2.
   \]

   Here \(\delta\) represents the transitions and \(\gamma\) the accepting states.
Example 2.4.7. (1) Data structures over infinite alphabets have received increasing attention in recent research. Infinite alphabets model the communication of values from infinite data types such as nonces [160], channel names [130], process identifiers [72], URLs [66], or data values in XML documents [192]. There are numerous automata models for infinite alphabets; partial surveys are found in [220, 70]. Data languages are sets of words over an infinite alphabet where one can only observe letters up to a symmetry, e.g. the equality of a pair of letters, rather than the letters themselves. One principled way of handling infinite alphabets uses nominal sets (see Example 2.1.9). For example, Kaminski and Francez’ classical register automata (with non-deterministic update) [143] are equivalent to nominal orbit-finite automata [71], which in turn can be understood as coalgebras for a functor on Nom. To see this, first recall from [201] that the orbit of an element \( x \) of a nominal set \( X \) is the set \( \{ \pi \cdot x : \pi \in \text{Perm}(A) \} \), where \( A \) is the set of names. \( X \) is said to be orbit-finite if it has only finitely many orbits. A nondeterministic nominal orbit-finite automaton (nofa) over \( A \) consists of an orbit-finite nominal set \( S \) of states equipped with equivariant subsets \( i \) and \( O \) of initial and final states and an equivariant next state relation \( R \subseteq S \times A \times S \). In order to describe nofas as coalgebras recall from Example 2.1.9 the notion of support of elements of a nominal set \( X \). Furthermore, there is an action on the set of subsets of a nominal set \( X \) given by \( \pi \cdot Y = \{ \pi \cdot y : y \in Y \} \) for \( Y \subseteq X \). The subset \( Y \) is called finitely supported if \( Y \) has a finite support w.r.t. this action. Then we form the set \( \mathcal{P}_{fs}X \) of all finitely supported subsets and note that \( \mathcal{P}_{fs} \) is an endofunctor on \( \text{Nom} \) (in fact, \( \text{Nom} \) is a topos and \( \mathcal{P}_{fs}X \) is the power object of \( X \)). Now the above equivariant relation \( R \) can, equivalently, be described as an equivariant map \( S \to \mathcal{P}_{fs}(A \times S) \). Moreover, the equivariant subset \( O \subseteq S \) is, equivalently, described as an equivariant function \( o : S \to \{0,1\} \), where the codomain is equipped with the discrete nominal structure, i.e. \( \pi \cdot i = i \) for \( i = 0,1 \). Disregarding initial states, a nofa is thus a coalgebra for the functor \( FX = \{0,1\} \times \mathcal{P}_{fs}(A \times X) \) on \( \text{Nom} \).

(2) Deterministic orbit-finite nominal automata can also be described as coalgebras. They are the special case of nofas where the next state relation is actually an equivariant function \( S \times A \to S \). To see that they are coalgebras recall from [201] that for two nominal sets \( X \) and \( Y \) we have an action on the set of all functions \( f : X \to Y \) given by \( (\pi \star f)(x) = \pi \cdot f(\pi^{-1} \cdot x) \). A function \( f : X \to Y \) is finitely supported if it has a finite support w.r.t. this action. Equivalently, there exists a finite subset \( A \subseteq A \) supporting \( f \), i.e. such that \( \pi \cdot f(x) = f(\pi \cdot x) \) for all \( x \in X \) and all \( \pi \in \text{Perm}(A) \) which fix all elements of \( A \). The nominal set \( X^Y \) of all finitely supported maps from \( X \) to \( Y \) is the exponential in \( \text{Nom} \), i.e. \( \text{Nom} \) is cartesian closed. So equivariant maps \( Z \times Y \to X \) are in bijective correspondence with equivariant maps \( Z \to X^Y \) via (un)currying.

Hence, disregarding initial states once again, a deterministic orbit-finite nominal automaton on the orbit-finite nominal set \( S \) is described by two equivariant functions \( o : S \to \{0,1\} \) and \( t : S \to S^A \). These functions form a coalgebra for the functor \( FX = \{0,1\} \times X^A \) on \( \text{Nom} \).

Note that, in contrast to what is known in ordinary automata theory, deterministic orbit-finite nominal automata are strictly weaker than nofas (i.e. the class of data languages accepted by them is strictly contained in the class accepted by nofas).
(3) One of the hallmarks of nominal automata is their capability of explicitly storing the current input letter (in a register) for comparing it with a letter read later. This is realized by binding transitions. Regular nondeterministic nominal automata (RNNA) [213] are a model with binding transitions. Their coalgebraic description involves the binding functor $A[X]$ (see Example 2.1.9) and the functor $\mathcal{P}_{ufs}$ mapping a nominal set $X$ to the set of uniformly finitely supported subsets of $X$, i.e. those $Y \subseteq X$ such that $\bigcup_{y \in Y} \text{supp}(y)$ is finite [234]. RNNAs are the coalgebras on orbit-finite state sets $S$ for the functor $F_X = \{0, 1\} \times \mathcal{P}_{ufs}(A \times X) \times \mathcal{P}_{ufs}(A[X])$. They can be equipped with either global or local freshness semantics, i.e. names in a binding transition can either be viewed as fresh w.r.t. the word read so far or as fresh w.r.t. boundedly many names currently stored in memory. Under global freshness semantics, RNNAs are essentially equivalent to Bollig et al.’s session automata [72]. Under local freshness semantics, RNNAs embed into the standard register automata model [143]. They retain a reasonable level of expressivity, in particular not only allow any number of ‘registers’ but can express languages that are not acceptable by deterministic or unambiguous register automata such as ‘some letter occurs twice’. Crucially, RNNAs, unlike unrestricted register automata, nevertheless have an inclusion problem that is decidable and in fact of elementary complexity (exponential space, or more precisely parametrized polynomial space); see [213] for details.

(4) A deterministic variant of RNNAs was considered by Kozen et al. [157]. These are coalgebra for the functor $F_X = \{0, 1\} \times X^A \times [A]X$ on $\text{Nom}$.

**General Properties of the category $\text{Coalg}_F$**

**Notation 2.4.8.** Given an endofunctor $F$ of $\mathcal{A}$ the corresponding endofunctor of $\mathcal{A}^{\text{op}}$ is denoted by $F^{\text{op}}$: it is defined by $F^{\text{op}}A = FA$ on objects and it sends a morphism $f : X \to Y$ of $\mathcal{A}^{\text{op}}$ to $Ff : FY \to FX$ in $\mathcal{A}$, i.e. to $Ff : FX \to FY$ in $\mathcal{A}^{\text{op}}$.

**Lemma 2.4.9.** The category of coalgebras for $f : \mathcal{A} \to \mathcal{A}$ is dual to the category of algebras for $F^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{A}^{\text{op}}$.

**Proof.** Indeed, the objects are the same: an $F$-coalgebra $\alpha : A \to FA$ in $\mathcal{A}$ is precisely an $F^{\text{op}}$-algebra $a : FA \to A$ in $\mathcal{A}^{\text{op}}$. Given another coalgebra $b : B \to FA$, a homomorphism of coalgebras $h : (A, \alpha) \to (B, \beta)$ for $F$ is precisely a homomorphism of algebras $h : (B, \beta) \to (A, \alpha)$ for $F^{\text{op}}$. \qed

**Remark 2.4.10.** This lemma is trivial, but important: it implies that every statement about algebras has a dual statement about coalgebras. For example, Proposition 2.1.10 yields the following fact:

If $\mathcal{A}$ has finite coproducts, then finite coproducts of coalgebras are formed on the level of $\mathcal{A}$. In particular, the initial coalgebra is simply $0 \to F0$. (Terminal coalgebras are much more interesting!)

Lemma 2.4.9 does not say that the category of coalgebras for $f : \mathcal{A} \to \mathcal{A}$ is dual to the category of algebras for the same functor $F$. In fact, this will almost always turn out to be false!

However, dually to Corollary 2.1.12 and Corollary 2.1.13 we have the following facts.
Corollary 2.4.11. If $A$ has an initial object $0$, then the initial coalgebra is $0 \to F0$.

Corollary 2.4.12. A homomorphism $h: (A, \alpha) \to (B, \beta)$ is epic in $\text{Coalg } F$ iff $h$ is epic in $A$.

Remark 2.4.13. (1) For monomorphisms, the situation is not as simple in $\text{Coalg } F$ as in $\text{Alg } F$ (cf. Corollary 2.1.13). However, every coalgebra homomorphism $h: (A, \alpha) \to (B, \beta)$ with $h$ monic in $A$ is monic in $\text{Coalg } F$. Thus it represents a subobject of $(B, b)$ (cf. Remark 2.1.15). When one speaks about subcoalgebras of a coalgebra $(B, \beta)$, we always understand those represented by coalgebra homomorphisms carried by monic morphisms of $A$.

(2) Suppose that $F: \mathcal{A} \to \mathcal{A}$ preserves monomorphisms. Then given a coalgebra $\alpha: A \to FA$, every subobject in $A$ carries at most one coalgebra structure making $m$ a subcoalgebra. Indeed, if the following two squares commute, then $\beta_1 = \beta_2$ since $Fm$ is monic and $Fm \cdot \beta_1 = \alpha \cdot m = Fm \cdot \beta_2$.

(3) Moreover, the order on subcoalgebras is the same as that of subobjects we saw in Remark 2.1.15. Indeed, let $n: C \hookrightarrow A$ be another subobject that carries a (unique) subcoalgebra of $(A, \alpha)$. Then $m \leq n$ holds in $A$ iff there exists $k: B \to C$ such that $m = n \cdot k$. But then $k$ is automatically a coalgebra homomorphism, as we see from the following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow{m} & & \downarrow{Fm} \\
A & \xrightarrow{\alpha} & FA
\end{array}
\]

Indeed, the left- and right-hand parts commute, and so do the lower square and the outside. Thus, the upper square commutes when postcomposed by $Fn$, which implies that this part commutes as desired.

2.5 Terminal coalgebras

Due to the duality between coalgebras in $\mathcal{A}$ and algebras in $\mathcal{A}^{\text{op}}$ which we saw in Lemma 2.4.9, everything we said about initial algebras in Section 2.2 dualizes:

Definition 2.5.1. A coalgebra for $F$ is terminal if it admits a unique homomorphism from every coalgebra. We denote it by $\nu F$ or $\nu X.FX$ and the coalgebra structure by $\tau: \nu F \to F(\nu F)$. 
Lemma 2.5.2 [163]. Every terminal coalgebra of an endofunctor is a fixed point.

A converse similar to Theorem 2.2.6 for set functors is not known for terminal coalgebras, and it is unlikely that one is possible. In fact, there exist two set functors that coincide on all sets (but not all maps) such that one has a terminal coalgebra and one does not, see [21].

Example 2.5.3. (1) The power-set functor does not have a terminal coalgebra, since it has no fixed points (see Example 2.2.7(1)).

(2) The terminal coalgebra for $F X = X + 1$ (cf. Example 2.4.2(1)) is the set $\mathbb{N}^\tau$, with the “predecessor” map as coalgebra structure:

\[
\ast \quad 0 \quad 1 \quad 2 \quad \cdots \quad \infty
\]

where $\ast$ denotes the element of 1. Indeed, for every $F$-coalgebra, i.e. a system $A$ with termination, the behavior defines a homomorphism $b_A: A \to \mathbb{N}^\tau$: If a state $x$ has behavior $n > 0$, then its next state $x'$ has behavior $n - 1 = \tau(n)$. If $x$ has behavior $\infty$, then so does the next state. Finally, $x$ is terminal iff $b_A(x) = 0$. This shows that the square defining homomorphisms commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A + 1 \\
\downarrow b_A & & \downarrow b_A + \text{id} \\
\mathbb{N}^\tau & \xrightarrow{\tau} & \mathbb{N}^\tau + 1
\end{array}
\]

It is further easy to see that $b_A$ is the unique homomorphism.

(3) Coalgebras for the functor $F X = \Sigma \times X$

can be viewed as dynamical systems with outputs in $\Sigma$: Given a coalgebra

\[
\alpha = (\alpha_0, \alpha_1): A \to \Sigma \times A,
\]

then $\alpha_1: A \to A$ determines next states and $\alpha_0: A \to \Sigma$ determines outputs.

Every state $q \in A$ emits an infinite stream of outputs $x_0 = \alpha_0(q), x_1 = \alpha_0\alpha_1(q), x_2 = \alpha_0\alpha_1\alpha_1(q), \ldots$. This defines a function

\[
b_A: A \to \Sigma^\omega
\]

into the set $\Sigma^\omega$ of all infinite streams. This set carries itself the structure of a coalgebra

\[
\tau = (\text{head}, \text{tail}): \Sigma^\omega \to \Sigma \times \Sigma^\omega.
\]

Here $\text{head}$ assigns to a stream $(x_n)_{n \in \mathbb{N}}$ its head $x_0$, and $\text{tail}$ its tail $(x_{n+1})_{n \in \mathbb{N}}$. The following square commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{(\alpha_1, \alpha_0)} & \Sigma \times A^\omega \\
\downarrow b_A & & \downarrow \text{id} \times (b_A)^\omega \\
\Sigma^\omega & \xrightarrow{(\text{head}, \text{tail})} & \Sigma \times \Sigma^\omega
\end{array}
\]
So $b_A$ is a homomorphism.

It is easy to verify that no other homomorphism exists from $A$ to $\Sigma^\omega$. We thus proved that $\Sigma^\omega$ is the terminal coalgebra:

$$\nu X. \Sigma \times X = \Sigma^\omega.$$ 

(4) A slight variation: adding terminal states means that we work with the functor

$$FX = \Sigma \times X + 1.$$ 

In a coalgebra, every state which is not terminal has a next state and emits an output in $\Sigma$. The behavior of a state is either a finite word (if the execution path encounters a terminal state) or an infinite stream. The set of all these streams is denoted by

$$\Sigma^\infty = \Sigma^* + \Sigma^\omega.$$ 

Again, we have a coalgebra structure

$$\tau : \Sigma^\infty \to \Sigma \times \Sigma^\infty + 1$$

taking the empty word to the right-hand summand, and all nonempty strings to the pair $\langle \text{head}, \text{tail} \rangle$. And again behavior defines a unique homomorphism into $(\Sigma^\infty, \tau)$. Thus we proved that

$$\nu X. \Sigma \times X + 1 = \Sigma^\infty.$$

**Remark 2.5.4.** The preceding examples demonstrate the important role that the terminal coalgebra $\nu F$ has: for systems “of type $F$” the elements of $\nu F$ are all possible behaviors of states. Given a system with the state set $A$, the unique coalgebra homomorphism from $A$ to $\nu F$ assigns to every state its behavior. The next example is a classical case:

**Example 2.5.5.** Terminal automaton. The terminal coalgebra for the functor $FX = \{0, 1\} \times X^\Sigma$ of deterministic automata, see Example 2.4.2(4) is

$$\nu X. \{0, 1\} \times X^\Sigma = \mathcal{P}\Sigma^*,$$

the set of all formal languages over $\Sigma$. Recall that $\mathcal{P}\Sigma^*$ can itself be considered as an automaton: a language $L \subseteq \Sigma^*$ is accepting iff it contains the empty word $\varepsilon$. Given a state (i.e. a language) $L$ and an input $s \in \Sigma$ the next state is defined by the Brzozowski derivative

$$s^{-1}L = \{w \in \Sigma^* : sw \in L\}.$$ 

This yields a coalgebra structure $\tau : \mathcal{P}\Sigma^* \to F(\mathcal{P}\Sigma^*)$.

For every automaton $A$ and every state $x \in A$ we form the language $L(x)$ accepted by $x$ in $A$. Observe that

1. $x$ is accepting iff $L(x)$ contains $\varepsilon$, i.e. iff $L(x)$ is accepting in $\mathcal{P}\Sigma^*$, and
2. for every input $s \in \Sigma$ if $y$ denotes the next state of $x$, then

$$L(y) = s^{-1}L(x)$$
From (a) and (b) it is easy to see that the above language-function $L: A \rightarrow \mathcal{P}\Sigma^*$ is a coalgebra homomorphism, i.e. the square below commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & \{0, 1\} \times A^{\Sigma} \\
\downarrow L & & \downarrow \text{id} \times L^{\Sigma} \\
\mathcal{P}\Sigma^* & \xrightarrow{\tau} & \{0, 1\} \times (\mathcal{P}\Sigma^*)^{\Sigma}
\end{array}
$$

Here $L^{\Sigma}$ is the function $p \mapsto L \cdot p$, for all $p \in A^{\Sigma}$. It is also easy to verify the uniqueness of $L$.

**Remark 2.5.6.** The functor $FX = \{0, 1\} \times (\mathcal{P}X)^{\Sigma}$ used for modelling non-deterministic automata as coalgebras does not have a terminal coalgebra, just as $\mathcal{P}$ does not have one. However, since in applications one is usually interested in finite non-deterministic automata, one may of course replace $\mathcal{P}$ above by the finite power-set functor $\mathcal{P}_f$. Then the resulting functor does have a terminal coalgebra, and we will discuss it when we present the terminal coalgebra for $\mathcal{P}_f$ in Theorem 4.5.7. For the moment let us just note that the terminal semantics does not provide the usual language semantics of non-deterministic automata. This can be achieved by working over the category of sets and relations, see Example 5.1.27.

**Example 2.5.7.** (1) Terminal Moore automata. Moore automata are coalgebras for $FX = \Gamma \times X^{\Sigma}$, see Example 2.4.2(6). The terminal coalgebra is the coalgebra $\Gamma^{\Sigma^*}$ of all functions from $\Sigma^*$ to $\Gamma$ (formal power series). Again $\Gamma^{\Sigma^*}$ is a Moore automaton: given a state $f: \Sigma^* \rightarrow \Gamma$ and an input $s \in \Sigma$, the corresponding output is $f(\varepsilon)$ and the next state is $\lambda w. f(sw)$.

(2) Analogously, for Mealy automata presented by the functor $FX = (\Gamma \times X)^{\Sigma}$: here terminal coalgebra can be described as the set of all causal stream functions from $\Sigma^\omega$ (the set of all streams in $\Sigma$) to $\Gamma^\omega$. These are functions $f: \Sigma^\omega \rightarrow \Gamma^\omega$ such that for every stream $x \in \Sigma^\omega$ the $n$-th element of $f(x)$ depends only on the first $n$ elements of $x$. The coalgebra structure assigns to every causal stream function $f$ and every input $a \in \Sigma$:

(a) the causal stream function $f_a$ given by $f_a(x) = \text{tail}(f(ax))$ and

(b) the output $\text{head}(f(ay))$ for any stream $y$.

(Note that this is well-defined since the head of the causal stream function $f$ only depends on the head of its argument). This is how Rutten [210] described the terminal coalgebra. Another description is presented in Example 2.5.11(6) below.

**Example 2.5.8** (Terminal $\Sigma$-coalgebra). Generalizing all of the above examples, consider coalgebras for a polynomial endofunctor $H_{\Sigma}$, see Example 2.4.3. The terminal coalgebra is pleasantly analogous to the initial algebra (consisting of all finite $\Sigma$-trees, see Example 2.2.11). Recall that trees are always considered up to isomorphism. We shall show that

$$
\nu H_{\Sigma} = \text{all } \Sigma\text{-trees}.
$$
The coalgebra structure $\tau^{-1} : \nu H_{\Sigma} \to H_{\Sigma} (\nu H_{\Sigma})$ is the inverse of the tree tupling map $\tau$ from Definition 2.2.12. It assigns to every $\Sigma$-tree

\[
\begin{tikzpicture}
  \node (root) at (0,0) {$\sigma$};
  \node (t0) at (-1,-1) {$t_0$};
  \node (t1) at (0,-1) {$t_1$};
  \node (tn-1) at (1,-1) {$t_{n-1}$};
  \draw (root) -- (t0);
  \draw (root) -- (t1);
  \draw (root) -- (tn-1);
  \node at (-2,-2) {$\ldots$};
\end{tikzpicture}
\]

the pair $(\sigma, (t_0, \ldots, t_{n-1}))$.

**Theorem 2.5.9.** The terminal coalgebra for a polynomial functor $H_{\Sigma}$ is the coalgebra of all $\Sigma$-trees with coalgebra structure inverse to the tree-tupling.

**Proof.** Let us denote by $T_{\Sigma}$ the set of all $\Sigma$-trees, and let $(A, \alpha)$ be a coalgebra for $H_{\Sigma}$. We are to define a homomorphism $h$ as shown below:

\[
\begin{array}{c}
A \\
\downarrow^h \\
T_{\Sigma} \\
\downarrow^{\tau^{-1}} \\
H_{\Sigma} T_{\Sigma}
\end{array}
\]

Since $\tau^{-1}$ is the inverse of tree tupling, the $\Sigma$-tree $t_x = h(x)$ must fulfill the following equation

\[
t_x = \begin{tikzpicture}
  \node (root) at (0,0) {$\sigma$};
  \node (t0) at (-1,-1) {$t_{y_0}$};
  \node (t1) at (0,-1) {$t_{y_{n-1}}$};
  \draw (root) -- (t0);
  \draw (root) -- (t1);
  \node at (-2,-2) {$\ldots$};
\end{tikzpicture}
\]

where $\alpha(x) = (\sigma, y_0, \ldots, y_{n-1})$. (2.11)

This determines the trees $t_x$ for all $x \in A$ uniquely. Indeed, let us use their description as partial functions on $N^*$ as in Remark 2.2.13. The values $t_x(w)$ are defined simultaneously for all $x$ by recursion over $w \in N^*$ as follows:

\[
t_x(\varepsilon) = \sigma \\
t_x(iw) = \begin{cases} t_{y_i}(v) & \text{if } i < n \\ \text{undefined} & \text{else} \end{cases}
\]

(2.12)

This recursion determines a unique function $h : A \to T_{\Sigma}$, $h(x) = t_x$ for which (2.11) holds. Thus, $h$ is the desired unique homomorphism. \qed

**Remark 2.5.10.** Note that the unique homomorphism $h : (A, \alpha) \to (\nu H_{\Sigma}, \tau^{-1})$ assigns to every element $x \in A$ its unravelling as a $\Sigma$-tree using the coalgebras structure $\alpha$. In fact, this is precisely the contents of the recursive definition (2.12). We shall therefore speak of $h(x) = t_x$ as the tree expansion of $x \in A$. 

51
Examples 2.5.11. (1) $FX = X + 1$ corresponds to one constant and one unary operation. A $\Sigma$-tree is either a single path of length $n \in \mathbb{N}$, or a single infinite path. This yields $\nu X.X + 1 = \mathbb{N}^\Sigma$, see Example 2.5.3(2).

(2) $FX = \Sigma \times X$ corresponds to the signature $\Sigma$ with unary operations only. The corresponding trees have no leaves, thus, they are simply infinite paths labelled in $\Sigma$. This yields $\nu X.\Sigma \times X = \Sigma^\omega$, see Example 2.5.3(3).

(3) Analogously $FX = \Sigma \times X + 1$: the additional constant is a (unique) labelling of a leaf. Thus the corresponding tree is a finite or infinite path labelled in $\Sigma$, therefore $\nu X.\Sigma \times X + 1 = \Sigma^\infty$ as in Example 2.5.3(4).

(4) The functor $FX = X \times X + 1$ is the polynomial functor associated to the signature with one binary operation symbol and one constant. Thus $\nu F$ may be identified with the set of all (finite and infinite) binary trees.

(5) Let $A$ and $B$ be fixed sets, and assume that $A$ has size $n$. Consider $FX = B \times X^A$; this functor $F$ corresponds to $H_\Sigma$ where the signature $\Sigma$ consists of $n$-ary operations indexed by $B$. An $F$-coalgebra is a dynamic system with $n$ inputs and with outputs using symbols of $B$. Recall from Remark 2.2.10(4) that $A^*$ represents the complete $n$-ary tree. Therefore, the set of $\Sigma$-trees for this signature is exactly $B^{\Sigma^*}$ (This is the set of functions from words on $A$ into $B$.) This is the terminal coalgebra; we leave the description of the coalgebra structure to the reader.

(6) As a special case of the last item, consider deterministic automata as coalgebras for $FX = \{0,1\} \times \Sigma^*$ from Example 2.4.2(3). As we have just seen, the terminal coalgebra is $\{0,1\}^{\Sigma^*}$. Owing to the general correspondence between subsets of a given sets and maps into $\{0,1\}$, the terminal coalgebra may be described as the set $\mathcal{P}\Sigma^*$ of formal languages on $\Sigma$ as in Example 2.5.5.

(7) For Mealy automata regarded as coalgebras for $FX = (\Gamma \times X)^\Sigma$ we saw a description of the terminal coalgebra in Example 2.5.7(2). Another description is obtained as follows: the functor $FX \cong \Gamma^\Sigma \times X^\Sigma$ is polynomial for the signature of $n$-ary operations (for $\Sigma$ of size $n$) indexed by $\Gamma^\Sigma$. Thus the terminal coalgebra consists of all labellings of the complete $n$-ary tree $\Sigma^*$ by labels in $\Gamma^\Sigma$. Every such tree is given by a function from $\Sigma^*$ to $\Gamma^\Sigma$, or, equivalently, a function from $\Sigma^+ = \Sigma \times \Sigma^*$ to $\Gamma$:

$$\nu X.(\Gamma^\Sigma \times X^\Sigma) = \Gamma^{\Sigma^+}.$$ 

The coalgebra structure assigns to every function $g: \Sigma^+ \to \Gamma$ and every input $a \in \Sigma$ the function $g(a-): \Sigma^+ \to \Gamma$ and the output $g(a) \in \Gamma$. This coalgebra $\Gamma^{\Sigma^+}$ is indeed isomorphic to that of all causal stream functions (see Example 2.5.7(2)) by the following canonical isomorphism. Let $g: \Sigma^+ \to \Gamma$ be a function and define $\tilde{g}: \Sigma^\omega \to \Gamma^\omega$ by assigning to every stream $a_0a_1a_2 \cdots$ in $\Sigma$ the stream

$$g(a_0) \ g(a_0a_1) \ g(a_0a_1a_2) \ \ldots.$$ 

Then $\tilde{g}$ is obviously a causal stream function, and it is easy to verify that this yields the desired isomorphism.
2.5 Terminal coalgebras

(8) Let $A$ be a set and consider the endofunctor on $\text{Set}$ given by $FX = A \times X \times X$. This is the polynomial functor for the signature with a binary operation symbol for every $a \in A$. So we know from Theorem 2.5.9 that $\nu F$ is carried by all labellings of the infinite binary tree by elements of $\mathcal{A}'$.

However, a different (and very useful) isomorphic description of the terminal coalgebra was given by Grabmayer et al. [118, Prop. 26] (see also Kupke and Rutten [205] for a related result). Consider the set $A^\omega$ of streams over $A$ and equip it with the coalgebra structure

$$\langle \text{head}, \text{odd}, \text{even} \rangle : A^\omega \to A \times A^\omega \times A^\omega,$$

where for a stream $s = (s_0, s_1, s_2, s_3, \ldots)$ we have $\text{head}(s) = s_0$, $\text{odd}(s) = (s_1, s_3, s_5, \ldots)$ and $\text{even}(s) = (s_2, s_4, s_6, \ldots)$. Then, this is a terminal $F$-coalgebra. To see this consider the bijection $z : \omega \to \{0, 1\}^*$ recursively defined by

$$z(0) = \varepsilon \quad z(2n + 1) = 0z(n) \quad z(2n + 2) = 1z(n).$$

More explicitly, to obtain $z(n)$ for $n > 0$, take the binary representation of $n + 1$, drop the leading 1, and reverse the sequence. This map is Eilenberg’s reversed bijective interpretation; see [92, p. 116].

Now labellings of the infinite binary tree in $A$ may be identified with functions $\{0, 1\}^* \to A$, and streams over $A$ are functions $\omega \to A$. Thus, precomposition with $z$ yields an isomorphism between the sets $\nu F$ above and $A^\omega$. One then verifies that this yields an isomorphism of coalgebras. Analogously, $A^\omega$ carries a terminal coalgebra for every functor $FX = A \times X^k$ for $k \in \mathbb{N}$ [118]. Its structure is $\langle \text{head}, p \rangle A^\omega \to A \times (A^\omega)^k$, where $p(s)(i) = (s_i, s_{i+1}, s_{2i+1}, \ldots)$ for $1 \leq i \leq k$.

We shall revisit this example in Chapter 10 when we study the behaviour of finite coalgebras, which in this case turn out to be the so-called automatic sequences in $A^\omega$ [49].

**Example 2.5.12.** Terminal weighted automaton. Weighted automata are coalgebras for the functor $FX = S \times X^\Sigma$ on the category $S$-$\text{Mod}$ of semimodules for the semiring $S$ (see Example 2.4.5). The terminal coalgebra for $F$ is carried by the set

$$\nu F = S^\Sigma^*$$

of formal power series (or weighted languages) with the obvious componentwise $S$-semimodule structure. The coalgebra structure $\langle o, t \rangle : S^\Sigma^* \to S \times (S^\Sigma^*)^\Sigma$ is given by $o(L) = L(\varepsilon)$ and $t(L)(a) = \lambda w. L(aw)$.

Given a coalgebra $\alpha : S^n \to S \times (S^n)^\Sigma$ the unique coalgebra homomorphism is the map $h : S^n \to S^{\Sigma^*}$ mapping each $i \in S^n$ to the weighted language accepted by the weighted automaton $(i, (M_a)_{a \in \Sigma}, o)$ corresponding to $(S^n, \alpha)$ with initial vector $i$. Indeed, it is easy to prove that $h(i)$ is the weighted language $L : \Sigma^* \to S$ with $L(w) = i M_a o$ where $i$ is considered as a row vector, $o$ as a column vector, and $M_a$ is the obvious extension of the $M_a$ to words, i.e. $M_e$ is the $n \times n$ identity matrix and $M_{aw} = M_a M_w$.

**Example 2.5.13.** Let us consider terminal coalgebras in $\text{CPO}_\bot$.

1. The terminal coalgebra for $FX = X_\bot$ is the poset

$$\nu X.X_\bot = \mathbb{N}^\top$$
of natural numbers extended by the top element $\infty$ with coalgebra structure inverse to the algebra structure of Example 2.2.17(1).

(2) The terminal coalgebra for $FX = (X \times X)_{\perp} + 1_{\perp}$ consists of all binary trees. Again, the coalgebra structure is inverse to tree-tupling of Example 2.2.17(2).

**Example 2.5.14.** Let us consider terminal coalgebras in $\text{Nom}$, the category of nominal sets (see Example 2.1.9).

(1) For the functor $FX = \{0, 1\} \times X^{A}$, for which orbit-finite deterministic automata are coalgebras (see Example 2.4.7(2)), the terminal coalgebra is carried by the set of all finitely supported languages over $A$. That is, one takes the nominal set $P_{\text{fs}}(\hat{A}^{*})$ and equips it with the obvious automaton structure $\langle o, t \rangle : P_{\text{fs}}(A^{*}) \to \{0, 1\} \times P_{\text{fs}}(A^{*})^{A}$ where $o(L) = 1$ if $L$ contains the empty word and $t$ is given by $t(L)(a) = a^{-1}L = \{w : aw \in L\}$ (similarly as for ordinary deterministic automata). It then easy to verify that this coalgebra structure is equivariant and indeed forms the terminal coalgebra.

(2) For the functor $FX = \{0, 1\} \times X^{A} \times [A]X$ the terminal coalgebra has been described by Kozen et al. [157, Thm. 4.2]. One forms the set $\hat{A} = A + \{a : a \in A\}$ and then takes words over this set modulo $\alpha$-equivalence. More precisely, we the $\alpha$-equivalence $\equiv_{\alpha}$ on $\hat{A}^{*}$ is generated by $wau \equiv_{\alpha} w bv$ if $\langle a \rangle u = \langle b \rangle v$ in the abstraction set $[A]|(\hat{A}^{*})$ (see Example 2.1.9). One writes $[w]_{\alpha}$ for the $\alpha$-equivalence class of a word $w \in \hat{A}^{*}$. The terminal coalgebra for $F$ is carried by the nominal set

$$\nu F = P_{\text{fs}}(\hat{M}), \quad \text{where } \hat{M} = \hat{A}^{*}/\equiv_{\alpha}.$$ 

Its elements are called bar languages [213]. The coalgebra structure is given by

$$\tau = \langle \tau_{1}, \tau_{2}, \tau_{3} \rangle : P_{\text{fs}}\hat{M} \to \{0, 1\} \times (P_{\text{fs}}\hat{M})^{A} \times [A](P_{\text{fs}}\hat{M}),$$

where $\tau_{1}(L) = 1$ iff $L$ contains $[\varepsilon]_{\alpha}$, $\tau_{2}(L)(a) = \{[w]_{\alpha} : [aw]_{\alpha} \in L\}$ and $\tau_{3}(L) = \langle a \rangle \{[w]_{\alpha} : [aw]_{\alpha} \in L\}$.

(3) Let us now consider the functor $FX = A + X \times X + [A]X$ on $\text{Nom}$ whose initial algebra is carried by the nominal set $L$ of $\lambda$-terms modulo $\alpha$-equivalence (see Example 2.2.18(1)). Recall from Remark 2.2.13 that terms can be represented by terms; similarly one can move from terms to trees over a signature, one moves from $\lambda$-terms to (finite and infinite) $\lambda$-trees considering variables from $A$ as constant operation symbols, application as a binary operation symbol and $\lambda x . -$ for every variable $x$, as a unary operation symbol. Kurz et al. [162] have shown that the terminal coalgebra $\nu F$ is carried by the nominal set of all $\alpha$-equivalence classes of $\lambda$-trees with finitely many free variables (but, possibly, infinitely many bound ones).

Similarly, for any functor on $\text{Nom}$ arising from a binding signature in the sense of [100].

**Example 2.5.15.** Similarly to Example 2.5.14(3) the terminal coalgebra for $FX = V + X \times X + \delta X$ on the category $\text{Set}^{\partial}$ of sets in context (see Example 2.1.8) is carried by the presheaf of all $\lambda$-trees modulo $\alpha$-equivalence [39, Thm. 2.10], i.e. for every context $\Gamma = \{x_{1}, \ldots, x_{n}\}$ one has

$$\nu F(\Gamma) = \text{all } \alpha\text{-equivalence classes of finite and infinite } \lambda\text{-trees with free variables } x_{1}, \ldots, x_{n} \text{ (but, possibly, infinitely many bound variables).}$$
Remark 2.5.16. Initial algebras are related to terminal coalgebras in a canonical way: suppose that both exist for an endofunctor $F$. Then the inverse of the coalgebra structure $\tau: \nu F \to F(\nu F)$ makes $\nu F$ an algebra. Hence, we have a unique algebra homomorphism $m: (\mu F, \iota) \to (\nu F, \tau^{-1})$. (We can play the dual game and consider $\mu F$ as a coalgebra: that would lead to the same morphism $m$, which is indeed also a coalgebra homomorphism.)

We will later see that for all set functors

1. this morphism is monic, and
2. if $\nu F$ exists, so does $\mu F$.

Thus, an initial algebra is always a subalgebra of the terminal coalgebra (when the latter is considered as an algebra). Observe that this holds in all the examples of the present section.

2.6 Corecursion and bisimulation

We have seen the connection of initiality, recursion, and induction: recursion in its most basic form on numbers may be considered as an application of initiality. Proof by induction is thus an application of the minimality of initial algebras.

In this section, we study the dual concepts. Whereas recursion deals with functions out of an initial algebra, corecursion is a definition principle for functions into a terminal coalgebra. This is the first topic in this section. After seeing examples, we then turn to bisimulation principles allowing us to prove interesting assertions about corecursively defined functions.

Definition 2.6.1. Let $F$ be an endofunctor with a terminal coalgebra. Given an object $A$, we say that a morphism $f: A \to \nu F$ is corecursively specified if there exists a (specification) morphism $\alpha: A \to FA$ such that $f$ is a coalgebra homomorphism, i.e. $\tau \cdot f = Ff \cdot \alpha$.

Example 2.6.2. Corecursive specification of addition [208]. We consider systems with termination as coalgebras for the set functor $FX = X + 1$. The terminal coalgebra is given by natural numbers extended by $\infty$: $\nu F = N^\top$ with coalgebra structure $\tau = \text{pred}$.

The corresponding coalgebra homomorphism $\text{add}: N^\top \times N^\top \to N^\top$ is the desired
addition function, since the square below commutes:

\[
\begin{array}{ccc}
\mathbb{N}^\tau \times \mathbb{N}^\tau & \xrightarrow{\alpha} & \mathbb{N}^\tau \times \mathbb{N}^\tau + 1 \\
\Downarrow{\text{add}} & & \Downarrow{\text{add + id}} \\
\mathbb{N}^\tau & \xrightarrow{\tau} & \mathbb{N}^\tau + 1
\end{array}
\]

(2.13)

Thus, \( \alpha \) is a corecursive specification of addition.

**Example 2.6.3.** Corecursive specification of streams. Recall from Example 2.5.3(3) that

\[ \nu X. \Sigma \times X = \Sigma^\omega \]

(1) To specify corecursively a concrete stream as a function \( f: 1 \rightarrow \Sigma^\omega \) we need a specification \( \alpha: 1 \rightarrow F(1) \cong \Sigma \), i.e. an element \( x \in \Sigma \). It is easy to see that the corresponding stream is simply \((x, x, x, \ldots)\) as a homomorphism from 1 to \( \Sigma^\omega \).

(2) What about non-constant streams, say,

\((x, y, x, y, x, y, \ldots)\)?

We can corecursively specify this stream by using pairs of streams, \( f: 2 \rightarrow \Sigma^\omega \) where \( 2 = \{0, 1\} \). The specification

\[ \alpha: \{0, 1\} \rightarrow \Sigma \times \{0, 1\} \quad \text{with} \quad \alpha(0) = (x, 1) \quad \text{and} \quad \alpha(1) = (y, 0) \]

defines the following coalgebra homomorphism

\[
\begin{array}{ccc}
\{0, 1\} & \xrightarrow{\alpha} & \Sigma \times \{0, 1\} \\
\Downarrow{h} & & \Downarrow{\text{id} \times h} \\
\Sigma^\omega & \xrightarrow{(\text{head}, \text{tail})} & \Sigma \times \Sigma^\omega
\end{array}
\]

such that the streams \( h(0) \) and \( h(1) \) have heads

\[ \text{head} \cdot h(0) = x \quad \text{and} \quad \text{head} \cdot h(1) = y \]

and tails that satisfy

\[ \text{tail} \cdot h(0) = h(1) \quad \text{and} \quad \text{tail} \cdot h(1) = h(0). \]

Obviously, this determines \( h \) as follows:

\[ h(0) = (x, y, x, y, \ldots) \quad \text{and} \quad h(1) = (y, x, y, x, \ldots). \]
2.6 Corecursion and bisimulation

(3) We next want to specify the zipping function which assigns to a pair of streams 
\( \overrightarrow{x} = (x_0, x_1, x_2, \ldots) \) and \( \overrightarrow{y} = (y_0, y_1, y_2, \ldots) \) the stream

\[
\text{zip}(\overrightarrow{x}, \overrightarrow{y}) = (x_0, y_0, x_1, y_1, \ldots)
\]

The corecursive specification describes the head and tail of \( \text{zip}(\overrightarrow{x}, \overrightarrow{y}) \). Obviously

\[
\text{head} \cdot \text{zip}(\overrightarrow{x}, \overrightarrow{y}) = x_0 = \text{head}(\overrightarrow{x}). \quad (2.14)
\]

Moreover, \( \text{tail} \cdot \text{zip}(\overrightarrow{x}, \overrightarrow{y}) = (y_0, x_1, y_1, x_2, y_2, \ldots) \) fulfils

\[
\begin{align*}
\text{head} & (\text{tail} \cdot \text{zip}(\overrightarrow{x}, \overrightarrow{y})) = \text{head}(\overrightarrow{y}) \\
\text{tail} & (\text{tail} \cdot \text{zip}(\overrightarrow{x}, \overrightarrow{y})) = \text{zip}(\text{tail} \overrightarrow{x}, \text{tail} \overrightarrow{y}).
\end{align*}
\]

(2.15)

The desired coalgebra structure \( \alpha : \Sigma^\omega \times \Sigma^\omega \to \Sigma \times \Sigma^\omega \times \Sigma^\omega \) will have the form \( \alpha = (\alpha_{\text{head}}, \alpha_{\text{tail}}) \) for the following functions

\[
\begin{align*}
\alpha_{\text{head}} : & \Sigma^\omega \times \Sigma^\omega \to \Sigma \\
\alpha_{\text{tail}} : & \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega \times \Sigma^\omega
\end{align*}
\]

where (2.14) and (2.15) indicate that

\[
\begin{align*}
\alpha_{\text{head}}(\overrightarrow{x}, \overrightarrow{y}) & = \text{head}(\overrightarrow{x}) \\
\alpha_{\text{tail}}(\overrightarrow{x}, \overrightarrow{y}) & = (\overrightarrow{y}, \text{tail}(\overrightarrow{x})).
\end{align*}
\]

Indeed, the function \( \text{zip} \) clearly makes the following squares commutative:

\[
\begin{array}{ccc}
\Sigma^\omega \times \Sigma^\omega & \xrightarrow{\text{zip}} & \Sigma^\omega \\
\downarrow \alpha_{\text{head}} & & \downarrow \text{id} \\
\Sigma^\omega & \xrightarrow{\text{head}} & \Sigma
\end{array}
\quad
\begin{array}{ccc}
\Sigma^\omega \times \Sigma^\omega & \xrightarrow{\text{zip}} & \Sigma^\omega \times \Sigma^\omega \\
\downarrow \alpha_{\text{tail}} & & \downarrow \text{zip} \\
\Sigma^\omega & \xrightarrow{\text{tail}} & \Sigma^\omega
\end{array}
\]

Thus, \( \text{zip} \) is uniquely determined by \( \alpha = (\alpha_{\text{head}}, \alpha_{\text{tail}}) \) since the pairing the above two squares yields the commutative square below:

\[
\begin{array}{ccc}
\Sigma^\omega \times \Sigma^\omega & \xrightarrow{\alpha} & \Sigma \times \Sigma^\omega \times \Sigma^\omega \\
\downarrow \text{zip} & & \downarrow \text{id} \times \text{zip} \\
\Sigma^\omega & \xrightarrow{(\text{head}, \text{tail})} & \Sigma \times \Sigma^\omega
\end{array}
\]

We have seen primitive recursion in its categorical form in Theorem 2.3.3. Dually, we have

**Theorem 2.6.4** (Primitive Corecursion). Let \( F \) be an functor with a terminal coalgebra on a category with finite coproducts. Then for every morphism \( \alpha : X \to F(X + \nu F) \) there exists a unique morphism \( h : X \to \nu F \) such that the square below commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & F(X + \nu F) \\
\downarrow h & & \downarrow F[\text{id}, \text{id}_F] \\
\nu F & \xrightarrow{\tau} & F(\nu F)
\end{array}
\]

57
Example 2.6.5. For the set functor $FX = X \times X + 1$, the terminal coalgebra $\nu F$ is carried by the set of all binary trees (see Example 2.5.11(4)). Fix a tree $t \in \nu F$. We use primitive corecursion to define the function $h: \nu F \to \nu F$ that takes a tree $u$ and substitutes $t$ for all the leaves in $u$. To achieve this we define $\alpha: \nu F \to F(\nu F + \nu F)$ by

$$\alpha = \left(\nu F \xrightarrow{\tau} \nu F \times \nu F + 1 \xrightarrow{\text{inl} \times \text{inl}, f} (\nu F + \nu F) \times (\nu F \times \nu F) + 1\right),$$

where $f(*) = \text{inl} \cdot (\text{inr}(t_1), \text{inr}(t_2))$ if $t$ has the maximum subtrees $t_1$ and $t_2$ and $f = \text{inr}$ if $t$ is the single-node tree. Then from the commutativity of the square in Theorem 2.6.4 we see that $h$ is the desired substitution function.

The point of all of this is that we have solution principles that go beyond what we get from terminal coalgebras alone.

Coinduction Principle By dualizing the Induction Principle (see Lemma 2.3.6) we could formulate a coinduction principle by stating that homomorphisms into the terminal coalgebra exist uniquely. Instead, we introduce the concept of a bisimulation here and formulate that principle accordingly.

Bisimulation is an important concept in the theory of coalgebra, somewhat dual to that of congruence for algebras. In the case of set functors, a bisimulation between coalgebras $(A, \alpha)$ and $(B, \beta)$ is a relation $R \subseteq A \times B$ such that both projections $\pi_A: R \to A$ and $\pi_B: R \to B$ become coalgebra homomorphisms for some coalgebra structure on $R$.

In general categories, relations between two objects $A$ and $B$ are subobjects of $A \times B$. As explained in Remark 2.1.15, they are represented by monomorphisms $r: R \to A \times B$. Or, equivalently, by collectively monic spans of morphisms

$$\xymatrix{ & R \ar[dl]_{r_A} \ar[dr]^{r_B} & \\
A \ar[rr] & & B }$$

Indeed, the span is collectively monic iff $\langle r_A, r_B \rangle: R \to A \times B$ is a monomorphism.

Definition 2.6.6. Let $(A, \alpha)$ and $(B, \beta)$ be coalgebras for $F$. By a bisimulation between them we mean a relation $\langle r_A, r_B \rangle: R \to A \times B$ for which a coalgebra structure on $R$ exists turning both $r_A$ and $r_B$ into coalgebra homomorphisms. If the two coalgebras are equal, we speak about a bisimulation on $(A, \alpha)$.

The importance of this notion is that pairs of states related by some bisimulation intuitively have the same behavior. And continuing with this intuition, the largest bisimulation between two coalgebras should be exactly the relation that relates two states iff they have the same behavior. We want to mention some examples here.

Examples 2.6.7. (1) Given two systems with termination (see Example 2.4.2(1)) $\alpha: A \to A + 1$ and $\beta: B \to B + 1$, a bisimulation is a relation $R \subseteq A \times B$ such that whenever $x R y$ then
2.6 Corecursion and bisimulation

(a) \( x \) is terminal iff \( y \) is, and
(b) if \( x \) and \( y \) are nonterminal, then \( x' \sim R \sim y' \) for the next states \( x' \) of \( x \) and \( y' \) of \( y \).

Indeed, if (a) and (b) hold, we can define a coalgebra structure \( \gamma : R \to R + 1 \) by sending a pair \( (x, y) \) in \( R \) to the right-hand summand of \( FR = R + 1 \) if \( x \) is terminal, else \( \gamma(x, y) = (x', y') \) for the above next states. The projections of \( R \) into \( A \) and \( B \) are clearly homomorphisms. Conversely, every bisimulation is easily seen to satisfy (a) and (b).

The behavior function \( b_A : A \to \mathbb{N}_+ \) of Example 2.5.3(2) fulfils for every bisimulation \( R \) that
\[
   x \sim R \sim y \quad \text{implies} \quad b_A(x) = b_A(y).
\]

Conversely, define a relation between coalgebras \( A \) and \( B \) by “having the same behavior”. This is an example of a bisimulation. Consequently, this is the largest bisimulation between \((A, \alpha)\) and \((B, \beta)\).

(2) Let \( A \) and \( B \) be deterministic automata as coalgebras for the set functor \( FX = \{0,1\} \times X^\Sigma \), see Example 2.4.2(3). A bisimulation \( R \subseteq A \times B \) is a relation such that for \( x \sim R \sim y \) we have
(a) \( x \) is accepting iff \( y \) is, and
(b) \( x' \sim R \sim y' \) whenever for some input \( s \in \Sigma \), \( x' \) is the next state of \( x \) and \( y' \) the next state of \( y \) under \( s \).

It is easy to see that this implies that the language accepting by \( x \) in \( A \) is equal to that accepting by \( y \) in \( B \).

Conversely, the relation “accepting the same language” is a bisimulation between \( A \) and \( B \) – thus, it is characterized as the largest bisimulation.

(3) The name bisimulation stems from the theory of labelled transition systems (see Example 2.4.2(6)) where we recover precisely Milner’s notion of a strong bisimulation [186]:
given a set \( \Sigma \) of actions and two LTS’s, \( A \) and \( B \) over \( \Sigma \), a relation \( R \) between \( A \) and \( B \) is a bisimulation iff for every \( x \sim R \sim y \) and every action \( s \in \Sigma \) we have
(a) given a transition \( x \xrightarrow{s} x' \) in \( A \), there exists a transition \( y \xrightarrow{s} y' \) in \( B \) with \( x' \sim R \sim y' \), and
(b) given a transition \( y \xrightarrow{s} y' \) in \( B \), there exists and edge \( x \xrightarrow{s} x' \) in \( A \) with \( x' \sim R \sim y' \).

It is easy to see that this coincides, for the functor \( FX = (\mathcal{P}X)^\Sigma \) (equivalently, for \( FX = \mathcal{P}(\Sigma \times X) \)), with above concept of bisimulation.

(4) Given graphs \((V_1, E_1)\) and \((V_2, E_2)\) as coalgebras for \( \mathcal{P} \) (see Example 2.4.4), a bisimulation is a relation \( R \subseteq V_1 \times V_2 \) with the following properties: whenever \( x \sim R \sim y \),
(a) for every edge \( (x, x') \in E_1 \) there exists an edge \( (y, y') \in E' \) with \( y \sim R \sim y' \), and
(b) for every edge \( (y, y') \in E_2 \) there exists edge \( (x, x') \in E_1 \) with \( x \sim R \sim x' \).

Remark 2.6.8. (1) The diagonal subobject of \( A \times A \) is the relation \( \Delta_A \) represented by \( id_A, id_A : A \to A \). For every coalgebra \((A, \alpha)\) this relation is clearly a bisimulation on \((A, a)\). Any bisimulation not contained in \( \Delta_A \) is called proper. Observe that a relation \( r_1, r_2 : R \to A \) is proper iff \( r_1 \neq r_2 \).
2 Algebras and Coalgebras

(2) Another bisimulation on \((A, \alpha)\) is given by an arbitrary parallel pair of coalgebra homomorphisms

\[ h_1, h_2 : (B, \beta) \to (A, \alpha). \]

More precisely, given such a pair in \(\text{Coalg} F\), where \(F\) is a set functor, the relation \(R \subseteq A \times A\) of all the pairs \((h_1(x), h_2(x))\), \(x \in B\), is a bisimulation.

Indeed, for the following epimorphism \(e : B \to R\) given by \(e(x) = (h_1(x), h_2(x))\) choose a splitting \(m : R \to B\), i.e. we have \(e \cdot m = \text{id}\). Since the projections \(r_1, r_2 : R \to B\) fulfil \(h_i = r_i \cdot e\), thus, \(h_i \cdot m = r_i\), the following diagram commutes for \(i = 1\) and 2:

\[
\begin{array}{ccc}
R & \xrightarrow{m} & B \\
\downarrow{r_i} & & \downarrow{\beta} \\
A & \xrightarrow{\alpha} & FA \\
\end{array}
\]

\[
\begin{array}{ccc}
& FB & \xrightarrow{Fe} FR \\
\downarrow{Fh_i} & & \downarrow{Fr_i} \\
& B & \\
\end{array}
\]

Consequently, \(r_i\) are homomorphisms for the coalgebra structure \(Fe \cdot \beta \cdot m\) on \(R\).

(3) In the case where a bisimulation \(R \subseteq A \times B\) is the graph of a function \(f : A \to B\), we call it a functional bisimulation. Observe that, for example, coalgebra homomorphisms between automata are precisely the functional bisimulations: see the description of homomorphisms in Example 2.4.2(3). The same holds for non-deterministic automata, systems with termination and graphs (as coalgebras of \(\mathcal{P}\)). In all these examples coalgebra homomorphisms are precisely the functional bisimulations.

We now formulate the Coinduction Principle by using the concept of proper bisimulation.

**Lemma 2.6.9 (Coinduction Principle).** If \(F\) is an endofunctor with a terminal coalgebra, then no proper bisimulation on \(\nu F\) exists.

**Proof.** Let \(r = \langle r_1, r_2 \rangle : R \to \nu F \times \nu F\) be a bisimulation. Then \(R\) has a coalgebra structure such that \(r_1, r_2 : R \to \nu F\) are coalgebra homomorphisms. Since \(\nu F\) is terminal, this proves \(r_1 = r_2\). Equivalently, \(R\) is not proper. \(\square\)

**Remark 2.6.10.** Lemma 2.6.9 shows that bisimulations are a sound proof principle to establish behavioural equivalence of states. The latter notion is defined for set functors \(F\) as follows: given two \(F\)-coalgebras \((A, \alpha)\) and \((B, \beta)\) a pair of element \(x \in A\) and \(y \in B\) is called behaviourally equivalent if there exists a pair of coalgebra homomorphisms \(h : (A, \gamma) \to (C, \gamma)\) and \(k : (B, \beta) \to (C, \gamma)\) such that \(h(x) = k(y)\). Equivalently, \(x\) and \(y\) are merged by the unique coalgebra homomorphisms into \(\nu F\) (if it exists).

As bisimulations are not a core topic of this book, we do not discuss the completeness of the bisimulation proof method at length. Let us just remark that it fails in general: there exist functors where behaviourally equivalent states need not be related be a bisimulation. However, if the functor \(F\) preserves weak pullbacks then every pair of behaviourally equivalent states is related by a bisimulation. For details see [5, 208].
Example 2.6.11. Addition on \( \mathbb{N}^\top = \nu X.X + 1 \) specified in Example 2.6.2 has the usual properties. We repeat from [208] coinductive proofs of some of them. Recall the coalgebra structure \( \tau: \mathbb{N}^\top \to \mathbb{N}^\top + 1 \) given by predecessor. We denote by \( s: \mathbb{N}^\top \to \mathbb{N}^\top \) the successor function, which is the restriction of the inverse of \( \tau \). Let us write \( x \oplus y \) in lieu of \( \text{add}(x, y) \).

(1) We prove by coinduction that addition \( n \oplus k \) satisfies the equations by which addition on \( \mathbb{N} \) is usually defined by recursion:

\[
0 \oplus k = k \\
s(n) \oplus k = s(n \oplus k).
\]

For the first equation we define the relation \( R = \{(0 \oplus k, k) \mid k \in \mathbb{N}^\top\} \). If we prove that \( R \) is a bisimulation, we are ready: then \( R = \triangle \) and our equality is proved. We verify the two conditions in Example 2.6.7(1). Indeed, we have that 0 \( \oplus k \) is terminating iff \( k \) is so (equivalently, \( k = 0 \)). For \( k \neq 0 \) we have \( \tau(0 \oplus k) = 0 \oplus \tau(k) R \tau(k) \).

To prove the second equation, let \( R \) be the relation on \( \mathbb{N}^\top \) consisting of all pairs \( (s(n) \oplus k, s(n \oplus k)) \) and all \( (n, n) \). We show that \( R \) is a bisimulation. First observe that the desired conditions hold for each pair \( (n, n) \) in \( R \). The remaining pairs do not contain any terminal states, and we have

\[
\tau(s(n) \oplus k) = \tau(s(n)) \oplus k = (n \oplus k) R (n \oplus k) = \tau(s(n \oplus k)).
\]

This proves that \( R \) is a bisimulation.

(2) Let us verify the following equation

\[
n \oplus s(k) = s(n) \oplus k.
\]

To prove this by coinduction, let \( R \) be the relation on \( \mathbb{N}^\top \) containing the pairs \( (n \oplus s(k), s(n) \oplus k) \) and all \( (n, n) \). Again, the two conditions are trivially true for all pairs \( (n, n) \), and the remaining ones are non-terminating. We use that \( \tau(s(n)) = n \) for all \( n \in \mathbb{N}^\top \) and that \( s(\tau(n)) = n \) for all \( n \neq 0 \) to verify that, for \( n = 0 \) we have

\[
\tau(n \oplus s(k)) = n \oplus \tau(s(k)) = n \oplus k R n \oplus k = \tau(s(n)) \oplus k = \tau(s(n) \oplus k).
\]

and similarly, for \( n \neq 0 \) we have

\[
\tau(n \oplus s(k)) = (\tau(n) \oplus s(k)) R (\tau(s(n)) \oplus k) = \tau(s(n)) \oplus k = \tau(s(n) \oplus k).
\]

This proves that \( R \) is a bisimulation.

(3) Addition is commutative. Indeed, let \( R \) be the relation of all pairs \( (n \oplus k, k \oplus n) \). To see that \( R \) is a bisimulation, note first that the only pair in \( R \) containing terminating states is \( (0 \oplus 0, 0 \oplus 0) \). For the remaining states we have, for \( n = 0 \) and \( k \neq 0 \) that

\[
\tau(n \oplus k) = (n \oplus \tau(k)) R (\tau(k) \oplus n) = \tau(k \oplus n),
\]
and for \( n \neq 0 \neq k \) we have
\[
\begin{align*}
\tau(n \oplus k) &= \tau(n) \oplus k \\
R(k \oplus \tau(n)) &= s(\tau(k)) \oplus \tau(n) = \tau(k) \oplus s(\tau(n)) = \tau(k) \oplus n = \tau(k \oplus n),
\end{align*}
\]
where we use item (2) as a lemma. Thus \( R \) is a bisimulation, and the commutativity is proved.

**Example 2.6.12** [208]. Recall the streams \((x, x, x, \ldots)\), \((y, y, y, \ldots)\) and \((x, y, x, y, \ldots)\) corecursively specified in Example 2.6.3 that we now denote by \( \bar{x}, \bar{y} \) and \( \bar{xy} \), respectively,

We will prove
\[
\text{zip}(\bar{x}, \bar{y}) = \bar{xy}
\]
by coinduction. Let \( R \) be the relation on \( \Sigma^\omega = \nu X.X \times \Sigma \) consisting of just two pairs:
\[
\text{zip}(\bar{x}, \bar{y}) R \bar{xy} \quad \text{and} \quad \text{zip}(\bar{y}, \bar{x}) R \bar{yx}.
\]

We prove that this is a bisimulation. Define a coalgebra structure on \( R \) by the following dynamic system \( \langle \delta, \gamma \rangle : R \to \Sigma \times R \) with outputs, see Example 2.5.3(3), where the output is denoted by \( \mapsto \):
\[
\begin{array}{c}
\langle \text{zip}(\bar{x}, \bar{y}), \bar{xy} \rangle \\
\gamma
\end{array}
\]

Both projections \( r_1, r_2 : R \to \Sigma^\omega \) are easily seen to be coalgebra homomorphisms. Thus, \( R \subseteq \Delta \) which proves the desired equality.

**Cofree coalgebras** At the close of Section 2.2, we saw that free algebras generalize initial algebras, and also that the free algebra construction leads to a monad. We now pursue the same topic, this time on the dual side. Here is the motivation for this. We often view a coalgebra \( \alpha : A \to FA \) as a “black box”: we cannot observe its states directly and therefore do not have a full information about next states. A terminal coalgebra gives us the observable part: for the given homomorphism \( h : A \to \nu F \) we do not know the state \( x \), but we know its behavior \( h(x) \). An improved view is provided by an additional output functions \( g : A \to \Gamma \). Here \( \Gamma \) is a set of colors, and whenever a cofree coalgebra \( \Gamma^* \) (as defined below) exists, then the corresponding homomorphism \( \bar{g} : A \to \Gamma^* \) gives us visible information about the colour of current state \( x \).

**Definition 2.6.13.** By a cofree \( F \)-coalgebra on an object \( \Gamma \) (of colors) in \( \mathcal{A} \) is meant a coalgebra
\[
\tau : \Gamma^* \to FT\Gamma^*
\]

Definition 2.6.13.}
homomorphism $\bar{g}: A \to \Gamma_\sharp$ with $g = \varepsilon_{\Gamma} \cdot \bar{g}$

Example 2.6.14. Consider dynamical systems as coalgebras of $FX = X + 1$. Let $A$ be a system with an additional output function $g: A \to \{0, 1\}$. Every state $x \in B$ defines a finite or infinite sequence of states: we start in $x$ and continue moving, stopping only when a terminating state is reached. We do not know the sequence $x = x_0, x_1, x_2 \ldots$ of states, but we do see the sequence of colors $g(x_0), g(x_1), g(x_2) \ldots$.

So if our interpretation of $A_\sharp$ above is correct, we should have

\[\{0, 1\}_\sharp = \{0, 1\}^\infty\]

the set of finite and infinite binary streams. This is indeed true, and it is a consequence of the following.

**Proposition 2.6.15.** The cofree $F$-coalgebra on $\Gamma$ is precisely the terminal coalgebra for the endofunctor $F(-) \times \Gamma$.

This is dual to Proposition 2.2.20. Applied to $FX = X + 1$ this tells us that the cofree algebra

\[\{0, 1\}_\sharp = \nu X.X \times \{0, 1\} + \{0, 1\}\]

is the terminal $\Sigma$-coalgebra, where $\Sigma$ has two unary operations and two constants. This terminal $\Sigma$-coalgebra consists of all $\Sigma$-trees. These are unary trees in which all nodes (inner ones as well as leaves) are labelled by 0 or 1. And this is isomorphic to $\{0, 1\}^\infty$.

**Definition 2.6.16.** An endofunctor $F$ is called a covarietor if every object generates a cofree coalgebra for $F$. The corresponding comonad is denoted by $F_\sharp$.

**Example 2.6.17.** (1) $FX = X + 1$ is a covarietor with

\[\Gamma_\sharp = \Gamma^\infty\]

This follows from $F(-) \times \Gamma = H_\Sigma$ where $\Sigma_0 = \Sigma_1 = \Gamma$ and $\Sigma_n = \emptyset$ for $n > 1$.

(2) $FX = X \times X + 1$ is a covarietor. The coalgebra $\Gamma_\sharp$ consists of all binary trees with nodes labelled in $\Gamma$. This follows from $F(-) \times \Gamma = H_\Sigma$ where $\Sigma_0 = \Gamma = \Sigma_2$ and otherwise $\Sigma_n = \emptyset$.

**Proposition 2.6.18.** For every covarietor $F$ the cofree comonad on $F$ is $F_\sharp$. Conversely, if $A$ is a cocomplete category, then every endofunctor generating a cofree comonad is a covarietor.

This is dual to Theorem 2.2.24.
2.7 Summary of this chapter

As this long chapter draws to a close, here are the points that readers should be most confident of, and let us also raise some general points about the material.

The main theme of this chapter has been the description of initial algebras and terminal coalgebras of endofunctors of $\text{Set}$ and $\text{CPO}_\perp$, particularly of polynomial endofunctors. Initial algebras $\mu F$ are important for a generic form of induction and recursion. Terminal coalgebras $\nu F$ formalize the concept of behaviour of a state: given a state-based system on a set $A$ of states (whose dynamics is expressed as a coalgebra for $F$) then the unique homomorphism from $A$ to $\nu F$ assigns to each state its behaviour. Terminal coalgebras are also important as the codomains for corecursive definitions. For the corresponding proof principle, coinduction, the concept of a bisimulation plays a crucial rôle. These themes will persist through the book.
3 Finitary Iteration

We show how to obtain an initial algebra for an endofunctor by iterating that endofunctor on the initial object. This can be seen as the categorical version of the famous Kleene Fixpoint Theorem for continuous functions on cpos that we recall first. The iterations considered here are countable, the more general case (generalizing the Knaster-Tarski Fixpoint Theorem) is treated in Chapter 6. We concentrate more on the dual construction: the terminal coalgebra as a limit of iterations of the endofunctor on the terminal object. Most of this chapter is devoted to examples. In later chapters of this book, we take up other topics: more on representations of terminal coalgebras, more on constructions that either use uncountable iterations or else use countable iteration in connection with extra order-theoretic or metric structure.

3.1 Initial-algebra chain

We recall the Kleene Fixpoint Theorem and present its categorical version: the initial algebra for an endofunctor as a colimit of iterations on the initial object. We show on concrete examples how this yields initial algebras for “well-behaved” endofunctors. For example, this yields a new view on the initial algebra for a polynomial functor associated to a signature $\Sigma$ described in Chapter 2 as the algebra of all finite $\Sigma$-labelled trees.

Throughout this section we assume that a category $\mathcal{A}$ and an endofunctor $F$ on $\mathcal{A}$ are given. We assume that $\mathcal{A}$ has an initial object $0$ and denote by $!: 0 \to X$ the unique morphism to a given object $X$.

**Theorem 3.1.1 (Kleene).** Let $A$ be a cpo with a least element $\bot$. Then every continuous endofunction $F$ has a least fixed point

$$\mu F = \bigvee_{n<\omega} F^n(\bot).$$

**Proof.** First, an induction on $n < \omega$ shows that $F^n(\bot) \leq F^{n+1}(\bot)$. So $\{F^n(\bot) : n < \omega\}$ is an $\omega$-chain. Write $\mu F$ for its join. By continuity, $F(\mu F) = \bigvee_n F(F^n(\bot))$. It is easy to check that $\bigvee_n F^n(\bot) = \bigvee_n F^{n+1}(\bot)$, and so $F(\mu F) = \mu F$. Thus, we have a fixed point of $F$. If $Fx \leq x$, then an easy induction on $n$ shows that $F^n(\bot) \leq x$; hence $\mu F \leq x$ as well.

**Remark 3.1.2.** More generally, a pre-fixed point of $F$ is an element $x$ with $Fx \leq x$; the argument above proves that $\mu F$ is also the least pre-fixed point of $F$.

In this book, we are not really interested in Kleene’s Theorem but in generalizations of it, and in dualizations of those generalizations, etc. Figure 3.1 shows how we generalize...
3 Finitary Iteration

<table>
<thead>
<tr>
<th>order-theoretic concept</th>
<th>category theoretic generalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>preorder ((A, \leq))</td>
<td>category (\mathcal{A})</td>
</tr>
<tr>
<td>(x \leq y) and (y \leq x)</td>
<td>(A) and (B) are isomorphic objects</td>
</tr>
<tr>
<td>least element 0</td>
<td>initial object 0</td>
</tr>
<tr>
<td>monotone (F: A \to A)</td>
<td>functor (F: \mathcal{A} \to \mathcal{A})</td>
</tr>
<tr>
<td>pre-fixed point: (Fx \leq x)</td>
<td>(F)-algebra: (f: FA \to A)</td>
</tr>
<tr>
<td>(\omega)-chain</td>
<td>functor from ((\omega, \leq)) to (\mathcal{A})</td>
</tr>
<tr>
<td>(F) is continuous</td>
<td>(F) preserves (\omega)-colimits</td>
</tr>
<tr>
<td>least pre-fixed point: (Fx \leq x)</td>
<td>initial (F)-algebra: (\iota: F(\mu F) \to \mu F)</td>
</tr>
</tbody>
</table>

Figure 3.1: Generalizing Kleene’s Theorem to categories

the order-theoretic concepts in Kleene’s Theorem to the level of categories. In each line, the order-theoretic concept on the left is a special case of the category theoretic concept to its right. (To see this, recall that a pre-order \((P, \leq)\) is exactly a category in which every homset is either empty or singleton set.) Of special interest is the generalization of the completeness condition on the poset and the continuity condition on the function.

Definition 3.1.3. By the initial-algebra \(\omega\)-chain of an endofunctor \(F: \mathcal{A} \to \mathcal{A}\) is meant

\[
0 \xrightarrow{!} F0 \xrightarrow{F} F^20 \xrightarrow{F^2} \cdots \xrightarrow{F^{n+1}} F^n0 \xrightarrow{F^n} F^{n+1}0 \xrightarrow{F^{n+1}} \cdots
\]  \hspace{1cm} (3.1)

Incidentally, the reason we use a hyphen in initial-algebra chain is to avoid the suggestion that this chain is initial in any category of “algebra chains.” (In fact, we have no such chains in this book.)

Notation 3.1.4. The above diagram gives a functor from \((\omega, \leq)\) to \(\mathcal{A}\). Such functors are in general called \(\omega\)-chains in \(\mathcal{A}\) and their colimits are called \(\omega\)-colimits.

A colimit of (3.1) is denoted by \(C\) with the colimit cocone \(c_n: F^n0 \to C, n < \omega\).

We must mention that later in Chapter 6 we shall need transfinite iterations of the initial-algebra chain. But in this section, we only consider the finite iterations as in (3.1).

A cocone of the initial-algebra \(\omega\)-chain is an object \(A\) of \(\mathcal{A}\) together with a family of morphisms \(a_n: F^n0 \to A\) such that the triangles below commute:

\[
\begin{array}{c}
F^n0 \\
\downarrow a_n \\
A \\
\end{array} \quad F^{n+1}0 \xrightarrow{F^{n+1}} F^{n+1}0 \\
\downarrow a_{n+1} \\
A
\end{array}
\]

for all \(n < \omega\).

A colimit of the initial-algebra \(\omega\)-chain is a cocone \(c_n: F^n0 \to C\) over it with the universal property that if \(a_n: F^n0 \to A\) is any cocone, then there is a unique factorizing morphism \(f: C \to A\) i.e. such that for all \(n < \omega\), \(a_n = f \cdot c_n\).
3.1 Initial-algebra chain

**Construction 3.1.5.** Every \( F \)-algebra \((A, \alpha)\) induces a canonical cocone \( \alpha_n : F^n 0 \to A \) of the initial-algebra \( \omega \)-chain as follows: \( \alpha_0 : 0 \to A \) is unique (since 0 is initial) and

\[
\alpha_{n+1} = (FF^n 0 \xrightarrow{F\alpha_n} FA \xrightarrow{\alpha} A).
\] (3.2)

The cocone property, \( \alpha_n = \alpha_{n+1} \cdot F^n \), is easy to verify by induction. We call this cocone the **cocone induced by** \((A, \alpha)\).

**Remark 3.1.6.** (1) Homomorphisms of algebras preserve the induced cocones: given a homomorphism \( h \) from \((A, \alpha)\) to \((B, \beta)\), i.e. \( h \cdot \alpha = \beta \cdot Fh \), it follows that \( h \cdot \alpha_n = \beta_n \).

This is trivial for \( n = 0 \), and the induction step is easy:

\[
\begin{align*}
\alpha_{n+1} = (h \cdot \alpha) \cdot F\alpha_n = (\beta \cdot Fh) \cdot F\alpha_n = \beta \cdot F\beta_n &= \beta_{n+1}.
\end{align*}
\]

(2) Let \( c_n : F^n 0 \to \mu F \) be the colimit of the initial-algebra \( \omega \)-chain. Applying \( F \) to each object and morphism in (3.1) yields another \( \omega \)-chain

\[
F0 \xrightarrow{F1} F20 \xrightarrow{F21} F30 \to \cdots \xrightarrow{F31} Fn+10 \xrightarrow{Fn+11} Fn+20 \to \cdots
\]

which obviously has the same colimit as (3.1).

This leads to the following result:

**Theorem 3.1.7** [8]. Let \( \mathcal{A} \) be a category with initial object 0 and with colimits of \( \omega \)-chains. If \( F : \mathcal{A} \to \mathcal{A} \) preserves \( \omega \)-colimits, then it has an initial algebra

\[
\mu F = \colim_{n<\omega} F^n 0.
\]

**Remark 3.1.8.** Specifically, we have the colimit cocone \( Fc_n : F^{n+1} 0 \to F(\mu F) \) of the chain (3.3). Let \( \iota : F(\mu F) \to \mu F \) be the unique morphism such that the following triangles commute:

\[
\begin{array}{ccc}
F(F^n 0) & = & F^{n+1} 0 \\
\downarrow Fc_n & & \downarrow c_{n+1} \\
F(\mu F) & \xrightarrow{\iota} & \mu F \\
\end{array}
\]

(n < \( \omega \)). (3.4)

Then \((\mu F, \iota)\) is an initial \( F \)-algebra. Moreover, for any algebra \((A, \alpha)\), the unique algebra morphism \( h : \mu F \to A \) is the factorization induced by the canonical cocone:

\[
h \cdot c_n = \alpha_n \quad \text{for all } n < \omega.
\]

**Proof.** In this proof we denote by \( \mu F \) the colimit above. The hypothesis that \( F \) preserves \( \omega \)-colimits implies that \( Fc_n : F^{n+1} 0 \to F(\mu F) \) is a colimit cocone of (3.3). We have another cocone of (3.3), namely \( c_{n+1} : F^{n+1} 0 \to F(\nu F) \). Hence there is a unique morphism \( \iota : F(\mu F) \to \mu F \) with the property (3.4).
3 Finitary Iteration

We claim that the \( F \)-algebra \( (\mu F, \iota) \) is initial. To check this, let \((A, \alpha)\) be any \( F \)-algebra. Consider the induced cocone \( \alpha_n : F^n 0 \to A \) from Construction 3.1.5. It is easy to check that \( F\alpha_n : F^{n+1} 0 \to FA \) is a cocone of (3.3). Let \( h : \mu F \to A \) be the unique factorization morphism i.e. \( h \cdot c_n = \alpha_n \) for all \( n < \omega \).

(a) We prove that \( h : (\mu F, \iota) \to (A, \alpha) \) is a homomorphism. In order to see that \( h \cdot \iota = \alpha \cdot Fh \), we check that both are mediating morphisms for the cocone \( \alpha_{n+1} : F^{n+1} 0 \to A \). That is, we check that for all \( n \),

\[
(h \cdot \iota) \cdot Fc_n = \alpha_{n+1} = (\alpha \cdot Fh) \cdot Fc_n.
\]

For the first assertion, \( h \cdot (\iota \cdot Fc_n) = h \cdot c_{n+1} = \alpha_{n+1} \). For the second, consider the diagram below:

\[
\begin{array}{ccc}
F^{n+1} 0 & \xrightarrow{\alpha_{n+1}} & A \\
F \downarrow & & \downarrow \alpha \\
Fc_n & \xrightarrow{F\alpha_n} & FA \\
F(\mu F) & \xrightarrow{Fh} & FA
\end{array}
\]

The upper right-hand triangle commutes by (3.2). The other one commutes by definition of \( h \). So the square commutes, showing that indeed \( (\alpha \cdot Fh) \cdot Fc_n = \alpha_{n+1} \). This verifies that \( h \) is a homomorphism.

(b) To prove that \( h \) is unique, suppose that \( k : \mu F \to A \) is also a homomorphism: \( k \cdot \iota = \alpha \cdot Fk \). We show that \( k \) is also factorization of the induced cocone, i.e. that \( k \cdot c_n = \alpha_n \); then the uniqueness of \( h \) implies that \( k = h \). For \( n = 0 \), \( k \cdot c_n \) and \( \alpha_n \) are both morphisms with domain 0, so they are the same. Assuming that \( k \cdot c_n = \alpha_n \), we see that

\[
\begin{align*}
k \cdot c_{n+1} &= k \cdot \iota \cdot Fc_n \quad \text{by definition of } \iota \\
&= \alpha \cdot F(k \cdot c_n) \quad \text{since } k \text{ is a homomorphism} \\
&= \alpha \cdot F\alpha_n \quad \text{by induction hypothesis} \\
&= \alpha_{n+1} \quad \text{by (3.2)}
\end{align*}
\]

This concludes the proof. \( \square \)

Remark 3.1.9. To obtain an initial algebra for an endofunctor \( F : \mathcal{A} \to \mathcal{A} \), it is not really necessary that \( \mathcal{A} \) have colimits of all \( \omega \)-chains or that \( F \) preserve all \( \omega \)-colimits. It is sufficient to assume that the colimit of the initial-algebra \( \omega \)-chain exists and that \( F \) preserve this colimit. That is, these are the only facts about \( \mathcal{A} \) and \( F \) that are used in the proof of Theorem 3.1.7. In many cases, it is just as easy to verify the stronger requirements that we stated in Theorem 3.1.7 than it is to verify the special cases used in the proof. But this is not always the case, and we shall see examples in topological and measure-theoretic settings where the sufficient conditions hold but the stronger ones do not.

3.2 Examples of initial algebras

We present examples of initial algebras obtained by finite iteration as in Theorem 3.1.7. We discuss these at some length because the same functors will appear throughout this
3.2 Examples of initial algebras

**Example 3.2.1.** (1) The functor $FX = X + 1$. Here 1 is a terminal object. In Example 2.2.7(3), we considered this as a functor on Set and found that $\mu F$ is the set of natural numbers.

In an arbitrary category $\mathcal{A}$ with binary coproducts and a terminal object 1 the initial algebra for $FX = X + 1$ was called by Lawvere [165] the natural numbers object, NNO for short. Thus an NNO is an object $N$ together with a morphism $\iota: N + 1 \rightarrow N$ (or, equivalently, a pair of morphisms $N \rightarrow N$ and $1 \rightarrow N$) universal among such morphisms. In more detail, given an object $A$ and morphisms $\alpha_0: 1 \rightarrow A$ and $\alpha_1: A \rightarrow A$, there exists a unique morphism $h: N \rightarrow A$ such that the square below commutes:

$$
\begin{array}{c}
N + 1 \\
\downarrow h + \text{id}
\end{array} \xrightarrow{\iota} \begin{array}{c} N \\
\downarrow h
\end{array} \quad \begin{array}{c} A + 1 \\
\downarrow [\alpha_0, \alpha_1]
\end{array} \xrightarrow{\ \ } \begin{array}{c} A
\end{array}
$$

The functor $FX = X + 1$ is easily seen to preserve $\omega$-colimits. As such, it has an initial algebra obtained as the colimit of the initial-algebra $\omega$-chain formed by the left-hand coproduct injections

$$
0 \longrightarrow 1 \longrightarrow 1 + 1 \longrightarrow (1 + 1) + 1 \longrightarrow \cdots
$$

More precisely, $\mathcal{A}$ has an NNO iff a colimit of this $\omega$-chain exists. When $\mathcal{A} = \text{Set}$, we may identify its $n$-th term with the natural number $n = \{0, \ldots, n - 1\}$ and obtain an $\omega$-chain of inclusion maps. The colimit is the set $\mathbb{N}$ of natural numbers, with the colimit morphisms $c_n: n \rightarrow \mathbb{N}$ inclusion maps. The algebra structure

$$
\iota: \mathbb{N} + 1 \rightarrow \mathbb{N}
$$

is the isomorphism taking 0 in the right-hand summand, 1, to $0 \in \mathbb{N}$, and the left summand $\mathbb{N}$ to itself via the successor function. This follows easily from $\iota \cdot Fc_n = c_{n+1}$ for all $n < \omega$. This is a new explanation of Example 2.2.7(2).

(2) We consider now the category $\text{Pos}$ of posets and monotone functions. Here coproducts are disjoint unions of partially ordered sets, i.e. with elements of different coproduct components incomparable. The above endofunctor takes a poset and adds a fresh element incomparable to the elements of $X$. Then $F^n0$ can be identified with the set $\{0, \ldots, n - 1\}$ discretely ordered, and the NNO is again $\mathbb{N}$ with the same structure as in (3.6) and discretely ordered. Observe that, in accordance to Example 2.2.9 $\mathbb{N} = \mathbb{N} \cdot 1$ in $\text{Pos}$.

There is also the functor $FX = X_\perp$ adding a new least element to a poset. Here the initial chain has the following form

$$
0 \longrightarrow \perp \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots
$$
3 Finitary Iteration

We can describe $F^n 0$ as the linear order $0 < 1 < 2 < \cdots < n - 1$ and $\mu F$ is $\mathbb{N}$ with the usual order and the algebra structure $\iota$ with $\iota(\bot) = 0$ and $\iota(n) = n + 1$.

Similarly, we have the endofunctor $FX = X^+$ adding a new top element. Here $\mu F$ is the set of natural numbers with the reverse of the usual order: $0 > 1 > 2 > \cdots$.

(3) For the analogous endofunctors on CPO see Example 2.2.17. In contrast, the NNO in this category is uninteresting: the functor $FX = X + 1$ is the identity functor with initial algebra $0 = \{\bot\}$.

Notation 3.2.2. (1) Let $\mathbf{MS}$ be the category of 1-bounded metric spaces and non-expanding maps. That is, objects are sets endowed with a metric $d: X \times X \to [0, 1]$, i.e. a function which satisfies

(a) $d(x, y) = 0$ iff $x = y$
(b) $d(x, y) = d(y, x)$
(c) $d(x, z) \leq d(x, y) + d(x, z)$ (the triangle inequality)

Morphisms are non-expanding functions, i.e. functions $f: X \to X'$ such that for all $x, y \in X$,

$$d'(f(x), f(y)) \leq d(x, y).$$

(2) We also have the full subcategory $\mathbf{CMS}$ of complete metric spaces; i.e. such that every Cauchy sequence has a limit.

Example 3.2.3. We now consider $FX = X + 1$ as an endofunctor of $\mathbf{MS}$. Coproducts are disjoint unions with distance 1 between points in different summands. The functor $FX = X + 1$ now has as an initial chain (3.5). That, it has as the initial algebra the set $\mathbb{N}$ of natural numbers, but as a discrete space: distinct points have distance 1. The same is true for $F_X = X + 1$ as an endofunctor of $\mathbf{CMS}$. Indeed, in the initial-algebra $\omega$-chain, we see that $F^n 0$ is the discrete space of $n$ elements, and the NNO, which is the colimit, is the discrete space on $\mathbb{N}$. The algebra structure (3.4) is the same as in Example 3.2.1.

Example 3.2.4. The situation changes when we scale the metric by $\frac{1}{2}$ (or by any other constant between 0 and 1). Let $\frac{1}{2}: \mathbf{MS} \to \mathbf{MS}$ scale a space by $\frac{1}{2}$. We now consider $GX = \frac{1}{2}X + 1$.

The initial-algebra $\omega$-chain of $G$ is the following chain of inclusions:

$$
\begin{align*}
0 \bullet & \quad 0 \bullet \longrightarrow 1 \quad 0 \bullet \;
\end{align*}
$$

Here 0 represents the unique element of the right-hand summand of

$$G^{n+1} 0 = \frac{1}{2} (G^n 0) + 1$$
3.2 Examples of initial algebras

and the element \(i\) of \(G^n0\) is represented by \(i + 1\) in \(G^{n+1}0\). We see that \(G^n0\) is the space \(\{0, 1, \ldots, n - 1\}\) with the following metric:

\[
d(i, i) = 0 \quad \text{and} \quad d(i, j) = 2^{-\min(i,j)} \quad \text{if} \ i \neq j.
\]

The colimit of this chain is \(\mu G = \mathbb{N}\) with the above metric.

**Example 3.2.5.** Let us consider the set functor \(FX = X \times X + 1\). When dealing with this functor and related ones, it is often useful to adopt a graphical notation. We shall draw \((x, y) \in X \times X \hookrightarrow FX\) as the ordered tree

```
          x
         / \n        y
```

We start with \(F^00 = \emptyset\). Now we picture the elements of \(F^i0\) for \(i = 1, 2,\) and \(3\) using \(\bullet\) to denote the element of 1:

- \(F^10: \bullet\)
- \(F^20: \bullet, \bullet, \bullet\)
- \(F^30: \bullet, \bullet, \bullet, \bullet, \bullet, \bullet\)

In general, \(F^n0\) is the set of all binary trees of height less than \(n\), and the connecting maps \(F^n0 \rightarrow F^{n+1}0\) are the inclusions.

Then the carrier of the initial algebra \(\mu F\) may be taken to be the union \(\bigcup_{n<\omega} F^n0:\)

\[
\mu F = \text{all finite binary ordered trees}.
\]

The structure map

\[
\iota: (\mu F \times \mu F) + 1 \rightarrow \mu F
\]

takes the unique element of 1 to the single-node tree, and the left-hand component is given by *tree tupling*, i.e.

```
          t_1
         /   \n        t_2
```

This is a new explanation of Example 2.2.15(1).

**Example 3.2.6.** (1) We return to the category \(\text{MS}\) introduced in Notation 3.2.2. In this category, products \(X \times X'\) are cartesian products with the *maximum metric*:

\[
d((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}.
\]
Thus, $FX = X \times X + 1$ can be considered as an endofunctor on $MS$. Its initial-algebra $\omega$-chain is

$$0 \rightarrow (1 \times 1) + 1 \rightarrow ((1 \times 1) + 1) \times ((1 \times 1) + 1) + 1 \rightarrow \cdots$$

the same as that of Example 3.2.5, but with each set $F^n0$ taken to be the discrete space: all distances between distinct points are 1. The colimit $\mu F$ is then, not surprisingly, the space of all binary trees equipped with the discrete metric.

(2) The situation changes when we scale the distances by half. This leads to the functor

$$GX = \frac{1}{2}(X \times X) + 1.$$ 

Here $G^{n+1}0$ consists of pairs of ordered trees $(t_1, t_2) \in G^n0 \times G^n0$ and of the single-node tree, whose distance from any $(t_1, t_2)$ is 1. And the distance between $(t_1, t_2)$ and $(s_1, s_2)$ is one half of the maximum of the distances $d(t_i, s_i), i = 1, 2$, in $G^n0$. From this it follows that $G^n0$ can be described as the set of all binary trees of height less than $n$ with the metric

$$d(t, u) = \begin{cases} 
2^{-k} & \text{if } t \neq u \\
0 & \text{if } t = u 
\end{cases} \quad (3.8)$$

for the least number $k$ such that $t$ and $u$ have different cuttings at level $k$.

The colimit in $MS$ is the space of all finite binary ordered trees with the above metric (3.8).

(3) When we turn from $MS$ to $CMS$, the situation changes because for $G$ above the colimit in $MS$ is not complete. In fact, under the metric (3.8) every infinite binary tree $t$ yields a Cauchy sequence $t_0, t_1, \ldots$, where $t_k$ cuts $t$ at level $k$. It turns out that the colimit of the initial-algebra $\omega$-chain $G^n0$ (of isometric embeddings) in $CMS$ is obtained from the colimit in $MS$ by forming the Cauchy completion. And this Cauchy completion is

$$\mu G = \text{all (finite and infinite) ordered binary trees},$$

with the metric (3.8) above. We shall see a general reason for this in Section 5.2 below.

**Example 3.2.7.** Consider the endofunctor $FX = M \times X$, where $M$ is a fixed object in one of our categories.

(1) As an endofunctor on $Set$, $\mu F = \emptyset$. The same holds in $Pos$, $MS$, and $CMS$.

(2) We next consider the situation in $CPO_\perp$. For a cpo $M$ the initial algebra $\omega$-chain has the following form:

$$\{\perp\} \rightarrow M \times \{\perp\} \rightarrow M \times M \times \{\perp\} \rightarrow \cdots$$

The tuples in each factor are ordered componentwise, and the connecting maps add to every $n$-tuple the element $\perp$ in the $n + 1$-st coordinate. The colimit in $CPO_\perp$ can be described as the cpo $M^\omega$ of all streams of elements of $M$ ordered componentwise. Shortly:

$$\mu X.(M \times X) = M^\omega.$$
The algebra structure $M \times M^\omega \rightarrow M^\omega$ adjoins a new head to a stream in the evident manner.

**Example 3.2.8.** (1) For finitary signatures $\Sigma$, the polynomial set functor $H_\Sigma$ of Example 2.1.3(4) preserves colimits of $\omega$-chains. We apply Theorem 3.1.7 to get a short proof that the initial chain yields

$$\mu H_\Sigma = \text{the algebra of finite } \Sigma\text{-trees};$$

cf. Proposition 2.2.14. (For infinitary signatures we describe $\mu H_\Sigma$ in Example 6.1.15(2) below.) In fact, we can identify $H_\Sigma^1(\emptyset) = H_\Sigma^0 = \Sigma^0$ with the set of all $\Sigma$-trees of height 0, i.e. one-node trees labelled by an element of $\Sigma^0$. And

$$H_\Sigma^2(\emptyset) = H_\Sigma \Sigma^0 = \Sigma^0 + \prod_{k>0} \Sigma_k \times \Sigma_0^k$$

with the set of all $\Sigma$-trees of height at most 1. More generally, $H_\Sigma^n(\emptyset)$ can be identified with the set of $\Sigma$-trees of height less than $n$. We obtain the chain of inclusion maps

$$H_\Sigma^0(\emptyset) \hookrightarrow H_\Sigma^1(\emptyset) \hookrightarrow H_\Sigma^2(\emptyset) \hookrightarrow \cdots$$

whose colimit (union) is the set of all finite $\Sigma$-trees.

The algebra structure $\iota : H_\Sigma(\mu H_\Sigma) \rightarrow \mu H_\Sigma$ of (3.4) is tree-tupling: $\iota$ maps an element $(\sigma, (t_i)_{i<k})$ of $\Sigma_k \times (\mu H_\Sigma)^k$ to the $\Sigma$-tree with root labelled by $\sigma$ (k-ary) and with the children $t_0, \ldots, t_{k-1}$ from left to right (cf. Definition 2.2.12).

(2) As a concrete example, consider a finite set $V = \{v_1 \ldots v_n\}$ (of boolean variables). Recall the concept of a binary decision tree: it is a finite binary ordered tree whose inner nodes are labelled by variables $v_i$ and leaves are labelled by 1 (for “true”) or 0 (for “false”). We see that the polynomial functor

$$FX = \{0, 1\} + V \times X \times X$$

has the initial algebra

$$\mu F = \text{all binary decision trees.}$$

For a connection with binary decision diagrams see Example 9.3.15(5).

**Example 3.2.9.** The finite power set endofunctor $\mathcal{P}_f : \text{Set} \rightarrow \text{Set}$ will be of special interest in this book. It preserves $\omega$-colimits, thus we can apply Theorem 3.1.7.

(1) Its initial sequence is given by the inclusion functions of the following sets

$$\emptyset \hookrightarrow \{\emptyset\} \hookrightarrow \{\emptyset, \{\emptyset\}\} \hookrightarrow \cdots$$

where $V_0 = \emptyset$, and $V_{n+1} = \mathcal{P}V_n$. The colimit of this sequence is the union $V_\omega = \bigcup_{n<\omega} V_n$, see Example 2.2.7(4).
3 Finitary Iteration

(2) An alternative important representation of $\mu P_t$ is by finite extensional trees. (Recall from Remark 2.2.10 that trees are considered up to isomorphism.) An unordered tree is extensional if for every pair of children of a given node the two corresponding subtrees are different (non-isomorphic). Every unordered tree has an extensional quotient obtained by successively identifying children of a given node with the same subtrees. See also Section 4.5 for more on extensional trees.

Observe that if $P^0\emptyset$ is represented by the singleton tree, then we have a natural representation of the elements of $P^n\emptyset$ by all finite extensional trees of height less than $n$: the tree representing of a set $\{x_1, \ldots, x_n\}$ has $n$ children (representing $x_i$):

$$P^1\emptyset = \{\emptyset\}$$ is represented by $\bullet$

$$P^2\emptyset = \{\emptyset, \{\emptyset\}\}$$ is represented by $\bullet, \bullet$

$$P^3\emptyset = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$ is represented by $\bullet, \bullet, \bullet, \bullet$

Etc. Therefore, we obtain

$$\mu P_t = \text{all finite extensional trees}$$

with the algebra structure $P_t(\mu P_t) \to \mu P_t$ given by tree tupling.

Example 3.2.10. A bag is a pair $(X, b)$, where $X$ is a set and $b: X \to \mathbb{N}$ is a function such that for all but finitely many $x$ we have $b(x) = 0$. (These are also called finite multisets and the number $b(x)$ is called the multiplicity of $x$ in the bag.) The size of $(X, b)$ is $\sum_{x \in X} b(x)$. A morphism of bags $f: (X, b) \to (Y, c)$ is a function $f: X \to Y$ such that the multiplicity of every $y \in Y$ is the sum of multiplicities in $f^{-1}(y)$; in symbols: $c(y) = \sum_{f(x) = y} b(x)$ for all $y \in Y$.

The bag functor $\mathcal{B}: \textbf{Set} \to \textbf{Set}$ takes a set $X$ to the set of all bags on $X$, and to every function $f: X \to Y$ it assigns the function $\mathcal{B}f: \mathcal{B}X \to \mathcal{B}Y$ given by

$$\mathcal{B}(f)(b) = \lambda y. \sum_{x \in f^{-1}(y)} b(x).$$

This functor preserves $\omega$-colimits, and so Theorem 3.1.7 applies. The set $\mathcal{B}\emptyset$ has a single element which we represent as the single-node tree. Given a representation of $\mathcal{B}^n\emptyset$ by (unordered) trees, represent $\mathcal{B}^{n+1}\emptyset$ as follows: every bag consisting of trees $t_1, \ldots, t_n \in \mathcal{B}^n\emptyset$ with multiplicities $k_1, \ldots, k_n$ is represented by the tree having $k_i$ children given by $t_i$ for $i = 1, \ldots, n$. Thus, $\mathcal{B}\emptyset$ is the set of the following trees

$$\bullet, \bullet, \bullet, \bullet, \ldots$$
3.2 Examples of initial algebras

These are all unordered trees of height at most 1. It is easy to see that $\mathcal{B}^0\emptyset$ are all finite unordered trees of height at most 2, etc.

The colimit of the initial-algebra $\omega$-chain is again the union. The initial algebra $\mu\mathcal{B}$ can thus be described as follows:

$$\mu\mathcal{B} = \text{all finite unordered trees}.$$ 

The algebra structure $\iota: \mathcal{B}(\mu\mathcal{B}) \to \mu\mathcal{B}$ is tree-tupling with multiplicities, i.e. $\iota$ maps a bag $\{(t_1, \ldots, t_n), b\}$ of finite unordered trees to the unordered tree that joins $b(t_i)$ copies of $t_i$, $i = 1, \ldots, n$, with a new common root.

Example 3.2.11. Generalizing several of the above examples, we recall the concept of an analytic functor introduced by Andr´ e Joyal [141, 142]: given a group $G$ of permutations on $k = \{0, \ldots, k-1\}$, we denote by $X^k/G$ the set of orbits under the action of $G$ on $X^k$ by coordinate interchange, i.e. the set of equivalence classes

$$X^k/G = \{x \sim_G y : (x_1, \ldots, x_k) \sim_G (y_1, \ldots, y_k) \text{ for each } p \in G\}$$

for the least equivalence $\sim_G$ with $(x_1, \ldots, x_k) \sim_G (y_1, \ldots, y_k)$. This defines a set functor taking $f: X \to Y$ to the obvious function $X^k/G \to Y^k/G$ derived from $f^k$. The analytic functors are precisely the coproducts of functors of the form $X^k/G$. More precisely, an analytic functor $F$ is of the form

$$F_X = \coprod_{\sigma \in \Sigma} X^k/G_k,$$

where $\Sigma$ is a finitary signature, $k$ the arity of $\sigma \in \Sigma$ and $G_k$ a group of permutations on $k$. Thus, every polynomial functor $H_\Sigma$ is analytic, and an important example of an analytic functor is the bag functor, which can be expressed as follows:

$$\mathcal{B}X = \coprod_{k \in \mathbb{N}} X^k/S_k,$$

where $S_k$ is the symmetric group of all permutations on $k$. In contrast, $\mathcal{P}_f$ is not analytic.

For every analytic functor $F$ the initial-algebra $\omega$-chain yields a quotient of the algebra $\mu H_\Sigma$ of Example 3.2.8.

$$\mu F = \mu H_\Sigma/\sim$$

where $\sim$ is the least equivalence such that for every tree $t \in \mu H_\Sigma$, every node $x$ of $t$ labelled by $\sigma \in \Sigma$ and every permutation $g$ in the group $G_\sigma$ we have $t \sim t'$, where $t'$ is obtained from $t$ by permuting the children of $x$ according to $g$.

In fact, the proof is completely analogous to Example 3.2.8. We identify $F^10 = \Sigma_0$ with one-point trees labelled in $\Sigma_0$. And since $F^{n+1}0 = \coprod_{\sigma \in \Sigma} (F^n0)^k/G_\sigma$, where $\sigma$ is a $k$-ary symbol, we can identify, for every $n \in \mathbb{N}$, the set $F^n0$ with the quotient of the set of all $\Sigma$-trees of height less than $n$ modulo the above equivalence $\sim$. We again obtain a chain of inclusion maps whose colimit (union) is $F_\Sigma/\sim$. This is illustrated by the case of the bag functor $\mathcal{B}$ in Example 3.2.10.
Example 3.2.12. Here is an example of an endofunctor $F$ on a category with a rather interesting initial algebra. Let $\mathrm{BiP}$ be the category of bipointed sets: objects are sets with distinguished elements $\bot$ and $\top$ that are required to be different. A morphism in $\mathrm{BiP}$ is a function preserving the distinguished points. There is a binary operation $\oplus$ on $\mathrm{BiP}$ taking $(X, \bot_X, \top_X)$ and $(Y, \bot_Y, \top_Y)$ to the disjoint union $X + Y$, identifying $\top_X$ and $\bot_Y$, and then using as distinguished points $\bot_X$ and $\top_Y$. Define an endofunctor on $\mathrm{BiP}$ by

$$FX = X \oplus X.$$  

Its initial-algebra $\omega$-chain starts with the initial object $0 = \{0, 1\}$ where $0 = \bot_0$ and $1 = \top_0$. Then $F0$ can be represented by $\{0, \frac{1}{2}, 1\}$ where $\frac{1}{2}$ represents the equivalence class $\{\bot_1, \top_0\}$. In this manner the initial chain is the following chain of injections

$$\begin{array}{cccc}
\top = 1 & \leftrightarrow & \top = 1 & \leftrightarrow \\
\bot = 0 & & 1/2 & & 1/4 & \leftrightarrow \\
1/2 & & 1/4 & & \vdots & \\
0 & F0 & FF0 & \\
\end{array}$$

The colimit of this chain is its union $D = \text{all dyadic rational numbers in } [0, 1]$.

That means that $D$ consists of all numbers $\frac{k}{2^n}$, $k = 0, \ldots, 2^n$. The algebra structure $\iota: D \oplus D \to D$ takes $y$ in the left-hand part to $\frac{y}{2}$ and in the right-hand one to $\frac{y+1}{2}$. Indeed, $F$ preserves the above colimit, thus $D$ is the initial algebra for $F$. To see this, observe that the connecting map $j: D \to FD$ from $\text{colim} F^n 0$ to $F(\text{colim} F^n 0)$ is $x \mapsto 2x$ on $[0, \frac{1}{2}]$ and $x \mapsto 2x - 1$ on $[\frac{1}{2}, 1]$. This maps $j$ is bijective, and its inverse $j^{-1}: FD \to D$ is the structure of the initial algebra. Related examples are discussed in Section 15.3.

Example 3.2.13. Let us mention also some negative examples. An endofunctor of $\text{Set}$ need not have an initial algebra, as we have seen in Example 2.2.7. The endofunctor $F: X \mapsto X^N + 1$ does have an initial algebra, as we will see in Chapter 4, but its initial-algebra $\omega$-chain does not converge. In a similar manner to what we have seen in Example 3.2.5, $F^00$ can be represented by all countably branching ordered trees of height less than $n$. But $\bigcup F^n0$ is not a fixed point of $F$. We will see in Chapter 6 that an initial-algebra chain of uncountable length is needed for this functor.

Continuing with a discussion of examples of initial algebras obtained by $\omega$-iteration on categories other than $\text{Set}$, we mention a result implying that for some special categories, every endofunctor has an initial algebra obtained that way.

**Definition 3.2.14** [102]. A category is called algebraically complete if every endofunctor has an initial algebra.

**Theorem 3.2.15** [10]. The categories $\text{Set}_c$ (countable sets and functions), $\text{Rel}_c$ (countable sets and relations), and $K \text{-Vec}_c$ (countably-dimensional vector spaces over a field $K$ and
3.3 Terminal-coalgebra chain

Linear functions) are algebraically complete. Moreover, every endofunctor $F$ has the initial algebra

$$\mu F = \operatorname{colim} F^0.$$

**Remark 3.2.16.** (1) Every complete lattice is algebraically complete by the classical fixed-point theorem of G. Birkhoff.

(2) Among categories with products, there are essentially no other examples:

**Theorem 3.2.17 [19].** Every algebraically complete category with products is a preorder.

**Proof.** The following beautiful proof was provided by Peter Freyd [102]: suppose $\mathcal{A}$ has products but is not a preorder. That is, some hom-set $\mathcal{A}(A, B)$ has at least two elements. Consider the endofunctor $F$ obtained by the following composite

$$F = \mathcal{A}(A, -) \xrightarrow{\mathcal{A}(A, -)} \text{Set} \xrightarrow{\text{Set}(-, 2)} \text{Set}^\text{op} \xrightarrow{B(-)} \mathcal{A}.$$

(Here $B(-)$ denotes the functor taking a set $M$ to the power $B^M$.) This functor $F$ can be expressed as $FD = B^{S(D)}$ where $S(D) = 2^{\mathcal{A}(A, D)}$. It does not have fixed points. In fact, assuming $D \cong FD$, we conclude that $\mathcal{A}(A, D)$ is isomorphic to $\mathcal{A}(A, FD) \cong \mathcal{A}(A, B^{S(D)}) \cong \mathcal{A}(A, B)^{S(D)}$.

But the cardinality of the right-hand side is at least

$$2^{S(D)} \cong 2^{\mathcal{A}(A, D)}$$

a contradiction: for every set $X$ the cardinality of $2^X$ is larger than that of $X$ – apply this to $X = \mathcal{A}(A, D)$.

### 3.3 Terminal-coalgebra chain

The construction of the terminal coalgebra in this section is another categorical version of the Kleene Fixpoint Theorem: a terminal coalgebra for an endofunctor is constructed as a limit of iterations of the functor on the terminal object. An important example is the terminal coalgebra for a polynomial functor which is described as the algebra of all (finite and infinite) $\Sigma$-trees. Throughout this section we assume that a category $\mathcal{A}$ with a terminal object $1$ is given. We denote by $!: X \to 1$ the unique morphism from an object $X$.

It is straightforward to dualize the general results of the last section. One dualizes initial objects 0 to terminal objects 1, $\omega$-chains to $\omega^\text{op}$-chains (that is, functors from $\omega^\text{op}$ to $\mathcal{A}$), colimits to limits, and the initial-algebra $\omega$-chain of Definition 3.1.3 as follows:

**Definition 3.3.1.** By the terminal-coalgebra $\omega^\text{op}$-chain of an endofunctor $F : \mathcal{A} \to A$ is meant the $\omega^\text{op}$-chain

$$1 \leftrightarrow F1 \leftrightarrow F^21 \leftrightarrow F^31 \cdots \leftrightarrow F^{n-1}1 \leftrightarrow F^n1 \leftrightarrow F^{n+1}1 \leftrightarrow \cdots \text{ (3.11)}$$
3 Finitary Iteration

Construction 3.3.2. Dually to Construction 3.1.5 every coalgebra \( \alpha : A \to FA \) induces a canonical cone over the terminal-coalgebra \( \omega^{\text{op}} \)-chain of \( F \) by induction: \( \alpha_0 : A \to 1 \) is uniquely determined and \( \alpha_{n+1} = F\alpha_n \cdot \alpha : A \to F^{n+1}1 \).

Remark 3.3.3. Dually to Remark 3.1.6, homomorphisms \( h : (A,\alpha) \to (B,\beta) \) of coalgebras preserve the induced cones:

\[ \alpha_n = \beta_n \cdot h \quad \text{for all } n < \omega. \]

It is worthwhile putting down the dual statement of Theorem 3.1.7. This was first explicitly formulated by Barr [58]:

Theorem 3.3.4. Let \( \mathcal{A} \) be a category with terminal object 1 and with limits of \( \omega^{\text{op}} \)-chains. If \( F : \mathcal{A} \to \mathcal{A} \) preserves limits of \( \omega^{\text{op}} \)-chains, then it has the terminal coalgebra

\[ \nu F = \lim_{n \in \omega^{\text{op}}} F^n1. \]

Remark 3.3.5. (1) It is sufficient to assume that \( F \) preserves the limit of its terminal-coalgebra chain (cf. Remark 3.1.9).

(2) Let \( \ell_n : \nu F \to F^n1(n < \omega) \) be the limit cone. The coalgebra structure

\[ \tau : \nu F \to F(\nu F) \]

is the unique morphism with

\[ F\ell_n \cdot \tau = \ell_{n+1} \quad (n < \omega) \]

This is dual to Remark 8.1.11.

We revisit the examples from Section 3.2.

Examples 3.3.6. The functor \( FX = X + 1 \) on \( \text{Set} \). The terminal-coalgebra \( \omega^{\text{op}} \)-chain is

\[ 1 \leftarrow 1 + 1 \leftarrow 1 + 1 + 1 \leftarrow \cdots \]

The \( n \)th object may be identified with \( n + 1 = \{0,1,\ldots,n\} \), and the connecting function \( F^n1 = F_n : n + 2 \to n + 1 \) is then given by \( F_n(i) = i \) for \( i \leq n \), and \( F_n(n + 1) = n \). The limit of this chain is the set of all \( \omega \)-tuples \( (x_0,\ldots,x_n,\ldots) \) with \( F_n(x_{n+1}) = x_n \) for every \( n \). One such tuple is \( \top = (0,1,2,\ldots) \). Every other tuple is of the form \( (0,1,\ldots,k,k,\ldots,k,\ldots) \) which we may identify with \( k \). Thus we describe the terminal coalgebra as

\[ \nu X.(X + 1) = \mathbb{N}^\top, \]

the set of natural numbers with an element \( \top \) added. The coalgebra structure \( \tau : \mathbb{N}^\top \to \mathbb{N}^\top + 1 \) has \( \top \) as a fixed point, sends 0 to the point in the right-hand summand of \( \mathbb{N}^\top + 1 \), and is otherwise the predecessor function on \( \mathbb{N} \). This is a new explanation of Example 2.5.3(2).
Example 3.3.7. (1) The functor $F X = X + 1$ on $\text{Pos}$. Using the same argument as above, we see that the terminal coalgebra is $\mathbb{N}^\top$ with the discrete order. The structure is the same map as in Example 3.3.6 above.

The functor $F X = X_\perp$ (cf. Example 2.1.7(1)) has $\nu F = \mathbb{N}^\top$ with the usual order $0 < 1 < 2 < \cdots < \infty$ and the same structure as before, the argument is analogous.

(2) The functor $F X = X + 1$ on $\text{CPO}_\perp$ essentially is the identity functor, whence $\nu F = 1 = \{\bot\}$. The functor $F X = X_\perp$ has as terminal coalgebra the same poset as in point (1) above. This is not surprising: $\text{CPO}_\perp$ is closed in $\text{Pos}$ under limits (but not colimits).

(3) On the category $\text{MS}$ of 1-bounded metric spaces, the above argument shows that the terminal coalgebra of the same functor $F X = X + 1$ is again $\mathbb{N}^\top$, this time with a discrete metric. The structure is the same as we have seen. The same is true for $\text{CMS}$.

When we change the functor to $G X = \frac{1}{2} X + 1$ as in Example 3.2.4, the terminal-coalgebra $\omega^{op}$-chain has essentially the same spaces that we saw in that example since $G \emptyset = 1$, and the connecting map $G^{n+1} 1 \to G^n 1$ merges $n$ and $n + 1$ and is identity else. We again get $\mathbb{N}^\top$ as the terminal coalgebra but with the metric shown in Example 3.2.4.

Example 3.3.8. (1) Let $\mathcal{A}$ have countable products. The functor $F X = M \times X$ where $M$ is a fixed (but arbitrary) object of $\mathcal{A}$. We identify each object with its product with 1. Thus we can write the terminal-coalgebra chain as

$$1 \leftarrow M \leftarrow M \times M \leftarrow M \times M \times M \leftarrow \cdots$$

The connecting morphisms are all the projections onto the left-most factors. We obtain as a limit the countable power $M^\omega$ with the limit projections $\ell_n : M^\omega \to M^n$ given by the left-hand projections of $M^\omega \cong M^n \times \bigsqcup_{i \geq n} M$. For example, if $\mathcal{A} = \text{Set}$, then $M^\omega$ is the set of all streams on $M$, also known as the infinite words on $M$:

$$\nu X.(M \times X) = M^\omega.$$ 

The coalgebra structure $M^\omega \to M \times M^\omega$ is $(\text{head}, \text{tail})$, where

$$\text{head}(a_1, a_2, a_3 \ldots) = a_1 \quad \text{and} \quad \text{tail}(a_1, a_2, a_3 \ldots) = (a_2, a_3 \ldots)$$

This is a special case of Theorem 2.5.9 for $\Sigma$ consisting of unary operations indexed by $M$: a $\Sigma$-tree in an infinite path labelled in $M$, i.e. a steam on $M$.

(2) As a concrete example, take $M = \mathbb{R}$, the set of real numbers. A coalgebra for $\mathbb{R} \times X$ is a stream automaton, i.e. a dynamic system with real outputs, see Example 2.5.3(3).

Here are two examples in which the output is represented by a label of the output arrow
Finitary Iteration

One way to represent real valued streams is by taking the set $A$ of real analytic functions, see [199]. Here one considers $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $x$ there is an open interval around 0 on which the $n$-th derivative $f^{(n)}$ is defined, and $f$ agrees with its Taylor series. The coalgebra structure on $A$ is given by

$$f(x) \mapsto (f(0), f'(x))$$

for every analytic function. This coalgebra is isomorphic to a subcoalgebra of the stream coalgebra that we saw above; to every analytic function $f(x)$ one associates the stream of coefficients of the Taylor series of $f(x)$, i.e. $A$ is isomorphic to the subcoalgebra of those streams $\sigma$ in $\mathbb{R}^\omega$ such that

$$\sum_{i=0}^{\infty} \frac{\sigma_i}{i!} x^i < \infty.$$

The above two stream automata thus present analytic functions by corecursion. In the coalgebra on the left, we have a function whose value at 0 is $r$ and equal to its own derivative: $f(x) = re^x$. On the right, we obtain four functions:

$$\sin x, \cos x, -\sin x, -\cos x.$$

Example 3.3.9. For every signature $\Sigma$, the terminal-coalgebra $\omega^{op}$-chain for the polynomial functor $H_\Sigma$ of Example 2.1.3 can be described as follows. Denote by $\Sigma_\bot = \Sigma + \{\bot\}$ the signature $\Sigma$ extended by a new constant symbol $\bot$. For every $n < \omega$ we have:

$$H^n_\Sigma 1 = \text{all } \Sigma_\bot\text{-trees of height at most } n \text{ with all leaves at height } n \text{ labelled by } \bot.$$

To see this, let us identify $1 = H^0_\Sigma 1$ with the set consisting of the singleton tree labelled by $\bot$. Then $H^1_\Sigma 1 = \prod \Sigma_k \times 1^k$ can be identified with the set of all trees of height $\leq 1$ whose root is labelled in $\Sigma$ and (for a root label of arity $k$) whose $k$ leaves are labelled by $\bot$. Analogously $H^2_\Sigma 1 = \prod \Sigma_k \times (H_\Sigma 1)^k$ are trees $t$ with a root labelled in $\Sigma_k$ and the $k$ subtrees are trees in $H^1_\Sigma 1$; thus all leaves of $t$ of depth 2 have label $\bot$ etc.

Moreover, the connecting map

$$H^{n+1}_\Sigma \rightarrow H^n_\Sigma 1$$

simply cuts the $\bot$-labelled leaves away and relabels all leaves of depth $n$ by $\bot$. This yields a new proof of Theorem 2.5.9:

**Theorem 3.3.10.** The terminal coalgebra for the polynomial functor $H_\Sigma$ is the set $T_\Sigma$ of all ordered $\Sigma$-trees, with the coalgebra structure given by the inverse to tree tupling.

**Proof.** We have described $H^n_\Sigma 1$ above as the set of all $\Sigma_\bot$-trees of height at most $n$ with leaves of depth $n$ labelled by $\bot$. Since $H_\Sigma$ preserves limits of $\omega^{op}$-chains, we know that

$$\nu H_\Sigma = \lim_{n<\omega} H^n_\Sigma 1.$$

(1) We prove that the limit of the chain $H^n_\Sigma 1$ (with the connecting maps $\nu_{n+1,n} : H^{n+1}_\Sigma 1 \rightarrow H^n_\Sigma 1$ given by cutting at level $n$ as above) is the set $T_\Sigma$ with the limit cone $\nu : T_\Sigma \rightarrow H^0_\Sigma 1$.
3.3 Terminal-coalgebra chain

given, again, by cutting at level \( n \) and relabelling all leaves at level \( n \) by \( \bot \). In other words, we need to show that for every collection of trees \( t_n \in H_\Sigma^1 \) which is compatible:

\[
v_{n,m}(t_n) = t_m \quad \text{for all } n \geq m \text{ finite},
\]

there exists a unique tree \( t \in T_\Sigma \) with \( \ell_n(t) = t_n \) for all \( n < \omega \). Existence: let \( t \) be the unique \( \Sigma \)-tree which on levels smaller than \( n \) agrees with \( t_n \) for every \( n \). More precisely, the root of \( t \) has the same label as \( t_n \) for all \( n \geq 1 \), the level one labels are the same as in \( t_n \) for all \( n \geq 2 \), etc. Obviously \( \ell_n(t) = t_n \) for all \( n \) finite. Uniqueness: if \( \ell_n(t) = t_n \), then \( t \) and \( t_n \) agree on all levels smaller than \( n \).

(2) The coalgebra structure \( \tau : T_\Sigma \to H_\Sigma T_\Sigma \) is, following Remark 3.3.5, the unique morphism with \( \ell_{m+1} = H_\Sigma \ell_m \cdot \tau \) for all \( m \). Since the tree tupling \( \bar{\tau} : H_\Sigma T_\Sigma \to T_\Sigma \) clearly fulfils \( \ell_{m+1} \cdot \bar{\tau} = H_\Sigma \ell_m \) (because to cut a tree tupling at level \( m+1 \) is the same as cutting the maximum subtrees at level \( m \) and then performing a tree tupling), we conclude \( \tau = \bar{\tau}^{-1} \).

Example 3.3.11. The functor \( FX = \{0,1\} \times X^A \) whose coalgebras are deterministic automata (see Example 2.5.11) yields

\[
F^n1 \cong \text{sets of words of length } < n \text{ in } A^*.
\]

More precisely, let \( A \) have \( k \) elements and let \( \Sigma \) be the signature with two \( k \)-ary operation symbols. Then \( F \cong H_\Sigma \). To give a tree in \( F^n1 \), i.e. a \( \Sigma_\bot \)-tree of height \( \leq n \) with leaves labelled in by \( \bot \), means precisely to label the nodes of a complete \( k \)-ary (see Remark 2.2.10(4)) tree of height \( \leq n - 1 \) by 0 or 1. This, is equivalent to giving a set of words of length \( < n \) to the set formed by the prefixes of length \( < m \) of words from \( M \).

A limit of the resulting terminal-coalgebra \( \omega^{op} \)-chain is the set of all formal languages \( P \subseteq A^* \). The limit cone \( \ell_n : \mathcal{P}A^* \to F^n1 \) takes a formal language \( L \subseteq A^* \) to the set of all prefixes of length \( n - 1 \) of words in \( L \).

Remark 3.3.12. In Chapter 6, we shall consider a more general notion of signature which allows for infinitary operations, see Example 6.1.15(2). Everything which we said in Example 3.3.9 holds for this more general notion of a signature.

Example 3.3.13. For every analytic functor \( F \) (see Example 3.2.11) the terminal coalgebra is obtained by the terminal-coalgebra \( \omega^{op} \)-chain. In fact, it is easy to derive from the formula (3.9) that analytic functors preserve limits of \( \omega^{op} \)-chains (actually, they preserve cofiltered limits \cite{142}). We will see a description of this terminal coalgebra in Example 4.3.30(3) below. The following example is a special case.

Example 3.3.14. The bag functor \( B \) of Example 3.2.10 has the terminal coalgebra

\[
\nu B = \text{all finitely branching unordered trees}.
\]

In fact, we can identify \( B^n1 \), analogously as in Example 3.3.9, with all unordered finitely branching trees of height \( \leq n \) whose leaves at level \( n \) are labelled by \( \bot \). The limit \( \nu B \) is then as stated.
Example 3.3.15. The finite power-set functor $\mathcal{P}_f$ is an example of a set functor which does not preserve limits of $\omega^{\text{op}}$-chains, and, moreover, as we demonstrate in Section 4.4, the limit of its terminal-coalgebra $\omega^{\text{op}}$-chain is not the terminal $\mathcal{P}_f$-coalgebra.

We will describe the terminal coalgebra for $\mathcal{P}_f$ below in several different ways (see Section 4.5).

Example 3.3.16. Consider the functor $FX = X \oplus X$ on the category of BiP of bipointed sets. Recall from Example 3.2.12 that the initial $F$-algebra is carried by the set of all dyadic rationals in $[0, 1]$. Freyd [104] proved that the terminal $F$-coalgebra is the real unit interval $[0, 1]$ with the structure $t: [0, 1] \to [0, 1] \oplus [0, 1]$ given by $x \mapsto 2x$ on $[0, \frac{1}{2}]$ and $x \mapsto 2x - 1$ on $[\frac{1}{2}, 1]$. We discuss this example and related ones in more detail in Section 15.3.

Remark 3.3.17. Many-sorted sets are used in many applications. These lead to endofunctors of the category $\text{Set}^S$ of $S$-sorted sets. In Example 2.1.6, we have seen the corresponding polynomial endofunctors on $\text{Set}^S$. Theorem 3.3.10 (and its proof) hold for these more general polynomial functors. Let us illustrate the initial-algebra $\omega$-chain and the terminal-coalgebra $\omega^{\text{op}}$-chain for a concrete 2-sorted signature:

Example 3.3.18. Consider finite lists of symbols over (arbitrary) alphabets. Here we work with two sorts $a$ (alphabet) and $l$ (list) and consider the signature $\Sigma$ with the following operation symbols

\[
\begin{align*}
\varepsilon &: l \quad \text{(empty list)} \\
\gamma &: a l \to l \quad \text{(concatenation)} \\
0, 1 &: \to a \quad \text{(constants 0 and 1)}
\end{align*}
\]

The corresponding polynomial functor $H_\Sigma$ has sorts

\[
(H_\Sigma X)_a = \{0, 1\} \quad \text{and} \quad (H_\Sigma X)_l = X_a \times X_l + \{\varepsilon\}.
\]

(1) Here is its initial-algebra chain:

<table>
<thead>
<tr>
<th>sort $a$</th>
<th>sort $l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \emptyset, \emptyset \rangle$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$H_\Sigma(\emptyset, \emptyset)$</td>
<td>$0, 1$</td>
</tr>
<tr>
<td>$H_\Sigma^2(\emptyset, \emptyset)$</td>
<td>$0, 1$</td>
</tr>
<tr>
<td>$H_\Sigma^3(\emptyset, \emptyset)$</td>
<td>$0, 1$</td>
</tr>
</tbody>
</table>

etc. If we drop the superfluous $\varepsilon$ at the ending of words, we see that for $n \geq 1$

\[
H_\Sigma^n(\emptyset, \emptyset) = (\{0, 1\}, \coprod_{k<n} \{0, 1\}^k)
\]

The connecting maps are inclusions. Thus, the colimit is

\[
\mu H_\Sigma = (\{0, 1\}, \{0, 1\}^\omega).
\]
3.3 Terminal-coalgebra chain

(2) We next describe the terminal-coalgebra $\omega^{\text{op}}$-chain:

<table>
<thead>
<tr>
<th>sort $a$</th>
<th>sort $l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td>$\bot_a$</td>
</tr>
<tr>
<td>$H_{\Sigma}^1(1, 1)$</td>
<td>0, 1</td>
</tr>
<tr>
<td>$H_{\Sigma}^2(1, 1)$</td>
<td>0, 1</td>
</tr>
<tr>
<td>$H_{\Sigma}^3(1, 1)$</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

etc. Again dropping $\varepsilon$ at the end of words, we see that sort $l$ of $H_{\Sigma}^n(1, 1)$ consists of all words $w \bot_a \bot_l$ where $w$ is a binary word of length $n - 1$, and all binary words $w$ of lengths $0, 1, \ldots, n - 1$. The connecting map $v_{n+1,n}: H_{\Sigma}^{n+1} \rightarrow H_{\Sigma}^n$ maps each word $w \bot_a \bot_l$ by dropping the last letter of $w$: analogously for binary words $w \neq \varepsilon$, it drops the last letter. The limit of this $\omega^{\text{op}}$-chain has the sort $l$ which consists of (a) all $u \bot_a \bot_l$ where $u$ is an infinite binary word and (b) all finite binary words. By dropping the superfluous $\bot_a \bot_l$ here, we get

$$\nu H_{\Sigma} = (\{0, 1\}, \{0, 1\}^* + \{0, 1\}^\omega).$$

We thus see that a $\Sigma$-tree has one of the following forms: for sort $a$ we just have two trees

$$a, 0 \quad \text{and} \quad a, 1$$

For sort $l$ we have the finite trees

$$l, \varepsilon \quad \text{and} \quad l, \gamma$$

representing the finite binary words $w = i_1 \cdots i_n$, plus the infinite trees

representing the infinite binary words $i_1 i_2 i_3 \cdots$. 

83
3 Finitary Iteration

**Example 3.3.19.** Recursive domain equations involving mutual recursion can often be solved by using polynomial endofunctors of many-sorted sets. For example, suppose that we need sets $X$ and $Y$ satisfying

$$X ≅ X \times Y + 1 \quad \text{and} \quad Y ≅ X + X.$$  

We form the polynomial endofunctor $F$ on $\text{Set} \times \text{Set}$ given by

$$F(X, Y) = (X \times Y + 1, X + X).$$

It clearly preserves colimits of $\omega$-chains, thus, it has an initial algebra which is a colimit of $F^n(0, 0)$. By Lambek’s Lemma 2.2.5 the components of the initial algebra form an (initial) solution of the above recursive equations.

**Theorem 3.3.20.** For every many-sorted signature $\Sigma$ the polynomial endofunctor $H_\Sigma$ has

1. the initial algebra $\mu H_\Sigma = F_\Sigma$, all finite $\Sigma$-trees with the algebra structure given by tree tupling, and
2. the terminal coalgebra $\nu H_\Sigma = T_\Sigma$, all $\Sigma$-trees, with the coalgebra structure given by the inverse of tree tupling.

The proof is completely analogous to that in Proposition 2.2.14 and Theorem 3.3.10.

**Remark 3.3.21.** Kozen [156] introduces coalgebras for a special class of polynomial functors $F$ on the category of many-sorted sets that arise from graphs representing type declarations of a programming language with sum and product types. He also presents an interesting equivalent description of the category of $F$-coalgebras and of the terminal $F$-coalgebra, in particular. However, the type declarations considered do not allow one to capture all polynomial functors on (many-sorted) sets.

### 3.4 Summary of this chapter

We started the chapter with a construction of the initial algebra $\mu F$ of an endofunctor as a colimit of the $\omega$-chain $F^n 0$. In short, $\mu F = \text{colim } F^n 0$. This holds whenever $F$ preserves $\omega$-colimits. We saw many examples. To mention one by way of review, for a polynomial endofunctor $F = H_\Sigma$ on $\text{Set}$, $F^n 0$ can be represented by $\Sigma$-trees of height less than or equal to $n$, and this yields $\mu H_\Sigma = \text{all finite } \Sigma$-trees.

Dually, a construction of the terminal coalgebra $\nu F$ as a limit of the $\omega^{\text{op}}$-chain of iterated application of $F$ to the terminal object 1 is obtained whenever $F$ preserves $\omega^{\text{op}}$-limits. For polynomial functors, this shows that $\nu H_\Sigma$ consists of all $\Sigma$-trees, and for the functor $F$ representing deterministic automata with input alphabet $\Sigma$, we see that $\nu F$ consists of all formal languages over $\Sigma$. Note however, that preservation of $\omega^{\text{op}}$-limits is a much more restrictive condition than preservation of $\omega$-colimits; for example, the finite power-set functor satisfies the latter but not the former.
4 Finitary Set Functors

The focus of Chapters 1-3 has been initial algebras, terminal algebras, and their construction via finite iteration. A lot of the rest of our book builds on the fundamental results in those chapters, and the examples there are critical for our subject. But we pause in this chapter to discuss terminal coalgebras which are not obtainable via finitary iteration.

The first example of this phenomenon is the finite power set functor $P_f: \text{Set} \to \text{Set}$ first seen in Example 1.3.2. We shall see in Example 4.4.5 that the limit of the terminal-coalgebra $\omega^{\text{op}}$-chain from Definition 3.3.1 is not preserved by the functor. Therefore Theorem 3.3.4 does not apply. We present several description of the terminal coalgebra for $P_f$ and of the limit of the terminal-coalgebra $\omega^{\text{op}}$-chain in Section 4.5. More generally, we shall study finitary set functors in this chapter, since this is the natural setting for most of the results. This is the central motivation of the chapter. We shall see several lines of investigation, leading to several descriptions of the terminal coalgebra of finitary set functors.\footnote{For finitary functors, the initial algebra construction from Chapter 3 does apply. We thus have very little to say about initial algebras in this chapter; Proposition 4.3.22 is the only relevant result.}

We start in Section 4.1 with very general material on limits in a base category and a coalgebra category. We continue with weakly terminal coalgebras in Section 4.2: the word “weakly” indicates that we weaken the definition of terminal by dropping the uniqueness of homomorphisms. We also discuss congruences and precongruences. We pass from weakly terminal coalgebras to terminal coalgebras by taking the quotient modulo the largest congruence. This congruence is related to equation presentations of functors, which we study in Section 4.3.

Section 4.4 goes back to the analysis of the terminal-coalgebra $\omega^{\text{op}}$-chain and to an extension of that chain. We present Worrell’s construction of the terminal coalgebra as a limit of “twice” the finitary iterations. That is, we iterate to $\omega + \omega$ rather than $\omega$. We also present a number of results relating the limit of the terminal-coalgebra $\omega^{\text{op}}$-chain to $\nu F$.

4.1 Limits and colimits of algebras and coalgebras

The main material in this chapter begins with Section 4.3. But we have two preliminary sections for use there and elsewhere in the remainder of this book. In this section, we present some basic facts about algebras and coalgebras for arbitrary endofunctors. We prove that the forgetful functor of the category of $F$-coalgebras creates (1) all colimits and (2) all limits that are preserved by $F$.\footnote{For finitary functors, the initial algebra construction from Chapter 3 does apply. We thus have very little to say about initial algebras in this chapter; Proposition 4.3.22 is the only relevant result.}
Recall, e.g. from Mac Lane’s book [164], that a diagram scheme $D$ is a small category, and a functor $G: B \rightarrow A$ is said to create colimits of scheme $D$ provided that for every diagram $D: D \rightarrow B$ the following holds: given a colimit cocone of $GD$ in $A$, say $c_d: GDd \rightarrow C$ ($d \in \text{obj} D$), then there exists a unique cocone of $D$ in $B$ that $G$ maps to $(c_d)$, and, moreover, that cocone is a colimit of $D$ in $B$.

We apply this to the forgetful functor $U: \text{Coalg } F \rightarrow A$, $U(A, \alpha) = A$.

Here, $F: A \rightarrow A$ is any endofunctor. That $U$ creates colimits implies that colimits of $F$-coalgebras are formed on the level of $A$. More precisely, given a diagram $D: D \rightarrow \text{Coalg } F$ of coalgebras with $Dd = (A_d, \alpha_d)$, $U$ creates the colimit of $D$ if for a colimit $c_d: A_d \rightarrow C$ of $UD$ in $A$ we have a unique coalgebra structure $\gamma: C \rightarrow FC$ such that every $c_d$ is a coalgebra homomorphism $c_d: (A_d, \alpha_d) \rightarrow (C, \gamma)$. Moreover, the last cocone is a colimit of $D$ in $\text{Coalg } F$.

**Proposition 4.1.1.** For every endofunctor $F$ on $A$, the forgetful functor $U: \text{Coalg } F \rightarrow A$ creates colimits.

**Proof.** Given a diagram $D: D \rightarrow \text{Coalg } F$ and a colimit cocone $(c_d: A_d \rightarrow C)$ of $UD$ in $A$, we need to find a unique coalgebra structure $\gamma: C \rightarrow FC$ such that each of the following squares commute:

$$
\begin{array}{ccc}
A_d & \xrightarrow{\alpha_d} & FA_d \\
\downarrow{c_d} & & \downarrow{Fc_d} \\
C & \xrightarrow{\gamma} & FC
\end{array}
$$

For that it is sufficient to verify that the family of morphisms $\alpha_d \cdot Fc_d$ forms a compatible cocone of the diagram $UD$; then $\gamma$ is uniquely determined by the universal property of the colimit cocone $(c_d)$. Indeed, for every morphism $h: d \rightarrow d'$ in $D$ we have

$$F_{c_{d'}} \cdot \alpha_{d'} \cdot UDh = Fc_{d'} \cdot F(Uh) \cdot \alpha_d = Fc_d \cdot \alpha_d,$$

using that $Dh: (A_d, \alpha_d) \rightarrow (A_{d'}, \alpha_{d'})$ is a coalgebra homomorphism and that $c_{d'} \cdot UDh = c_d$. Thus we obtain the desired unique coalgebra structure $\lambda$.

It remains to verify the universal property of the cocone $c_d: (A_d, \alpha_d) \rightarrow (C, \gamma)$ in $\text{Coalg } F$. Let $f_d: (A_d, \alpha_d) \rightarrow (B, \beta)$ be a cocone of $D$. Then $f_d: A_d \rightarrow B$ is a cocone of $UD$. Thus there exists a unique morphism $f: C \rightarrow B$ of $A$ with $f \cdot c_d = f_d$ for all $d \in \text{obj } D$. To see that $f$ is a homomorphism, consider the diagram below:

$$
\begin{array}{ccc}
A_d & \xrightarrow{\alpha_d} & FA_d \\
\downarrow{c_d} & & \downarrow{Fc_d} \\
C & \xrightarrow{\gamma} & FC \\
\downarrow{f} & & \downarrow{Ff} \\
B & \xrightarrow{\beta} & FB
\end{array}
$$

(4.1)
The left-hand and right-hand parts, the upper square, and the outside of the diagram clearly commute. Thus so does the lower square when precomposed by every colimit injection $c_d$. Since the colimit injections $c_d$ are collectively epic, this proves that $\alpha$ is a coalgebra homomorphism, as desired. \hfill \Box

**Examples 4.1.2.** (1) As a first (easy) application of Proposition 4.1.1, suppose that $\mathcal{A}$ has coproducts. Given two coalgebras $\alpha: A \to FA$ and $\beta: B \to FB$, there is a unique coalgebra structure on $A + B$ such that the coproduct injections $\text{inl}: A \to A + B$ and $\text{inr}: B \to A + B$ are coalgebra homomorphisms. Indeed, this coalgebra structure is

$$A + B \xrightarrow{\alpha + \beta} FA + FB \xrightarrow{[\text{Fin}, \text{Finr}]} F(A + B).$$

(2) Also, let us mention one application that we shall need in the next section. Suppose that $\mathcal{A}$ has coequalizers, and that we have a parallel pair $f, g: (A, \alpha) \to (B, \beta)$ of coalgebra homomorphisms. Let $c: B \to C$ be the coequalizer in the base category. Then there is a unique coalgebra structure $C \to FC$ so that $c$ becomes a coalgebra homomorphism and (the main point) that we have a coequalizer in $\text{Coalg } F$.

(3) A homomorphism in $\text{Coalg } F$ in epic iff the underlying morphism in epic (in the base category). This follows from the fact that $e$ is epic iff the the pushout along itself is formed by isomorphisms.

We emphasize that creation of colimits does not make any assumptions on the existence of colimits in $\mathcal{A}$, and it holds for all endofunctors and diagrams. But we do have the following easy consequence of Proposition 4.1.1 which is worth stating on its own:

**Corollary 4.1.3.** If $F$ is any endofunction on a cocomplete category $\mathcal{A}$, then $\text{Coalg } F$ is also cocomplete.

**Remark 4.1.4.** Dually, all limits of $F$-algebras are formed on the level of $\mathcal{A}$. More precisely, creation of limits is the dual concept of creation of colimits, and for every endofunctor $F: \mathcal{A} \to \mathcal{A}$ the forgetful functor $\text{Alg } F \to \mathcal{A}$ creates limits.

For limits of coalgebras the situation is more involved, as they are not always formed on the level of $\mathcal{A}$. However, limits which $F$ preserves are:

**Proposition 4.1.5.** For every endofunction $F$ on $\mathcal{A}$ preserving limits of the diagram scheme $\mathcal{D}$, the forgetful functor $U: \text{Coalg } F \to \mathcal{A}$ creates limits of that scheme.

**Proof.** Given is a diagram $D: \mathcal{D} \to \text{Coalg } F$ with $Dd = (A_d, \alpha_d)$ and a limit of $UD$ with limit projections $\pi_d: L \to A_d$ ($d \in \text{obj } \mathcal{D}$).

We need to prove that there exists a unique coalgebra structure $\lambda: L \to FL$ making each $\pi_d$ a homomorphism.

$$\begin{array}{ccc}
L & \xrightarrow{\lambda} & FL \\
\pi_d & \downarrow & \downarrow \text{Fin} \\
A_d & \xrightarrow{\alpha_d} & FA_d
\end{array}$$
and that, moreover, $\pi_d: (L, \lambda) \to (A_d, \alpha_d)$ form the limit of $D$. Since $F$ preserves the limit of $UD$, the diagram $FUD$ has the limit cone given by the morphisms $F\pi_d: FL \to FA_d$, $d \in \text{obj } D$. We show that $\alpha_d \cdot \pi_d$ form a compatible cone of that diagram. Indeed, for every morphism $f: d \to d'$ of $D$ we have

$$FUDh \cdot \alpha_d \cdot \pi_d = \alpha_d' \cdot UDh \cdot \pi_d = \pi_d' \cdot \alpha_d' \cdot \pi_d'$$

using that $Df: (A_d, \alpha_d) \to (A_d', \alpha_d')$ is a coalgebra homomorphism and that $UDh \cdot \pi_d = \pi_d'$. Thus we obtain the desired unique coalgebra structure $\lambda$.

It remains to verify that the cone $\pi_d: (L, \lambda) \to (A_d, \alpha_d)$ is a limit of $D$ in $\text{Coalg } F$. Let $f_d: (B, \beta) \to (A_d, \alpha_d)$ be a cone of $D$. Then $f_d: B \to A_d$ is a cone of $UD$. Thus there exists a unique morphism $f: B \to L$ such that $\pi_d \cdot f = f_d$ for all $d \in \text{obj } D$. To see that $f$ is a homomorphism, we argue dually as for Diagram (4.1). Consider the following diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{\lambda} & FL \\
\downarrow & & \downarrow Ff \\
B & \xrightarrow{\beta} & FB \\
\downarrow \pi_d & & \downarrow F\pi_d \\
A_d & \xrightarrow{\alpha_d} & FA_d \\
\end{array}
\]

and use that the family of morphisms $F\pi_d$ is collectively monic.

**Example 4.1.6.** Let us underscore the need for $F$ to preserve the limit in Proposition 4.1.5. Consider $FX = X + 1$ on $\text{Set}$. The limit of the empty diagram is the terminal object, $1$. Yet the terminal coalgebra is not of the form $1 \to F1$.

**Remark 4.1.7.** Dually, colimits of algebras are not always formed on the level of $\mathcal{A}$. However, for every endofunctor $F$ preserving colimits of a diagram scheme $D$, the forgetful functor $\text{Alg } F \to \mathcal{A}$ creates colimits of that diagram.

### 4.2 Weakly terminal coalgebras

Let $F: \mathcal{A} \to \mathcal{A}$ be an endofunctor. An $F$-coalgebra $(A, \alpha)$ is **weakly terminal** if for every coalgebra $(B, \beta)$ there is at least one homomorphism $(B, \beta) \to (A, \alpha)$. Such coalgebras are the topic of this section. It turns out that the terminal coalgebra for a functor $F$ is the quotient of (any) weakly terminal coalgebra. A beautiful example is the coalgebra of all unordered finitely branching trees, which is weakly terminal for $\mathcal{P}_f$.

**Examples 4.2.1.** (1) The terminal coalgebra for the identity functor $\text{Id}: \text{Set} \to \text{Set}$ is the singleton coalgebra. A coalgebra $\alpha: A \to A$ is weakly terminal iff $\alpha$ has a fixed point.

(2) If $(A, \alpha)$ is a weak terminal coalgebra and $f: (A, \alpha) \to (B, \beta)$ a coalgebra homomorphism, then $(B, \beta)$ is again weakly terminal.

(3) We next consider the finite power-set functor $\mathcal{P}_f: \text{Set} \to \text{Set}$. Recall from Example 2.4.2(4) that coalgebras for $\mathcal{P}_f$ are the finitely branching graphs. We consider the set
4.2 Weakly terminal coalgebras

$D$ of all finitely branching trees; these trees are rooted, and unordered, and as always we take trees up to isomorphism. This insures that the trees under consideration indeed form a set rather than a proper class. We endow $D$ with the coalgebra structure $\delta: D \to \mathcal{P}_1 D$ which takes a tree $t$ to the (finite) set of all maximum proper subtrees of $t$.

To clarify matters, here is an example. Let $b \in D$ be the full infinite binary tree. Then in the coalgebra $D$, $b$ is its own child; in fact, it is its only child: $\delta(b) = \{b\}$. The same remark applies to the infinite chain $u$, or to any infinite tree having the same number of children at every node.

We argue that the coalgebra $(D, \delta)$ is weakly terminal for $\mathcal{P}_f$. Every finitely branching graph, considered as a coalgebra $(A, \alpha)$, admits a canonical coalgebra homomorphism to $(A, \alpha) \to (D, \delta)$: it assigns to every node $x \in A$ the tree unfolding of $x$, which we denote by $A_x$. The nodes of this tree are the (finite) paths $\overrightarrow{p} = p_0 \to p_1 \to \cdots \to p_n$ starting at $x$ and following the edges in $A$. We have an edge from $\overrightarrow{p}$ to $\overrightarrow{q}$ in $A_x$ iff $\overrightarrow{q}$ extends the path $\overrightarrow{p}$ by an edge of $A$:

The root of the tree $A_x$ is given by the node $x$, considered as a path of length 0. For example, if $A = 1$ consists of a single loop, then $A_x$ is an infinite path. Note that the trees $A_y$, for edges $x \to y$ in $A$ are precisely the maximum proper subtrees of $A_x$:

\[
\{t: t \in \delta(A_x) \text{ in } D\} = \{A_y: y \in \alpha(x) \text{ in } A\}.
\]

This states that $x \mapsto A_x$ is a coalgebra homomorphism $(A, \alpha) \to (D, \delta)$.

However, $(D, \delta)$ is not terminal: $1$ has many homomorphisms to $(D, \delta)$; for example we can map it to either of the infinite path or the complete binary tree. In Example 4.2.10 below, we shall see a quotient of $D$ which is a terminal $\mathcal{P}_f$-coalgebra.

(4) The power-set functor $\mathcal{P}: \text{Set} \to \text{Set}$ does not have a weakly terminal coalgebra. Indeed, we have seen that $\mathcal{P}$ has no terminal coalgebra, and from a weakly terminal coalgebra we can always construct a terminal one – hence a fixed point – as we prove in Theorem 4.2.8.

**Notation 4.2.2.** Let $f: X \to Y$ in Set. The kernel equivalence of $f$ is the equivalence relation

\[\ker(f) = \{(x, x'): f(x) = f(x')\} \subseteq X \times X.\]

Categorically speaking $\ker(f)$ is the (domain of the) pullback of $f$ along itself.

**Remark 4.2.3.** (1) Dually to Remark 2.1.15, quotient objects of an object $A$ of a category $\mathcal{A}$ are represented (uniquely up to isomorphism of the codomain) by epimorphisms $e: A \to \bar{A}$.

(2) For an object $A$ in a category $\mathcal{A}$, the quotients of $A$ are ordered by factorization: given quotients $e: A \to \bar{A}$ and $f: A \to A^*$ we write $e \leq f$ iff $e$ factorizes through $f$:

\[
\begin{array}{ccc}
A & \xrightarrow{e} & \bar{A} \\
\downarrow f & & \downarrow \\
& A^* \\
\end{array}
\]
Thus for $\mathcal{A} = \text{Set}$ the quotients of a set $A$ are ordered by their cardinality as usual, but equivalence relations on $A$, representing quotients are ordered dually: $A/\sim$ is smaller or equal to $A/\approx$ iff $\approx$ is contained in $\sim$. (The smallest quotient thus has codomain 1 whenever $A \neq \emptyset$.)

(3) Given a pushout of epimorphisms $f_1, f_2$:

\[
\begin{array}{ccc}
A & \xrightarrow{f_1} & A_1 \\
\downarrow{f_2} & & \downarrow{g_1} \\
A_2 & \xrightarrow{g_2} & B
\end{array}
\]

it follows that $g_1, g_2$ are epimorphisms, too. Thus, the quotient of $A$ represented by $e = g_i \cdot f_i : A \rightarrow B$ is the meet of $f_1$ and $f_2$ in the poset of quotients of $A$.

(4) More generally, a meet of a collection of quotients $f_i : A \rightarrow A_i (i \in I)$ is represented by their wide pushout.

Remark 4.2.4. We apply Remark 4.2.3(1) to $\text{Coalg} F$: a quotient coalgebra is represented by a homomorphism carried by an epimorphism in the base category (cf. Example 4.1.2(3)).

In other words, a quotient $e : A \rightarrow \bar{A}$ in the base category $\mathcal{A}$ represents a quotient coalgebra of $(A, \alpha)$ iff there exists a (necessarily unique) coalgebra structure $\bar{\alpha} : \bar{A} \rightarrow F\bar{A}$ making $e$ a coalgebra homomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow{e} & & \downarrow{Fe} \\
A & \xrightarrow{\bar{\alpha}} & F\bar{A}
\end{array}
\]

In the case where $\mathcal{A} = \text{Set}$, we often identify the epimorphism $e$ with its kernel equivalence. The kernel equivalence of a quotient coalgebra structure is called a congruence. We write $A/\sim$ for the coalgebra $(\bar{A}, \bar{\alpha})$ above.

Example 4.2.5. Let $G$ be a finitely branching graph, considered as a coalgebra for $\mathcal{P}_1$. A congruence on $G$ is then precisely an equivalence relation on $G$ which is also a graph bisimulation (cf. Example 2.6.7(4)). This is an equivalence relation $R \subseteq G \times G$ such that whenever $xRy$, then

\[
\text{for every child } x' \text{ of } x \text{ there is some child } y' \text{ of } y \text{ such that } x'Ry', \text{ and}
\]

\[
\text{for every child } y' \text{ of } y \text{ there is some child } x' \text{ of } x \text{ such that } x'Ry'. \tag{4.2}
\]

Remark 4.2.6. We will not be using the general concept of a bisimulation on an $F$-coalgebra in this section. But we remind readers familiar with the concept that whenever $F$ preserves weak pullbacks, the largest bisimulation on a coalgebra is always an equivalence relation, and hence a congruence (see Rutten [208, Corollary 5.6]).

Remark 4.2.7. (1) Quotient coalgebras of $(A, \alpha)$ are ordered as shown in Remark 4.2.3(2). For quotient coalgebras $e : (A, \alpha) \rightarrow (\bar{A}, \bar{\alpha})$ and $f : (A, \alpha) \rightarrow (A^*, \alpha^*)$ we write $e \leq f$ if
4.2 Weakly terminal coalgebras

$e$ factorizes through $f$; i.e. there is $g: A^* \to \bar{A}$ with $e = g \cdot f$. It follows that $g$ is a coalgebra homomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow{f} & & \downarrow{Ff} \\
A^* & \xrightarrow{\alpha^*} & FA^* \\
\downarrow{g} & & \downarrow{Fg} \\
\bar{A} & \xleftarrow{\bar{\alpha}} & F\bar{A}
\end{array}
\]

Indeed, the lower square commutes because it does when precomposed with the epimorphism $f$.

(2) Given a coalgebra $(A, \alpha)$, a meet of its quotient coalgebras (in the poset of quotients of $A$ in the base category) is a quotient coalgebra (see Remark 4.2.3(4)).

**Theorem 4.2.8.** Let $F$ be an endofunctor on a cocomplete and co-well-powered category. Let $(A, \alpha)$ be a weakly terminal $F$-coalgebra. Then there is a smallest quotient coalgebra $e: (A, \alpha) \to (\bar{A}, \bar{\alpha})$, and $(\bar{A}, \bar{\alpha})$ is a terminal coalgebra for $F$.

**Proof.** Since the base category $\mathcal{A}$ is co-well-powered, $A$ has only a set of quotients, and the corresponding lattice is complete, because meets exist via wide pushouts.

By Remark 4.2.7(2), the meet of all quotient coalgebras (in the poset of quotients of $A$) is a quotient coalgebra, and this is the smallest quotient coalgebra $e: (A, \alpha) \to (\bar{A}, \bar{\alpha})$. Moreover, $(\bar{A}, \bar{\alpha})$ is obviously, a weakly terminal coalgebra (since $(A, \alpha)$ is, cf. Example 4.2.1(2)). It remains to prove that if $f, g: B \to \bar{A}$ are two coalgebra homomorphisms, then $f = g$. To this end, take the coequalizer $k$ of $f$ and $g$ in $\mathcal{A}$. Then since $f$ and $g$ are coalgebra homomorphisms, so is $k$ (see Example 4.1.2). Being a coequalizer, $k$ is an epimorphism. Thus, $k \cdot e$ is a quotient coalgebra of $(A, \alpha)$. But the choice of $e$ as the least quotient coalgebra implies that $e \leq k \cdot e$. So $e = x \cdot k \cdot e$ for some $x$. It follows that $x \cdot k = \text{id}$; thus $k$ is an epi and a split mono, whence an isomorphism. Therefore $f = g$, as desired. \qed

**Remark 4.2.9.** Note that this argument is a special instance of the (dual of) the classical characterization of the existence of an initial object [164, Theorem V.6.1] which is at the heart of Freyd’s Adjoint Functor Theorem.

**Example 4.2.10.** The terminal coalgebra for $\mathcal{P}_t$ is the quotient $D/\approx$ of the coalgebra $D$ of all finitely branching trees (see Example 4.2.1(3)) modulo the greatest congruence. To describe $\approx$, recall from Example 3.2.9 that the terminal chain is given by

$\mathcal{P}_t^n 1 = \text{all extensional trees of height } \leq n$.

Given a tree $t \in D$, considered as a coalgebra (see Example 2.4.4), the canonical cone from $t$ to $\mathcal{P}_t^n 1$ of Construction 3.3.2 assigns to $t$ the following tree $\partial_n t$. First we take the cutting of $t$ at level $n$. This is the set of nodes whose depth is at most $n$, considered as an induced subgraph of $t$. Then we take the extensional quotient $\partial_n t$ of this cutting. We
show a tree $t$ and four of its cuttings below:

```
  t

  \partial_0 t  \partial_1 t  \partial_2 t  \partial_3 t  \ldots
```

We mention in Example 4.3.27 that the greatest congruence $\approx$ of $D$ is defined by $t \approx s$ iff $\partial_n t = \partial_n s$ holds for all $n$.

**Example 4.2.11.** Here is another weakly terminal coalgebra for $\mathcal{P}_I$, due to Aczel [3].

A pointed graph, i.e. a pair $(G, g)$ consisting of a graph $G$ and a node $g$ in it, is called *reachable* if every node $x$ admits a path from $g$ to $x$. Let $W$ be a set of reachable pointed graphs representing all such graphs up to (point preserving) isomorphism. This set $W$ has a $\mathcal{P}_I$-coalgebra structure $\alpha: W \to \mathcal{P}_I W$ assigning to every pointed graph $(G, g)$ the set of all pointed graphs $(H, h)$ where $g \to h$ is an edge of $G$ and $H$ is the subgraph of $G$ determined by the set of nodes of $G$ reachable from $h$. Every coalgebra $A$ for $\mathcal{P}_I$ has a homomorphism into $W$ assigning to every node $x$ the representative of the reachable subgraph of $A$ determined by $x$. Thus, $\nu \mathcal{P}_I$ is the quotient coalgebra $W/\sim$ for the greatest congruence on this coalgebra $W$. This is given by $(G, g) \sim (H, h)$ iff there is a graph bisimulation between the graphs $G$ and $H$ which relates $g$ to $h$.

At this point, we have presented a construction of the terminal coalgebra that differs from the iterative construction which we saw in Chapter 3. Although taking a quotient of a weakly terminal coalgebra seems easier than taking a limit, we frequently want to have information about the terminal coalgebra which is not readily available from its description as a quotient. This is addressed in the next section, when we turn to presentations of finitary set functors and their terminal coalgebras.

### 4.3 Presentation of finitary set functors

In this section we turn to the main notion of this chapter, *finitary* endofunctors on $\text{Set}$. Intuitively, a functor on sets is finitary if its behaviour is completely determined by its action on *finite* sets and functions. For a general functor, this intuition is captured by the following definition:

**Definition 4.3.1.** A directed diagram in a category $\mathcal{A}$ is a diagram $D: \mathcal{D} \to \mathcal{A}$ where $\mathcal{D}$ is a directed poset, i.e. $\mathcal{D}$ is nonempty and every pair of elements has an upper bound.

A functor is called *finitary* if it preserves directed colimits, i.e. colimits of directed diagrams.
4.3 Presentation of finitary set functors

Remark 4.3.2. (1) Equivalently, filtered colimits can be used in lieu of directed ones, see Theorem B.3.1.
(2) Note that \(\omega\)-chains are a special kind of directed diagram. Hence, every finitary functor preserves \(\omega\)-colimits (cf. Notation 3.1.4).
(3) In the case where \(\mathcal{A} = \text{Set}\), a more intuitive and equivalent concept is finite boundedness defined as follows.

Definition 4.3.3 [30]. A set functor \(F\) is finitely bounded if for each set \(X\) and each \(x \in FX\), there exists a finite subset \(M \subseteq X\) such that \(x \in Fi[M]\), where \(i: M \hookrightarrow X\) is the inclusion map.

Proposition 4.3.4 [30, Cor. 3.3]. A set functor is finitely bounded iff it is finitary.

Examples 4.3.5. (1) Constant functors and the identity functor are clearly finitary.
(2) Finitary set functors are closed under taking coproducts and colimits, composition, and finite products.
(3) It follows that polynomial set functors \(H_{\Sigma}\) for (finitary) signatures \(\Sigma\) are finitary.
(4) The finite power-set functor \(P_f\) is finitary, but the full power-set functor \(P\) is not (cf. Example 4.3.13).

Just as quotient objects of an object \(A\) are represented by epimorphisms with domain \(A\), quotient functors of a given endofunctor \(H\) are represented by natural epi-transformations with domain \(H\). That is, natural transformations with all components epic. We use the notation \(\varepsilon: H \rightarrow F\) to say that \(F\) is a quotient functor of \(H\) via \(\varepsilon\). An important special case are the set functors that are quotients of a polynomial functor \(H_{\Sigma}\) associated signature \(\Sigma\) (see Definition 2.1.4): these functors have an equational presentation using the signature \(\Sigma\).

The main results of this section deal with functors \(F\) for which there is an epitransformation from a polynomial functor, so we have \(\varepsilon: H_{\Sigma} \rightarrow F\). This turns out to be an important situation to study, and the present chapter is even named for it.

The following was proved for set functor by Gumm and Schröder [122, Lemma 2.3(iii)]

Lemma 4.3.6. Let \(H\) be an endofunctor with a weakly terminal coalgebra \(\tau: T \rightarrow HT\). Let \(\varepsilon: H \rightarrow F\) be a quotient functor, and assume that every component \(\varepsilon_X\) is a split epimorphism. Then \(F\) has the following weakly terminal coalgebra:

\[
T \xrightarrow{\tau} HT \xrightarrow{\varepsilon_T} FT.
\]

Proof. Let \(\alpha: A \rightarrow FA\) be any \(F\)-coalgebra. Since \(\varepsilon_A: HA \rightarrow FA\) is a split epimorphism, we can choose a morphism \(m: FA \rightarrow HA\) such that \(\varepsilon_A \cdot m = \text{id}\). So we obtain an \(H\)-coalgebra \(m \cdot \alpha: A \rightarrow HA\). We thus have an \(H\)-coalgebra homomorphism \(h: (A, m \cdot \alpha) \rightarrow (T, \tau)\), and we show that \(h\) is also an \(F\)-coalgebra homomorphism from \((A, \alpha)\) to \((T, \varepsilon_T \cdot \tau)\).
Indeed, consider the diagram below:

\[
\begin{array}{ccc}
A & \xrightarrow{m \cdot \alpha} & HA & \xrightarrow{\varepsilon_A} & FA \\
\downarrow h & & \downarrow Hh & & \downarrow Fh \\
T & \xrightarrow{\tau} & HT & \xrightarrow{\varepsilon_T} & FT
\end{array}
\]

The top commutes because \(\varepsilon_A \cdot m = \text{id}\), the square on the left does by definition of \(h\), and the one on the right by naturality of \(\varepsilon\). Thus the outside commutes, showing \(h\) to be an \(F\)-coalgebra homomorphism.

**Definition 4.3.7.** By a *presentation* of a set functor \(F\) we mean a signature \(\Sigma\) and a natural epi-transformation \(\varepsilon: H\Sigma \to F\). If \(\Sigma\) is *finitary* (see Definition 2.1.4) we call the presentation *finitary*.

**Remark 4.3.8.** Note that every natural transformation \(\varepsilon: H \to F\) induces a functor \(\text{Coalg } H \to \text{Coalg } F\) by post-composition:

\[(A \xrightarrow{\alpha} HA) \mapsto (A \xrightarrow{\alpha} HA \xrightarrow{\varepsilon A} FA).\]

In particular, for every presentation \(\varepsilon: H\Sigma \to F\) a coalgebra \((A, \alpha)\) for \(H\Sigma\) can be considered as the coalgebra \((A, \varepsilon_A \cdot \alpha)\) for \(F\).

**Corollary 4.3.9.** For every set functor \(F\) with a presentation \(\varepsilon: H\Sigma \to F\), the terminal coalgebra for \(H\Sigma\) (of all ordered \(\Sigma\)-trees) is weakly terminal for \(F\). Consequently, \(F\) has a terminal coalgebra given by the smallest quotient coalgebra of the \(F\)-coalgebra of all \(\Sigma\)-trees.

Indeed, this follows from Lemma 4.3.6 and Theorem 4.2.8.

**Remark 4.3.10.** Corollary 4.3.9 above stems from Barr [58] and also Gumm and Schröder [122, Lem. 2.3(iv)]. Both consider a more general situation: an epi-transformation \(\varepsilon: H \to F\) where \(H\) is any functor which has a terminal coalgebra (cf. Lemma 4.3.6). We use this in Example 4.3.11 below.

**Example 4.3.11.** The coalgebra \(D\) of all unordered finitely branching trees is weakly terminal for \(\mathcal{P}_f\) (see Example 4.2.1(3)), and here is a different proof. Let \(\mathcal{B}\) be the bag functor (see Example 3.2.10). There is an epi-transformation \(\varepsilon: \mathcal{B} \to \mathcal{P}_f\) mapping a bag to the set of its elements. Naturality is easy to see, as is the surjectivity of each \(\varepsilon_X\). As a splitting of any \(\varepsilon_X\) choose the map that maps a finite subset of \(S \subseteq X\) to the bag of \(S\) (each occurring once). Now by Lemma 4.3.6, we obtain that \(D\), being the terminal coalgebra for \(\mathcal{B}\), is weakly terminal for \(\mathcal{P}_f\).

In addition, we can also apply Lemma 4.3.6 to the presentation of \(\mathcal{P}_f\) in Example 4.3.15(2) to see that the algebra \(T\Sigma\) of all ordered finitely branching trees is weakly terminal.

**Proposition 4.3.12.** Every finitary set functor has an initial algebra and a terminal coalgebra.
This result highlights our interest in finitary functors in this chapter. The initial algebra part comes in Proposition 4.3.22, and the terminal coalgebra result in Theorem 4.3.26.

**Example 4.3.13** [45, Prop. III.4.3]. For a signature $\Sigma$ the polynomial functor $H_\Sigma$ is finitary if $\Sigma$ is a finitary signature. (This last condition means that all of the arities in $\Sigma$ are finite. But $\Sigma$ might have infinitely many operation symbols, see Definition 2.1.4.) The finite power set functor $P_f$ is finitary, but the full power set functor $P$ is not.

**Proposition 4.3.14.** A set functor has a finitary presentation iff it is finitely bounded.

*Proof.* (1) Let $F$ be a finitely bounded functor and let $\Sigma$ be the finitary signature defined by

$$\Sigma_n = Fn \quad (n < \omega)$$

(where $n$ is, as usual, considered to be the set $\{0, 1, \ldots, n-1\}$ for every $n < \omega$). Define a natural transformation $\varepsilon: H_\Sigma \rightarrow F$ to have the following components

$$\varepsilon_X: \prod_{n<\omega} \Sigma_n \times X^n \rightarrow FX :$$

given a pair $(\sigma, f) \in \Sigma_n \times X^n$, the function $f: n \rightarrow X$ yields a function $Ff: Fn \rightarrow FX$ and we put

$$\varepsilon_X(\sigma, f) = Ff(\sigma).$$

Let us verify the naturality square for an arbitrary function $h: X \rightarrow Y$:

$$\prod_{n<\omega} \Sigma_n \xrightarrow{\varepsilon_X} FX \quad \prod_{n<\omega} \Sigma_n \xrightarrow{id \times h^n} \prod_{n<\omega} \Sigma_n \times Y^n \xrightarrow{\varepsilon_Y} FY$$

The upper passage takes $(\sigma, f)$ to $Fh(Ff(\sigma)) = F(h \cdot f)(\sigma)$. The lower one takes it to $\varepsilon_Y(\sigma, h \cdot f) = F(h \cdot f)(\sigma)$.

Moreover, $\varepsilon_X$ is an epi-transformation since for every element $x \in FX$ there exists a natural number $n$ and an $n$-element subset $i: n \rightarrow X$ such that $x$ lies in the image of $Fi$. Let $\sigma \in Fn$ be such that $x = Fi(\sigma)$. Then we have

$$x = \varepsilon_X(\sigma, i).$$

Thus $\varepsilon$ is a presentation of $F$.

(2) If $\varepsilon: H_\Sigma \rightarrow F$ is a finitary presentation of $F$, then we prove that $F$ is a finitely bounded functor. Fix a set $X$. For every element $x \in FX$, choose $y \in H_\Sigma X$ with $\varepsilon_X(y) = x$. Then $y$ lies in the component $X^n$ of $H_\Sigma X$ corresponding to some $n$-ary operation $\sigma$ of $\Sigma$. Use the naturality square on $y: n \rightarrow X$:
Appendix B. For a presentation of

\[ H \]

where \( H \) denotes the element of 1. Our main interest in this functor is that it provides interesting (counter)examples; see Appendix B. For a presentation of (−)\(^3\) we take \( \Sigma \) to have three binary symbols, say \( \sigma, \tau, \) and \( \varrho \). Then we define \( \varepsilon : H_\Sigma \rightarrow (−)\(^3\) \) by:

\[ \varepsilon_X(\sigma(a,b)) = (a, a, b), \varepsilon_X(\tau(a,b)) = (a, b, a), \text{ and } \varepsilon_X(\varrho(a,b)) = (b, a, a). \]

(5) The countable power-set functor \( \mathcal{P}_c \) (of all countable subsets) is not finitary. But it has an infinitary presentation given by an \( \omega \)-ary operation \( \sigma \) and a constant \( c \). The natural transformation \( \varepsilon : H_\Sigma \rightarrow \mathcal{P}_c \) has components \( \varepsilon_X \) given by \( (x_i)_{i \in \mathbb{N}} \mapsto \{x_i\}_{i \in \mathbb{N}} \) and \( c \mapsto \emptyset \).
4.3 Presentation of finitary set functors

Remark 4.3.16. The presented functor \( F \) is determined (up to natural isomorphism) by the kernel equivalences of \( \varepsilon_X : H_\Sigma X \to FX \) for all sets \( X \). We can describe the kernel equivalence of \( \varepsilon_X \) as a set of pairs of elements of \( H_\Sigma X = \prod_{n\in\mathbb{N}} \Sigma_n \times X^n \) (see Definition 2.1.4). Each such element is, for some \( \sigma \in \Sigma_n \), an \( (n+1) \)-tuple \((\sigma, x_0, \ldots, x_{n-1})\). We write this as the term \( \sigma(x_0, \ldots, x_{n-1}) \), and we write the pairs contained in the kernel equivalence of \( \varepsilon_X \) as term equations

\[
\sigma(x_0, \ldots, x_{n-1}) = \tau(y_0, \ldots, y_{m-1}),
\]

where \( \sigma \in \Sigma_n \), \( \tau \in \Sigma_m \) and all \( x_i \) and \( y_j \) are elements of \( X \). These equations have the following special form:

Definition 4.3.17. A term over \( \Sigma \) is called basic if it contains precisely one symbol from \( \Sigma \). That is, it has the form \( \sigma(x_0, \ldots, x_{n-1}) \) for some \( \sigma \in \Sigma_n \) and \( n \) variables \( x_i \). The variables might contain repeats.

A basic equation over \( X \) is a pair of basic terms with variables from the set \( X \).

Note that \( H_\Sigma X \) is thus the set of all basic terms over \( X \).

Notation 4.3.18. Fix a countably infinite set \( V \) whose elements are called variables. Let \( \Sigma \) be a signature, and let \( \mathcal{E} \) be a set of basic equations over \( V \). We define a quotient functor \( H_\Sigma /\mathcal{E} \) of \( H_\Sigma \) as follows: given a set \( X \), form the smallest equivalence \( \equiv_{\varepsilon_X} \) on \( H_\Sigma X \) such that

\[
\sigma(s(v_0), \ldots, s(v_{n-1})) \equiv_{\varepsilon_X} \tau(s(w_0), \ldots, s(w_{m-1})),
\]

for every \( s : V \to X \) and \( \sigma(x_0, \ldots, x_{n-1}) = \tau(y_0, \ldots, y_{m-1}) \) in \( \mathcal{E} \). In words, we think of \( s \) as a substitution of elements of \( X \) for the variables in \( V \), and we identify all substitution instances of all pairs of basic equations. The functor \( H_\Sigma /\mathcal{E} \) maps a set \( X \) to the set \( H_\Sigma X /\equiv_{\varepsilon_X} \) of equivalence classes of basic terms over \( X \). To a function \( f : X \to Y \) it assigns the function defined by

\[
[\sigma(s(v_0), \ldots, s(v_{n-1}))] \mapsto [\sigma(f(s(v_0)), \ldots, f(s(v_{n-1}))]].
\]

Proposition 4.3.19 [6, Thm. 3.9]. Every finitary set functor \( F \) can be presented by a finitary signature \( \Sigma \) and a set \( \mathcal{E} \) of basic equations in the sense that \( F \) and \( H_\Sigma /\mathcal{E} \) are naturally isomorphic functors.

Examples 4.3.20. (1) The functor \( \mathcal{P}_3 \) is presented by the signature \( \Sigma \) with one constant \( c \) and one binary operation symbol \( \ast \) modulo the equation \( x \ast y = y \ast x \) expressing commutativity of \( \ast \).

(2) \( \mathcal{P}_f \) is presented by the signature \( \Sigma = \{\sigma_n\}_{n<\omega} \) with \( \sigma_n \) of arity \( n \)(see Example 4.3.15(2)), and all basic equations

\[
\sigma_n(x_0, \ldots, x_{n-1}) = \sigma_m(y_0, \ldots, y_{m-1})
\]

for which the sets \( \{x_0, \ldots, x_{n-1}\} \) and \( \{y_0, \ldots, y_{m-1}\} \) are equal. For example, one of the equations is \( \sigma_2(x, y) = \sigma_3(x, y, x) \).
4 Finitary Set Functors

(3) Let \( FX = \coprod_{\sigma \in \Sigma} X^k / G_k \) be an analytic functor (see Example 3.2.11), where \( k \) is the arity of \( \sigma \) and \( G_k \) is the given group of permutations on \( k \). Then \( F \) is presented by the signature \( \Sigma \) and the following basic equations:

\[
\sigma(x_1, \ldots, x_k) = \sigma(x_{p(1)}, \ldots, x_{p(k)}) \quad \text{for all } \sigma \in \Sigma \text{ and all } p \in G_\sigma.
\]

(4) We have seen the Aczel-Mendler \((-)^3_2\) functor in Example 4.3.15(4). This functor is presented by the two basic equations

\[
\sigma(x, x) = \tau(x, x) \quad \text{and} \quad \tau(x, x) = \varrho(x, x).
\]

We now consider algebras of set functors in connection with functor presentations.

**Remark 4.3.21.** (1) Given a presentation \( F \cong H_{\Sigma}/\mathcal{E} \), the category of \( F \)-algebras is equivalent to the variety of all \( \Sigma \)-algebras satisfying \( \mathcal{E} \) in the usual sense.

(2) Conversely, every variety of \( \Sigma \)-algebras presented by basic equations is equivalent to \( \text{Alg} F \) for some finitary set functor \( F \). This is proved by Adámek and Trnková [45, Chapter III].

Given a presentation \( \varepsilon: H_{\Sigma} \to F \) of a set functor \( F \), we can consider the category \( \text{Alg} F \) of all algebras for \( F \) as a full subcategory of \( \text{Alg} H_{\Sigma} \), following Remark 4.3.21(1).

More precisely, we have a full embedding assigning to every algebra \( \alpha: FA \to A \) for \( F \) the corresponding \( \Sigma \)-algebra \( \alpha \cdot \varepsilon_A: H_{\Sigma}A \to A \) and defined on homomorphisms by \( f \mapsto f \).

Analogously to quotient coalgebras (see Remark 4.2.4) we have the concept of quotient algebra of \( \alpha: FA \to A \): it is an epimorphism \( e: A \to \bar{A} \) in \( \mathcal{A} \) for which an algebra structure \( \bar{\alpha} \) exists making it a homomorphism:

\[
\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
\downarrow{Fe} & & \downarrow{e} \\
F \bar{A} & \xrightarrow{\bar{\alpha}} & \bar{A}
\end{array}
\]

Since all set functors preserve epimorphisms, \( Fe \) is an epimorphism, and hence the structure morphism \( \bar{\alpha} \) is unique.

**Proposition 4.3.22** [25]. Given a presentation \( \varepsilon: H_{\Sigma} \to F \), the initial algebra \( \mu F \) is the largest quotient of the initial \( \Sigma \)-algebra lying in \( \text{Alg} F \).

More concretely,

\[
\mu F = \mu H_{\Sigma}/\sim,
\]

where for two finite \( \Sigma \)-trees \( s, t \) we have

\[
s \sim t \quad \text{iff } s \text{ and } t \text{ can be equated by finitely many applications of the basic equations for } \varepsilon.
\]

This means that \( s = t \) is derivable using the standard rules of equational logic.
Example 4.3.23. The initial algebra for $\mathcal{P}_3$ is the quotient of $\mu X. (X \times X + 1)$, the algebra of all finite binary ordered trees of Example 2.2.15, modulo the least congruence merging trees $x * y$ and $y * x$. Consequently, the linear ordering of children is simply forgotten:

$$\mu \mathcal{P}_3 = \text{all finite binary unordered trees.}$$

Remark 4.3.24. Given a presentation $\varepsilon: H_\Sigma \rightarrow F$ of a finitary set functor, then a similar description holds for the terminal coalgebra [25, Thm. 3.15]; we have

$$\nu F = T_\Sigma/\approx,$$

where $\approx$ is the congruence of finite and infinite applications of the basic equations. Of course, we have to make clear what is meant by “infinite application” of basic equations:

Notation 4.3.25. Recall from Theorem 3.3.10 that the limit projection $\ell_n: T_\Sigma = \nu H_\Sigma \rightarrow H_{\Sigma + 1}$ assigns to every $\Sigma$-tree $t$ the cutting of $t$ at level $n$, where all leaves of that level are relabelled by $\bot$. Thus $\ell_n(s)$ is a finite $\Sigma_\bot$-tree where $\Sigma_\bot$ is the signature $\Sigma$ with an additional nullary symbol $\bot$. Given a presentation $\varepsilon: H_\Sigma \rightarrow F$, we define an equivalence relation $\approx$ on $T_\Sigma$ as follows: for two $\Sigma$-trees $s$ and $t$ we put

$$s \approx t \quad \text{iff for all } n < \omega \text{ we have } \ell_n(s) \sim \ell_n(t). \quad (4.4)$$

Here $\sim$ is the congruence of (4.3) for the quotient functor $\varepsilon_X + \{\bot\}: H_{\Sigma + 1}X \rightarrow FX + \{\bot\}$.

Recall from Corollary 4.3.9 that the terminal coalgebra $\tau: T_\Sigma \rightarrow H_\Sigma T_\Sigma$ (of all ordered $\Sigma$-trees) is a weakly terminal coalgebra for $F$.

Theorem 4.3.26 [25]. Given a presentation $\varepsilon: H_\Sigma \rightarrow F$ of a finitary set functor $F$, the terminal coalgebra of $F$ is a quotient of the $F$-coalgebra $T_\Sigma$ of all $\Sigma$-trees, modulo the equivalence $\approx$ above; in symbols:

$$\nu F = T_\Sigma/\approx.$$ 

Example 4.3.27. For the finite power-set functor $\mathcal{P}_f$ consider the presentation

$$\varepsilon_X: X^* \rightarrow \mathcal{P}_f X$$

of Example 4.3.15. Since $\Sigma$ has one $n$-ary operation for every $n$, we know that the terminal coalgebra $T_\Sigma$ is (isomorphic to) the coalgebra of all finitely branching ordered trees. For finite trees $t$ and $u$, the equivalence $\sim$ of (4.3) is easily seen to be

$$t \sim u \quad \text{iff the extensional quotients of } t \text{ and } u \text{ are equal.}$$

(Recall our convention that isomorphic trees are identified.) By applying (4.4) we conclude that for arbitrary trees $t$ and $u$ in $T_\Sigma$ we have in the notation of Example 4.2.10

$$t \approx u \quad \text{iff } \partial_n t = \partial_n u \text{ for all } n < \omega.$$ 

We conclude that $\nu \mathcal{P}_f = T_\Sigma/\approx$. 

99
4 Finitary Set Functors

Remark 4.3.28. The equivalence above ignores the linear order on the children of nodes. Therefore, the example above can be formulated using the coalgebra $D$ of all unordered finitely branching trees of Example 4.2.1(3): denoting by $t \approx u$ the above equivalence for unordered trees, we have

$$\nu \mathcal{P}_f = D/\approx$$

This description of $\nu \mathcal{P}_f$ is due to Barr [58].

Remark 4.3.29. There is an interesting connection of the last result to the terminal-coalgebra $\omega^{\text{op}}$-chain. Let $\varepsilon : H \Sigma \rightarrow F$ be a presentation. Firstly, $\varepsilon$ induces a natural transformation $\gamma$ from the terminal-coalgebra $\omega^{\text{op}}$-chain of $H \Sigma$ to that of $F$ by induction: $\gamma_0 = \text{id}_1$ and

$$\gamma_{n+1} = (H \Sigma H_1 H_2 \xrightarrow{H \Sigma \varepsilon_1} F H_1 H_2 \xrightarrow{F \gamma_n} F F_1) \quad (4.5)$$

Hence we obtain the following commutative diagram:

It is not difficult to prove that $\gamma_n : H_1 \rightarrow F^n$ is an epimorphism whose kernel equivalence is given by the congruence $\sim$ of finite applications of basic equations from (4.3) restricted to trees of height at most $n$. Thus, for every $s, t \in T_\Sigma$ we have

$$s \approx t \iff \gamma_n \cdot \ell_n(s) = \gamma_n \cdot \ell_n(t) \text{ for all } n < \omega,$$

where $\ell_n$ cuts trees at level $n$ and $\gamma_n$ is the quotient of finite application of basic equations.

Example 4.3.30. (1) We continue Example 4.3.23 where $F = \mathcal{P}_3$. Here is a pair of infinite trees which are equivalent in $T_\Sigma$:

Indeed, we can check that this pair is related by $\approx$, since we have

$$\bot \sim \bot 
\bot \sim \bot 
\bot \sim \bot$$

$\bot \sim \bot$
(2) For the finite power-set functor \( P_f \) recall the presentation \( \varepsilon_X : X^* \to P_fX \) of Example 4.3.15(1). Here we have, for example,

\[
\begin{align*}
\sigma_1 & \approx \\
\sigma_1 & \sigma_2 \\
\sigma_3 & \\
\sigma_4 & \\
\sigma_5 & \sigma_6 \\
\sigma_7 & \sigma_8 \\
\sigma_9 & \sigma_{10} \sigma_{11} \\
\vdots & \vdots \\
\end{align*}
\]

similarly as in (1) above.

(3) We have introduced analytic functors \( F \) on \( \text{Set} \) in Example 3.2.11. By Theorem 4.3.26 we have a direct description of \( \nu F \): Let \( FX = \coprod_{\sigma \in \Sigma} X^k/G_k \) be an analytic functor, where \( k \) is the arity of \( \sigma \) and \( G_k \) is the given group of permutations on \( k \). Then the terminal coalgebra is the quotient

\[ \nu F = T_\Sigma/\approx \]

of the \( \Sigma \)-tree coalgebra modulo the equivalence \( \approx \) analogous to \( \sim \) of (4.3) but allowing infinitely many permutations of children of nodes; i.e. \( \nu F \) is the coalgebra of all \( \Sigma \)-trees modulo permutations of children of any \( \sigma \)-labelled node (using elements of the permutation group associated with \( \sigma \)).

(4) In particular, for the bag functor \( \mathcal{B}X = \coprod_{n \in \mathbb{N}} X^n/S_n \) (cf. Example 3.2.10), the terminal coalgebra \( \nu \mathcal{B} \) is carried by the set of all unordered finitely branching trees (we saw in Example 4.2.1(3) that the same set carries a weakly terminal \( P_f \)-coalgebra). In fact, the corresponding polynomial functor is \( H_\Sigma X = X^* \). We know that \( \nu H_\Sigma \) is the coalgebra of all finitely branching trees. Hence \( \nu \mathcal{B} \) is the quotient coalgebra of \( \nu H_\Sigma \) given by allowing arbitrary permutations of children of any \( \sigma \)-labelled node. This means that the carrier consists of all unordered trees.

\textbf{Remark 4.3.31.} We have seen congruences on a coalgebra in Remark 4.2.4. To determine whether or not a given relation is a congruence is often tedious, as is the task of coming up with a congruence that relates two points in a given coalgebra. The point of Theorem 4.3.33 below is to make it easier to give

\textbf{Notation 4.3.32.} For every set \( A \) and every relation \( R \subseteq A \times A \), we write \( A \xrightarrow{q_R} A/R \) for the the coequalizer of the evident projections \( R \rightrightarrows A \). Thus, \( A/R \) is the quotient set of \( A \) by the the smallest equivalence relation which includes \( R \), and \( q_R \) is the canonical map. Notice that with this notation every presentation of a functor \( F \) has a component \( \varepsilon_{A/R} : H_\Sigma(A/R) \to F(A/R) \).

\textbf{Theorem 4.3.33} [225]. Let \( \varepsilon : H_\Sigma \to F \) be a presentation. Let \((A, \alpha)\) be an \( H_\Sigma \)-coalgebra, so that \((A, \varepsilon_A \cdot \alpha)\) is an \( F \)-coalgebra, and let \( h : (A, \varepsilon_A \cdot \alpha) \to \nu F \) be the unique homo-
4 Finitary Set Functors

morphism. Given a relation $R$ with

$$R \subseteq \ker(\varepsilon_{A/R} \cdot Hq_R \cdot \alpha).$$

Then $R \subseteq \ker(h)$.

Theorem 4.3.33 gives a coinduction principle: in order to show that two elements of $A$ have the same image in the terminal $F$-coalgebra, it is enough to exhibit a relation $R$ on $A$ relating them and such that $R(a,b)$ implies $[\alpha(a)] \equiv_{\varepsilon_{A/R}} [\alpha(b)]$, where $[\alpha(a)]$ denotes $Hq_R(\alpha(a))$, and similarly for $[\alpha(b)]$. Here $\varepsilon$ is a set of basic equations which present $F$, and $\equiv_{\varepsilon_{A/R}}$ is from Notation 4.3.18.

Example 4.3.34. We revisit Example 4.3.30. We take $\Sigma$ to have a binary symbol $*$ (written in infix notation) and a constant $c$. The set $\varepsilon$ has just one basic equation: $x \ast y = y \ast x$. Consider the coalgebra is $(A, \alpha)$, where $A = \{a, b, z\}$, $\alpha(a) = a \ast z$, $\alpha(b) = z \ast b$, and $\alpha(z) = c$. In Example 4.3.30, we showed pictures of the images of $a$ and $b$ in $\nu H\Sigma$: call these $t_1$ and $t_2$. We also showed what it would take to prove that $R(t_1, t_2)$. To be sure, it would take an inductive argument that we did not give. But given that $t_1 \equiv t_2$, we see that $t_1$ and $t_2$ are identified in $\nu \mathcal{P}_3$. The point of this example is to do this using Theorem 4.3.33.

Consider the relation $R = \{(a, b)\}$ on $A$. Let us verify that $[\alpha(a)] \equiv_{\varepsilon_{A/R}} [\alpha(b)]$. We have

$$[a \ast z] = [a] \ast [z] \equiv_{\varepsilon_{A/R}} [z] \ast [a] = [z] \ast [b] = [z \ast b].$$

The "$\equiv_{\varepsilon_{A/R}}$" is due to the equation $x \ast y = y \ast x$. For the second "$=$", note that $A/R$ has two equivalence classes, and $q_R(a) = q_R(b)$. That is, $[a] = [b]$.

We conclude from this and Theorem 4.3.33 that for the $\mathcal{P}_3$-coalgebra $(A, \varepsilon_A \cdot \alpha)$, $a$ and $b$ have the same image in $\nu \mathcal{P}_3$.

Example 4.3.35. We return to analytic functors, see Examples 3.2.11 and 4.3.20(3). Consider the analytic functor $FX = X^3/G_3$ where $G_3 = A_3$, the three cyclic permutations of $\{1, 2, 3\}$. (For example, $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$ is cyclic, but $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$ is not cyclic.) So $F$ has a presentation $\varepsilon : H\Sigma \to F$, where $\Sigma$ consists of a single ternary operation symbols $\sigma$.

Consider the $H\Sigma$-coalgebra $(A, \alpha)$, where $A = \{a, b, c\}$, $\alpha(a) = \sigma(b) = \sigma(a, b, c)$, and $\alpha(c) = \sigma_3(c, b, a)$, and $\alpha(c) = \sigma_3(a, b, c)$. We claim that $b$ and $c$ have the same image in the terminal $F$-coalgebra. We use $R = \{(b, c)\}$, and then we verify:

$$\sigma(a, b, c) \equiv \sigma(b, c, a) \quad \text{(using a cyclic permutation)}$$

$$\equiv \sigma(c, b, a) \quad \text{(since } b \equiv c, c \equiv b, \text{ and } a \equiv a).$$

The point again is that this argument is simpler than one which uses the cuttings of the images of $b$ and $c$ in the terminal $F$-coalgebra.

4.4 Iterating the terminal-coalgebra chain to $\omega + \omega$

When a functor $F$ preserves the limit of its terminal coalgebra $\omega^{\text{op}}$-chain, then this limit is a terminal coalgebra (see Theorem 3.3.4). The starting point of this section is that
4.4 Iterating the terminal-coalgebra chain to $\omega + \omega$

this limit preservation does not always happen, even for finitary set functors such as $\mathcal{P}_1$.

However, Worrell [245] provided a construction of the terminal coalgebra that works for every finitary set functor. The proof presented here is simpler than the Worrel’s original one. Actually, he proved a more general result about accessible endofunctors on Set that we present later (cf. Theorem 11.3.10).

We begin with a general point about set functors due to Trnková, and then an application to finitary set functors. Recall that an intersection of a family of subobjects $s: S_i \hookrightarrow X$ ($i \in I$) is their wide pullback. We use the suggestive notation $\bigcap_{i \in I} s_i$ for intersections.

**Proposition 4.4.1** [232]. Every set functor preserves finite nonempty intersections.

We prove this in Proposition B.2.1(5) on page 410.

**Example 4.4.2.** Empty intersections need not be preserved. For example, let $\mathcal{P}_1'$ be the functor obtained from $\mathcal{P}_1$ by changing just the value at $\emptyset$ to $\mathcal{P}_1'[\emptyset] = \emptyset$. That is, $\mathcal{P}_1'$ maps a set to the set of its nonempty subsets. Then $\mathcal{P}_1'$ does not preserve the (empty) intersection of the two maps $t, f: 1 \rightarrow 2$.

**Proposition 4.4.3.** Every finitary set functor preserves nonempty intersections.

**Proof.** We use the following notation: for every subset $s: S \hookrightarrow X$ we write $x \in FS$ to indicate that there exists $x' \in FS$ with $x = FS(x')$. Note that

$$\text{if } S' \hookrightarrow S \hookrightarrow X \text{ and } x \in FS', \text{ then also } x \in FS. \quad (4.6)$$

Let $s_i: S_i \hookrightarrow X$ ($i \in I$) be a family of subsets whose intersection

$$s = \bigcap_{i \in I} s_i: S = \bigcap_{i \in I} S_i \hookrightarrow X$$

is nonempty (this implies $X \neq \emptyset$, of course). Given $x \in FX$ with $x \in FS_i$ for all $i \in I$, it is our task to show that $x \in FS$. Since $F$ is finitary, we have for every $i \in I$ a finite subset $t_i: T_i \hookrightarrow S_i$ such that $x \in FT_i$. Let us write $u_i = s_i \cdot t_i: T_i \hookrightarrow X$. Without loss of generality we may assume that every $T_i$ is nonempty and that the intersection $u = \bigcap_{i \in I} u_i: T = \bigcap_{i \in I} T_i \hookrightarrow X$ is nonempty. For if any $T_i$ were empty we can choose a nonempty $T_i'$ such that $T_i \hookrightarrow T_i' \hookrightarrow X$. Then $x \in FT_i$ implies $x \in FT_i'$ by (4.6), which means we can work with the nonempty $T_i'$ in lieu of $T_i$. Similarly, if the intersection $T$ were empty, then choose any element $y \in \bigcap S_i$ and define $T'_i = T_i \cup \{y\}$. Then $x \in FT_i$ implies $x \in FT_i'$ for all $i \in I$ by another application of (4.6).

Furthermore we can express $T$ as the intersection of a finite family of the subsets of $X$. More precisely, fix $j^* \in I$. Then $T = \bigcap_{i \in J} (T_i \cap T_{j^*})$. Being finite $T_{j^*}$ has only finitely many subsets. So there is a finite $J \subseteq I$ such that $T = \bigcap_{i \in J} (T_i \cap T_{j^*})$. For all $i$, $F(T_i \cap T_{j^*}) = F(T_i) \cap F(T_{j^*})$ by Proposition 4.4.1. So $x \in F(T_i \cap T_{j^*})$. Since $J$ is finite, we have

$$x \in \bigcap_{i \in J} F(T_i \cap T_{j^*}) = F \left( \bigcap_{i \in J} (T_i \cap T_{j^*}) \right) = FT.$$  

By (4.6), this implies that $x \in FS$, as desired. \qed
4 Finitary Set Functors

Notation 4.4.4. (1) Recall from Remark 3.3.5 the limit of the terminal $\omega^{op}$-chain and denote

$$V_\omega = \lim_{n \in \omega^{op}} F^n1$$

with limit projections $\ell_n: V_\omega \to F^n1$ for all $n \in \omega^{op}$. (4.7)

We obtain a unique map $m: FV_\omega \to V_\omega$ having the property that for all $n \in \omega^{op}$, the triangles below commute:

$$\xymatrix{FV_\omega \ar[rr]^m \ar[dr]_{l_n} & & V_\omega \\
 & F^{n+1} \ar[ru]_{\ell_{n+1}}}_{(4.8)}$$

(2) We put $V_{\omega+1} = FV_\omega$, $V_{\omega+2} = FV_{\omega+1}$, etc. and form the following $\omega^{op}$-chain:

$$\xymatrix{V_\omega \ar[l]_{m} \ar[r]^{Fm} & V_{\omega+1} \ar[l]_{m} \ar[r]^{Fm} & V_{\omega+2} \ar[l]_{m} \cdots}_{(4.9)}$$

Its limit is denoted by $V_{\omega+\omega} = \lim_{n<\omega} V_{\omega+n}$ with the limit cone $\tilde{\ell}_n: V_{\omega+\omega} \to V_{\omega+n}$, for $n < \omega$. (In Definition 6.2.1 we introduce $V_i$ for all ordinal numbers $i$. The limit projections $\tilde{\ell}_n$ are denoted by $v_{\omega+\omega}, v_{\omega+n}$ there.)

We know that $V_\omega$ is the terminal $F$-coalgebra whenever the limit in (4.7) above is preserved by $F$ (see Theorem 3.3.4); equivalently, whenever $m$ is invertible. We shall soon prove that for every finitary set functor $F$, $m$ is a split monomorphism. Thus, $V_{\omega+\omega}$ is the intersection of the chain (4.9) of monomorphisms. Moreover, we obtain that $V_{\omega+\omega} = \nu F$. But first let us record that $m$ is not always invertible:

Example 4.4.5 [20]. For the finite power-set functor, the morphism $m$ is not epic (whence not invertible). Assuming the contrary, we obtain the following property of elements of $V_\omega$: for every $x \in V_\omega$ there exists $y \in P_f V_\omega$ such that $x = m(y)$, whence $\ell_{n+1}(x) = P_f \ell_n(y)$ for all $n \in \omega^{op}$. Since $P_f \ell_n$ assigns direct images, we know that $\ell_{n+1}(x)$ does not have more elements than the finite set $y$.

Now let $x_n \in P^1_f1$ be the largest element (w.r.t. set inclusion) for every $n \in \omega^{op}$. The connecting maps $P^1_f1!$ of the terminal-coalgebra $\omega^{op}$-chain are all epic and monotone w.r.t. subset inclusion, being direct image maps. Hence, we have $x_n = P^1_f1(x_{n+1})$ for all $n$. Thus, there exists a unique $x \in V_\omega$ such that $\ell_n(x) = x_n$ for all $n$. However, since the number of elements of the $x_n$ is unbounded, $\ell_{n+1}(x)$ does not have the above property.

Lemma 4.4.6. Let $\ell_n: L \to L_n$ be a limit cone of an $\omega^{op}$-chain in Set. For every finite subset $s: S \to L$ there exists $n$ such that $\ell_n \cdot s$ is a monomorphism.

Proof. Since the limit cone is collectively monic, for every pair of distinct elements in $S$ we have $k \in \omega$ such that $l_k$ distinguishes that pair. Let $n$ be the maximum of all $k$’s (ranging over $S \times S$).

Lemma 4.4.7 [245]. Let $F$ be a finitary set functor. Then the cone $F\ell_n: FV_\omega \to F^{n+1}1$ is collectively monic, and $m: FV_\omega \to V_\omega$ in (4.8) is a split monomorphism.
4.4 Iterating the terminal-coalgebra chain to $\omega + \omega$

Proof. (1) If $F$ is constantly $\emptyset$, the statement is trivial. If not, then $FX \neq \emptyset$ for every $X \neq \emptyset$ (since, given $X$, for every set $Y$ we have a map from $Y$ to $X$, thus, one from $FY$ to $FX$.) Consequently, $\nu F = \emptyset$. Indeed, from $\nu F = \emptyset$ we conclude $m = \text{id}_\emptyset$ is invertible, thus $\nu F = \emptyset$. This implies that every coalgebra $A$ is empty (due to the homomorphism $A \to \nu F = \emptyset$). In contradiction, since $F1 \neq \emptyset$, we have at least one coalgebra $1 \to F1$.

(2) We now prove that $(F\ell_n)$ is a collectively monic cone: Assuming that for all $k$, $F\ell_k(x) = F\ell_k(y)$, we show that $x = y$. By the assumption that $F$ is finitary, there is a finite subset $s : S \to V_\omega$ and $x', y' \in FS$ such that $x = F\ell_s(x')$ and $y = F\ell_s(y')$. Without loss of generality we may assume that $S$ is nonempty. We see that

$$F(\ell_k \cdot F_s)(x') = F\ell_k(x) = F\ell_k(y) = (F\ell_k \cdot F_s)(y').$$

By Lemma 4.4.6, there exists some $n$ such that $\ell_n \cdot s$ is monic, thus $S \neq \emptyset$ implies that it is a split monomorphism. Thus, $F(\ell_n \cdot f)$ is monic, too. So $x' = y'$, and thus $x = y$.

(3) From the definition of $m$ in (4.8) it now follows that it is a monomorphism. It splits since $FV_\omega \neq \emptyset$. \qed

**Theorem 4.4.8** [245]. Every finitary set functor $F$ has the terminal coalgebra

$$\nu F = V_{\omega + \omega} = \lim_{n<\omega} V_{\omega+n}.$$  

The coalgebra structure $\tau$ is determined by the following commutative triangles

$$\begin{array}{ccc}
\nu F & \xrightarrow{\tau} & F(\nu F) \\
\downarrow \ell_{n+1} & & \downarrow F\ell_n \\
FV_{\omega+n} & \xrightarrow{m} & FV_{\omega+n+1}
\end{array}$$  

(4.10)

Proof. (1) We are going to prove in part (2) that $F$ preserves the limit $V_{\omega + \omega} = \lim_{n<\omega} V_{\omega+n}$. It then follows that there is a unique isomorphism $\tau$ for which the above triangles commute.

The proof that $(\nu F, \tau)$ is a terminal coalgebra is then analogous to the proof of Theorem 3.3.4. There we used the canonical cone $\alpha_n : A \to F^n1$ for a given coalgebra $\alpha : A \to FA$ defined in Construction 3.3.2. We extend this to a cone $\alpha_{\omega+n} : A \to V_{\omega+n}$ as follows. The cone $\alpha_n$ induces a unique morphism $\alpha_\omega : A \to V_\omega$ with $\ell_n \cdot \alpha_\omega = \alpha_n$ for all $n < \omega$, and given $\alpha_{\omega+n}$ we define $\alpha_{\omega+n+1} = F\alpha_{\omega+n} \cdot \alpha : A \to V_{\omega+n+1}$. To see that this is a cone for (4.9) it suffices to show that $\alpha_{\omega} = m \cdot \alpha_{\omega+1}$; the rest then follows by an easy induction. We prove the desired equation by postcomposing it by every limit projection $\ell_{n+1}, n < \omega$, and then use that those limit projections are collectively monic:

$$\begin{align*}
\ell_{n+1} \cdot \alpha_\omega &= \alpha_{n+1} & \text{(def. of } \alpha_\omega) \\
&= F\alpha_n \cdot \alpha & \text{(def. of } \alpha_{n+1}) \\
&= F\ell_n \cdot F\alpha_\omega \cdot \alpha & \text{(def. of } \alpha_\omega) \\
&= F\ell_n \cdot \alpha_{\omega+1} & \text{(def. of } \alpha_{\omega+1}) \\
&= \ell_{n+1} \cdot m \cdot \alpha_{\omega+1} & \text{(def. of } m)
\end{align*}$$

$$105$$
We now obtain a unique morphism \( \bar{\alpha} : A \to V_{\omega^+\omega} \) such that \( \bar{\alpha}_n = \alpha_{n+\omega} \) for all \( n < \omega \).

Then \( \bar{\alpha} : A \to V_{\omega^+\omega} \) is a coalgebra homomorphism: in the following diagram

\[
\begin{array}{c}
A & \xrightarrow{\alpha} & FA \\
\downarrow{\bar{\alpha}} & & \downarrow{F\alpha_{\omega+n}} \\
V_{\omega^+\omega} & \xrightarrow{F\bar{\alpha}} & FV_{\omega^+\omega}
\end{array}
\]

all inner parts commute, thus the outside does: recall that all \( F\bar{l}_n \) form a limit cone which is thus collectively monic. To verify uniqueness, let \( \tilde{\alpha} : A \to V_{\omega^+\omega} \) be another coalgebra homomorphism. In fact, one first proves by induction that for every \( n < \omega \) we have

\[
\alpha_n = (A \xrightarrow{\tilde{\alpha}} V_{\omega^+\omega} \xrightarrow{\bar{l}_n} V_\omega \xrightarrow{\ell_n} F^m 1).
\]

This shows that \( \bar{l}_0 \cdot \tilde{\alpha} = \alpha_\omega \) because both sides are merged by all limit projections \( \ell_n \).

Using this as the base case, another induction proof then shows that for all \( n < \omega \) we have

\[
\alpha_{\omega+n} = (A \xrightarrow{\tilde{\alpha}} V_{\omega^+\omega} \xrightarrow{\bar{l}_n} V_{\omega+n}).
\]

Thus the family \( \bar{l}_n \) merges \( \tilde{\alpha} \) and \( \alpha \), which proves \( \tilde{\alpha} = \bar{\alpha} \) (since \( \bar{l}_n \) forms a limit cone).

(2) The proof that \( F \) preserves the limit \( V_{\omega^+\omega} \) is trivial in the case where \( F \) is constant with value \( \emptyset \). Assuming the contrary, we have clearly \( F1 \neq \emptyset \), thus there is (at least one) coalgebra \( \alpha : 1 \to F1 \). This implies \( V_{\omega^+\omega} \neq \emptyset \) due to \( \tilde{\alpha} : 1 \to V_{\omega^+\omega} \), whence \( V_{\omega+n} \neq \emptyset \) for all \( n \).

From Lemma 4.4.7 we know that \( m \) is a split monomorphism, thus so are \( Fm, FFm \), etc. Thus the chain (4.9) is a decreasing chain of nonempty subobjects \( u_n : V_{\omega+n} \to V_\omega \) where \( u_0 = id_{V_\omega} \) and \( u_{n+1} = u_n \cdot F^n m \). Therefore, the limit \( V_{\omega^+\omega} \) is simply the intersection of these subobjects, more precisely, for the first projection \( \bar{l}_0 : V_{\omega^+\omega} \to V_\omega \) we have

\[
\bar{l}_0 = \bigcap_{n<\omega} u_n.
\]

Consequently, since \( V_{\omega^+\omega} \neq \emptyset \) and \( F \) preserves nonempty intersections by Proposition 4.4.3, it preserves the limit \( V_{\omega^+\omega} \). □

**Corollary 4.4.9.** For every finitary set functor \( F \) we have a weakly terminal coalgebra \( V_\omega \) given by any splitting of \( m : FV_\omega \to V_\omega \).

**Proof.** Given \( \bar{m} : V_\omega \to FV_\omega \) with \( \bar{m} \cdot m = id \) we observe that \( \bar{l}_0 : (V_{\omega^+\omega}, \tau) \to (V_\omega, \bar{m}) \) of Notation 4.4.4 is a coalgebra homomorphism:

\[
\begin{array}{c}
V_{\omega^+\omega} \xrightarrow{\tau} FV_{\omega^+\omega} \\
\downarrow{\bar{l}_0} & & \downarrow{F\bar{l}_0} \\
V_\omega \xrightarrow{\bar{m}} FV_\omega
\end{array}
\]
4.4 Iterating the terminal-coalgebra chain to $\omega + \omega$

The upper triangle commutes by definition of $\tau$. For the lower one use $\bar{l}_0 = m \cdot \bar{l}_1$ and multiply this equation by $\bar{m}$. Since $V_{\omega+\omega}$ is terminal this implies $V_\omega$ is weakly terminal by Example 4.2.1(2).

**Remark 4.4.10.** The limit $V_\omega$ is an algebra for $F$ in a canonical sense with the structure $m: FV_\omega \to V_\omega$ in (4.8). Theorem 4.4.8 implies that for a finitary set functor the terminal coalgebra, considered as an algebra, is a cartesian subalgebra of $V_\omega$, i.e. the following square is a pullback:

$$
\begin{array}{ccc}
F(\nu F) & \xrightarrow{\tau^{-1}} & \nu F \\
\downarrow & & \downarrow \\
\bar{l}_0 & \xrightarrow{m} & V_\omega
\end{array}
$$

Here $\bar{l}_0$ is the first projection of $\nu F = \lim_{n<\omega} V_{\omega+n}$ (see Notation 4.4.4). The square commutes: combine $F\bar{l}_0 \cdot \tau = \bar{l}_1$ (see (4.10)) and $\bar{l}_0 = m \cdot \bar{l}_1$ to obtain

$$
m \cdot F\bar{l}_0 = m \cdot \bar{l}_1 \cdot \tau^{-1} = \bar{l}_0 \cdot \tau^{-1}.
$$

It is a pullback since $m$ is a monomorphism and $\tau^{-1}$ an isomorphism.

**The terminal-coalgebra chains of $H_\Sigma$ and $F$.** Our final topic in this section returns to presentations of finitary set functors and to the relation between the terminal-coalgebra chains of $H_\Sigma$ and $F$ which we saw in Remark 4.3.29. Recall the natural transformation $\gamma$ from the first chain to the second.

$$
\begin{array}{ccccccc}
1 & \xleftarrow{!} & H_\Sigma 1 & \xleftarrow{H_\Sigma 1} & H_\Sigma H_\Sigma 1 & \xleftarrow{H_\Sigma H_\Sigma 1} & \cdots \\
\downarrow & & \downarrow \gamma_1 = \varepsilon_1 & & \downarrow \varepsilon_{H_\Sigma 1} & & \\
\nu H_\Sigma & \xleftarrow{F!} & F1 & \xleftarrow{F!} & FF1 & \xleftarrow{FF!} & \cdots
\end{array}
$$

We extend $\gamma$ to $\gamma^*: \nu H_\Sigma \to V_\omega$ so that for all $n$, $\ell_n \cdot \gamma^* = \gamma_n \cdot p_n$, where $p_n: \nu H_\Sigma \to H_\Sigma 1$ is the canonical projection. It is easy to check that the following square commutes:

$$
\begin{array}{ccc}
\nu H_\Sigma & \xleftarrow{\tau^{-1}} & H_\Sigma(\nu H_\Sigma) \\
\downarrow & & \downarrow \varepsilon_{\nu H_\Sigma} \\
\gamma^* & F(\nu H_\Sigma) & \xleftarrow{F \gamma^*} \\
\downarrow & & \downarrow m \\
V_\omega & \xleftarrow{m} & FV_\omega
\end{array}
$$

Indeed, this is yet another argument using the fact that the cone $(\ell_n)$ is collectively monic. In addition, this fact follows from stronger facts which we shall see below on solution morphisms in corecursive algebras. In general, $\gamma^*$ is not epic. However, there is a natural assumption that implies this condition, as we now present.
4 Finitary Set Functors

Definition 4.4.11. A function \( f: X \to Y \) is finite-to-one if the inverse image of every \( y \in Y \) is finite. A set functor \( F \) preserves finite-to-one morphisms if \( Ff \) is finite-to-one whenever \( f \) is. A presentation \( \varepsilon: H_\Sigma \to F \) of a finitary functor is finite-to-one if every component \( \varepsilon_X \) is finite-to-one.

Examples 4.4.12. We revisit the presentations in Example 4.3.20.

1. The presentations of analytic functors are finite-to-one. Indeed, any presentation with the following two properties is finite-to-one: \( H \) has only finitely many symbols of each arity \( n \), and all basic equations relate terms of the same arities.

2. The presentation of \( P_f \) is not finite-to-one: for all sets \( A \) and all \( a \in A \), \( \varepsilon_A \) merges \( \sigma_1(a) \), \( \sigma_2(a,a) \), \( \sigma_3(a,a,a) \), etc. Moreover, it follows from Theorem 4.4.14 that \( P_f \) has no finite-to-one presentation.

Lemma 4.4.13. Let \( \varepsilon: H_\Sigma \to F \) be a finitary finite-to-one presentation of the finitary set functor \( F \). Then the following hold:

1. The functor \( F \) preserves finite-to-one functions.
2. Each \( \gamma_n: H_\Sigma^n \to F_\Sigma^n \) is finite-to-one.
3. The map \( \gamma^*: \nu H_\Sigma \to V_\omega \) is an epimorphism.

Proof. Every polynomial functor preserves finite-to-one functions, easily. A naturality argument shows that \( F \), too, preserves finite-to-one functions. An easy induction on \( n \) using (4.5) shows that \( \gamma_n \) is finite-to-one.

The last part is the most important. Let \( x \in V_\omega \) and write \( x_n \) for \( \ell_n(x) \). Construct a tree \( T \) as follows. Let

\[
T = \gamma_0^{-1}(x_0) \cup \gamma_1^{-1}(x_1) \cup \cdots \cup \gamma_n^{-1}(x_n) \cup \cdots
\]

We assume that the unions above are all pairwise disjoint. So \( T \) is the disjoint union of the inverse images of the points \( x_n \) under the maps \( \gamma_n \). Note that \( \gamma_n^{-1}(x_n) \subseteq H_n^{\otimes 1} \).

Put an arrow from \( t \) to \( u \) provided that for some \( n \), \( t \in \gamma_n^{-1}(x_n) \), \( u \in \gamma_{n+1}^{-1}(x_{n+1}) \), and \( H_n^{\otimes 1}(u) = t \). We think of \( T \) as the tree of attempts to build an element of \( y \in \nu H \) such that \( \gamma_n(\ell_n^H(y)) = x_n \) for all \( n \). Notice that \( T \) is a finitely branching tree, since each \( \gamma_n \) is finite-to-one. And \( T \) has infinitely many nodes, since each \( \ell_n \) is surjective. By König’s Lemma [148], there is some infinite branch through \( T \). This branch corresponds to some \( y \in \nu H \) with the property mentioned above: \( \gamma_n(\ell_n^H(y)) = x_n \). It follows that \( \gamma^*(y) = x \).

Theorem 4.4.14 [58]. Let \( F \) be a finitary set functor which has a finite-to-one presentation. Then \( \nu F = V_\omega \).

Proof. Consider \( m: FV_\omega \to V_\omega \) from Notation 4.4.4. We have \( \gamma^* \cdot \tau^{-1} = m \cdot F \cdot \varepsilon \cdot \nu H \). By Lemma 4.4.13, \( \gamma^* \cdot \tau^{-1} \) is an epimorphism. It follows that \( m \), too, is epic. It then follows from Lemma 4.4.7 that \( m \) is an isomorphism, and we are done by Theorem 3.3.4.

Corollary 4.4.15. For every analytic set functor \( F \) we have \( \nu F = V_\omega \). 

108
Remark 4.4.16. The converse of Theorem 4.4.14 does not hold in general. There are finitary set functors $F$ with $\nu F = V_\omega$ that have no finite-to-one presentation. For example, the nonempty finite powerset functor $P'_f$ (see Example 4.4.2) satisfies $\nu P'_f = V_\omega = 1$ since $P'_f1 = 1$. However, it has no finite-to-one presentation. For suppose that $\varepsilon : H_\Sigma \to P'_f$ is any finite-to-one presentation, let $\bar{\Sigma}$ extend the signature $\Sigma$ by the constant symbol $c$, and extend $\varepsilon$ to $\bar{\varepsilon} : H_{\bar{\Sigma}} \to P_f$ by $\bar{\varepsilon}_X(c) = \emptyset$ for every set $X$. Then $\bar{\varepsilon}$ is a finite-to-one presentation of $P_f$ contradicting the fact that $\nu P_f$ is not $V_\omega$.

4.5 Finite Power-Set Functor: the Terminal Coalgebra vs. the $\omega^{op}$-Limit

The finite power-set functor $P_f$ is fascinating when it comes to (1) its terminal coalgebra $\nu P_f$ as well as (2) the limit $V_\omega$ of its terminal-coalgebra $\omega^{op}$-chain which contains $\nu P_f$ as a proper subset (cf. Example 4.4.5).

This can be seen from the number of papers devoted to various descriptions of these two coalgebras, see e.g. [1, 20, 23, 58, 244, 245]. Before turning to our descriptions, let us recall the initial chain of $P_f$ from Example 3.2.9(2):

$$P^n_f \emptyset = \text{extensional trees of height} < n.$$ 

The terminal-coalgebra $\omega^{op}$-chain is quite analogous. Indeed, since $1 \cong P_\emptyset$, we have $P^n_f1 \cong P^{n+1}_f0$. We thus have

$$P^n_f1 = \text{all extensional trees of height} \leq n.$$ 

The connecting maps

$$v_{n+1,n} : P^{n+1}_f \to P^n_f1, \quad t \mapsto \partial_n t$$ 

are given by cutting the trees at level $n$ and forming the extensional quotient, see Example 4.2.10. Indeed, this is trivially true for $n = 0$, and the induction step is easy to see.

**A first description**: $\nu P_f$ is the set of all finitely branching trees modulo $\approx$. Let $D$ be the set of finitely branching trees from Example 4.2.1(3). We know that $D$ is a weakly terminal coalgebra. Incidentally, $D$ is distinct from $V_\omega$. More precisely, endowed with the cutting maps into $P^n_f \emptyset$ for $n < \omega$, $D$ is not the limit of the terminal-coalgebra $\omega^{op}$-chain. Recall from Example 4.3.27 the congruence $\approx$ on $D$ with $t \approx u$ iff $\partial_n(t) = \partial_n(u)$ holds for all $n < \omega$. We know from Remark 4.3.28 that $\nu P_f = D/\approx$.

**A second description**: $V_\omega$ is the set of sequences of finite extensional trees. We can describe $F$ as the set of all sequences $(t_n)_{n < \omega}$ of finite extensional trees, where $t_n$ has height at most $n$, and for all $n$ such that $t_n = \partial_n(t_{n+1})$. The correctness of this description follows from the fact that $V_\omega$ is the limit of the $\omega^{op}$-chain $(P^n_f1)_{n < \omega}$ above.
Example 4.5.1. (1) The following sequence of finite trees $t_n$:

```
  t_0  t_1  t_2  t_3  ...
```

gives an element of $V_\omega$.

(2) Consider the following infinite tree $t$

```
  ...
```

Then the cuttings $\partial_n t$, $n < \omega$, are precisely the trees $t_n$ above. This single tree $t$ represents the element of $V_\omega$ given by (1) above. (We will use this representation in the subsequent descriptions of $V_\omega$.)

A third description: $\nu P_f$ and $V_\omega$ are the sets of strongly extensional trees that are finitely branching and compactly branching, respectively. Here we first present a description of $\nu P_f$ due to Worrell [245]. The main concept that Worrell introduced is tree bisimulation: this is more special than graph bisimulation (see Example 4.2.5) since the goal is that for every tree by factoring modulo the greatest bisimulation we obtain a tree again.

All trees in this discussion are unordered. We use the notation $t_x$ for the subtree of $t$ rooted in the node $x$.

Definition 4.5.2 [245]. (1) A tree bisimulation between two trees $t$ and $u$ is a graph bisimulation $R$ such that

(a) the roots of $t$ and $u$ are related; the roots are not related to other nodes; and

(b) whenever two nodes are related, their parents are also related.

Two trees are called tree bisimilar if there is a tree bisimulation between them.

(2) A tree $t$ is called strongly extensional if every tree bisimulation on it is a subrelation of the diagonal relation $\Delta = \{(x, x) : x \in t\}$. Said differently: $t$ is strongly extensional iff distinct children $x$ and $y$ of the same node define subtrees $t_x$ and $t_y$ which are not tree bisimilar.

Remark 4.5.3. (1) Every strongly extensional tree is clearly extensional.

(2) Consider the following trees (where the right-hand one has $n$ children for the $n$-th
vertex in the breadth-first search):

Both are extensional: the left-hand one is strongly extensional, the right-hand one is not, since the relation relating all nodes of the same depth is a tree bisimulation on it.

(3) It is trivial to prove that every composition and every union of tree bisimulations is again a tree bisimulation. In addition, the opposite relation of every tree bisimulation is a tree bisimulation: if $R$ is a tree bisimulation from $t$ to $u$, then $R^{op}$ is a tree bisimulation from $u$ to $t$. Consequently, the largest tree bisimulation on every tree is an equivalence relation.

(4) Observe that the notion of tree bisimulation is different from the usual graph bisimulation. For example, the picture below

depicts a strongly extensional tree, but there is a graph bisimulation relating the two leaves.

**Proposition 4.5.4.** Let $t$ be a finite tree. Then $t$ is extensional iff it is strongly extensional.

**Proof.** Let $t$ be extensional, and let $R$ be a tree bisimulation on it. We claim that if $x R y$, then the corresponding subtrees $t_x$ and $t_y$ are equal. First notice that every node of $t_x$ must be related by $R$ to some node of $t_y$ (to see this, use induction on the depth of nodes, i.e. their distance from the root) and vice versa. Thus, $t_x$ and $t_y$ have the same height, $n$ say. We now prove $t_x = t_y$ by induction on $n$. For $n = 0$, the result is obvious because the nodes of height 0 are leaves. Assume our result for $n$, and let $x$ and $y$ be related by $R$ and of height $n + 1$. Then by the induction hypothesis and extensionality of $t$, for every child $x'$ of $x$ there is a unique child $y'$ of $y$ and $t_{x'} = t_{y'}$; and vice-versa. This implies that $t_x = t_y$.

It now follows that if $t$ is an extensional tree, then $t$ must be strongly extensional.

**Definition 4.5.5.** The *strongly extensional quotient* $\tilde{t}$ of a tree $t$ is the quotient tree of $t$ modulo its largest tree bisimulation.

**Lemma 4.5.6.** Let $t$ and $u$ be trees.

(1) $\tilde{t}$ is strongly extensional, and it is tree bisimilar to $t$.

(2) If $t$ and $u$ are strongly extensional and related by a tree bisimulation, then $t = u$. 

111
Proof. In (1), it is easy to see that $\bar{t}$ is strongly extensional. Moreover, the canonical quotient map $t \rightarrow \bar{t}$ may be considered as a relation. As such, it is a tree bisimulation.

For (2), let $R$ be a tree bisimulation between $t$ and $u$. By Remark 4.5.3, $R^{\text{op}} \cdot R$ is a tree bisimulation on $t$, whence $R^{\text{op}} \cdot R \subseteq \Delta$ by strong extensionality. But every node of $t$ is related to at least one node of $u$ (use induction on the depth of nodes, i.e., their distance from the root) implying that $R^{\text{op}} \cdot R = \Delta$. Similarly, $R \cdot R^{\text{op}} = \Delta$. Thus, $R$ (is a function and it) is an isomorphism of trees, and we identify such trees.

Now recall from Remark 4.3.28 the coalgebra $D$ of all unordered finitely branching trees, and the terminal coalgebra $D/\approx$ of $\mathcal{P}_f$. Since a subtree of a strongly extensional tree is strongly extensional, the set $D_0$ consisting of the strongly extensional trees is a subcoalgebra of $D$.

**Theorem 4.5.7** [245]. The subcoalgebra $D_0$ of $D$ is a terminal coalgebra for $\mathcal{P}_f$.

**Proof.** We prove that $D_0$ is isomorphic to a coalgebra that we know to be terminal, $D/\approx$. For that we need to verify that given trees $t,u \in D$, then for their strongly extensional quotients we have

$$t \approx u \quad \text{iff} \quad \bar{t} = \bar{u}.$$ 

Then the map $[t] \rightarrow \bar{t}$ is then an isomorphism from $D/\approx$ to $D_0$.

$(\Rightarrow)$ If $t \approx u$ we prove that $t$ and $u$ are tree bisimilar. Then, by Remark 4.5.3, it follows that $\bar{t}$ and $\bar{u}$ are tree bisimilar, which implies that $\bar{t} = \bar{u}$ by Lemma 4.5.6. Recall $\partial_n$ from Example 4.2.10.

Define $R \subseteq t \times u$ by relating nodes $x \in t$ and $y \in u$ iff they have the same depth $n$ and for every $m \geq n$ the node of $\partial_m t$ corresponding to $x$ equals the node of $\partial_m u$ corresponding to $y$. Using that $t$ and $u$ are finitely branching, it is not difficult to prove that $R$ is a tree bisimulation. We leave the details to the reader.

$(\Leftarrow)$ Conversely, suppose we have $\bar{t} = \bar{u}$. Then we get, by composing with the quotient maps, a tree bisimulation $R \subseteq t \times u$. By restricting $R$ to all pairs of nodes of depth at most $n$ we obtain a tree bisimulation $R_n$ between the cuttings of $t$ and $u$ at level $n$. Then also the extensional quotients $\partial_n t$ and $\partial_n u$ are tree bisimilar. Since $\partial_n t$ and $\partial_n u$ are strongly extensional by Proposition 4.5.4, we see that $\partial_n t = \partial_n u$. Thus, $t \approx u$. $\square$

**Example 4.5.8.** The terminal coalgebra of $\mathcal{P}_f$ is uncountable. For every subset $A \subseteq \mathbb{N}$ let $t_A$ be the tree obtained from the infinite path

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$$

by adding, for every $i \in A$, a new leaf as a child of $i$. Each $t_A$ belongs to $\nu \mathcal{P}_f$ since it is finitely branching and strongly extensional. For $A \neq B$, $t_A$ and $t_B$ are non-isomorphic. Thus $|\nu \mathcal{P}_f| \geq |\mathcal{P}\mathbb{N}|$. And since every finitely branching tree is isomorphic to a tree whose nodes are natural numbers, the cardinality of the set of trees is at most the cardinality of the set of pairs of natural numbers. Thus, $|\nu \mathcal{P}_f| = |\mathcal{P}\mathbb{N}| = 2^{\aleph_0}$. 

112
Example 4.5.9. Returning to Theorem 4.5.7 and strongly extensional trees, similar descriptions arise for the terminal coalgebra of related set functors. For example, consider the functor $F X = \{0,1\} \times (\mathcal{P} X)^{\Sigma}$, viz. the finitary version of the type functor of non-deterministic automata, see Example 2.4.2(5). In order to describe $\nu F$ one considers finitely branching trees whose nodes are labelled in $\{0,1\}$ and whose edges are labelled in the input alphabet $\Sigma$. The notion of tree bisimulation then needs to be adjusted as follows: it is a bisimulation of labelled transition systems (cf. Example 2.6.7(3)) such that the roots are related, roots are not related to other nodes, and every pair of related nodes have the same depth and the same label in $\{0,1\}$. The notion of a strongly extensional tree is then defined as before, and it is not difficult to see that $\nu F$ is the coalgebra of all strongly extensional trees.

Note that this means that the unique map from an $F$-coalgebra into the terminal one thus provides the behaviour of states modulo bisimilarity, i.e. taking into account the non-deterministic branching of the given automaton. This semantics is therefore different from the usual language semantics of non-deterministic automata. We shall see in Example 5.1.27 that by modelling non-deterministic automata as coalgebras for a functor on the category of sets and relations one obtains their usual language semantics similarly as for deterministic automata in Example 2.5.5 via the terminal coalgebra.

We now present an analogous description of $V^\omega$ by strongly extensional compactly branching trees. Worrell introduced in [245] the following pseudometric $d$ on the class of all strongly extensional trees:

$$d(z,u) = \inf \{2^{-n} : n < \omega \text{ with } \partial_n z = \partial_n u\}. \quad (4.13)$$

Consequently $t$ and $u$ have distance 0 iff $t \approx u$, that is $\partial_n t = \partial_n u$ for all $n < \omega$. The resulting pseudometric space is compact because for every $\epsilon > 0$ we have a finite number of $\epsilon$-balls covering it. Indeed, choose $n$ with $2^{-n} < \epsilon$ and take the (finite) set $A$ of all extensional trees of height at most $n$. Then every tree $t$ satisfies $\partial_n t \in A$ and $d(t,\partial_n t) \leq 2^{-n} < \epsilon$.

Consequently, a set $M$ of strongly extensional trees is compact iff the corresponding set of $\approx$-classes is closed (in the space of all strongly extensional trees modulo $\approx$). Explicitly: $M$ is compact iff for every collection $t_n \in M$ ($n < \omega$) and every strongly extensional tree $s$ with $\partial_n s = \partial_n t_n$ ($n < \omega$) there exists a tree $t \in M$ with $\partial_n s = \partial_n t$ for all $n < \omega$.

A strongly extensional tree $t$ is compactly branching if for every node $x$ the set of all maximum proper subtrees of $t_x$ is compact.

Corollary 4.5.10 [245]. The limit $V^\omega$ can be described as the set of all compactly branching strongly extensional trees.

A fourth description: $V^\omega$ is the set of strongly extensional saturated trees.

$$V^\omega = \text{all saturated, strongly extensional trees.}$$

Definition 4.5.11 [23]. A tree $t$ is called saturated provided that for every node $x$ has the following property: given a tree $s$ for which $x$ has children $y_n$ ($n < \omega$) such that $\partial_n(s) = \partial_n(t_{y_n})$, it follows that $x$ has a child $y$ with $s \approx t_y$. 

113
4 Finitary Set Functors

Example 4.5.12. (1) Every finitely branching tree is clearly saturated.
(2) The following tree of Example 4.5.1:

is not saturated. If $s$ denotes the single-path infinite tree, then the root of $t$ has children $y_n$ with $\partial_n(s) = \partial_n(ty_n)$, however, no child $y$ fulfils $s \approx ty$.

In contrast, the following tree is saturated (observe also that $t \approx t'$):

It turns out that a strongly extensional tree is saturated iff it is compactly branching:

Theorem 4.5.13 [23]. The limit $V_\omega$ can be described as follows:

$$V_\omega = \text{all saturated, strongly extensional trees.}$$

Other descriptions of $V_\omega$. Let us mention further descriptions due to Abramsky [1]:

(1) Cauchy completion of $\mu \mathcal{P}_t$. In Section 6.2 we will see that the initial algebra carries a canonical metric whose Cauchy completion is a metric space on the set $V_\omega$.

(2) Ideal completion of $\mu \mathcal{P}_t$. In Section 6.2 we will also see a canonical partial order on the initial algebra whose ideal (= free CPO) completion is carried by $V_\omega$.

(3) All maximal consistent theories of the modal logic $K$. [23, Theorem 5.11] proves that saturated trees precisely correspond to modally saturated trees defined by Fine [99] (when modal logic is taken without atoms). Moreover, we have obtained the following descriptions [23, Proposition 5.7, Theorem 5.9]:

$$V_\omega = \text{all maximal consistent theories in } K, \text{ and}$$

$$\nu \mathcal{P}_t = \text{all hereditarily finite theories in } K.$$ 

(4) Terminal coalgebra of a modified Hausdorff functor. In Example 5.2.25 we will present an endofunctor on the category CMS of complete metric spaces whose terminal coalgebra is carried by $V_\omega$.

Remark 4.5.14. (1) The pseudometric $d$ in (4.13) is a metric when restricted to $V_\omega = \text{all saturated, strongly extensional trees.}$ Indeed, for such trees $t$ and $u$ we have: $t \approx u$ implies $t = u$. (Recall that trees are considered up to isomorphism.)
(2) When $V_\omega$ is described by all saturated, strongly extensional trees and $\nu \mathcal{P}_t$ by the finitely branching ones, what is the canonical monomorphism $\bar{l}_0: \nu \mathcal{P}_t \to V_\omega$ of Remark 4.4.10? This is just the inclusion map. Indeed, the monomorphism $m$ of Notation 4.4.4 represents the subobject of $V_\omega$ of all trees finitely branching at the root. Analogously $\mathcal{P}_t m: \mathcal{P}_t^2 V_\omega \to \mathcal{P}_t V_\omega$ represents the subobject of all trees finitely branching on levels 0 and 1, etc. And $\bar{l}_0$, which is the intersection of these subobjects, thus represents all finitely branching trees in $V_\omega$.

4.6 Summary of this Chapter

In the present chapter we first proved some useful facts about constructions of (co)algebras for general endofunctors $F$: colimits of coalgebras (and dually limits of algebras) are canonically formed on the level of the base category, and the same is true for those limits of coalgebras that $F$ preserves (and dually for colimits of $F$-algebras). We then discussed quotient coalgebras and proved that whenever a weakly terminal coalgebra is given, then the smallest quotient coalgebra yields $\nu F$.

The main topic of this chapter was a construction of terminal coalgebras for finitary set functors. We presented a new proof of Worrell’s result [244] that $\nu F$ is a limit of the terminal-coalgebra chain after $\omega + \omega$ steps. We also observed that the object $V_\omega$ obtained after the first $\omega$ steps of that chain is a weakly terminal coalgebra. For the finite power-set functor $\mathcal{P}_t$ that limit was studied by a number of authors, and we presented various descriptions of $\nu \mathcal{P}_t$ and $V_\omega$ in the last section.
5 Finitary Iteration in Enriched Settings

In Chapter 3 we saw two constructions: the initial algebra as a colimit of (finitary) iterations and the terminal coalgebra as a limit of (finitary) iterations. Then in Chapter 4 we saw ways to “get our hands on” some of the terminal coalgebras which are obtained by iteration, namely the terminal coalgebras of finitary functors on Set. In this chapter, we strike out in a different direction by considering categories other than Set. In two important types of base categories, namely those enriched over complete partial orders and over complete metric spaces, it turns out that (under mild conditions on the endofunctor) the two constructions from Chapter 3 coincide. Moreover, they yield a fixed point that is canonical in the sense of the following definition (recall from Lambek’s Lemma that the structure morphism of an initial algebra has an inverse):

Definition 5.0.1. Let \( F : \mathcal{A} \to \mathcal{A} \) be an endofunctor on a category \( \mathcal{A} \). A canonical fixed point of \( F \) is an initial algebra \( a : FA \to A \) such that \( a^{-1} : A \to FA \) is a terminal coalgebra.

Almost none of the examples of initial algebras and terminal coalgebras in Set are canonical fixed points. (The exceptions are for the constant functors.) But we have seen some examples when we considered CPO\( \perp \) (complete partial orders with least element) and CMS (complete metric spaces). For example, a canonical fixed point of \( FX = X \perp \) on CPO\( \perp \) was presented in Examples 2.2.17(1) and 3.3.6. It is \( \mathbb{N}^\perp \), the natural numbers with an additional largest element \( \top \). What the CPO\( \perp \) and CMS settings have in common is a nice theory of approximation. In the case of CPO\( \perp \) approximation is given in terms of joins of \( \omega \)-chains, and in the case of CMS it is limits of Cauchy sequences. In this chapter, we are going to work with functors which preserve this additional approximation structure. We also typically work on a pointed category: this means that the initial object 0 is the same as the terminal object 1. We call 0 = 1 a zero object. In such a category we may lay the initial \( \omega \)-chain on top of the terminal \( \omega_{\text{op}} \)-chain, as follows:

\[
0 = 1 \xrightarrow{e} F1 \xrightarrow{Fe} F^21 \xrightarrow{F^2e} \ldots
\]

Here \( e \) is by initiality of 0 and \( \hat{e} \) is by finality of 1. Since clearly \( \hat{e} \cdot e = \text{id} \), the two chains are already related to each other. It turns out that under suitable hypotheses, we have a limit-colimit coincidence: the colimit of the initial \( \omega \)-chain exists and coincides with the limit of the terminal \( \omega_{\text{op}} \)-chain. For CPO\( \perp \) this is a classical result by Smyth and Plotkin [224] that we shall present in Section 5.1 (see Theorem 5.1.23). The case of CMS was first studied by America and Rutten [50], and we consider it in Section 5.2. In both settings, the “approximation structure” is taken to apply to the morphisms in the

117
category rather than the objects, and this is why we are really working with enriched categories in this chapter.

This chapter has two themes: the canonical fixed points which we just mentioned, but also fixed points of functors involving *mixed variance*. This topic is related to semantic models for the lambda calculus and other calculi. We are not going to discuss these applications in any detail. But we do want to foreshadow some of the ideas that will be prominent in the chapter. The search for interesting models of the lambda calculus essentially becomes a problem of the following form: can we find a cartesian closed category $\mathcal{A}$ and some object $X$ satisfying $X \cong [X,X]$? In this discussion,

$$[X,Y]$$

is the object in the category corresponding to the homset $\mathcal{A}(X,Y)$ which the cartesian closed structure provides. We would especially want constructions in concrete categories, and the methods should be general enough that they allow us to solve related equations such as $X \cong [X,X] + A$ for all objects $A$. It is clear that $\mathcal{A}$ cannot be $\text{Set}$, since the only way to have $X \cong [X,X]$ in $\text{Set}$ is for $X$ to have just one element. So one would want other categories. Indeed, the kinds of categories and functors used in our study are those that have canonical fixed points via the limit-colimit coincidence.

**Organization of the chapter.** The two themes of this chapter are each interesting and will be presented somewhat separately. Section 5.1 discusses canonical fixed points in categories enriched over complete partial orders. We present full detail in this section because our proofs are shorter than those in the literature, and also because this body of material does not seem to have been presented comprehensively in other books. Following this, we turn in Section 5.2 to canonical fixed points in the metric setting rather than the order-theoretic one. We close the chapter with a discussion of domain equations in Section 5.3. Section 5.2 may be read on its own, but some of the ideas in Section 5.1 are used there. Section 5.3 may be read independently of Section 5.2.

### 5.1 Canonical fixed points in $\text{CPO}$-enriched categories

Recall the category

$$\text{CPO}$$

of complete partial orders (see Example 2.1.7(1)) and continuous maps. The objects in the category need not have a least element $\bot$.

The category $\text{CPO}$ is cartesian closed: the internal hom-objects $[X,Y]$ are the hom-sets $\text{CPO}(X,Y)$ ordered pointwise. We denote by $\bigsqcup f_i$ the join of an $\omega$-chain formed by $f_i: X \to Y$, $i < \omega$.

**Definition 5.1.1.** A category $\mathcal{A}$ is $\text{CPO}$-enriched if its hom-sets come with a $\text{CPO}$ structure, and composition is continuous in both variables. That is, if $f_1 \subseteq f_2: A \to B$, then $h \cdot f_1 \cdot g \subseteq h \cdot f_2 \cdot g$ for all $h$ and $g$ composable with $f_1$ and $f_2$. And given an $\omega$-chain $f_i: X \to Y$, $h \cdot (\bigsqcup f_i) \cdot g = \bigsqcup h \cdot f_i \cdot g$. 

118
5.1 Canonical fixed points in CPO-enriched categories

Examples 5.1.2. (1) CPO is, of course, CPO-enriched using the above pointwise ordering on hom-sets.
(2) CPO⊥ is also CPO-enriched (see Example 2.1.7(2)).
(3) Analogously to cpos a poset is called a dcpo (directed-complete partial order) if every
directed subset has a join. The corresponding morphisms are the \( \Delta \)-continuous functions,
i.e. functions preserving directed joins. The category DCPO⊥ of directed-complete partial
orders with least element is again CPO-enriched.
(4) The category Pfn of sets and partial functions is CPO-enriched by inclusion: for
partial functions \( f_1, f_2 : X \rightarrow Y \) we put \( f_1 \sqsubseteq f_2 \) iff whenever \( f_1(x) \) is defined so is \( f_2(x) \)
and \( f_1(x) = f_2(x) \), i.e. the graph of \( f_1 \) is a subset of the graph of \( f_2 \). The least element
of Pfn(\( X, Y \)) is the nowhere defined function, and \( \bigsqcup f_i \) is the set-theoretic union for every
\( \omega \)-chain \( (f_i)_{i<\omega} \).
(5) The category Rel of sets and relations is also CPO-enriched by inclusion.
(6) CLat is the category of complete lattices with maps preserving joins and meets. This
category has two CPO-enriched structures. One is where we use the pointwise order in
each homset, and the other uses the dual of this order.
(7) If \( \mathcal{A} \) is CPO-enriched, then so is \( \mathcal{A}^{\text{op}} \). We order \( \mathcal{A}^{\text{op}}(Y, X) \) with the same relation as in \( \mathcal{A}(Y, X) \).
(8) A product of CPO-enriched categories is CPO-enriched with respect to the compo-
nentwise ordering.

Definition 5.1.3 [218]. In a CPO-enriched category, a morphism \( e : X \rightarrow Y \) is called an
embedding if there exists a morphism \( \hat{e} : Y \rightarrow X \) with
\[
\hat{e} \cdot e = \text{id}_X \quad \text{and} \quad e \cdot \hat{e} \sqsubseteq \text{id}_Y.
\] (5.2)

Let us check that the morphism \( \hat{e} \) is uniquely determined by (5.2). Suppose that
\( e^* \cdot e = \text{id}_X \) and \( e \cdot e^* \sqsubseteq \text{id}_Y \). Then \( \hat{e} = e^* \cdot e \cdot \hat{e} \sqsubseteq e^* \). Similarly, \( e^* \sqsubseteq \hat{e} \). Thus \( \hat{e} = e^* \). And
\( \hat{e} \) is called the projection for \( e \). A pair \((e, \hat{e})\) is called an embedding-projection pair.

Examples 5.1.4. (1) In CPO, the embeddings are precisely those monomorphisms \( e : X \rightarrow Y \)
such that for every \( y \in Y \) there exists a largest \( x \in X \) with \( e(x) \sqsubseteq y \). In fact,
this condition allows us to define \( \hat{e} : Y \rightarrow X \) by choosing this largest \( x \) as \( \hat{e}(y) \). Then
\( \hat{e} \cdot e = \text{id}_X \) (since \( e \) is one-to-one), and \( e \cdot \hat{e} \sqsubseteq \text{id}_Y \). The verification that the condition is
also necessary is trivial.
(2) It follows from (1) that in the category CLat of complete lattices, every monomorphism
is an embedding.
(3) Every monomorphism in Pfn is an embedding: monomorphisms are total injective
functions, and for these, \( \hat{e} \) is the (partially defined) inverse function \( e^{-1} \). Identifying
functions with their graph relations, \( \hat{e} \) is just the converse of \( e \).
(4) Every split monomorphism in Rel is an embedding: split monomorphisms are precisely
those relations \( e : X \rightarrow Y \) which are injective maps, and for these \( \hat{e} = e^{\text{op}} \).

Observation 5.1.5. Suppose that \( \mathcal{A} \) is a CPO-enriched category in which every hom-set
$\mathcal{A}(X,Y)$ has a least element $\perp_{XY}$.

(1) A general example of an embedding is a coproduct injection $v_i$ of an arbitrary coproduct $X = \bigsqcup_{i \in I} X_i$; indeed, $\hat{v}_i : X \to X_i$ has components $id_{X_i}$ and $\perp_{X_iX_j}$ for $j \neq i$. In particular, the morphisms with domain 0 are always embeddings.

(2) The name “projection” stems from the dual of (1): for every product $X = \prod_{i \in I} X_i$ the projection $\pi_i$ has the form $\hat{e}_i$ where $e_i : X_i \to X$ has components $id_{X_i}$ and $\perp_{X_iX_j}$.

**Notation 5.1.6.** The objects of a CPO-enriched category $\mathcal{A}$ together with the embeddings form a category $\mathcal{A}^E$.

The point is that a composite $e \cdot f$ of embeddings is itself an embedding with

$$\hat{e} \cdot \hat{f}.$$  \hfill (5.3)

**Remark 5.1.7.** Let $\mathcal{A}$ be a CPO-enriched category in which every hom-set also has meets of decreasing $\omega$-chains. For example, the categories CPO, Pfn, Rel and CLat in Example 5.1.2 have this property. Then $\mathcal{A}^E$ is a CPO-enriched category, since given an $\omega$-chain $e_i$ of embeddings in $\mathcal{A}(X,Y)$, the join $\bigsqcup_i e_i$ is an embedding with projection given by the meet $\bigcap_i \hat{e}_i$.

**Observation 5.1.8.** If $e : X \to Y$ is an embedding in $\mathcal{A}$, then in $\mathcal{A}^{op}$, $\hat{e} : X \to Y$ is an embedding. So the dual category of $\mathcal{A}^E$ is the category whose objects are those of $\mathcal{A}$ and whose morphisms are the projections.

The following lemma, due to Smyth and Plotkin [224], could look technical at first glance. Its intend is to consider cocones on chains of embeddings and to connect the global property of a given cocone (being a colimit) and the local property mentioned in (5.5) below. This connection is exploited in all the rest of the results in this section.

**Basic Lemma 5.1.9.** Let $\mathcal{A}$ be a CPO-enriched category. Consider an $\omega$-chain of embeddings in $\mathcal{A}$

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots$$ \hfill (5.4)

Let $c_i : E_i \to C$ be a cocone. Then the following are equivalent:

(1) $(c_i)$ is a colimit cocone.

(2) Each $c_i$ is an embedding, the composites $c_i \cdot \hat{c}_i$ form an $\omega$-chain in $\mathcal{A}(C,C)$, and

$$\bigsqcup_i c_i \cdot \hat{c}_i = id_C.$$  \hfill (5.5)

**Proof.** (1) $\Rightarrow$ (2): For every $i$ we have the following cocone of the shortened chain with
5.1 Canonical fixed points in CPO-enriched categories

codomain $E_i$:

\[
\begin{array}{c}
E_i \xrightarrow{e_i} E_{i+1} \xrightarrow{e_{i+1}} E_{i+2} \xrightarrow{e_{i+2}} \cdots \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\hat{e}_i \quad \quad \hat{e}_{i+1} \\
\downarrow \quad \quad \downarrow \\
E_i+1 \\
\end{array}
\]

Since the shortened chain has the same colimit as the original chain, namely $C$, there exists a unique morphism $\hat{c}_i : C \to E_i$ such that the triangle on the left below commutes for all $k \geq i \geq 0$:

\[
\begin{array}{c}
E_k \xrightarrow{e_k} \cdots \xrightarrow{e_{i+1}} E_i \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
C \xrightarrow{c_i} E_i \\
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{c}
E_i \xrightarrow{e_i} \cdots \xrightarrow{e_{k-1}} E_k \\
\downarrow \quad \quad \downarrow \\
C \xrightarrow{c_k} E_k \\
\end{array}
\]

In particular

\[
\hat{c}_i \cdot c_i = \text{id}_{E_i}. \tag{5.8}
\]

We verify the triangles on the right in (5.7) commute by induction on $n = k - i$. (5.8) is the base case. Assuming the commutativity when the difference is $n - 1$, we get it when the difference is $n$:

\[(e_k \cdot e_{k-1} \cdot \cdots \cdot e_{i+1}) \cdot e_i = \hat{c}_k \cdot c_{i+1} \cdot e_i = \hat{c}_k \cdot c_i.\]

Comparing the left-hand triangles in (5.7) for $\hat{c}_i$ and $\hat{c}_{i+1}$, we see that given $k \geq i + 1$ we have

\[
\hat{c}_i \cdot c_k = \hat{c}_i \cdot \hat{c}_{i+1} \cdot c_k.
\]

Now $(c_k)_{k \geq i+1}$ is a colimit cocone and therefore a jointly epic family. Thus, we conclude

\[
\hat{c}_i = \hat{c}_i \cdot \hat{c}_{i+1}. \tag{5.9}
\]

Hence, the morphisms $c_i \cdot \hat{c}_i$ form a chain in $\mathcal{A}(C, C)$:

\[
c_i \cdot \hat{c}_i = (c_{i+1} \cdot e_i) \cdot (\hat{e}_i \cdot \hat{c}_{i+1}) \sqsubseteq c_{i+1} \cdot \hat{c}_{i+1}
\]

due to $e_i \cdot \hat{e}_i \sqsubseteq \text{id}$. We next prove that the join of this chain is $\text{id}_C$, thus also proving that $c_i \cdot \hat{c}_i \sqsubseteq \text{id}$; this inequality together with (5.8) establishes that $c_i$ is an embedding. To see that $\text{id}_C$ is the desired join it is sufficient to prove that for every $k$

\[
\left( \bigsqcup_i c_i \cdot \hat{c}_i \right) \cdot c_k = c_k.
\]

For this, we need only show that for $i \geq k$, $c_i \cdot \hat{c}_i \cdot c_k = c_k$. Using the triangle on the right in (5.7) and interchanging $k$ and $i$, we see that

\[
c_i \cdot \hat{c}_i \cdot c_k = c_i \cdot e_i \cdot e_{i-1} \cdot \cdots \cdot e_k = c_k.
\]
We have proved (5.5). This verifies all parts of (2) in our lemma.

(2) \implies (1): Suppose we are given a cocone \(b_i : E_i \to B\) of the given chain:

\[
\begin{array}{c}
C \\
\downarrow e_0 \downarrow e_1 \downarrow e_2 \cdots \\
E_0 \\
b_0 \downarrow b_1 \downarrow b_2 \\
B
\end{array}
\]

We first observe that the morphisms \(b_i \cdot \hat{c}_i\) form a chain in \(\mathcal{A}(C, B)\). Since the \(c_i\)'s are embeddings and \(c_i = c_{i+1} \cdot e_i\), from (5.3) we get (5.9). Thus

\[
\begin{align*}
b_i \cdot \hat{c}_i &= (b_{i+1} \cdot e_i) \cdot (\hat{c}_i \cdot \hat{c}_{i+1}) = b_{i+1} \cdot (e_i \cdot \hat{c}_i) \cdot \hat{c}_{i+1} \subseteq b_{i+1} \cdot \hat{c}_{i+1}.
\end{align*}
\]

Define

\[
b = \bigsqcup_i (b_i \cdot \hat{c}_i) : C \to B. \tag{5.10}\]

We show that this is the desired factorization: we fix \(k\) and show that \(b \cdot c_k = b_k\). For this, we may restrict the join to \(i \geq k\). So we shall show that

\[
b \cdot c_k = \bigsqcup_{i \geq k} (b_i \cdot \hat{c}_i \cdot c_k) = b_k. \tag{5.11}\]

Recall that \((c_i)\) is a cocone. Thus for \(i \geq k\), the top triangle below commutes:

\[
\begin{array}{c}
E_k \downarrow e_k \downarrow e_{k+1} \downarrow e_{i-1} \cdots \downarrow e_i \\
C \downarrow \hat{c}_i \downarrow \hat{c}_{i+1} \downarrow \cdots \\
E_i \\
b_i \downarrow b \\
B
\end{array}
\]

The other triangles clearly commute, and so the outside does too. And as \((b_i)\) is a cocone, we have

\[
b_i \cdot \hat{c}_i \cdot c_k = b_i \cdot (e_{i-1} \cdots \cdot e_k) = b_k.
\]

This for all \(i \geq k\) shows (5.11). We have shown that \(b\) is a morphism through which the cocone \((b_i)\) factorizes.

The factorization is unique: Given \(b' : C \to B\) with \(b' \cdot c_i = b_i\) for all \(i < \omega\), we have, due to (5.5)

\[
b' = b' \cdot \bigsqcup_i c_i \cdot \hat{c}_i = \bigsqcup_i b_i \cdot \hat{c}_i = b,
\]

which completes the proof. \(\square\)
Remark 5.1.10. If every \( b_i, i < \omega \) is an embedding, then so is the morphism \( b \) of (5.10). Indeed, we have \( c_i = c_{i+1} \cdot c_i \) and \( \hat{b}_i = \hat{e}_i \cdot \hat{b}_{i+1} \). Thus
\[
c_i \cdot \hat{b}_i = c_{i+1} \cdot (e_i \cdot \hat{e}_i) \cdot \hat{b}_{i+1} \subseteq c_{i+1} \cdot \hat{b}_{i+1}.
\]
We define \( \hat{b} \) to be \( \bigsqcup_i c_i \cdot \hat{b}_i \). This is a projection for \( b \):
\[
b \cdot \hat{b} = (\bigsqcup_i b_i \cdot \hat{c}_i) \cdot (\bigsqcup_i c_i \cdot \hat{b}_i) = \bigsqcup_i b_i \cdot \hat{c}_i \cdot c_i \cdot \hat{b}_i = \bigsqcup_i b_i \cdot \hat{b}_i \subseteq \text{id}_B.
\]
and also \( \hat{b} \cdot b = \text{id}_C \), due to (5.5):
\[
\hat{b} \cdot b = \bigsqcup_i c_i \cdot \hat{b}_i \cdot b_i \cdot \hat{c}_i = \bigsqcup_i c_i \cdot \hat{c}_i = \text{id}_C.
\]

Remark 5.1.11. Although formulated for \( \omega \)-chains only, the Basic Lemma holds for \( \lambda \)-chains for all limit ordinals \( \lambda \) (in categories enriched so that hom-sets are posets with joins of all chains and composition preserves joins of chains). This is immediately seen from the proof above.

We draw several corollaries from this Basic Lemma.

Corollary 5.1.12. Consider an \( \omega \)-chain of embeddings in a CPO-enriched category \( \mathcal{A} \)
\[
E_0 \xrightarrow{\hat{e}_0} E_1 \xrightarrow{\hat{e}_1} E_2 \xrightarrow{\hat{e}_2} \ldots
\]
Let \( (c_i) \) be a colimit cocone. Then \( (\hat{c}_i) \) is a limit cone of the \( \omega^{\text{op}} \)-chain of projections \( \hat{c}_i : E_{i+1} \to E_i \).

Proof. First, note that the morphisms \( c_i \) are embeddings by the Basic Lemma 5.1.9, and so the notation \( \hat{c}_i \) makes sense. We also know that \( \bigsqcup_i c_i \cdot \hat{c}_i = \text{id}_C \). Now recall that the category \( \mathcal{A}^{\text{op}} \) is also CPO-enriched. In this category, we have a chain of embeddings
\[
E_0 \xleftarrow{\hat{c}_0} E_1 \xleftarrow{\hat{c}_1} E_2 \xleftarrow{\hat{c}_2} \ldots
\]
each \( \hat{c}_i \) is also an embedding, \( \hat{c}_i \cdot c_i \) is a chain, and \( \bigsqcup_i \hat{c}_i \cdot c_i = \text{id}_C \). Thus by the Basic Lemma 5.1.9 again, \( (\hat{c}_i) \) is a colimit cocone in \( \mathcal{A}^{\text{op}} \) of \( (\hat{c}_i) \). Moving from \( \mathcal{A}^{\text{op}} \) back to \( \mathcal{A} \), we see that \( (\hat{c}_i) \) is a limit cone in \( \mathcal{A} \) of \( (\hat{c}_i) \), as desired.

Remark 5.1.13. Thus in a CPO-enriched category we have a limit-colimit coincidence for \( \omega \)-chains of embeddings (or, equivalently, for \( \omega^{\text{op}} \)-chains of projections in the sense of Definition 5.1.3). A colimit cone of an \( \omega \)-chain of embeddings yields a limit cone of the corresponding \( \omega^{\text{op}} \)-chain of projections, and vice versa.

Corollary 5.1.14. If a CPO-enriched category has colimits of \( \omega \)-chains of embeddings, then \( \mathcal{A}^{\text{op}} \) has colimits of all \( \omega \)-chains.

Proof. Let \( (e_i) \) be an \( \omega \)-chain in \( \mathcal{A}^{\text{op}} \) and let \( (c_i) \) be a colimit cocone in \( \mathcal{A} \) for the underlying \( \omega \)-chain in \( \mathcal{A} \). Then by the Basic Lemma 5.1.9, each colimit morphism \( c_i \) is itself an embedding. And given a cocone \( (b_i) \) for \( (e_i) \) in \( \mathcal{A}^{\text{op}} \), the factorization morphism \( b \) is an embedding by Remark 5.1.10.
At this point, we need an important definition in this subject. The concept of local continuity is important because it is typically easier to check than the preservation of \( \omega \)-colimits.

**Definition 5.1.15.** A functor \( F : \mathcal{A} \to \mathcal{B} \) between CPO-enriched categories is called **locally continuous** if it is CPO-enriched, i.e. it preserves the order of hom-sets, as well as joins of \( \omega \)-chains: given \( f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots \) in \( \mathcal{A}(X,Y) \), then \( F(\bigsqcup_n f_n) = \bigsqcup_n Ff_n \).

For the following examples recall from Example 2.1.7(1) that coproducts in CPO are disjoint unions, and products are given by cartesian products with coordinate-wise order. More generally, a limit of a diagram in CPO is the limit of the underlying diagram in \( \text{Set} \) with the largest order making every limit projection monotone (we leave the easy proof of this fact to the reader).

**Examples 5.1.16.** Here are some examples of locally continuous endofunctors on CPO-enriched categories.

1. \( \text{Id} \) is locally continuous, and so is every constant functor.
2. A composite, product or coproduct of locally continuous functors is locally continuous; this is easy to prove, see also Barr [57].
3. In analogy to the polynomial endofunctors on \( \text{Set} \), given a collection \( \Sigma = (\Sigma_n)_{n<\omega} \) of cpos, we denote by \( H_{\Sigma} \) the endofunctor on CPO given by \( H_{\Sigma}X = \bigsqcup_{n<\omega} \Sigma_n \times X^n \). Its local continuity follows from (1) and (2).
4. The functor \( FX = X^\perp \) on CPO from Example 2.1.7(1) is locally continuous.

**Example 5.1.17.** Ideal completion is an example of a natural endofunctor that is not locally continuous. For every poset \( P \) the ideal completion is the free dcpo on it. By an ideal is meant a nonempty directed down-set in \( P \). The ideal completion \( \text{Idl}(P) \) is the poset of all ideals ordered by inclusion. In particular, every element \( x \in P \) defines the prime ideal \( \downarrow x = \{y \in P : y \leq x\} \), and \( x \leq y \) holds in \( P \) iff \( \downarrow x \subseteq \downarrow y \) in \( \text{Idl}(P) \). Therefore, \( P \) may be identified with the subposet of \( \text{Idl}(P) \) given by the prime ideals. The universal property is that every monotone map from \( P \) to a cpo has a unique continuous extension to a continuous map from \( \text{Idl}(P) \) to that cpo, see Abramsky and Jung [2].

Denote by \( I \) the endofunctor on CPO mapping a cpo \( X \) to \( \text{Idl}(X) \) and a continuous map \( f : X \to Y \) to the continuous extension \( \text{Idl}(X) \to Y \) composed with the embedding \( Y \hookrightarrow \text{Idl}(Y) \). This endofunctor preserves order on homsets, but it is not locally continuous. To see this, recall the poset \( \mathbb{N}^T \) from Example 3.3.7(1). Let \( f_i : \mathbb{N}^T \to \mathbb{N}^T \) be \( n \mapsto \min(n,i) \). Then \( \bigsqcup_i f_i = \text{id} \). But \( \text{Idl}(\mathbb{N}^T) \) is the poset \( 0 < 1 < 2 < \cdots < T < T' \). Clearly, \( \text{Idl}(\text{id})(T') = \text{id}(T') = T' \). However, \( \text{Idl}(f_i)(T') = \{0,1,\ldots,i\} \), and so \( \bigsqcup_i \text{Idl}(f_i)(T') = T' \). This example is from [111, Example IV-5.12].

**Remark 5.1.18.** Most of the functors found in domain theory are locally continuous. Here are some further examples.

1. Consider \( F : \text{CPO}^{op} \times \text{CPO} \to \text{CPO} \) defined as follows: \( F(D,E) = [D,E] \) is the set of CPO morphisms as a CPO object. On
5.1 Canonical fixed points in CPO-enriched categories

morphisms, \((f, g): (D, E) \to (X, Y)\), define \(F(f, g)\) as the function which takes \(u: D \to E\) to \(f \cdot u \cdot g\). The category \(\text{CPO}^{\text{op}} \times \text{CPO}\) is CPO-enriched due to Example 5.1.2(7), (8). Moreover, \(F\) is easily seen to be locally continuous.

(2) In contrast, the functor \(F: \text{Pfn}^{\text{op}} \times \text{Pfn} \to \text{Pfn}\) defined by \(F(D, E) = \text{Pfn}(D, E)\) and \(F(f, g)\) sending \(u\) to \(f \cdot u \cdot g\) is not locally continuous: \(F(f, g)\) is a total function. Thus, if \(f \subseteq f'\) but \(f' \nsubseteq f\), we have \(F(f, g) \nsubseteq F(f', g)\).

(3) On the category of \(\omega\)-algebraic \(\text{CPO}'s\), the Plotkin powerdomain operation (i.e. the formation of a free semilattice in that category) extends to a locally continuous functor (see e.g. Knijnenburg [151]).

We have another corollary to the Basic Lemma, connecting local continuity to the preservation of \(\omega\)-colimits.

**Corollary 5.1.19.** Let \(\mathcal{A}\) be a CPO-enriched category. Then every locally continuous endofunctor on \(\mathcal{A}\) preserves colimits of \(\omega\)-chains of embeddings.

**Proof.** Let \(F: \mathcal{A} \to \mathcal{A}\) be locally continuous. Consider an \(\omega\)-chain of embeddings in \(\mathcal{A}\)

\[
E_0 \xrightarrow{c_0} E_1 \xrightarrow{c_1} E_2 \xrightarrow{c_2} \cdots
\]

Let \(c_i: E_i \to C\) be a colimit cocone. Since \(F\) preserves the order on homsets, each \(Fc_i\) is an embedding and \(\tilde{F}c_i = Fc_i\). Each composite \(Fc_i \cdot \tilde{F}c_i = F(c_i \cdot \tilde{c}_i)\). Thus the composites \(Fc_i \cdot \tilde{F}c_i\) form a chain in \(\mathcal{A}(FC, FC)\), and their join is \(\text{id}_{FC}\) using local continuity of \(F\):

\[
\bigsqcup_i Fc_i \cdot \tilde{F}c_i = F\left(\bigsqcup_i c_i \cdot \tilde{c}_i\right) = F\text{id}_C = \text{id}_{FC}.
\]

By Basic Lemma 5.1.9, \((Fc_i)\) is a colimit cocone. \(\square\)

We now restrict attention to certain CPO-enriched categories pertaining to least elements. We should point out that our next definition has a certain subtlety: this is not the same thing as being enriched over \(\text{CPO}_{\perp}\).

**Definition 5.1.20.** A strict CPO-enriched category is a nonempty CPO-enriched category \(\mathcal{A}\) with a least element \(\perp_{XY}\) in every hom-CPO \(\mathcal{A}(X, Y)\) such that for all \(f: Y \to Y'\) we have

\[
f \cdot \perp_{XY} = \perp_{XY'}\quad\text{and}\quad \perp_{YZ} \cdot f = \perp_{YZ}.
\]

**Example 5.1.21.** The category \(\text{CPO}_{\perp}\) of complete partial orders with least element and continuous functions that are strict, i.e. preserve that least element, is a strict CPO-enriched category. Also, \(\text{DCPO}_{\perp}\), \(\text{Pfn}\) and \(\text{Rel}\) are strict CPO-enriched.

**Lemma 5.1.22** [57]. Every strict CPO-enriched category with \(\omega\)-colimits has a zero object \(0 = 1\).

**Proof.** Choose any object \(X\) and form the \(\omega\)-chain whose objects are all equal to \(X\) and whose connecting morphisms are \(\perp_{XX}\). The colimit \(c_n: X \to C\) then fulfills \(c_n = c_{n+1} \cdot \perp_{XX} = \perp_{XC}\) for all \(n\). Thus \(C\) is initial: for every object \(Y\) the unique morphism
from $C$ is $\perp_{CY}$. Indeed, given $f: C \to Y$ then $f \cdot c_n = f \cdot \perp_{CX} = \perp_{XY} = \perp_{CY} \cdot e_n$ for all $n$, thus, $f = \perp_{CY}$ since the colimit injections are jointly epic. And $C$ is terminal: the unique morphism from $Y$ to $C$ is $\perp_{YC}$ because clearly $\perp_{CC} = \text{id}_C$, hence, given $f: Y \to C$ we have $f = \text{id}_C \cdot f = \perp_{CC} \cdot f = \perp_{YC}$. \hfill \Box

The next result provides a sufficient condition for the existence of canonical fixed points. It is the centerpiece of this section.

**Theorem 5.1.23** [224]. Let $\mathcal{A}$ be a strict CPO-enriched category with $\omega$-colimits. Every locally continuous endofunctor $F: \mathcal{A} \to \mathcal{A}$ has a canonical fixed point

$$\mu F = \nu F = \text{colim} F^n 0.$$

**Proof.** The unique morphism $e: 0 \to F0$ is an embedding with projection $\widehat{e} = \perp_{F0,0}$. Indeed, $\widehat{e} \cdot e = \text{id}$ is clear, and $e \cdot \widehat{e} = e \cdot \perp_{F0,0} = \perp_{F0,F0} \sqsubseteq \text{id}_{F0}$. In (5.1), we have the initial-algebra chain and the terminal-coalgebra chain of $F$ together, and all the pairs of morphisms are embedding-projection pairs. Let $c_i: F^i 1 \to C$ be a colimit cocone of the initial-algebra chain. By Corollary 5.1.19, $F$ preserves the colimit of its initial-algebra chain. Moreover, by Corollaries 5.1.12 and 5.1.19, $(\widehat{c}_i)$ is a limit cone of the terminal-coalgebra chain of $F$, and $(F\widehat{c}_i)$ is a limit cone of the same chain, but again without the first term.

As a result, the initial algebra $\mu F$ and the terminal coalgebra $\nu F$ exist. The structure $\iota$ of the initial algebra is, by Remark 3.1.8, defined by $\iota \cdot Fc_i = c_{i+1}$. Following equation (5.10) this yields

$$\iota = \bigsqcup_{i \geq 1} c_i \cdot F\widehat{c}_{i-1}.$$

Dually, the structure $\tau$ of the terminal coalgebra fulfils

$$\tau = \bigsqcup_{i \geq 1} Fc_{i-1} \cdot \widehat{c}_i.$$

It is our task to prove that $\iota$ is inverse to $\tau$:

$$\iota \cdot \tau = \bigsqcup_{i \geq 1} c_i \cdot F(\widehat{c}_{i-1} \cdot c_{i-1}) \cdot \widehat{c}_i \quad \tau \cdot \iota = \bigsqcup_{i \geq 1} F(c_i \cdot \widehat{c}_{i-1}) \cdot (\bigcup_{i \geq 1} F\widehat{c}_i)$$

$$= \bigsqcup_{i \geq 1} c_i \cdot \widehat{c}_i \quad = \bigsqcup_{i \geq 1} Fc_i \cdot F\widehat{c}_i$$

$$= \text{id} \quad = \text{id}$$

We have used the Basic Lemma once again. This proves $\tau = \iota^{-1}$. \hfill \Box

**Remark 5.1.24.** Instead of assuming all $\omega$-colimits in $\mathcal{A}$, it is sufficient in Theorem 5.1.23 above to assume a zero object $0 = 1$ and colimits of $\omega$-chains of embeddings. Also, instead of assuming that $F$ be locally continuous, we could assume the weaker condition that it preserve $\omega$-colimits of embeddings.

**Example 5.1.25.** (1) Continuous algebras with a binary operation and a constant considered in CPO consist of a cpo $A$, a continuous operation $\alpha_0: A \times A \to A$, and
an element of $A$. As we have previously observed, CPO has limits, coproducts, and furthermore, it has colimits of chains of monomorphisms preserved by the forgetful functor into Set. A continuous algebra is given by a single morphism $A \times A + 1 \to A$, viz. the structure of an algebra for the endofunctor $FX = X \times X + 1$.

This functor preserves colimits of $\omega$-chains of monomorphism, thus $\mu F = \text{colim}_{n<\omega} F^n 0$ (see Theorem 3.1.7). As in Example 3.2.5 the underlying set is that of all finite binary trees with the discrete order (since $F$ preserves discrete posets).

The functor $F$ also preserves limits of $\omega^{\text{op}}$-chains. Thus, $\nu F = \lim_{n<\omega} F^n 1$ (see Theorem 3.3.4). As in Theorem 3.3.10 this is the set of all binary trees, again with the discrete order.

(2) Let us now consider continuous algebras with a binary operation and a constant in CPO$\perp$, where operations are required to be continuous (but not necessarily strict), and morphisms are the strict continuous homomorphisms. A non-strict binary operation on a poset $A$ can be expressed by a strict continuous functions from $(A \times A)_{\perp}$ to $A$, where $(-)_{\perp}$ is the lifting, see Example 2.1.7(2). Recall that coproducts in CPO$\perp$ are formed by identifying in the disjoint union the bottom elements. Thus, the structure of a continuous algebra is expressed by a single morphism $\alpha : (A \times A)_{\perp} + 1_{\perp}$, i.e. the structure of an algebra for the endofunctor $FX = (X \times X)_{\perp} + 1_{\perp}$ from Example 2.1.7(2). It is easy to see that $F$ is locally continous so the above theorem it has a canonical fixed point $\mu F = \nu F = \text{colim}_{n<\omega} F^n 0$.

Let us represent the initial object of CPO$\perp$ by $\{\perp\}$ where $\perp$ is considered as a single node labelled tree. Given a tree representation of $F^n 0$, represent pairs $(t_1, t_2)$ in $F^n 0 \times F^n 0$ as binary trees with maximum subtrees $t_1$ and $t_2$. Then it is easy to see that $F^n 0$ is the cpo of all binary trees of height at most $n$ ordered by

$t \sqsubseteq t'$ iff $t$ is a cutting of $t'$ at some level.

The connecting maps $F^n 0 \hookrightarrow F^{n+1} 0$ are the inclusions. It follows that the canonical fixed point is

$\mu F = \nu F = \text{all binary trees},$

ordered as above.

(3) Generalizing the previous example, for every finitary signature $\Sigma$ we consider continuous algebras in CPO$\perp$. Operations are continuous and homomorphisms are the strict continuous $\Sigma$-homomorphisms. They are precisely the $F$-algebras for the endofunctor given by

$FX = \prod_{\sigma \in \Sigma} (X^n)_{\perp},$

where $n$ denotes the arity of $\sigma \in \Sigma$. This functor is locally continous. Its canonical fixed point is $T_\Sigma$, the algebra of all $\Sigma$-trees, with the following order [40]. Given $t \in T_\Sigma$ as a partial function on $\mathbb{N}^*$ as explained in Remark 2.2.13, let $\hat{t} : \mathbb{N}^* \to (\prod_{n \in \mathbb{N}} \Sigma_n)_{\perp}$, where the coproduct on the right is discretely ordered, be the total function with $\hat{t}(w) = \perp$ if $t(w)$ is undefined and $\hat{t}(w) = t(w)$ else. Then for two trees $s, t \in T_\Sigma$ put

$s \sqsubseteq t$ iff $\hat{s}(w) \sqsubseteq \hat{t}(w)$ for all $w \in \mathbb{N}^*$. 127
Remark 5.1.26. In our next example we work in the strict CPO-enriched category Rel of sets and relations. Coproducts in this category are disjoint unions (as in Set). Since Rel is self-dual, products are also given by disjoint unions with the projections the opposite relations to the coproduct injections. Moreover, an embedding in Rel is precisely an injective function $e: A \to B$ with the projection $\hat{e}$ the opposite relation. Colimits of $\omega$-chains of embeddings are the unions, again as in Set.

Example 5.1.27 [127]. We have discussed deterministic automata in Examples 2.4.2(3) and 2.5.5, and we have seen non-deterministic automata as coalgebras over Set in Example 2.4.2(5). Here we consider them as coalgebras over the category Rel. Recall that non-deterministic automata (ndas) are defined in the same way as deterministic ones, except that the transition functions $\delta_s: S \to S$ for every $s \in \Sigma$ are replaced by transition relations $\delta_s \subseteq S \times S$. Thus, in the category Rel, an nda is given by an object $S$ of states, a morphism $S \to 1$ relating the accepting states to the single element of 1 and a morphism $\delta: S \to \Sigma \times S$ (where $\times$ denotes cartesian product) relating a state $x \in S$ to all pairs $(s, y)$ with $(x, y) \in \delta_s$. We can combine this to obtain a single morphism

$$\alpha: S \to 1 + \Sigma \times S,$$

whence ndas are precisely the coalgebras for the functor $F: \text{Rel} \to \text{Rel}$ given by the coproduct of 1 and $\Sigma$ copies of $\text{Id}$, shortly $F X = 1 + \Sigma \times S$. The initial-algebra chain

$$
\emptyset \xrightarrow{1} 1 \xrightarrow{F1} 1 + \Sigma \xrightarrow{FF1} 1 + \Sigma + \Sigma \times S \xrightarrow{F^3} \ldots
$$

coincides with that of the corresponding endofunctor on Set, since the connection relations are simply the inclusion maps. Thus, the colimit is $\Sigma^*$. It follows from Theorem 5.1.23 that the canonical fixed point of $F$ is $\mu F = \nu F = \Sigma^*$.

It is easy to see that a coalgebra homomorphism from an nda $\alpha: S \to 1 + \Sigma \times S$ to an nda $\alpha': 1 + \Sigma S'$ is a relation $R: S \to S'$ such that

1. a state $x \in S$ is accepting iff $x R x'$ for some accepting state $x' \in S'$, and
2. for every triple $(x, s, x') \in S \times \Sigma \times S'$ the following statements are equivalent
   - there is a transition $x \xrightarrow{s} y$ in $S$ with $y R x'$, and
   - there is a transition $z \xrightarrow{s} x'$ in $S'$ with $x R z$.

For example, for every nda $(S, \alpha)$ the unique coalgebra homomorphism $R: S \to \Sigma^*$ to the terminal coalgebra relates every state $x$ to the language $L(x)$ it accepts, i.e. we have $x R w$ iff $w \in L(x)$.

In contrast, we saw in Example 4.5.9 that the semantics of non-deterministic automata considered them as coalgebras over Set yields the behaviour of states modulo bisimilarity rather than the usual language semantics.

We have seen the finite power set functor $\mathcal{P}_f: \text{Set} \to \text{Set}$ at several points, and we gave many descriptions of its terminal coalgebra. So it should be interesting to consider analogs of $\mathcal{P}_f$ on Rel.
5.1 Canonical fixed points in CPO-enriched categories

Example 5.1.28. (1) The functor $P_!$ has a lifting $P_! : \text{Rel} \to \text{Rel}$ assigning to every morphism $r : X \to Y$, i.e. a relation $r \subseteq X \times Y$, the relation $P_! r \subseteq P_! X \times P_! Y$ that consists of all pairs $(A, B)$ of finite sets such that

$$\text{for every } a \in A \text{ there exist } b \in B \text{ with } (a, b) \in r.$$  \hfill (5.12)

It is easy to see that this is a well-defined locally continuous functor on $\text{Rel}$. Indeed, suppose that $r_n : X \to Y$, $n < \omega$, is an $\omega$-chain of relations and $r = \bigcup_{n<\omega} r_n$. Then a pair $(A, B) \in P_! X \times P_! Y$ lies in $P_! r$ iff there exists an $n < \omega$ such that $(A, B)$ lies in $P_! r_n$, using the fact that $A$ and $B$ are finite.

Therefore $P_!$ has a canonical fixed point by Theorem 5.1.23 which is a colimit of its initial-algebra chain. Since this is a chain of embeddings and colimits of such chains in $\text{Rel}$ are the same as in $\text{Set}$, we conclude that

$$\mu P_! = \nu P_! = V_{\omega},$$

i.e. the hereditarily finite sets (see Example 2.2.7(1)).

(2) There are other locally continuous liftings of $P_!$ to $\text{Rel}$. For example, the lifting $P'_!$ obtained by changing (5.12) to

$$\text{for every } b \in B \text{ there exists } a \in A \text{ with } (a, b) \in r,$$

or $P''_!$ defined such that $P''_! r$ is the intersection of $P'_! r$ and $P'_! r$. The canonical fixed points of $P'_!$ and $P''_!$ are given by $V_{\omega}$, again.

Remark 5.1.29. In some applications one uses categories such as $\text{CPO}_*$ whose objects are complete partial orders with $\bot$ and morphisms are continuous maps (not necessarily preserving $\bot$). Observe that this means that composition of morphisms is left-strict: $\bot \cdot f = \bot$, but not necessarily right-strict:

Definition 5.1.30. A CPO-enriched category is called left-strict if every hom-set has a least element $\bot$ and every morphism $f$ fulfils $\bot \cdot f = \bot$.

Example 5.1.31. Smyth-Plotkin’s Theorem 5.1.23 does not extend to left-strict CPO-enriched categories. Indeed, the identity functor on $\text{CPO}_*$ does not have an initial algebra although it is locally continuous. (The “natural candidate” $\text{id}_{\bot}$ is not an initial algebra: consider any algebra with two fixed points.) For that matter, $\text{CPO}_*$ does not have an initial object. This shows that Lemma 5.1.22 requires the strict CPO-enrichedness. However, Freyd [103] proved a weaker result:

Proposition 5.1.32. Let $\mathcal{A}$ be a left-strict CPO-enriched category. If a locally continuous functor $F$ has an initial algebra, then $\mu F$ is a canonical fixed point.

Proof. Let $\iota : FI \to I$ be an initial algebra. For every coalgebra $\alpha : A \to FA$, we prove that a unique homomorphism into $(I, \iota^{-1})$ exists.

(1) Existence. The endomap $g$ of $\mathcal{A}(A, I)$ given by $h \mapsto \iota \cdot Fh \cdot \alpha$ is obviously continuous, hence, it has a fixed point $h : A \to I$ with $\iota^{-1} \cdot h = F h \cdot \alpha$ by the Kleene Theorem 3.1.1. This is a coalgebra homomorphism. Observe that $h = \bigsqcup g^i(\bot)$ by Kleene’s Theorem.
(2) Uniqueness. First notice that for \( \mathcal{A}(I, I) \) we have an analogous endomorphism \( k \mapsto \iota \cdot Fk \cdot \iota^{-1} \). Since \( I \) is initial, the only fixed point is \( k = \text{id}_I \). And this is the least fixed point, thus, \( \text{id}_I = \bigoplus_i u_i \) where \( u_0 = \bot \) and \( u_{i+1} = \iota \cdot Fu_i \cdot \iota^{-1} \). Now suppose that \( h' \colon (A, \alpha) \to (I, \iota^{-1}) \) is any coalgebra homomorphism. Next we show that for every \( i \) we have
\[
u_i \cdot h' = \varrho_i(\bot).
\]
We verify this by induction: \( u_0 \cdot h' = \bot \cdot h' = \bot = \varrho_0(\bot) \) and
\[
u_i+1 \cdot h' = \iota \cdot Fu_i \cdot \iota^{-1} \cdot h' = \iota \cdot Fu_i \cdot Fh' \cdot \alpha = \iota \cdot F\varrho_i(\bot) \cdot \alpha = \varrho_{i+1}(\bot).
\]
Consequently,
\[
u_i+1 \cdot h' = \bigoplus_i \varrho_i(\bot).
\]

One of the themes of this section has been canonical fixed points. We saw in Theorem 5.1.23 conditions on a category and a functor that insure the existence of a canonical fixed point. Looking ahead to Section 5.2, we shall see conditions there that insure unique fixed points. So one might ask whether the fixed points given by Theorem 5.1.23 are unique. As it happens, they are not (see Example 5.1.28 below). However if we assume an extra condition on the fixed point, we do have a uniqueness result.

**Theorem 5.1.33.** The canonical fixed point of a locally continuous endofunctor \( F \) on \( \text{CPO}_\bot \) is the only fixed point \( FX \sim X \) with the property that \( \text{id}_X \) is the least algebra endomorphism of \( X \).

Moreover, it is the only fixed point with the property that the \( \text{id}_X \) is the only strict algebra endomorphism. For the proof, see Abramsky and Jung [2, Section 5].

We also might note another fact about the canonical fixed points coming from Theorem 5.1.23.

**Theorem 5.1.34.** Under the conditions in Theorem 5.1.23, the algebra morphism from the canonical fixed point \( \mu F \) to any fixed point of \( F \) is an embedding.

**Proof.** Let \( \alpha \colon FA \to A \) be an isomorphism. Recall the cocone \( \alpha_n \colon F^n0 \to A \) from (3.2): \( \alpha_0 \colon 0 \to A \) is given by initiality, and \( \alpha_{n+1} = \alpha \cdot F\alpha_n \). The algebra morphism \( h \colon \mu F \to A \) is the colimit morphism for this cocone. So our result follows from Corollary 5.1.14 as soon as we show that each \( \alpha_n \) is an embedding. We also have a coalgebra \( \alpha^{-1} \colon A \to FA \) and hence a canonical cone \( \beta_n \colon A \to F^{n+1} = F^n0 \). This cone fulfills \( \beta_0 \colon A \to 1 \) by finality, and \( \beta_{n+1} = F\beta_n \cdot \alpha^{-1} \). An easy induction shows that \( \alpha_n = \beta_n \) for all \( n \). So indeed each \( \alpha_n \) is an embedding.

**Remark 5.1.35.** A higher-order variation of Theorem 5.1.23 was established in Adámek [11] where cpo’s are generalized as follows: A category \( \mathcal{A} \) is called Scott complete if it has
(1) filtered colimits,
(2) an initial object,
(3) limits of diagrams with a cone, and
5.2 CMS-enriched categories

(4) a set of finitely presentable objects (i.e. objects X whose hom-functor \( \mathcal{A}(X, -) \) is finitary) whose closure under filtered colimits is \( \mathcal{A} \).

We obtain a 2-category of Scott-complete categories, using as 1-cells the functors preserving filtered colimits and the initial object, and as 2-cells the natural transformations.

**Theorem 5.1.36** [11]. Every locally continuous 2-endofunctor of the category of Scott complete categories has a canonical fixed point.

**Remark 5.1.37.** We have seen how to obtain initial algebras and terminal coalgebras for locally continuous endofunctors on strict CPO-enriched categories. Surprisingly, for all set functors \( F \) with \( F\emptyset \neq \emptyset \) and preserving \( \omega^{\text{op}} \)-limits it turns out that the initial-algebra chain and the terminal-coalgebra chain converge, and the terminal coalgebra also carries a structure of a CPO. It is obtained from the initial algebra as a free ideal completion (see Example 5.1.17). Indeed, let \( u: \emptyset \rightarrow 1 \) be the unique morphism. Then there exists a unique \( \bar{u}: \colim_{n<\omega} F^n\emptyset \rightarrow \lim_{n<\omega} F^n1 \) such that \( F^nu: F^n\emptyset \rightarrow F^n1 \) has the form \( \ell_n: \bar{u} \cdot c_n \) (see Notation 3.1.4 and Remark 3.3.5) for every \( n \). Choose a point \( p: 1 \rightarrow F\emptyset \) and put

\[
eq_n = (F^n1 \xrightarrow{F^n p} F^{n+1}\emptyset \xrightarrow{c_{n+1}} \colim_{n<\omega} F^n\emptyset \xrightarrow{\bar{u}} \nu F).
\]

Then the partial order \( \sqsubseteq \) on \( \nu F \) is given as follows for \( x \neq y \):

\[
x \sqsubseteq y \text{ iff } x = e_n \cdot \ell_n(y) \text{ for some } n < \omega.
\]

This is a cpo with the properties mentioned above [13].

In Section 6.4 we prove a (non-finitary) generalization of the main result: every stable functor on \( \text{DCPO}_\perp \) with a fixed point has a canonical one (see Corollary 6.4.8).

### 5.2 CMS-enriched categories

Complete partial orders, used in Section 5.1, express the idea of how much information is contained in an approximation of a solution of a recursive specification. But we can also measure the distance of various approximations, and apply the idea of a metric space in lieu of a poset.

For example, recall that the terminal coalgebra for the polynomial functor \( H_\Sigma \) is the set of all \( \Sigma \)-labelled trees, as explained by Theorem 3.3.10. Arnold and Nivat [52] studied this set using the following natural metric: the distance of distinct trees \( t \) and \( s \) is

\[
d(s, t) = 2^{-n} \text{ for the least level } n \text{ at which } t \text{ and } s \text{ differ. \hspace{1cm} (5.13)}
\]

This metric space is complete, i.e. every Cauchy sequence converges.

In this section we consider categories enriched over the category \( \text{CMS} \) of complete metric spaces, and prove a result analogous to that of Section 5.1: every locally contracting endofunctor has a canonical fixed point. In fact, a stronger and surprising result is true: such a functor has, up to isomorphism, a unique fixed point. Thus every fixed point is canonical.
Complete metric spaces were first applied by de Bakker and Zucker [87], and their method was further developed by America and Rutten in their seminal paper [50] on this topic. In both of these papers, the morphisms are a little different from the non-expansive maps: they are the retraction-embedding pairs \((e, \hat{e})\) consisting of an isometric embedding \(e: X \to Y\) together with a non-expanding map \(\hat{e}: Y \to X\), such that \(\hat{e} \cdot e = \text{id}_X\).

**Notation 5.2.1.** We denote by CMS the category of complete metric spaces with distances bounded by 1 and non-expanding functions as morphisms; these are the functions \(f\) with the property that \(d(fx, fy) \leq d(x, y)\) for all \(x, y\).

**Definition 5.2.2.** A category \(\mathcal{A}\) is CMS-enriched if its hom-sets come equipped with a complete metric, and composition is non-expanding in both variables. This means that given morphisms \(f_1, f_2: A \to B\), then for all \(h\) and \(g\) composable with \(f_i\),

\[
d(h \cdot f_1 \cdot g, h \cdot f_2 \cdot g) \leq d(f_1, f_2).
\]

**Example 5.2.3.** (1) CMS is CMS-enriched with respect to the supremum metric

\[
d_{X,Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).
\]

(2) CMS\(_a\), the category of pointed complete metric spaces, i.e. objects are pairs \((X, x)\) where \(X\) is a complete metric space and \(x \in X\), and morphisms are and non-expanding maps preserving the distinguished point, is also CMS-enriched using the supremum metric.

(3) If \(\mathcal{C}\) is CMS-enriched then so is \(\mathcal{C}^{\text{op}}\), where the metric of \(\mathcal{C}^{\text{op}}(A, B)\) is that of \(\mathcal{C}(B, A)\).

(4) A product \(\mathcal{C}' \times \mathcal{C}''\) of CMS-enriched categories (with metrics \(d'_{A', B'}\) and \(d''_{A'', B''}\) on hom-sets, respectively) is CMS-enriched using the maximum metric: the distance of \((f', f'')\) and \((g', g'')\) is \(\max\{d'(f', g'), d''(f'', g'')\}\).

At the root of our study is the following classical result. In it, recall that a function \(f: X \to Y\) is contracting for some \(\varepsilon < 1\), if \(d(fx, fy) < \varepsilon \cdot d(x, y)\) for all \(x, y \in X\).

**Theorem 5.2.4** (Banach Fixed Point Theorem). Every contracting endofunction on a nonempty complete metric space has a unique fixed point.

**Definition 5.2.5** [50]. An endofunctor \(F\) of a CMS-enriched category \(\mathcal{A}\) is locally contracting if there exists \(\varepsilon < 1\) such that for all parallel pairs \(f, g: X \to Y\) we have

\[
d(Ff, Fg) \leq \varepsilon \cdot d(f, g).
\]

**Example 5.2.6.** (1) For every \(\varepsilon < 1\), the functor which takes a space \((X, d)\) and shrinks the metric by \(\varepsilon\), giving \((X, \varepsilon \cdot d)\) is contracting.

(2) Every composite, coproduct, and finite product of locally contracting endofunctors is locally contracting.
(3) Every polynomial functor $H_\Sigma$ on $\text{Set}$ has a \textit{contracting lifting} $H'_\Sigma$ to CMS. This means that the following square

$$
\begin{array}{ccc}
\text{CMS} & \xrightarrow{H'_\Sigma} & \text{CMS} \\
U \downarrow & & \downarrow U \\
\text{Set} & \xrightarrow{H_\Sigma} & \text{Set}
\end{array}
$$

commutes, where $U$ is the forgetful functor taking a metric space to its set of points. This follows from (1) and (2) since polynomial functors are formed from $\text{Id}$ by finite products and coproducts. Using the constant $\varepsilon = \frac{1}{2}$ as in (1) above, we can describe the terminal coalgebra for $H'_\Sigma$ on CMS as the set of all $\Sigma$-trees equipped with the metric of (5.13) as we will see in Example 5.2.24(3) below. Moreover, if $\Sigma$ contains at least one constant symbol, then this is a canonical fixed point of $H'_\Sigma$.

(4) The Hausdorff Functor. This is the endofunctor $H$ of CMS assigning to a metric space $(X,d)$ the space $\mathcal{H}(X,d)$ of all nonempty compact subsets. Its metric $\bar{d}$ was introduced by Hausdorff who also proved that it yields a complete metric space. It is defined by

$$
\bar{d}(A,B) = \max(\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)).
$$

for every pair of compact sets $A$ and $B$. Here $d(a,B)$ denotes, as usual, $\inf_{b \in B} d(a,b)$. For morphisms $f: (X,d) \to (X',d')$ we define $\mathcal{H}(f): A \mapsto f[A]$. The resulting functor $\mathcal{H}$ is locally non-expanding, but not locally contracting. However, we will see in Section 11.3 that it is finitary.

By modifying the Hausdorff functor by halving all distances as in (3), we obtain a contracting endofunctor.

We are going to present the proof that every contracting endofunctor has a canonical fixed point with all details. As in the CPO-enriched case, this material has not been presented so comprehensively before. Once again, for CMS the main ideas stem from America and Rutten [50]. Their results were extended in [42], and we sharpen them up a bit. First a surprising result about unicity of fixed points:

\textbf{Theorem 5.2.7.} Let $\mathcal{A}$ be a CMS-enriched category with nonempty hom-sets. Every fixed point of a locally contracting endofunctor on $\mathcal{A}$ is canonical.

\textbf{Proof.} (1) Let $\alpha: FA \to A$ be a fixed point of a locally contracting endofunctor $F$ with contraction factor $\varepsilon$. We prove that this is the initial algebra of $F$.

Given an algebra $\beta: FB \to B$, define an endofunction $k$ of $\mathcal{A}(A,B)$ as follows:

$$
k: (A \xrightarrow{f} B) \mapsto (A \xrightarrow{\alpha^{-1}} FA \xrightarrow{\beta} FB) \xrightarrow{\beta} B\).$$

Then $k$ is $\varepsilon$-contracting: for every parallel pair $f,g: A \to B$ we have

$$
d(k(f),k(g)) = d(\beta \cdot Ff \cdot \alpha^{-1}, \beta \cdot Fg \cdot \alpha^{-1}) \leq d(Ff,Fg) \leq \varepsilon \cdot d(f,g)
$$

133
since composition is non-expanding and $F$ is $\varepsilon$-contracting. Thus $k$ has a fixed point:

$$h : A \to B \text{ with } h = \beta \cdot F h \cdot \alpha^{-1}.$$  

This implies that $h$ is an algebra homomorphism. Conversely, every algebra homomorphism $(\alpha \cdot h = \beta \cdot F h)$ is a fixed point of $k$. By Banach’s Theorem 5.2.4 $(A, \alpha)$ is an initial algebra.

(2) The proof that $\alpha^{-1} : A \to FA$ is a terminal coalgebra is by duality: if $\mathcal{C}$ is a CMS-enriched category, then so is $\mathcal{C}^{\text{op}}$ (using the same metric on hom-sets). And $F^{\text{op}}$ is a contracting endofunctor. By applying (1) to it we see that $(A, \alpha^{-1})$ is an initial algebra for $F^{\text{op}}$, i.e. a terminal coalgebra for $F$.

**Notation 5.2.8.** Let $\mathcal{A}$ be a CMS-enriched category. We denote by $\mathcal{A}^E$ the category of all objects of $\mathcal{A}$, where the morphisms from $A$ to $B$ are all pairs $(e, \hat{e})$ of morphisms in $\mathcal{A}$, with $e : A \to B$, and $\hat{e} : B \to A$, and $\hat{e} \cdot e = \text{id}_A$:

$$\text{id} \bigcirc \begin{array}{c} e \cr \hat{e} \end{array} : A \longrightarrow B$$

Composition and identity morphisms in $\mathcal{A}^E$ are defined componentwise. If $\mathcal{A} = \text{CMS}$, then $\hat{e} \cdot e = \text{id}_A$ implies that $e$ is an isometric embedding. Thus CMS$^E$ is precisely the category $\mathcal{C}$ of America and Rutten [50].

One difference between the work in this section and the previous one is that in the case of CPO-enriched categories, each embedding $e$ had a unique projection $\hat{e}$. The uniqueness is lost in the CMS setting. And so the morphisms in categories $\mathcal{A}^E$ are pairs in the CMS setting, while in the CPO setting we were able to get by with only the embeddings.

**Definition 5.2.9.** An $\omega$-chain in the category $\mathcal{A}^E$

$$E_0 \xrightarrow{\hat{e}_0} E_1 \xrightarrow{\hat{e}_1} E_2 \xrightarrow{\hat{e}_2} \cdots$$

is called *contracting* if there exists $\varepsilon < 1$ such that for all $i$ we have

$$d(e_{i+1} \cdot \hat{e}_{i+1}, \text{id}_{E_{i+1}}) \leq \varepsilon \cdot d(e_i \cdot \hat{e}_i, \text{id}_{E_i}).$$

**Example 5.2.10.** Let $F$ be a locally contracting endofunctor on a CMS-enriched category with a terminal object 1 and such that $\mathcal{A}(1,F1)$ is nonempty. Given $e : 1 \to F1$, the chain

$$1 \xrightarrow{e} F1 \xrightarrow{Fe} F^21 \xrightarrow{F^2e} F^31 \cdots$$

is contracting. Indeed, if $\varepsilon$ is the contraction constant for $F$, then

$$d(F^{i+1}e \cdot F^{i+1}, \text{id}) = d(F^i(e \cdot F^i), F \text{id}) \leq \varepsilon \cdot d(F^i(e \cdot F^i), \text{id}).$$
**Basic Lemma 5.2.11.** Let \( \mathcal{A} \) be a CMS-enriched category. Consider a contracting chain

\[
E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots
\]  

in \( \mathcal{A} \). Let \( c_i : E_i \to C \) be a cocone of \( (e_i)_{i \in \mathbb{N}} \) in \( \mathcal{A} \). Then the following are equivalent:

1. \( (c_i) \) is a colimit cocone in \( \mathcal{A} \).
2. Each \( c_i \) is a split monomorphism and there exist splittings \( \hat{c}_i \) forming a cone of \( (\hat{e}_i) \) in \( \mathcal{A} \) with \( \lim_i (c_i \cdot \hat{c}_i) = \text{id}_C \).

**Proof.**

(1) First observe that the above condition \( d(e_i \cdot \hat{e}_i, \text{id}) \leq \varepsilon \cdot d(e_{i-1} \cdot \hat{e}_{i-1}, \text{id}) \) implies that for all \( i < k \) we have

\[
d(e_k \cdot \hat{e}_k, \text{id}) \leq \varepsilon^{k-i} \cdot d(e_i \cdot \hat{e}_i, \text{id}) \leq \varepsilon^k
\]

The proof is any easy induction on \( i \) where in the first step we use \( d(e_0 \cdot \hat{e}_0, \text{id}) \leq 1 \). (Recall that 1 is our bound on all distances).

(2) To prove \( 1 \Rightarrow 2 \) use, for every \( i \), the shortened \( \omega \)-chain \( E_i \xrightarrow{e_i} E_{i+1} \xrightarrow{e_{i+1}} E_{i+2} \xrightarrow{e_{i+2}} \cdots \) and define, analogously to the Basic Lemma 5.1.9, morphisms \( \hat{c}_i : C \to E_i \) by

\[
\hat{c}_i \cdot \hat{c}_i = \text{id}_{E_i} \quad \text{and} \quad \hat{c}_i \cdot c_k = \hat{e}_i \cdots \hat{e}_{k-1} \quad \text{for } k > i.
\]

Again, we obtain the formula (5.9) for all \( i \)

\[
\hat{c}_i = \hat{c}_i \cdot \hat{c}_{i+1}.
\]

The sequence \( c_i \cdot \hat{c}_i : C \to C \) is Cauchy: to prove this, we verify below that

\[
d(c_i \cdot \hat{c}_i, c_{i+1} \cdot \hat{c}_{i+1}) \leq \varepsilon^i \quad \text{for all } i,
\]

from which we conclude that the distance of \( c_i \cdot \hat{c}_i \) and \( c_k \cdot \hat{c}_k \), where \( k > i \), is at most

\[
\varepsilon^i + \varepsilon^{i+1} + \cdots + \varepsilon^{k-1} \leq \sum_{j=i}^{\infty} \varepsilon^j = \frac{\varepsilon^i}{1-\varepsilon}.
\]

This converges to 0 as \( i \to \infty \). The desired inequality is clear from \( \hat{e}_i \cdot \hat{e}_{i+1} = \hat{e}_i \) and \( c_{i+1} \cdot e_i = c_i \):

\[
d(c_i \cdot \hat{c}_i, c_{i+1} \cdot \hat{c}_{i+1}) = d(c_{i+1} \cdot (e_i \cdot \hat{e}_i) \cdot \hat{c}_{i+1}, c_{i+1} \cdot \hat{c}_{i+1})
\leq d(e_i \cdot \hat{e}_i, \text{id}) \quad \text{by (5.14)}
\leq \varepsilon^i
\]

The rest is analogous to the Basic Lemma 5.1.9 again: we know that \( \lim_i (c_i \cdot \hat{c}_i) \) exists, and we derive that it is \( \text{id}_C \).
5 Finitary Iteration in Enriched Settings

(3) The proof of (2) ⇒ (1) is also analogous to the Basic Lemma 5.1.9. Given a cocone $b_i : E_i \to B$ of $(\hat{e}_i)$, the sequence $b_i \cdot \hat{c}_i : C \to B$ is Cauchy. This follows from

$$d(b_i \cdot \hat{c}_i, b_{i+1} \cdot \hat{c}_{i+1}) = d(b_{i+1} \cdot (e_i \cdot \hat{c}_i), b_{i+1} \cdot \hat{c}_{i+1}) \leq d(e_i \cdot \hat{c}_i, \text{id}) \leq \varepsilon^i$$

by (5.14)

Then $\lim_i (b_i \cdot \hat{c}_i)$ is the desired unique factorization morphism. \qed

The following corollary yields a limit-colimit coincidence in CMS-enriched categories analogous to the one we have seen in Remark 5.1.13.

**Corollary 5.2.12.** Given a chain (5.16) in $A^E$, a cone $(\hat{c}_i)$ of the $\omega$-chain $(\hat{e}_i)$ is a limit cone in $A$ iff each $\hat{e}_i$ is a split epimorphism and there exist splittings $c_i$ forming a cocone of $(e_i)$ with $\lim c_i \cdot \hat{c}_i = \text{id}_C$.

By Example 5.2.3(3), we can apply the Basic Lemma 5.2.11 to $A^{op}$.

**Corollary 5.2.13.** If a CMS-enriched category has colimits of $\omega$-chains, then so does $A^E$.

**Remark 5.2.14.** (1) The Basic Lemma also holds for the converging towers of America and Rutten [50]: These are sequences (5.15) such that for every $\varepsilon > 0$ there exists $N$ with the distance between $\text{id}_{E_m}$ and $e_n \cdot \ldots \cdot e_{m+1} \cdot \hat{e}_{m+1} \cdot \ldots \cdot \hat{e}_n$ at most $\varepsilon$ for all $m > n \geq N$. The proof is the same.

(2) In analogy to Theorem 5.1.23 we might now expect that contracting endofunctors on CMS-enriched categories have canonical fixed points. Unfortunately, this is not quite true:

(a) A locally contracting endofunctor need not have a terminal coalgebra: take the poset $\mathbb{N}^{op}$ as a category. It is trivially CMS-enriched. Every endofunctor is (trivially) contracting. But not every endofunctor, e.g. $s(x) = x + 1$, has a terminal coalgebra.

(b) A locally contracting endofunctor on CMS need not have a canonical fixed point: take $F(X, d) = (X, \frac{1}{2}d)$. Here $\mu F = \emptyset$ and $\nu F = 1$. (We might note in passing that on MS, this functor has other fixed points, such as the open interval $(0, 1)$.)

Nevertheless, for categories enriched over CMS* of Example 5.2.3(2) (which is analogous to the strictness required in the Smyth-Plotkin Theorem 5.1.23) canonical fixed points exist:

**Definition 5.2.15.** A strict CMS-enriched category $\mathcal{A}$ is a nonempty CMS-enriched category in which every hom-object $\mathcal{A}(X, Y)$ has a chosen element $\bot_{XY}$ such that for all morphisms $f : A \to B$ we have $f \cdot \bot_{XA} = \bot_{XB}$ and $\bot_{BY} \cdot f = \bot_{AY}$.

**Theorem 5.2.16.** Every locally contracting endofunctor of a strict CMS-enriched category with $\omega$-colimits has a canonical fixed point.

**Proof.** Every strict CMS-enriched category $\mathcal{A}$ has a zero object, the proof is completely analogous to Lemma 5.1.22. The rest of the proof is analogous to the proof of Theorem 5.1.23, using Basic Basic Lemma 5.2.11 and arguing with $\lim_i$ in lieu of $\sqcup_i$. \qed
5.2 CMS-enriched categories

Corollary 5.2.17. Every locally contracting endofunctor $F$ of CMS with $F\emptyset \neq \emptyset$ has a canonical fixed point.

Indeed, we modify $F$ to an endofunctor $\bar{F}$ on CMS, as follows: choose $a_0 \in F\emptyset$ and for every space $X$ put $a_X = Fh_X(a_0)$, where $h_X: \emptyset \to X$ is the unique morphism. Define $\bar{F}$ on objects by

$$\bar{F}(X, x) = (FX, a_X)$$

and on morphisms $f: (X, x) \to (Y, y)$ by $\bar{F}f = Ff$. This is well-defined due to $h_Y = f \cdot h_X$. Then $\bar{F}$ is locally contracting, hence, has a canonical fixed point. The same space is the canonical fixed point of $F$.

Remark 5.2.18. (1) Instead of assuming all $\omega$-colimits in Theorem 5.2.16, it is sufficient to assume a zero object and colimits of $\omega$-chain of split monomorphisms.

(2) For CMS-enriched categories the theorem above does not hold in general, but the following weaker result has a completely analogous proof:

Theorem 5.2.19. Let $\mathcal{A}$ be a CMS-enriched category with $\omega$-colimits. Let $F: \mathcal{A} \to \mathcal{A}$ be locally contracting with $\mathcal{A}(1,F1)$ nonempty. Then $F$ has a terminal coalgebra.

Remark 5.2.20. In the above proof of the existence of a canonical fixed point we did not fully use the power of $F$ being locally contracting: we only needed this for pairs consisting of $\text{id}_A$ and an endomorphism on $A$. This led America and Rutten [50] to the concept of a hom-contracting endofunctor $F$, i.e. there exists $\varepsilon < 1$ with

$$d(Ff, \text{id}_A) \leq \varepsilon \cdot d(f, \text{id}_A)$$

for all $f: A \to A$.

However, this is equivalent to being locally contracting whenever the following type of coproducts exists:

Definition 5.2.21. In a CMS-enriched category an $\varepsilon$-contracting copower of an object $A$ is an object $A +_\varepsilon A$ with morphisms $i_1, i_2: A \to A +_\varepsilon A$ universal is the following sense:

(1) $d(i_1, i_2) \leq \varepsilon$;

(2) given morphisms $f_1, f_2: A \to B$ with $d(f_1, f_2) \leq \varepsilon$, there exists a unique morphism $[f_1, f_2]: A +_\varepsilon A \to B$ with $f_k = [f_1, f_2] \cdot i_k$ for $k = 1, 2$;

(3) the distance of $\text{id}_{A +_\varepsilon A}$ and $[i_2, i_1]$ is at most $\varepsilon$: $d(\text{id}, [i_2, i_1]) \leq \varepsilon$.

Example 5.2.22. CMS has contracting copowers. Indeed, for $\varepsilon \geq 1$ these are just copowers. That is, disjoint unions

$$A + A = A \times \{1, 2\}$$

with the original distance on $A \times \{1\}$ as well as $A \times \{2\}$, and with $d((x, 1), (y, 2)) = 1$ for all $x, y \in A$.

The space $A +_\varepsilon A$ differs only in the last item, and then only for $x = y$: we put $d((x, 1), (x, 2)) = \varepsilon$.

Lemma 5.2.23. Every hom-contracting endofunctor of a CMS-enriched category with contracting coproducts is locally contracting.
5 Finitary Iteration in Enriched Settings

Proof. Let $F$ be hom-contracting with contraction factor $\varepsilon$. Given a parallel pair $f_1, f_2 : A \to B$, we prove $d(Ff_1, Ff_2) \leq \varepsilon \cdot d(f_1, f_2)$. To this end let $r = d(f_1, f_2)$, consider $g = [f_1, f_2] : A + r A \to B$ and put $\sigma = [i_2, i_1]$, which satisfies $\sigma \cdot i_1 = i_2$. Then we have

$$d(Ff_1, Ff_2) = d(Fg \cdot Ff_1, Fg \cdot Ff_2) \leq d(Ff_1, Ff_2) = d(Ff_1, F\sigma \cdot Ff_1) \leq d(id, F\sigma) \leq \varepsilon \cdot d(id, \sigma) \leq \varepsilon \cdot r = \varepsilon \cdot d(f_1, f_2) \quad \Box$$

Example 5.2.24. (1) Binary streams. Let us consider the functor $F(X, d) = 2 \times (X, \frac{1}{2} d)$ on CMS. Here, 2 is the two-point space $\{0,1\}$ with the discrete metric. Since $F\emptyset = \emptyset$, the initial algebra of $F$ is also $\emptyset$. Its terminal coalgebra is carried by the set $2^{\omega}$ of binary streams with the metric given by

$$d(s_0s_1s_2\cdots, t_0t_1t_2\cdots) = 2^i, \text{ where } i \text{ is least with } s_i \neq t_i,$$

for distinct streams $s_0s_1s_2\cdots$ and $t_0t_1t_2\cdots$. Note that the set $2^{\omega}$ is the terminal coalgebra for $FX = 2 \times X$ on $\text{Set}$ and also that every space $F^n1$ is the set $2^n$ of binary sequences of length $n$ with the metric given similarly. Notice also that the above metric is a special case of the metric (5.13) as $F$ is (isomorphic to) the lifting $H^\omega_{\Sigma}$ of the polynomial functor $H^\omega_{\Sigma}X = X + X$.

(2) We have discussed deterministic automata and formal languages in Example 2.5.5, and now we revisit the topic from the metric point of view. This involves the functor $FA = \{0,1\} \times X^A$ on $\text{Set}$. We lift this to CMS as in our previous examples: $\{0,1\}$ is the discrete space and $X^A$ carries the half of the maximum metric. We obtain as a terminal coalgebra the set $\mathcal{P}(A^*)$ of formal languages over $A$. The structure

$$(o, t) : \mathcal{P}(A^*) \to \{0,1\} \times \mathcal{P}(A^*)^A$$

is given by $(t(L))(a) = \{w : aw \in L\}$, and $o(L) = 1$ iff the empty word $\varepsilon$ belongs to $L$. The metric works by assigning to two different languages $L$ and $M$ the number $2^{-n}$, where $n$ is least so that there is a word of length $n$ in $(L \setminus M) \cup (M \setminus L)$.

(3) $\Sigma$-trees. Recall the contracting lifting $H^\omega_{\Sigma}$ of the polynomial endofunctor $H_{\Sigma}$ from Example 5.2.6.(3). That is, given elements of $H^\omega_{\Sigma}X$ of the same summand $\sigma \in \Sigma$, say $x = \sigma(t_i)$ and $x' = \sigma(t'_i)$, then $d(x, x') = \frac{1}{2} \max d(t_i, t'_i)$. It follows that the metric space $T^\omega_{\Sigma}$ of all $\Sigma$-trees with the Arnold-Nivat metric (5.13) is a fixed point of $H^\omega_{\Sigma}$; the usual tree-tupling $\alpha : H^\omega_{\Sigma}T^\omega_{\Sigma} \to T^\omega_{\Sigma}$ is an isomorphism. Indeed, consider two elements of $H^\omega_{\Sigma} = \Pi_{\sigma \in \Sigma} T^\omega_{\Sigma} : x = (t_i)_{i<\infty}$ in the summand $\sigma$ and $x' = (t'_i)_{i<\infty}$ in the summand $\sigma'$. If $\sigma \neq \sigma'$ the distances of $x, x'$ and $\alpha(x), \alpha(x')$ are both 1. If $\sigma = \sigma'$, then $d(x, x') = \frac{1}{2} 2^{-n}$ where $2^{-n} = \max d(t_i, t'_i)$ means that for some $i_0$ the trees $t_{i_0}, t'_{i_0}$ differ at level $n$. Then $(x), \alpha(x')$ differ first at level $n + 1$, thus $d(\sigma(x), \sigma(x')) \leq 2^{-n+1}$.
In fact, \( T_\Sigma \) is always the terminal coalgebra for \( H'_\Sigma \). And if \( \Sigma \) contains a constant symbol, \( T_\Sigma \) is the canonical fixed point of \( H'_\Sigma \).

If \( \Sigma \) consists of two \(|A|-ary operations), we get ((2)) above. Here \( T_\Sigma \) is the complete \(|A|-ary tree without leaves, labeled by \( \{0,1\} \). Since that complete tree can be identified with \( A^* \), we again get \( T_\Sigma = P(\Sigma^*) \) with the metric above.

As another concrete example consider \( \Sigma \) of one binary operation and one constant. The functor \( H_\Sigma X = X \times X + 1 \) with the maximum metric is not contracting: its initial algebra is as in \( \text{Set} \) with the discrete metric, the same holds about the terminal coalgebra. But the scaled functor \( H'_\Sigma X \) has \( T_\Sigma \), the binary trees (with the metric (5.13)) as the unique fixed point.

Example 5.2.25 \[1\]. Recall the limit \( F = \lim_{n<\omega} P^n \) from Section 4.5. We present a “natural” endofunctor on \( \text{CMS} \) whose terminal coalgebra is \( F \).

Denote by \( 1 \) the terminal endofunctor of \( \text{CMS} \) with value, say, \( \{\emptyset\} \), and recall the Hausdorff functor \( H \) from Example 5.2.6(3). Then the coproduct \( H + 1 \) assigns to every finite metric space \((X,d)\) a metric space on the set \( PfX \): every nonempty subset is of course compact, and the summand \( 1 \) takes care of the empty set. Let \( \tilde{H} : \text{CMS} \to \text{CMS} \) be the functor scaling the metric on \( H + 1 \) by \( \frac{1}{2} \). Then \( \tilde{H} \) is a locally contracting endofunctor and, again, \( \tilde{H}(X,d) \) has the underlying set \( PfX \) for all finite metric spaces \((X,d)\). Consequently, the terminal \( \omega^{\text{op}} \)-chain of \( \tilde{H} \) is a lifting of that of \( Pf \). Since \( \omega^{\text{op}} \)-limits in \( \text{CMS} \) are preserved by the forgetful functor to \( \text{Set} \), and since \( \nu\tilde{H} = \lim \tilde{H}^n 1 \), we conclude that \( \nu\tilde{H} \) is carried by \( F \) by Theorem 5.2.19. More precisely, considering \( \nu\tilde{H} \) as an algebra for \( \tilde{H} \) (by inverting the coalgebra structure), the underlying \( Pf \)-algebra is \( F \).

Later, in Theorem 11.3.12, we shall obtain a result that allows us to obtain a terminal coalgebra for the class of functors formed by coproducts, products and compositions of polynomial functors and the Hausdorff functor without any scaling, see Example 11.3.14.

Example 5.2.26. Every limit \( \lim X_n \) of an \( \omega^{\text{op}} \)-chain in \( \text{Set} \) (with projections \( \ell_n \)) carries a canonical complete metric as follows: given elements \( x \neq y \), put

\[
d(x,y) = 2^{-n} \quad \text{for the least } n \text{ with } \ell_n(x) \neq \ell_n(y).
\]

For every Cauchy sequence \((x_k)\) in the limit it is easy to see that there exists a subsequence \((y_m)\) satisfying \( d(y_m, y_{m+1}) \leq 2^{-(m+1)} \) for every \( m \). This means that the connection morphism \( f_n : X_{n+1} \to X_n \) fulfills \( f_n \cdot \ell_{n+1}(y_{m+1}) = \ell_n(y_{m+1}) = \ell_n(y_n) \). Thus, the elements \( \ell_n(y_n) \in X_n \) form a compatible sequence of elements of the given diagram. Therefore, there is a unique \( y \) in the limit with \( \ell_n(y) = \ell_n(y_n) \) for all \( n \), hence \( d(y,y_n) \leq 2^{-n} \), which means that \( \lim y_n = y \) whence \( \lim x_k = y \).

The following result was first proved by Barr [58] under the additional assumption that \( F \) also preserves \( \omega \)-colimits:
Theorem 5.2.27 (Adámek [13]). Let $F : \text{Set} \to \text{Set}$ fulfill $F\emptyset \neq \emptyset$, and assume that $F$ has the terminal coalgebra $\nu F = \lim_{n<\omega} F^n\emptyset$. Then the above complete metric space $\nu F$ is a Cauchy completion of the initial algebra.

Remark 5.2.28. (1) Birkedal, Støvring, and Thamsborg [67] considered a setting which is quite similar to our work in this section. They work with ultrametric spaces, where one strengthens the triangle inequality of metric spaces to the condition that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. For example, the above metric on $\lim X_n$ is always an ultrametric.

(2) Alessi, Baldan and Belle [48] study endofunctors of $\text{KMS}$, the full subcategory of $\text{CMS}$ on compact metric spaces. This category is obviously CMS-enriched, but fails to have $\omega^{\text{op}}$-limits, which makes the situation more complicated:

Theorem 5.2.19 has an analog. Let us call an endofunctor $F$ on $\text{KMS}$ locally contracting if there exists a constant $\varepsilon < 1$ with $d(Ff, Fg) \leq \varepsilon \cdot d(f, g)$ for every parallel pair $f, g$ of morphisms.

Theorem 5.2.29 [48]. Let $F : \text{KMS} \to \text{KMS}$ be locally contracting, and assume that $F0 \neq 0$. Then $F$ has a canonical fixed point.

5.3 Solving domain equations

Theorem 5.1.23 allows us to solve $X \sim = FX$ whenever $F$ is a locally continuous endofunctor on a strict CPO-enriched category with $\omega$-colimits. The solutions are not in general unique, but the canonical fixed point gives a workable notion of solution.

We are also interested in solving “equations with mixed variance” such as $D \sim = [D, D] + N_\bot$. Indeed, solving this kind of equation was Scott’s original motivation for much of the work on embedding-projection pairs, due to its connection with models of the untyped lambda-calculus. So work on this topic is, in effect, the source of embedding-projection pairs in domain theory and also the categorical formulation of the limit-colimit coincidence (see Scott [218] and Smyth and Plotkin [224]). The obstacle is that $[D, D]$ is not a functor of the form $\text{CPO}_\perp \times \text{CPO}_\perp \to \text{CPO}_\perp$. Indeed, it is contravariant in its first argument, so its type is $\text{CPO}^{\text{op}}_\perp \times \text{CPO}_\perp \to \text{CPO}_\perp$. But the theory which we have developed up to this point does not allow us to obtain a fixed point of such a functor, due precisely to the contravariance. As it happens, there are two ways to proceed. They are related, and we are going to present both and briefly touch on the relation afterwards.

Recall from Notation 5.1.6 the category $\mathcal{AE}$ of objects and embeddings.

Theorem 5.3.1. Let $\mathcal{A}$ be a strict CPO-enriched category with $\omega$-colimits of embeddings. Every locally continuous functor $F : \mathcal{A}^{\omega} \times \mathcal{A} \to \mathcal{A}$ defines an endofunctor $F^E$ on $\mathcal{AE}$ by

$$F^E X = F(X, X) \quad \text{and} \quad F^E e = F(\hat{e}, e),$$

which has an initial algebra $X \sim = F^E(X)$. It satisfies $F(X, X) \sim = X$ in $\mathcal{A}$.

Proof. It is easy check the functoriality of $F^E$. To verify that $F(\hat{e}, e)$ is an embedding, one checks that its projection is $F(e, \hat{e})$. 

140
5.3 Solving domain equations

First, note that $A^E$ has $\omega$-colimits, by Corollary 5.1.14. Moreover, we show that $F^E$ preserves $\omega$-colimits of embeddings:

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots$$

Let $c_i : E_i \to \mathcal{C}$ be a colimit cocone. By the Basic Lemma, $\bigsqcup c_i \cdot \hat{c}_i = \text{id}$. We consider

$$F(E_0, E_0) \xrightarrow{F(e_0, e_0)} F(E_1, E_1) \xrightarrow{F(e_1, e_1)} F(E_2, E_2) \xrightarrow{F(e_2, e_2)} \cdots,$$

and we show that $(F(\hat{c}_i, c_i))$ is a colimit cocone. By the Basic Lemma 5.1.9 again, we need to show that

$$\bigsqcup \hat{F}(\hat{c}_i, c_i) \cdot F(c_i, \hat{c}_i) = \text{id}.$$

For this, we use the local continuity:

$$\bigsqcup \hat{F}(\hat{c}_i, c_i) \cdot F(c_i, \hat{c}_i) = \bigsqcup \hat{F}(\hat{c}_i \cdot c_i, c_i \cdot \hat{c}_i)$$

$$= F(\bigsqcup \hat{c}_i, \bigsqcup c_i)$$

$$= F(\text{id}, \text{id})$$

$$= \text{id}.$$

From Theorem 3.1.7, we conclude that $X = \text{colim}(F^E)^n 0$ is an initial $F^E$-algebra. By Lambek’s Lemma, $X$ is a fixed point of $F^E$. Clearly, $X \cong F(X, X)$ in $\mathcal{A}$. This completes the proof.

We now turn to the second solution method.

**Theorem 5.3.2** [103]. Let $\mathcal{A}$ be a strict CPO-enriched category with $\omega$-colimits. Let $F : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{A}$ be locally continuous. Let $G$ be the endofunctor on $\mathcal{A}^{\text{op}} \times \mathcal{A}$ defined by

$$G(X, Y) = (F(Y, X), F(X, Y))$$

$$G(f, g) = (F(g, f), F(f, g))$$

for $f : X \to X'$ and $g : Y \to Y'$

Then $G$ has a fixed point of the form $(X, X)$, and for such $X$, we have $F(X, X) \cong X$.

**Proof.** This proof is based on an argument in Glimming and Ghani [114].

We define an isomorphism of categories

$$I : \text{Alg} G \to (\text{Coalg} G)^{\text{op}}.$$

Consider a $G$-algebra $A = ((Y, X), (\alpha, \beta))$, so that $\alpha : F(Y, X) \to Y$ and $\beta : F(X, Y) \to X$ are morphisms of $\mathcal{A}$. Then $IA$ is defined to be the $G$-coalgebra

$$(Y, X) \xrightarrow{(\beta, \alpha)} (F(Y, X), F(X, Y)).$$
5 Finitary Iteration in Enriched Settings

If $A' = (\alpha', \beta') : (F(Y', X'), F(X', Y')) \to (Y', X')$ is also a $G$-algebra, and if $(f, g) : A \to A'$ is an algebra morphism, then $I$ assigns to $(f, g)$ the $G$-coalgebra morphism $(g, f) : IA' \to IA$.

The category $\mathcal{A}^{\text{op}} \times \mathcal{A}$ is strict $\text{CPO}$-enriched and has $\omega$-colimits. $G$ is locally continuous. By Theorem 5.1.23, we have a canonical fixed point $(\alpha, \beta) : G(X, Y) \to (X, Y)$.

In particular, $(\alpha, \beta)^{-1} = (\alpha^{-1}, \beta^{-1}) : (X, Y) \to G(X, Y)$ is a terminal coalgebra, and since $I$ takes initial algebras to terminal coalgebras, $(\beta, \alpha) : (Y, X) \to G(Y, X)$ is also a terminal coalgebra. Thus, there is an isomorphism of $G$-coalgebras $(i, j) : (X, Y) \to (Y, X)$. In particular, $i$ and $j$ are invertible. All of the morphisms shown below are isomorphisms:

\[
\begin{array}{c}
(X, X) \xrightarrow{(\alpha, \beta)^{-1}} (X, Y) \xrightarrow{(\alpha^{-1}, \beta^{-1})} (Y, X) \\
\xrightarrow{(\alpha, \beta)} (F(X, Y), F(Y, X)) \xrightarrow{F(i, \text{id}, F(\text{id}, j))} (F(X, X), F(X, X)).
\end{array}
\]

The composites show that $(X, X) \cong G(X, X)$, and also $X \cong F(X, X)$, as desired.

**Example 5.3.3.** This example illustrates two ways to solve a system of domain equations in $\text{CPO}_\perp$. This category has coproducts given by disjoint union with least elements identified. Consider the following system:

\[
X \cong [Y, X] \quad Y \cong Z + D \quad Z \cong [Z, X],
\]

where $D$ is a fixed object of $\text{CPO}_\perp$.

First, we solve (5.17) following the method suggested by Theorem 5.3.1. Let

\[ A = \text{CPO}_\perp \times \text{CPO}_\perp \times \text{CPO}_\perp. \]

This category is strict $\text{CPO}$-enriched. We also have $\mathcal{E} = (\text{CPO}_{\perp})^3$. We express our system in terms of an endofunctor on $\mathcal{E}$:

\[(X, Y, Z) \mapsto ([Y, X], Z + D, [Z, X]).\]

For embeddings $f : X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$, the first component of $F(f, g, h)$ takes $p : Y \to X$ to $f \cdot p \cdot \tilde{f}$. In more detail, we take the contravariant functor $[Y, X]$ from Remark 5.1.18(1) and then apply the construction in Theorem 5.3.1. The third component is defined similarly, again using a projection $\tilde{g}$. The second component is $h + \text{id}_D$. Then one forms the colimit of the initial sequence of this functor to obtain the canonical fixed point. Call this fixed point $(X^*, Y^*, Z^*)$. Then we get the desired solution $X^* \cong [Y^*, X^*]$, $Y^* \cong Z^* + D$, and $Z^* \cong [Z^*, X^*]$. 

142
5.3 Solving domain equations

Second, let us solve (5.17) in a manner inspired by Theorem 5.3.2. The variables $X$, $Y$, and $Z$ become doubled: each gets a $+$ version and a $-$ version. Our equations now become

\[
\begin{align*}
X^+ &\cong [Y^-, X^+] & X^- &\cong [Y^+, X^-] \\
Y^+ &\cong Z^+ + D & Y^- &\cong Z^- + D \\
Z^+ &\cong [Z^-, X^+] & Z^- &\cong [Z^+, X^+]
\end{align*}
\]

We think of this as a locally continuous endofunctor $G: \mathcal{B} \to \mathcal{B}$, where

\[
\mathcal{B} = (\text{CPO}_\perp)^3 \times (\text{CPO}_\text{op})^3.
\]

and $G(X^+, Y^+, Z^+, X^-, Y^-, Z^-)$ is given by the equations above. Then Theorem 5.3.2 tells us that the canonical fixed point will be of the form $(X^*, Y^*, Z^*, X^*, Y^*, Z^*)$: the solutions for the “positive” and “negative” variables will agree. Moreover, the triple $(X^*, Y^*, Z^*)$ solves our original system (5.17).

**Example 5.3.4.** Scott’s model of the untyped $\lambda$-calculus. The formulas $t$ of the $\lambda$-calculus have the form

\[
t ::= k \mid x \mid tt \mid \lambda x.t
\]

where $k$ ranges through a set $K$ of constants, and $x$ through a countable set of variables. The meaning of $t_1t_2$ is “application”: we evaluate $t_2$ (a function) in $t_1$. The meaning of $\lambda x.t$ is “$\lambda$-abstraction”: this function takes a value $a$ and responds with $t[a/x]$, the term $t$ in which $x$ is substituted by $a$. Thus if $D$ is the set of all closed terms, we obtain an isomorphism

\[
D \cong K + D \times D + [D, D].
\]

No such set $D$ exists because $|[D, D]| > |D|$ whenever $D$ is not a singleton set.

Scott [218] decided to use a cartesian closed category with products and coproducts, and interpret the above equation in that category. He originally used continuous lattices, but Smyth and Plotkin [224] made it clear that $\text{CPO}_\perp$ is sufficient (and simpler). In fact, consider the flat cpo $K_\perp = K + \{\bot\}$ with all pairs in $K$ incomparable, in place of the set $K$. The category $\text{CPO}_\perp$ has the usual products $\times$, and coproducts $+$ are disjoint unions with bottom elements identified. Recall from Example 5.1.16 that $D \mapsto D \times D$ is locally continuous. We obtain a locally continuous functor

\[
F: \text{CPO}_\text{op} \times \text{CPO} \to \text{CPO} \quad \text{with} \quad F(X, Y) = K_\perp + Y \times Y + [X, Y].
\]

If $D$ is the initial algebra for $F^E$, see Theorem 5.3.1, then

\[
D \cong K_\perp + D \times D + [D, D]
\]

is a model of $\lambda$-calculus. (Observe that the “artificial” $\perp$ of $K$ disappears in the formation of coproduct.)
We conclude this section by mentioning a result which is parallel to Theorem 5.3.1 but in the metric setting. It can be used for getting fixed points of functors of mixed variance.

**Theorem 5.3.5** [50]. Let \( \mathcal{A} \) be a strict CMS-enriched category with colimits of \( \omega \)-chains. Every locally contracting endofunctor \( F: \mathcal{A} \times \mathcal{A}^{\text{op}} \to \mathcal{A} \) defines an endofunctor \( F^E \) of \( \mathcal{A}^E \) by

\[
F^E X = F(X, X) \quad \text{and} \quad F^E (e, \hat{e}) = F(\hat{e}, e),
\]

and \( F^E \) has an initial algebra \( X \simeq F^E(X) \). It satisfies \( F(X, X) \simeq X \) in \( \mathcal{A} \).

There is also an obvious parallel to Theorem 5.3.2, and with the same proof, mutatis mutandis.

### 5.4 Summary of this chapter

We have presented two closely related types of categories in which the terminal coalgebra and the initial algebra coincide for every “well-behaved” endofunctor: categories enriched over

- complete partial orders, where “well-behaved” are the locally continuous endofunctors,
- complete metric spaces, where “well-behaved” are the locally contracting endofunctors.

In both cases the main technical result was a coincidence of colimits of \( \omega \)-chains of embeddings (or of contracting \( \omega \)-chains of split subobjects, respectively) and limits of \( \omega^{\text{op}} \)-chains of the corresponding projections (or splittings, respectively).

Let us point out some of the sources that inspired the material of this chapter. The famous construction of a model of the untyped-\( \lambda \)-calculus presented by Scott in [218] applied \( \omega^{\text{op}} \)-limits and it is the starting point of our study. Scott used the category of continuous lattices and embedding-projection maps, and we shall see these below. Later, Smyth and Plotkin [224] introduced the concept of a locally continuous endofunctor and noticed that in the category of domains, the finitary constructions of the initial algebra and the terminal coalgebra coincide for these endofunctors, yielding a canonical fixed point. For more on this, see Abramsky and Jung [2]. Taylor proved the same result in the category of domains and adjoint pairs [227], and this was generalized further in the unpublished Ph.D. thesis of Velebil [239]. Adámek [11] proves a related result in which CPOs are generalized to finitely accessible categories and locally continuous functions to finitary functors.
6 Transfinite Iteration

This chapter deals with iterations of the initial-algebra chain which go into the transfinite. We generalize the objects $F^n0$ for $n \in \mathbb{N}$ to objects $W_i$ for all ordinals $i$ and prove that whenever the connecting map to $W_{i+1}$ is invertible, $W_i$ is the initial algebra. A dual result holds for terminal coalgebras: we generalize $F^n1$ to $V_i$ for all ordinals $i$ and obtain the terminal algebra from an invertible connecting map.

In order to help the reader understand this material, we provide a very short background discussion on ordinal numbers.

Background on ordinal and cardinal numbers and transfinite induction. Cardinal numbers are representatives of sets up to isomorphism: to every set $X$ one assigns a cardinal number $|X|$, called the cardinality of $X$, so that a set $Y$ is isomorphic (in Set) to $X$ iff $|X| = |Y|$. The set-theoretic choice of such representatives uses ordinal numbers (or, ordinals, for short). Recall that a well-ordered set is a poset with a total order (i.e. for every pair $x, y$ of elements in it, either $x \leq y$ or $y \leq x$) such that every nonempty subset contains a least element. For example, $\mathbb{N}$ is a well-ordered set but $\mathbb{Z}$ is not. For every well-ordered set, one obtains another well-ordered set by adding a new top element. Thus starting from the smallest (empty) well-ordered set we get the following well-ordered subsets of $\mathbb{N}$:

$$0 = \emptyset, \quad 1 = \{\emptyset\}, \quad 2 = \{\emptyset, \{\emptyset\}\}, \quad 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots \quad (6.1)$$

Recall that, in formal set theory, elements of every set are sets themselves. The proper definition of an ordinal is as follows:

**Definition 6.0.1.** An ordinal is a set which is well-ordered by the following relation

$$x \leq y \quad \text{iff} \quad x \in y \text{ or } x = y.$$ 

The first ordinals are the natural numbers in (6.1) above. Note that for every ordinal $i$ we obtain an ordinal $i \cup \{i\}$ denoted by $i + 1$. An ordinal is a successor ordinal if it is of the form $i + 1$ for some $i$. An ordinal which is neither 0 nor a successor ordinal is called a limit ordinal. The smallest limit ordinal is denoted by $\omega$; it is the set of natural numbers,

$$\omega = \{0, 1, 2, \ldots\}.$$ 

The sum $i + j$ of two ordinals is the ordinal $i \cup \{i + k : k \in j\}$. (Observe that $i + 1$ above is a special case.) Incidentally, the second limit ordinal $\omega + \omega$ plays an interesting role in work on the finitary set functors (see Example 6.2.3). This ordinal is a countable set. The first uncountable ordinal is denoted by $\omega_1$. 

145
Moreover, the ordinals themselves form a proper class which is itself well-ordered by the inclusion relation: \( i \leq j \) iff \( i \subseteq j \). Every element of an ordinal is an ordinal, and the union of every set of ordinals is an ordinal. This gives a well-ordered proper class \( \text{Ord} \).

**Remark 6.0.2.** We often use proofs by *transfinite induction*: to prove that a class \( C \) of ordinals contains all ordinals, it is sufficient to verify that 0 lies in \( C \) (the *base case*), if \( i \in C \), then \( i + 1 \in C \) (the *isolated step*), and that a limit ordinal \( i \) lies in \( C \) provided each ordinal \( j < i \) belongs to \( C \) (the *limit step*).

Analogously for an ordinal \( k \) (which is the set of all smaller ordinals): to prove that \( C \subseteq k \) is all of \( k \) one verifies 0 \( \in C \), if \( i \in C \) and \( i + 1 < k \), then \( i + 1 \in C \), and the limit step as above.

**Cardinals** are canonical representatives of isomorphism classes of sets: every set is in bijective correspondence with a unique cardinal. The set-theoretic definition is as follows:

**Definition 6.0.3.** A *cardinal* is an ordinal \( i \) such that whenever \( j \) is an ordinal which is isomorphic to \( i \) in \( \text{Set} \), then \( i \leq j \).

The smallest cardinals are the natural numbers (the finite cardinals), and then \( \omega \) itself. The infinite cardinals are listed in a sequence \( (\aleph_i) \), where \( i \) ranges through ordinals. So \( \aleph_0 = \omega \), then \( \aleph_1 = \omega_1 \) is the first uncountable cardinal, etc. For every cardinal \( \lambda \), the cardinal successor is denoted by \( \lambda^+ \). Thus, if \( \lambda = \aleph_i \), then \( \lambda^+ = \aleph_{i+1} \).

A function between ordinals \( f : i \rightarrow j \) is called *cofinal* if for every ordinal \( k < j \) there is some \( i' < i \) such that \( k < f(i') \). A cardinal \( \kappa \) is *regular* if it is infinite and there is no cofinal map from a smaller ordinal to \( \kappa \). A cardinal is *singular* if it is infinite and not regular. The smallest singular cardinal is \( \aleph_\omega \). For every ordinal \( i \), the cardinal \( \aleph_i^+ \) is regular.

**Remark 6.0.4.** (1) Recall that an ordinal \( i \) is the (linearly ordered) set of all ordinals smaller than \( i \). As such it is also a category.

(2) By an *i-chain* is a category \( \mathcal{C} \) is meant a functor \( C : i \rightarrow \mathcal{C} \). It consists of objects \( C_j \) for all ordinals \( j < i \) and (connecting) morphisms \( c_{j,j'} : C_j \rightarrow C_j' \) for all pairs \( j \leq j' < i \).

Analogously, an *Ord-chain in \( \mathcal{C} \) is a functor from \( \text{Ord} \) to \( \mathcal{C} \). In both cases we will simply speak of a (transfinite) *chain* whenever confusion is unlikely.

(3) A category \( \mathcal{C} \) has colimits of chains if for every ordinal \( i \) a colimit of every i-chain exists in \( \mathcal{C} \). (This does not include \( \text{Ord} \)-chains.) In particular, \( \mathcal{C} \) has an initial object since the ordinal 0 is the empty set.

(4) Dually, an *i-op-chain in \( \mathcal{C} \) is a functor \( C : i^{\text{op}} \rightarrow \mathcal{C} \). It is given by objects \( C_j \) \( (j < i) \) and morphisms \( c_{j',j} : C_j \rightarrow C_{j'} \) for all \( j \leq j' < i \). Moreover, an *Ord-op-chain is a functor \( C : \text{Ord}^{\text{op}} \rightarrow \mathcal{C} \).
6.1 The initial-algebra chain

Again, we will simply speak of a (transfinite) chain whenever confusion about the length or direction is unlikely.

(5) A category has limits of chains if for every ordinal \( i \) every \( i^{\text{op}} \)-chain has a limit. This includes the terminal object (the case \( i = 0 \)).

6.1 The initial-algebra chain

In this section, we pursue the transfinite iteration of the initial-algebra chain. The terminal-coalgebra chain is treated later, in Section 6.2. We begin with a famous result on fixed points of order-preserving maps on a chain-complete poset.

Let \( P \) be a chain-complete poset, i.e. every chain in it has a least upper bound. In particular \( P \) has a least element \( \bot \), the least upper bound of the empty chain. Furthermore, let \( f: P \rightarrow P \) an order-preserving map. Then we can define an ordinal-indexed sequence of \( f^i(\bot) \) as follows:

\[
f^0(\bot) = \bot \quad f^{j+1}(\bot) = f(f^j(\bot)) \quad \text{and} \quad f^j(\bot) = \bigsqcup_{i < j} f^i(\bot) \quad \text{for limit ordinals} \ j
\]

It is easy to verify that this is a chain in \( P \); i.e. \( f^i(\bot) \leq f^j(\bot) \) if \( i \leq j \).

**Theorem 6.1.1.** Let \( P \) be a chain-complete poset. Every order-preserving function \( f: P \rightarrow P \) has a least fixed point \( \mu f \). Moreover, for some ordinal \( j \) we have

\[
\mu f = f^j(\bot).
\]

**Proof.** Take \( i \) to be an ordinal larger than \(|P|\). For this \( i \), there must be some \( j < i \) such that \( f^j(\bot) = f^{j+1}(\bot) \). Indeed, this follows from Hartogs’ lemma, stating that for every set \( P \) there exists an ordinal \( i \) such that there is no injection \( i \mapsto P \). Thus, there are \( j < k < i \) such that \( f^j(\bot) = f^k(\bot) \). Since \( f^j(\bot) \leq f^{j+1}(\bot) \leq f^k(\bot) \) we have \( f^j(\bot) = f^{j+1}(\bot) \). Hence \( f^j(\bot) \) is a fixed point of \( f \). Let \( f(x) = x \). An easy transfinite induction shows that \( x \geq f^i(\bot) \) for all \( i \). This shows that \( f^j(\bot) \) is the least fixed point of \( f \).

We should attribute this theorem to Zermelo, since the mathematical content of the result appears in his 1904 paper [246] proving the Wellordering Theorem. Note that the well-known Knaster-Tarski Theorem has a stronger hypothesis (complete lattices) and a stronger conclusion (the fixed points form a complete lattice) [150]. There are also variations on Theorem 6.1.1, such as the result often called the Bourbaki-Witt Theorem: if \( f: P \rightarrow P \) has the property that \( f(x) \geq x \), then \( f \) has a fixed point.

**Remark 6.1.2.**

1. In all of these results, it is critical that \( P \) be a set. For consider the successor function on the ordered class \( \text{Ord} \) of ordinals. It has no fixed point.

2. Alternatively, one may assume in Theorem 6.1.1 that \( P \) be a dcpo (with bottom). This is not a stronger assumption because chain-complete posets are equivalently dcpos, as shown by Markowsky [172].

147
6 Transfinite Iteration

Remark 6.1.3. A pre-fixed point of a monotone function \( f \) is an element \( x \) with \( f(x) \leq x \). Note that \( \mu_f \) in Theorem 6.1.1 is also the least pre-fixed point of \( f \): whenever \( x \in P \) fulfils \( f(x) \leq x \), then \( \mu_f \leq x \). Moreover, by Lambek’s Lemma 2.2.5, a least pre-fixed point of any monotone function (functor) on any poset is a fixed point.

A category-theoretic generalization of the construction of the least fixed point in the proof of Theorem 6.1.1 was formulated by Adámek [8]. It was applied there to the functor \( F(–) + A \); in other words, the free \( F \)-algebra on an object \( A \) was considered instead of the initial \( F \)-algebra (see Proposition 2.2.20).

Definition 6.1.4 [8]. Let \( A \) be a category with colimits of chains (see Remark 6.0.4(3)). For an endofunctor \( F \) we define the initial-algebra chain \( W : \text{Ord} \to \mathcal{A} \). Its objects are denoted by \( W_i \) and its connecting morphisms by \( w_{ij} : W_i \to W_j \), \( i \leq j \in \text{Ord} \). They are defined by transfinite recursion

\[
\begin{align*}
W_0 &= 0, \\
W_{j+1} &= FW_j & \text{for all ordinals } j, \\
W_j &= \text{colim}_{i<j} W_i & \text{for all limit ordinals } j, \text{ and}
\end{align*}
\]

\[
\begin{align*}
w_{0,1} : 0 &\to W_0 \text{ is unique,} \\
w_{j+1,k+1} &= Fw_{j,k} : FW_j \to FW_k, \\
w_{i,j} (i < j) &\text{ is the colimit cocone for limit ordinals } j.
\end{align*}
\]

Remark 6.1.5. (1) Note that for finite ordinals \( n \) we have \( W_n = F^n 0 \) and the initial-algebra \( \omega \)-chain from Definition 3.1.3 is the beginning of the transfinite chain \( (W_i) \).

(2) We did not mention the connecting morphism

\[
w_{\omega, \omega+1} : W_\omega \to FW_\omega = W_{\omega+1}
\]

because it is uniquely determined by the universal property of the colimit \( W_\omega = \text{colim}_{n<\omega} W_n \). We have already seen this (in dual form) in Notation 4.4.4 (see Diagram (4.8)). Indeed, leaving out the first member 0 of that \( \omega \)-chain we still have the same colimit \( W_\omega = \text{colim}_{n+1<\omega} W_{n+1} \) with colimit injections \( w_{n+1,\omega} : W_{n+1} \to W_\omega \) \( (n+1 < \omega) \). We also see that the following morphisms

\[
w_{n+1,\omega+1} = Fw_{n,\omega} : W_{n+1} = FW_n \to FW_\omega
\]

form a cocone. Hence, we obtain \( w_{\omega, \omega+1} \) uniquely such that the following triangles commute:

\[
\begin{CD}
W_{n+1} @. FW_n @. FW_\omega \\
@| @VV{Fw_{n,\omega}} V \\
W_\omega @. FW_\omega @. W_{\omega+1}
\end{CD}
\]

(3) Proceeding as in item (2) we obtain \( w_{i,j} \) for all pairs of ordinals \( i \leq j \). In fact, the initial-algebra chain is determined by Definition 6.1.4 uniquely up to natural isomorphism as we prove in Remark 6.3.2.
We therefore speak of the initial-algebra chain of $F$. Notice that Definition 6.1.4 generalizes what we saw in (6.2) above for sequences associated with a given order-preserving endofunction $f: P \to P$.

**Example 6.1.6.** If $\mathcal{A}$ is a chain-complete poset and $f$ a monotone function on $\mathcal{A}$, then $W_i$ is precisely $f^i(\bot)$ from the proof of Theorem 6.1.1.

**Remark 6.1.7.** Our goal is to provide a category-theoretic generalization of Theorem 6.1.1. Since a monotone function on a poset is a functor, one may expect a result stating that on a category satisfying certain conditions, every functor has an initial algebra, which is to say that the category is algebraically complete (see Definition 3.2.14). However, Theorem 3.2.17 tells us that every algebraically complete category with products is a preorder. Therefore, we should not expect a sweeping categorification of Theorem 6.1.1, the way the main result on iterative constructions of initial algebras (Theorem 3.1.7) generalizes Kleene’s Theorem 3.1.1.

**Construction 6.1.8.** Analogously to Construction 3.1.5 every algebra $\alpha: FA \to A$ induces a canonical cocone

$$\alpha_i: W_i \to A \quad (i \in \text{Ord})$$

on the initial-algebra chain: this is the unique cocone with

$$\alpha_{i+1} = (W_{i+1} = FW_i \xrightarrow{F\alpha_i} FA \xrightarrow{\alpha} A)$$

for all ordinals $i$.

This is an easy definition by transfinite recursion. Indeed, $\alpha_0$ is unique since 0 is the initial object. The isolated step is given, and for any limit ordinal $j$ such that $\alpha_i$ ($i < j$) form a cocone, we use that $W_j = \text{colim}_{i<j} W_i$ to obtain $\alpha_i$ as the unique factorizing morphism with $\alpha_i \cdot w_{i,j} = \alpha_i$.

**Remark 6.1.9.** Homomorphisms of $F$-algebras $h: (A, \alpha) \to (B, \beta)$ preserve the canonical cocones: for every ordinal $i$ we have

$$\beta_i = h \cdot \alpha_i: W_i \to B.$$ 

This is easy to verify by transfinite induction.

**Definition 6.1.10.** We say that the initial-algebra chain of a functor $F$ converges in $\lambda$ steps if $w_{\lambda, \lambda+1}$ is an isomorphism. We say that the chain converges in exactly $\lambda$ steps if $\lambda$ is the least such ordinal.

If $w_{i,i+1}$ is an isomorphism, then so is $w_{i,j}$, for all $j > \lambda$. This is easy to prove by transfinite induction. The following proof of the converse is from Barr [57]:

**Proposition 6.1.11.** If $i < j$ and $w_{i,j}$ is an isomorphism, then so is $w_{i,i+1}$.

**Proof.** Since $w_{i,j}$ is an isomorphism, so is $Fw_{i,j} = w_{i+1,j+1}$. And as $w_{i+1,j+1} = w_{j,j+1} \cdot w_{i+1,j}$, we see that $w_{i+1,j}$ is a split monomorphism. In addition, $w_{i+1,j}$ is a split epimorphism, since $w_{i,j} = w_{i+1,j} \cdot w_{i,i+1}$. Hence $w_{i+1,j}$ is an isomorphism. Thus so is $w_{i,i+1} = w_{i+1,j} \cdot w_{i,j}$. \qed
Theorem 6.1.12 [8]. Let \( \mathcal{A} \) be a category with colimits of chains. If the initial-algebra chain of an endofunctor \( F \) converges in \( j \) steps, then \( W_j \) is the initial algebra with the algebra structure
\[
w^{-1}_{j,j+1} : FW_j \to W_j.
\]
The proof is analogous to that of Theorem 3.1.7:

**Proof.** Let \((A, \alpha)\) be any \( F \)-algebra. Denote by \( h : W_j \to A \) the component \( \alpha_j \) of the canonical cocone \( \alpha_i : W_i \to A \) (see Construction 6.1.8).

(1) In order to see that \( h \) is a homomorphism, note that the outside of the following diagram commutes:
\[
\begin{array}{c}
FW_j \\
\downarrow^{Fh=F\alpha_j} \\
FA \\
\downarrow^{\alpha} \\
A
\end{array}
\]
Indeed, the lower left-hand triangle commutes by the definition of \( \alpha_j+1 \), and the upper right-hand one by the fact that \((\alpha_i)\) is a cocone.

(2) To prove that \( h \) is unique, let \( k : W_j \to A \) be any homomorphism. Then \( h \cdot w_{i,j} = k \cdot w_{i,j} \) holds for all \( i \leq j \); the proof of this fact is by transfinite induction, as in Theorem 3.1.7. The case \( i = j \) yields \( k = h \).

Theorem 6.1.12 is the raison d’être of this section. The rest of the section studies the following topics:

1. sufficient conditions for the convergence of the initial-algebra chain;
2. examples of specific set functors and their convergence ordinals;
3. an examination of when convergence of the initial-algebra chain is necessary for the existence of the initial algebra.

**Corollary 6.1.13.** Let \( \mathcal{A} \) be a category with colimits of chains. Let \( F : \mathcal{A} \to \mathcal{A} \) preserve colimits of \( \lambda \)-chains for some ordinal \( \lambda \). Then the initial-algebra chain of \( F \) converges in \( \lambda \) steps. Therefore,
\[
\mu F = W_\lambda.
\]
The proof is completely analogous to that of Theorem 3.1.7: since \( W_\lambda = \operatorname{colim}_{i<\lambda} W_i \) implies \( FW_\lambda = \operatorname{colim}_{i<\lambda} FW_i \) with the colimit cocone \( Fw_{i,\lambda} = w_{i+1,\lambda+1} \), we have a unique morphism \( \iota : FW_\lambda \to W_\lambda \) with \( \iota \cdot Fw_{i,\lambda} = w_{i+1,\lambda} \) for all \( i \), and this is the inverse of \( w_{\lambda,\lambda+i} \).

**Remark 6.1.14.** (1) In Example 6.1.15 we use the concept of height of a tree (see Remark 2.2.10). Trees without an infinite path are called well-founded (cf. Chapter 7, where well-founded coalgebras are studied more generally). The height of a well-founded tree \( t \) is defined by
\[
h(t) = \sup\{h(t') + 1 : t' \text{ a maximum proper subtree of } t\}.
\]
If \( t \) has an infinite path, then we set \( h(t) = \infty \), a fixed value “larger than all ordinals”.

150
Every finite tree \( t \) is automatically well-founded. For such trees, \( h(t) \) is the length of the longest path from the root to a leaf. For example, the following infinite trees

\[
\begin{array}{c}
t:
\end{array}
\]

and

\[
\begin{array}{c}
t':
\end{array}
\]

have height \( h(t) = \omega \) and \( h(t') = \omega + 1 \), respectively.

(2) For a given ordinal \( i \), a tree has height at most \( i \) iff all its proper subtrees have heights less than \( i \).

**Example 6.1.15.** (1) Let us consider the set functor \( F X = X^\mathbb{N} + 1 \) whose algebras have a constant and an \( \mathbb{N} \)-ary operation. We will show that the initial algebra consists of all well-founded countably branching trees. To this end, we describe the initial-algebra chain as follows:

\[
W_i = \text{all countably branching trees of height } < i.
\]

The connecting maps are given by the inclusions \( W_i \hookrightarrow W_j \) for all \( i \leq j \). Recall that trees are always considered up to isomorphism. We will see that this chain converges in \( \omega_1 \) steps (where \( \omega_1 \) is the first uncountable ordinal).

In the first steps, \( W_0 = \emptyset \) since no tree has height less than 0, and \( W_1 = 1 \) is the singleton \( \{t_0\} \), where \( t_0 \) is the root-only tree (i.e. the unique tree of height 0). Next \( W_2 = 1 + W_1^\mathbb{N} \) has two elements: the tree \( t_0 \) (in the left-hand summand) and the sequence \( (t_0, t_0, t_0, \ldots) \) which is represented by the unique countably branching tree of height 1:

\[
\begin{array}{c}
t_1:
\end{array}
\]

In general, the equation \( W_{i+1} = 1 + W_i^\mathbb{N} \) states that \( W_{i+1} \) consists of \( t_0 \) and all the trees below:

\[
\begin{array}{c}
s_0
\end{array}
\]

where \( s_n \in W_i \) for every \( n \in \mathbb{N} \). Assuming that \( W_i \) consists of all countably branching trees of height less than \( i \), it follows from Remark 6.1.14(2) that \( W_{i+1} \) consists of all such trees of height less than \( i + 1 \).

The connecting map \( w_{01} : \emptyset \rightarrow W_1 \) is, of course, the (empty) inclusion map. The functor \( F X = X^\mathbb{N} + 1 \) clearly takes every inclusion map to an inclusion map. Thus, \( w_{12} = F w_{01} \), \( w_{23} = F w_{12} \) etc. are all inclusion maps. Consequently the colimit \( W_\omega = \text{colim}_{i<\omega} W_i \) is simply the union:

\[
W_\omega = \bigcup_{i<\omega} W_i = \text{all countably branching trees of finite height.}
\]
The initial-algebra chain continues: $W_{\omega+1}$ is a proper super-set of $W_{\omega}$: for example, the following tree

![Tree Diagram](Image)

where $t_n$ is the complete countably branching tree of height $n$, has height $\omega$; thus, it lies in $W_{\omega+1} \setminus W_{\omega}$.

It is easy to see that for every well-founded countably branching tree $t$, the height $h(t)$ is a countable ordinal; i.e. $h(t) < \omega_1$ for the first uncountable ordinal $\omega_1$. Consequently, the initial-algebra chain converges at this ordinal: $\mu F = W_{\omega_1}$. This set consists of all well-founded countably branching trees. The algebra structure $W_{\omega_1}^\omega + 1 \rightarrow W_{\omega_1}$ is given by tree-tupling: to a sequence $(t_0, t_1, \ldots)$ of trees in the left-hand summand it assigns the tree in (6.3).

(2) More generally, we return to polynomial functors $H_\Sigma$ on sets, see Example 2.1.5. Recall from Definition 2.2.12 and we saw in Proposition 2.2.14 that the initial algebra for $H_\Sigma$ is formed by all finite $\Sigma$-trees. We generalize this to signatures which allow for infinitary operations. Such a signature is a set $\Sigma$ of operation symbols $\sigma$, each with an arity $\text{ar}(\sigma)$, which is a (finite or infinite) cardinal number. For every set $X$ and every cardinal $\kappa$, $X^\kappa$ is the set of functions from $\kappa$ to $X$. The signature $\Sigma$ determines a polynomial functor $H_\Sigma$. It is defined on sets $X$ by

$$H_\Sigma X = \prod_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}$$

and similarly on maps.

The concept of a $\Sigma$-tree naturally generalizes to infinitary signatures, too: it is a tree labelled in $\Sigma$ such that every node with $k$ children (where $k$ is a cardinal) is labelled by a $k$-ary operation symbol.

Analogously to (1), one easily shows that the polynomial functor $H_\Sigma$ has the initial-algebra chain

$$W_i = \text{all } \Sigma\text{-trees of height } < i$$

and the initial algebra

$$\mu H_\Sigma = \text{all well-founded } \Sigma\text{-trees}.$$}

The algebra structure is given by tree-tupling.

(3) In contrast to Example 6.1.15(1), $X^\mathbb{N} + 1$ as an endofunctor on CPO is locally continuous. Thus its initial algebra is, by Theorem 5.1.23, $W_\omega$. It consists of all countably branching trees.

**Example 6.1.16.** (1) The power-set functor has no fixed point, thus, no initial algebra. Nonetheless, it has an interesting initial-algebra chain. It is given by the sets

$$W_0 = \emptyset, \quad W_{i+1} = \mathcal{P}W_i, \quad \text{and} \quad W_i = \bigcup_{j < i} W_j \quad \text{for limit ordinals } i$$
6.1 The initial-algebra chain

and the inclusion functions as connecting maps $W_i \hookrightarrow W_j$ for $i \leq j$. In set theory these sets are usually written $V_i$. They constitute a proper hierarchy: every ordinal $i$ belongs to $W_{i+1} \setminus W_i$. Due to the Foundation Axiom, every set lies in $W_{i+1}$ for some $i$. The least such $i$ is called the \textit{rank} of the set.

Of special interest is the set

$$W_\omega = HF$$

of all \textit{hereditarily finite sets}. We have seen in Example 2.2.7 that $HF$ is the initial algebra of $P_f$.

(2) Analogously, denote by

$$P_c$$

the subfunctor of $P$ of all countable subsets. The initial-algebra chain of $P_c$ yields the initial algebra in $\omega_1$ steps. Referring to the functor $P_c$ and its initial algebra chain, we have the algebra $W_{\omega_1} = HC$ of \textit{hereditarily countable sets}.

(3) Generalizing $P_f$ and $P_c$ above, let us define, for every cardinal $\lambda$, the subfunctor

$$P_\lambda$$

of $P$ by taking all subsets of cardinality less than $\lambda$. When $\lambda$ is regular and infinite, $P_\lambda$ preserves colimits of $\lambda$-chains. Hence,

$$\mu P_\lambda = W_\lambda.$$  

Indeed, the initial-algebra chain converges exactly at $\lambda$. To see this, assume it converges exactly at $i < \lambda$. Then for all $j < i$ we have $W_j \subsetneq W_i$ and we can choose $x_j \in W_i - W_j$. The set $X = \{x_j : j < i\}$ has cardinality less than $\lambda$. Thus, $X \in P_\lambda W_i = W_{i+1}$. It is easy to see that $X \not\in W_i$. So $W_i \neq W_{i+1}$, a contradiction.

We can also study the functor $P_\lambda$ when $\lambda$ is a singular cardinal. This functor does not preserve colimits of $\lambda$-chains. But $\lambda^+$ is regular (as it is for all infinite cardinals), and so the initial-algebra chain converges in $\lambda^+$ steps. Indeed, it cannot be less than $\lambda$, see the above argument, and it cannot by an ordinal $i$ with $\lambda \leq i < \lambda^+$. Such an ordinal would have a cofinal subset of size $< \lambda$, and it is easy to extend the argument above.

\textbf{Are all initial algebras obtainable by iteration?} At this point, we know that iteration is a powerful way of obtaining initial algebras. Thus we can ask whether it is \textit{necessary}. In other words, are all initial algebras obtainable by iteration? It turns out that the answer is “No, but in settings that are sufficiently \textit{Set}-like, Yes.” Spelling this out is our next goal.

\textbf{Definition 6.1.17 [231].} A class $\mathcal{M}$ of monomorphisms in a category $\mathcal{A}$ is called \textit{smooth} provided that it is closed under composition, contains all isomorphisms, and for every chain of monomorphisms in $\mathcal{M}$,

(1) a colimit exists and is formed by monomorphisms in $\mathcal{M}$, and

(2) the factorization morphism of every cocone of monomorphisms in $\mathcal{M}$ is again a monomorphism in $\mathcal{M}$.
Remark 6.1.18. (1) In particular, a category with a smooth class of monomorphisms has an initial object $0$ and the unique morphism $0 \to A$ is a monomorphism for every object $A$. Indeed, $0$ is the colimit of the empty chain of monomorphisms in $\mathcal{M}$.

(2) Note that a class $\mathcal{M}$ of monomorphisms containing the identities and closed under composition can be regarded as the subcategory of $\mathcal{A}$ given by all morphisms in $\mathcal{M}$. Then $\mathcal{M}$ is smooth iff the inclusion functor $\mathcal{M} \hookrightarrow \mathcal{A}$ creates colimits of chains.

(3) Monomorphisms are automatically “co-smooth” in the following sense: a limit of a chain of monomorphisms is formed by monomorphisms. Moreover, given a cone of monomorphisms, the unique factorizing morphism is monic.

(4) Condition (1) in Definition 6.1.17 does not imply condition (2) in general; see Remark 6.1.20.

Examples 6.1.19. (1) In the categories of sets, graphs, posets, and semigroups the class of all monomorphisms is smooth.

(2) In the category $\text{Pfn}$ of sets and partial functions, the classes of all monomorphisms and all the split monomorphisms both form a smooth class. (The latter are precisely the monomorphisms of $\text{Set}$.)

(3) In the category $K\text{-Vec}$ of vector spaces over a fixed field, every monomorphism splits, and consequently the class of all monomorphisms is smooth. Indeed, given a subspace $m: A \to B$ of a space $B$, there exists a complementary subspace $m': A' \to B$ with $B = A + A'$. Then $[\text{id}_A, 0]$ splits $m$. And chain colimits are formed as in $\text{Set}$.

(4) In $\text{CPO}$-enriched categories with colimits of $\omega$-chains of embeddings, the class of all embeddings is smooth: see Basic Lemma 5.1.9 and Corollary 5.1.14. However, the class of all monomorphisms is usually non-smooth. For example, in the category $\text{CPO}$ itself consider the cpo $\mathbb{N}^\top$ of natural numbers with a top element $\top$. The subcpo $C_n = \{0, \ldots, n\} \cup \{\top\}$, $n \in \mathbb{N}$, form an $\omega$-chain in $\text{CPO}$. Its colimit is $\mathbb{N}^\top \cup \{\infty\}$ where $n < \infty < \top$ for all $n \in \mathbb{N}$. The cocone formed by the inclusion maps $C_n \hookrightarrow \mathbb{N}^\top$ is formed by monomorphisms. However the factorizing morphism from $\text{colim} C_n$ to $\mathbb{N}^\top$ is not monic, as it merges $\infty$ and $\top$.

(5) In contrast, the collection of all monomorphisms is not smooth in the category of algebras for the functor $FX = X + 1$. The initial object is $\mathbb{N}$ (see Example 3.2.1). However, the unique homomorphism into the terminal $F$-algebra is clearly not monic.

(6) Analogously, monomorphisms are not smooth in the category of rings. Its initial object is the ring $\mathbb{Z}$ of integers, and there are non-injective homomorphisms whose domain is $\mathbb{Z}$.

Remark 6.1.20. (1) Epimorphisms are automatically “smooth”: a colimit of a chain of epimorphisms is formed by epimorphisms. Moreover, given a cocone of that chain formed by epimorphisms, the unique factorizing morphism is epic. This holds in all categories and is easy to prove.

(2) We have seen in Example 6.1.19 below that many everyday categories have smooth monomorphisms. In contrast, the dual property often fails. For example, in $\text{Set}$ this fails.

Here is an example of an $\omega^{op}$-chain of epimorphisms $a_n: A_{n+1} \to A_n$, $n < \omega$, in $\text{Set}$,
and a cone of epimorphisms \( b_n : B \rightarrow A_n, \ n < \omega, \) such that the factorizing morphism \( b : B \rightarrow \lim_{n<\omega} A_n \) is not epic.

Consider the set functor \( FX = X + 1, \) let \( A_n = n + 1 \) be the \( \omega^{op} \)-chain of Example 3.3.6; i.e. \( A_n = F^n1 \) for all \( n; \) and let \( B = \mathbb{N}. \) We have \( \lim A_n = \mathbb{N}^\top. \) The following cone

\[
\begin{align*}
b_n : \mathbb{N} & \rightarrow n + 1, \\
b_n(i) & = \min(i, n + 1) \quad \text{for all } i \in \mathbb{N}
\end{align*}
\]

consists of epimorphisms. But the factorizing morphism \( b : B \rightarrow A \) is not epic, it is the inclusion map \( b : \mathbb{N} \hookrightarrow \mathbb{N}^\top. \) (3) It is nevertheless true that in \( \text{Set} \) limits of chains of epimorphisms are formed by epimorphisms. Thus, in \( \text{Set}^{op}, \) for \( M \) the class of monomorphisms, we see that (1) holds in Definition 6.1.17 but not (2).

**Remark 6.1.21.** We recall the general definition of a subobject from Remark 2.1.15. Generalizing a bit, for an object \( A, \) an \( M \)-subobject is a subobject represented by a morphism \( m_B : B \rightarrow A \) in \( M. \) When the base category is well-powered, we write

\[
\left(Sub_M(A), \leq \right)
\]

for the poset of \( M \)-subobjects of \( A. \) If \( M \) is smooth, then \( Sub_M(A) \) has joins of chains. Equivalently, \( Sub_M \) is a dcpo.

We say that a functor \( F \) preserves \( M \)-monomorphisms provided that \( m \in M \) implies \( Fm \in M. \)

**Theorem 6.1.22** (Initial Algebra Theorem [231]). Let \( \mathcal{A} \) be a well-powered category, let \( M \) be a smooth class of monomorphisms, and let \( F \) preserve \( M \)-monomorphisms. Then the following are equivalent:

1. the initial-algebra chain converges.
2. \( F \) has an initial algebra.
3. \( F \) has a fixed point; i.e. an object \( A \cong FA. \)
4. \( F \) has an \( M \)-pre-fixed point; i.e. an object \( A \) with an \( M \)-monomorphism \( m : FA \rightarrow A. \)

Moreover, if (4) holds, then \( \mu F \) is a subobject of every \( M \)-pre-fixed point of \( F. \) This follows from the proof.

**Proof.** We know that (1) implies (2), and Lambek’s Lemma tells us that (2) implies (3). Clearly, (3) implies (4).

Here is a proof that (4) implies (1). First we define the function

\[
f : \text{Sub}_M(A) \rightarrow \text{Sub}_M(A)
\]

which maps a given \( M \)-subobject \( u : B \hookrightarrow A \) to

\[
f(u) = FA \xrightarrow{Fu} FA \xrightarrow{m} A.
\]

It is monotone by our assumption that \( F \) preserves \( M. \) Theorem 6.1.1 applies since the assumption of smoothness implies that \( \text{Sub}_M(A) \) has joins of chains; therefore it is a dcpo.
6 Transfinite Iteration

By Theorem 6.1.1, it has the least fixed point

$$\mu f = f^i(\bot)$$

for some ordinal \(i\).

The canonical cocone \(\alpha_j : W_j \rightarrow A\) (see Construction 6.1.8) satisfies

$$\alpha_j = f^j(\bot)$$

for all \(j \in \text{Ord}\).

This is easily verified by transfinite induction. Thus, from \(f(f^i(\bot)) = f^i(\bot)\) we conclude that \(\alpha_i\) and \(\alpha_{i+1}\) represent the same subobject of \(A\). Since \(\alpha_i = \alpha_{i+1} \cdot w_{i,i+1}\), it follows that \(w_{i,i+1}\) is invertible. So the initial-algebra chain converges. \(\square\)

**Remark 6.1.23.** We present a different proof of Theorem 6.1.22 in Theorem 7.3.15. That proof uses facts about recursive coalgebras developed in the next chapter.

We present several examples which show, *inter alia*, that none of the hypotheses in Theorem 6.1.22 can be left out.

**Example 6.1.24** [57]. Consider the category \(\text{Ord}^\top\), the ordinals with a new element \(\top\) added “on top”. This category is not well-powered. Let \(F : \text{Ord}^\top \rightarrow \text{Ord}^\top\) be given by \(F(\lambda) = \lambda + 1\), and \(F(\top) = \top\). Clearly, \(\top\) is the only fixed point of \(F\), and this is not \(W_i\) for any ordinal \(i\).

**Example 6.1.25.** Here is another example that the existence of an initial algebra does not imply convergence of the initial-algebra chain; the only assumption not fulfilled is the preservation \(M\)-monomorphisms. This example is a small variation on one from [45]. We use the category \(\text{Gra}\) of graphs and graph homomorphisms (i.e. functions preserving edges). \(\text{Gra}\) is well-powered, and the class of all monomorphisms is smooth. The terminal object in the category is a loop on one point; we write this as \(1\). The initial object, \(0\), is the empty graph. The *chromatic number* of a (loop-free) graph \(G\), denoted by \(\chi(G)\), is the smallest (finite or infinite) cardinal \(\kappa\) such that \(G\) has a homomorphism into \(C_\kappa\), the (loop-free) clique of size \(\kappa\). For example, the clique on the cardinal \(\kappa = \{i : i < \kappa\}\) has \(\chi(C_\kappa) = \kappa\). Note that if \(f : G \rightarrow H\) is a homomorphism, then \(\chi(G) \leq \chi(H)\). Recall from the beginning of this chapter that for every cardinal \(\kappa\) the cardinal successor is denoted by \(\kappa^+\).

Consider \(F : \text{Gra} \rightarrow \text{Gra}\) defined by

$$F(G) = \begin{cases} 1 & \text{if } G \text{ has loops} \\ C_{\kappa^+} & \text{if } G \text{ has no loops, where } \kappa = \chi(G) + \aleph_1. \end{cases}$$

If \(f : G \rightarrow H\) is a homomorphism and neither \(G\) nor \(H\) has loops, then \(Ff\) is the inclusion.

If \(G\) has loops, then so does \(H\), and \(Ff = \text{id}\). If \(G\) has no loops but \(H\) does, then \(Ff\) is the constant. The point of \(\kappa^+\) in the definition of \(F\) is that every \(F\)-algebra must have loops. Indeed, an \(F\)-algebra structure on a graph amounts to a choice of a loop. The graph \(1\) is a fixed point, and it is easy to see that \(\text{id}_1 : 1 \rightarrow 1\) is an initial algebra. But the initial-algebra chain does not converge: \(W_i = C_{\aleph_i}\) for every \(i > 0\) in \(\text{Ord}\).
Example 6.1.26. Pre-fixed points do not in general imply fixed points. Again, we use $\text{Set} \times \text{Set}$. This time, the endofunctor is

$$F(X, Y) = \begin{cases} (\emptyset, 1) & \text{if } X \neq \emptyset \\ (\emptyset, \mathcal{P}Y) & \text{if } X = \emptyset \end{cases}$$

It is defined on morphisms as expected, using $\mathcal{P}$ in the case where $X = \emptyset$. This functor has no fixed points for two reasons: first, $(\emptyset, Y)$ and $(\emptyset, \mathcal{P}Y)$ are never isomorphic, by Cantor’s Theorem (cf. Example 2.2.7(1)). Second, if $X \neq \emptyset$, then there exists no morphism from $(X, Y)$ to $F(X, Y) = (\emptyset, 1)$.

For every endofunctor on $\text{Set}$, the four conditions of Theorem 6.1.22 are equivalent. This is clear if $F\emptyset = \emptyset$. So suppose we have $F\emptyset \neq \emptyset$.

(1) We prove that for every pre-fixed point $\alpha: FA \to A$ with the canonical cocone $\alpha_j: W_j \to A$ we have that $\alpha_\omega$ is a monomorphism. Since $F\emptyset \neq \emptyset$, $FA$ and therefore $A$ are nonempty. Hence, we can choose $\alpha'$: $A \to FA$ with $\alpha' \cdot \alpha = \text{id}_{FA}$. We also define morphisms $u_n: A \to F^{n+1}0$ by induction: choose an arbitrary morphism $u_0: A \to F\emptyset$ and let $u_{n+1} = Fu_n \cdot \alpha'$. Then we obtain

$$w_{n,n+1} = u_n \cdot \alpha_n: F^n0 \to F^{n+1}0 \quad \text{for all } n < \omega$$

by induction: we clearly have $w_{0,1} = u_0 \cdot \alpha_0: \emptyset \to F\emptyset$ and for the induction step we compute:

$$w_{n+1,n+2} = Fu_{n+1} = Fu_n \cdot F\alpha_n = (Fu_n \cdot \alpha') \cdot (\alpha \cdot F\alpha_n) = u_{n+1} \cdot \alpha_{n+1}.$$ 

Given elements $x, y \in W_\omega$ merged by $\alpha_\omega$, we prove $x = y$. Since $W_\omega = \text{colim}_{n<\omega} F^n\emptyset$, we can choose $n < \omega$ and elements $x', y'$ of $F^n0$ mapped by $w_{n,\omega}$ to the given pair. Then $\alpha_n = \alpha_\omega \cdot w_{n,\omega}$ merges $x'$ and $y'$, thus, these elements are also merged by

$$(w_{n+1,\omega} \cdot u_n) \cdot \alpha_n = w_{n+1,\omega} \cdot w_{n+1,n+1} = w_{n,\omega},$$

which proves $x = y$.

(2) Now we prove that the initial-algebra chain converges. Since $\alpha_\omega$ is a monomorphism with nonempty domain, it splits. Thus, $F\alpha_\omega$ is also a monomorphism. This proves that $\alpha_{\omega+1} = \alpha \cdot F\alpha_\omega$ is a monomorphism. In this manner we see that all $\alpha_j$ with $j \geq \omega$ are monomorphisms, and then we argue as in Theorem 6.1.22. 

\begin{proof}
This is clear if $F\emptyset = \emptyset$. So suppose we have $F\emptyset \neq \emptyset$.

(1) We prove that for every pre-fixed point $\alpha: FA \to A$ with the canonical cocone $\alpha_j: W_j \to A$ we have that $\alpha_\omega$ is a monomorphism. Since $F\emptyset \neq \emptyset$, $FA$ and therefore $A$ are nonempty. Hence, we can choose $\alpha'$: $A \to FA$ with $\alpha' \cdot \alpha = \text{id}_{FA}$. We also define morphisms $u_n: A \to F^{n+1}0$ by induction: choose an arbitrary morphism $u_0: A \to F\emptyset$ and let $u_{n+1} = Fu_n \cdot \alpha'$. Then we obtain

$$w_{n,n+1} = u_n \cdot \alpha_n: F^n0 \to F^{n+1}0 \quad \text{for all } n < \omega$$

by induction: we clearly have $w_{0,1} = u_0 \cdot \alpha_0: \emptyset \to F\emptyset$ and for the induction step we compute:

$$w_{n+1,n+2} = Fu_{n+1} = Fu_n \cdot F\alpha_n = (Fu_n \cdot \alpha') \cdot (\alpha \cdot F\alpha_n) = u_{n+1} \cdot \alpha_{n+1}.$$ 

Given elements $x, y \in W_\omega$ merged by $\alpha_\omega$, we prove $x = y$. Since $W_\omega = \text{colim}_{n<\omega} F^n\emptyset$, we can choose $n < \omega$ and elements $x', y'$ of $F^n0$ mapped by $w_{n,\omega}$ to the given pair. Then $\alpha_n = \alpha_\omega \cdot w_{n,\omega}$ merges $x'$ and $y'$, thus, these elements are also merged by

$$(w_{n+1,\omega} \cdot u_n) \cdot \alpha_n = w_{n+1,\omega} \cdot w_{n+1,n+1} = w_{n,\omega},$$

which proves $x = y$.

(2) Now we prove that the initial-algebra chain converges. Since $\alpha_\omega$ is a monomorphism with nonempty domain, it splits. Thus, $F\alpha_\omega$ is also a monomorphism. This proves that $\alpha_{\omega+1} = \alpha \cdot F\alpha_\omega$ is a monomorphism. In this manner we see that all $\alpha_j$ with $j \geq \omega$ are monomorphisms, and then we argue as in Theorem 6.1.22. 

\end{proof}
6 Transfinite Iteration

Proof. For $\text{Set}$, use Theorem 6.1.27. For $\text{Pfn}$ and $K$-$\text{Vec}$, apply the Initial-Algebra Theorem 6.1.22 with $M$ the class of all split monomorphisms (which every endofunctor preserves).

Example 6.1.29 [46]. Fixed points do not in general imply initial algebras. Let $F: \text{Set} \times \text{Set} \to \text{Set} \times \text{Set}$ be given by

$$F(X, Y) = \begin{cases} (1, 1) & \text{if } X \neq \emptyset \\ (\emptyset, \mathcal{P}Y) & \text{if } X = \emptyset. \end{cases}$$

Given a morphism $(f, g): (X, Y) \to (X', Y')$ with $X = \emptyset$, then $F(f, g) = (\text{id}_\emptyset, \mathcal{P}g)$, and $F$ works in the evident way on the other morphisms, using the fact that $(1, 1)$ is the terminal object. Although $(1, 1)$ is a fixed point, $F$ has no initial algebra: the $F$-algebras $(\emptyset, \mathcal{P}Y) \to (\emptyset, Y)$ cannot be initial by Lambek’s Lemma, and algebras of the form $(1, 1) \to (X, Y)$ admit no homomorphisms to those of the form $(\emptyset, \mathcal{P}Y) \to (\emptyset, Y)$. However, $\text{Set} \times \text{Set}$ is well-powered and has smooth monomorphisms. Again, what is going wrong here is that this functor does not preserve monomorphisms.

Remark 6.1.30. The same general argument shows that in the category $\text{Rel}$, existence of a pre-fixed point implies the convergence of the initial-algebra chain. More precisely, let $M$ be the class of all injective functions (considered as morphisms in $\text{Rel}$) and suppose that $F: \text{Rel} \to \text{Rel}$ is an extension of a set functor $G: \text{Set} \to \text{Set}$. That means that for the inclusion functor $J: \text{Set} \to \text{Rel}$ we have a commutative square

$$\begin{array}{ccc} \text{Rel} & \xrightarrow{F} & \text{Rel} \\ J \uparrow & & \uparrow J \\ \text{Set} & \xrightarrow{G} & \text{Set} \end{array}$$

Note that $J$ is a left adjoint and so preserves all colimits. Thus, $M$ is a smooth class in $\text{Rel}$. Moreover, the initial-algebra chain extends from $\text{Set}$ to $\text{Rel}$ in the obvious way. Thus, for every endofunctor on $\text{Rel}$ which is an extension of a set functor, the conditions of the Initial-Algebra Theorem 6.1.22 are equivalent for the class $M$ of all injective functions.

The proofs of the following two theorems are rather technical. We therefore state them without proof.

Theorem 6.1.31 [46]. If the initial-algebra chain for a set functor converges in exactly $\lambda$ steps, then either $\lambda \leq 3$ or $\lambda$ is an infinite regular cardinal.

Example 6.1.32. Conversely, all such convergence ordinals are possible. Convergence in exactly $\lambda$ steps is illustrated by $\mathcal{P}_\lambda$ for regular cardinals $\lambda$. Convergence in exactly $i \leq 3$ steps is illustrated by the following functors $F_i$: $F_0 = \text{id}$, $F_1 = C_1$, the constant functor of value 1, and $F_2 = C_{01}$ differing from $C_1$ only by $C_{01}\emptyset = 2 = \{0, 1\}$. Define $F_3\emptyset = 3$ and for $X \neq \emptyset$ put $F_3X = \{A \subseteq X : |A| = 3 \text{ or } 0\}$. For morphisms $f: X \to Y$, $Ff$ is constantly $\emptyset$ if $X = \emptyset$; if $X \neq \emptyset$ put $Ff(A) = f[A]$ whenever $|f[A]| = 3$, else $Ff(A) = \emptyset$.

Theorem 6.1.33 [46]. For the category $\text{Set}^S$ of many sorted sets, whenever an endofunctor has an initial algebra, then the initial-algebra chain converges.
6.1 The initial-algebra chain

This concludes our discussion of the question on page 153 concerning the relation between convergence of the initial-algebra chain and the existence of initial algebras. Now we return to free algebras. Recall from Remark 2.2.23 that $F$ is called a varietor if free $F$-algebras exist.

**Corollary 6.1.34.** Let $F$ be an endofunctor of $\text{Set}$, $\text{Pfn}$, or $K\text{-Vec}$. Then $F$ is a varietor iff $F$ has arbitrarily large pre-fixed points: for every cardinal there exists a pre-fixed point of a larger cardinality.

**Proof.** Let $F$ be a varietor. Let $\kappa$ be a cardinal and let $|A| = \kappa$. By Proposition 2.2.20, $F(\cdot) + A$ has an initial algebra. So there is some set $X$ with $FX + A \cong X$, by Lambek’s Lemma 2.2.5. Hence $|FX + A| \leq |X|$. In particular, $|FX| \leq |X|$ and $|A| \leq |X|$.

In the other direction, assume that $F$ has arbitrarily large pre-fixed points. Let $A$ be a set, and let $X$ be a set with $\text{max}(|A|, |\mathbb{N}|) \leq |X|$ and $|FX| \leq |X|$. Then $|FX + A| \leq |X|$. (In $\text{Set}$ and $\text{Pfn}$, $FX + A$ is the disjoint union. In $K\text{-Vec}$, $FX + A = FX \times A$.) So there is a monomorphism $m : FX + A \to X$. By Corollary 6.1.28, the functor $F(\cdot) + A$ has an initial algebra. By Proposition 2.2.20, $F$ is a varietor.

An unfortunate fact [45]: the collection of all set functors possessing initial algebras is not well-behaved. There are set functors $F$, $G$ having initial algebras such that none of $F + G$, $F \times G$, or $G \cdot F$ has a fixed point. We defer the details to Example 6.2.17.

**Remark 6.1.35.** In contrast, postfixed points of set functors $F$ are much more frequent: unless the restriction of $F$ to nonempty sets is essentially constant (see Definition 6.1.36 below), all sufficiently large sets $X$ are postfixed points, i.e. they fulfill $|X| \leq |FX|$.

**Definition 6.1.36.** A functor is called *essentially constant* if it is naturally isomorphic to a constant functor.

The following result was first proved by Koubek [155, Thm. 1.3]. We present a new and simpler proof here.

**Theorem 6.1.37.** Let $F$ be a set functor. Then one of the following two alternatives holds:

1. All sufficiently large sets are postfixed points of $F$: there is a cardinal $\kappa$ such that $|FY| \geq |Y|$ for all $Y$ with $|Y| \geq \kappa$.
2. The restriction of $F$ to nonempty sets is almost constant.

**Proof.** For every set $X$, we write $t_X : X \to 1$ for the function.

Case 1: For some $X \neq \emptyset$, $Ft_X$ is not injective. We prove that the cardinal $\kappa = \text{max}(|X|, |\mathbb{N}|)$ has the desired property. Choose an element $x^* : 1 \to X$. Since $Ft_X$ is not injective, and $Ft_X \cdot Fx^* = \text{id}_{F1}$, $Fx^*$ is not surjective. Let $Y$ be any set with $|Y| \geq \kappa$. Since $Y$ is infinite, $|Y| = |X \times Y|$. Therefore there is a partition of $Y$ into $|Y|$ subsets of cardinality $|X|$:

$$Y = \bigcup_{i \in I} Y_i$$

with $|I| = |Y|$ and $|Y_i| = |X|$ for every $i \in I$. 

159
Since $Y$ is infinite, we can fix $y^* \in Y$ and add it to each $Y_i$ without changing cardinality. Choose injection $m_i : X \hookrightarrow Y$ with $m_i[X] = Y_i \cup \{y^*\}$. We can even arrange that $m_i(x^*) = y^*$, where $x^* \in X$ was chosen above. Then for $i \neq j$, the intersection $m_i \cap m_j$ is given by the pullback on the left below:

Since $F$ preserves finite nonempty intersections (see Proposition 4.4.1), we therefore obtain the pullback squares on the right. As shown above, $Fx^*$ is not surjective. Hence, we can choose $x^* \in FX$ that is not in the image of $Fx^*$. The squares on the right above being pullbacks, we see that for $i \neq j$, $Fm_i(x^*) \neq Fm_j(x^*)$. It follows that $|FY| \geq |I| = |Y|$, as desired.

Case 2: For all nonempty $X$, $t_X$ is injective. Then $Ft_X$ is an isomorphism, since $t_X$ is surjective, and (every) $F$ preserves surjections in $Set$. We conclude that item (2) holds: the assignment $\tau_X = Ft_X$ yields a natural isomorphism $\tau : F' \to G$, where $F'$ is the restriction of $F$ to nonempty sets, and $G$ is the constant functor with value $F1$. Naturality is due to the fact that for all $f : X \to Y$, $t_Y \cdot f = t_X$.

**Corollary** 6.1.38. Let $F$ be a set functor which is not essentially constant when restricted to nonempty sets. If $F$ has arbitrarily large pre-fixed points, it has arbitrarily large fixed points. In that case, $F$ is a varietor.

**Proof.** Under the hypothesis, we have some $\kappa$ such that for all $|X|$ with $|X| \geq \kappa$, $|X| \leq |FX|$. Fix a cardinal $\lambda \leq \kappa$, and let $|X| \geq \lambda$ have $|FX| \leq |X|$. Then $|FX| = |X|$.

**6.2 The terminal-coalgebra chain**

Functors from $\text{Ord}^{op}$, the dually ordered class of all ordinals, should be called *transfinite opchains*. We however follow the usual custom of calling them *transfinite chains*. Recall that a category $\mathcal{A}$ is said to have limits of chains if for every ordinal $i$ all diagrams $D : i^{op} \to \mathcal{A}$ have limits. This includes the existence of a terminal object 1 (the case $i = 0$).

The following definition is nothing else than the dual of the initial-algebra chain of Definition 6.1.4. It was formulated explicitly by Barr [58].

**Definition** 6.2.1. Let $\mathcal{A}$ be a category with limits of chains. For every endofunctor $F$ the *terminal-coalgebra chain* is the transfinite chain in $\mathcal{A}$ indexed by $\text{Ord}^{op}$, having
6.2 The terminal-coalgebra chain

objects $V_j$ and connecting morphism $v_{ji}$, $j \geq i$, defined by transfinite recursion as follows:

$$
V_0 = 1, \\
V_{j+1} = FV_j \quad \text{for all ordinals } j, \\
V_j = \lim_{i<j} V_i \quad \text{for all limit ordinals } j, \text{ and}
$$

- $v_{1,0}: F1 \to 1$ is unique,
- $v_{k+1,j+1} = Fv_{k,j}: FV_k \to FV_j$,
- $v_{ji} (j > i)$ is the limit cone for every limit ordinal $j$.

Just as in Definition 6.1.10, we say that the terminal-coalgebra chain converges in $\lambda$ steps if $v_{\lambda+1,\lambda}$ is an isomorphism. We similarly say that it converges in exactly $\lambda$ steps, if $\lambda$ is the least such ordinal, and we also speak about convergence ordinal likewise.

**Remark 6.2.2.** (1) Dually to the initial-algebra chain, Definition 6.2.1 determines the chain $(V_j)$ uniquely up to natural isomorphism.

(2) For finite ordinals $n$ we have $V_n = F^n1$, and we obtain the terminal-coalgebra $\omega^{\text{op}}$-chain of Definition 3.3.1.

(3) Note that $v_{\omega+1,\omega}: FV_\omega \to V_\omega$ is precisely the morphism $m$ in Notation 4.4.4; there the limit cone $v_{\omega+\omega,\omega+n}: V_{\omega+n} \to V_{\omega+n}$ was denoted by $\ell_n$.

(4) Dually to Construction 6.1.8 we have for every coalgebra $\alpha: A \to FA$ the canonical cone

$$
\alpha_i: A \to V_i \quad (i \in \text{Ord}^{\text{op}})
$$

of the terminal coalgebra chain satisfying $\alpha_{i+1} = F\alpha_i \cdot \alpha$ for all ordinals $i$.

**Example 6.2.3.** The terminal-coalgebra chain of every finitary endofunctor on Set converges in $\omega + \omega$ steps; see the proof of Theorem 4.4.8.

**Theorem 6.2.4** (Dual of Theorem 6.1.12). Let $\mathcal{A}$ be a category with limits of chains. If the terminal-coalgebra chain of a functor $F$ converges in $\lambda$ steps, then $V_\lambda$ is the terminal algebra with coalgebra structure $v_{\lambda,\lambda+1}^{-1}: V_\lambda \to FV_\lambda$.

**Corollary 6.2.5** (Dual of Corollary 6.1.13). Let $\mathcal{A}$ be a category with limits of chains. If an endofunctor $F$ preserves limits of $\lambda^{\text{op}}$-chains for some infinite ordinal $\lambda$, then the terminal-coalgebra chain converges in $\lambda$ steps, hence,

$$
\nu F = V_\lambda.
$$

**Example 6.2.6.** For polynomial functors $H_\Sigma$ (even the non-finitary ones of Example 6.1.15(2)) we conclude that the terminal coalgebra is constructed by the terminal-coalgebra $\omega^{\text{op}}$-chain:

$$
\nu H_\Sigma = V_\omega.
$$

Indeed, for all cardinals $\kappa$, the functors $X \mapsto X^\kappa$ preserve limits of $\omega^{\text{op}}$-chains. Hence, so do all polynomial functors. This follows from the fact that a coproduct of set functors preserving limits of $\omega^{\text{op}}$-chains preserves them, too. Recall that $\nu H_\Sigma$ is the coalgebra of all $\Sigma$-trees with the inverse of tree tupling as its structure; for finitary signatures we showed this in Theorem 2.5.9, and for non-finitary ones the proof is analogous.
At this point, we have seen one condition, preservation of limits of $\lambda^{\text{op}}$-chains, which implies an upper bound on the convergence ordinal of the terminal-coalgebra chain. However, this condition is not satisfied by many everyday functors. We therefore now turn to another, rather mild, condition. Recall that finitary set functors have the convergence ordinal $\omega + \omega$ (see Theorem 4.4.8). We are going to generalize this to accessible set functors below. Intuitively, a set functor is $\lambda$-accessible, where $\lambda$ is a fixed infinite regular cardinal, if its behaviour is completely determined by its action on sets of cardinality less than $\lambda$. For a general functor, this intuition is captured by the following definition.

**Definition 6.2.7.** Let $\lambda$ be a regular cardinal. A $\lambda$-directed diagram in a category $\mathcal{A}$ is a functor $D: \mathcal{D} \to \mathcal{A}$, where $\mathcal{D}$ is a $\lambda$-directed poset; that is, every subset of less than $\lambda$ elements has an upper bound. In particular, every $\lambda$-directed poset is nonempty. A endofunctor of $\mathcal{A}$ is $\lambda$-accessible if it preserves colimits of $\lambda$-directed diagrams.

For a set functor $F$ this is equivalent to being $\lambda$-bounded, defined as follows.

**Definition 6.2.8.** Let $\lambda$ be an infinite regular cardinal. A set functor $F$ is called $\lambda$-bounded if for each set $X$ and each $x \in FX$, there exists a subset $M \subseteq X$ of cardinality $|M| < \lambda$ such that $x \in Fi[FM]$, where $i: M \hookrightarrow X$ is the inclusion map.

**Proposition 6.2.9.** A set functor is $\lambda$-bounded iff it is $\lambda$-accessible.

A proof is presented in Section B.5.

**Examples 6.2.10.** (1) Finitary functors are precisely the $\aleph_0$-accessible ones.

(2) Let $\Sigma$ be a signature. Then the polynomial functor $H_\Sigma$ (see Example 6.1.15(2)) is $\lambda$-accessible iff all arities of operations symbols in $\Sigma$ are smaller than $\lambda$.

(3) The bounded power-set functor $P_\lambda X = \{M \subseteq X : |M| < \lambda\}$, a subfunctor of $P$, is $\lambda$-accessible. In contrast, $P$ is not $\lambda$-accessible for any $\lambda$.

We have seen that for a finitary set functor the terminal coalgebra $\nu F = V_{\omega+\omega}$ is the limit of the $\omega$-chain $V_{\omega+i}$ ($i < \omega$). The proof of the generalization to $\lambda$-accessible functors below is very similar. However, there is a crucial difference: the previous proof made use of the fact that finitary set functors preserve nonempty intersections (Proposition 4.4.3). In general, this fails for $\lambda$-accessible functors (see Example B.7.4 on page 421) and so the proof of the more general theorem below needs adjustment.

**Theorem 6.2.11 [245, Thm. 11].** The terminal-coalgebra chain of a $\lambda$-accessible set functor converges in $\lambda + \lambda$ steps:

$$\nu F = V_{\lambda+\lambda}.$$

**Proof.** If $F1 = \emptyset$, then $F$ is constant with value $\emptyset$, and the statements hold trivially. So we may assume $F1 \neq \emptyset$. Then all objects in the terminal-coalgebra chain are nonempty because there is (at least one) coalgebra $\alpha: 1 \to F1$ and therefore we have the maps $\alpha_i: 1 \to V_i$ for every ordinal $i$.

(1) We prove that all connecting morphisms $v_{j,i}$, for $j \geq i \geq \lambda$, are monic. It is sufficient to prove that $v_{\lambda+1,\lambda}$ is monic. Indeed then $v_{\lambda+1,\lambda}$ is a split monomorphism, hence
6.2 The terminal-coalgebra chain

\[ v_{\lambda+2,\lambda+1} = F v_{\lambda+1,\lambda} \] is monic, etc. We obtain by transfinite induction that all \( v_{i,\lambda} \) with \( i \geq \lambda \) are monic.

We verify that two distinct elements \( x \) and \( y \) of \( V_{\lambda+1} = F V_{\lambda} \) remain distinct under \( v_{\lambda+1,\lambda} \). Since \( F \) is \( \lambda \)-bounded, there exists a nonempty subset \( u: U \hookrightarrow V_{\lambda} \) such that \( |U| < \lambda \) and \( x \) and \( y \) lie in the image of \( F u \). In addition, \( (v_{\lambda,i})_{i<\lambda} \) is a limit cone, thus collectively monic. Hence, every pair of distinct elements of \( U \) remains distinct under \( v_{\lambda,i} \) for some \( i < \lambda \). From the fact that \( |U \times U| < \lambda \) we conclude that one \( i_0 < \lambda \) can be chosen for all distinct pairs in \( U \). (This is precisely where we use the regularity of \( \lambda \).) In other words, \( v_{\lambda,i_0} \cdot u \) is a monomorphism. It splits since \( U \neq \emptyset \), thus \( F v_{\lambda,i_0} \cdot F u \) is monic, which implies that \( F v_{\lambda,i_0} \) keeps \( x \) and \( y \) distinct. Moreover \( v_{\lambda+1,\lambda} \) also keeps them distinct, because

\[ F v_{\lambda,i_0} = v_{\lambda+1,i_0+1} = v_{\lambda,i_0+1} \cdot v_{\lambda+1,\lambda}. \]

(2) In order to prove that \( v_{\lambda+\lambda+1,\lambda+\lambda} \) is invertible, it suffices to prove that \( F \) preserves the limit \( V_{\lambda+\lambda} \) of \( (V_i)_{i<\lambda+\lambda} \). We can disregard the first \( \lambda \) members of that chain and obtain the same limit:

\[ V_{\lambda+\lambda} = \lim_{i<\lambda} V_{\lambda+i}. \]

The connecting morphisms

\[ v_i = v_{\lambda+i,\lambda}: V_{\lambda+i} \hookrightarrow V_{\lambda} \]

of that last \( \lambda \)-chain are monic by (1). So the limit is just the intersection \( V_{\lambda+\lambda} = \bigcap_{i<\lambda} V_{\lambda+i} \), and we write \( v_{\lambda} = v_{\lambda+\lambda,\lambda}: V_{\lambda+\lambda} \hookrightarrow V_{\lambda} \) for the monic connecting map. We will now prove that \( F \) preserves this intersection.

Given an element \( x \in F V_{\lambda} \) lying in the image of \( F v_i \), for all \( i < \lambda \), we have the task to prove that \( x \) lies in the image of \( F v_{\lambda+\lambda,\lambda} \). Using that \( F \) is \( \lambda \)-bounded, we choose a subset \( u: U \hookrightarrow V_{\lambda}, |U| < \lambda \), such that \( x \) lies in the image of \( F u \). Without loss of generality we may assume that \( u \cap v_i \) is nonempty; for otherwise we may pick any element from \( V_{\lambda+\lambda} \) and add it to \( U \). Since we have the inclusion maps \( v_{\lambda+\lambda,\lambda+i}: V_{\lambda+\lambda} \hookrightarrow V_{\lambda+i} \) we obtain that \( u \cap v_i \) is nonempty for every ordinal number \( i \). Then for \( u_i = u \cap v_i \) we know that \( x \in \text{im}(F u_i) \) for all \( i < \lambda \) because \( F \) preserves nonempty finite intersections by Proposition 4.4.1. However, \( (u_i) \) is a decreasing \( \lambda \)-chain of subsets of \( U \). Since \( |U| < \lambda \), this chain converges at some \( i_0 < \lambda \). It follows that \( u_{i_0} = u \subseteq v_{\lambda+\lambda,\lambda} \), whence indeed \( x \) lies in the image of \( F v_{\lambda+\lambda,\lambda} \).

\[ \square \]

**Remark 6.2.12** [245, Cor. 14]. The collection of set functors whose terminal-coalgebra chain converges in \( \lambda + \lambda \) steps contains all \( \lambda \)-accessible functors and all set functors preserving limits of \( \lambda^{op} \)-chains, and is closed under composition, coproduct, and limits of functors.

We present examples related to the convergence ordinals of various \( \lambda \)-accessible set functors.

**Example 6.2.13.** For every regular cardinal \( \lambda \), the functor \( \mathcal{P}_\lambda \) of Example 6.2.10(3) has a terminal coalgebra consisting of all strongly extensional \( \lambda \)-branching trees, i.e. every node has less than \( \lambda \) successors (see Definition 4.5.2). The coalgebra structure \( \nu \mathcal{P}_\lambda \hookrightarrow \)
\( \mathcal{P}_\lambda(\nu \mathcal{P}_\lambda) \) is the inverse map of tree-tupling. This was proved for \( \lambda = \aleph_0 \) by Worrell [245] and for general \( \lambda \) by Schwencke [217].

The terminal-coalgebra chain for \( \mathcal{P}_\lambda \) may be presented in terms of a generalization of the saturated trees which we saw in Definition 4.5.11. This leads to the following results, proved in [23, Thm. 3.17]: the terminal-coalgebra chain of \( \mathcal{P}_\lambda \) converges in exactly \( \lambda + \omega \) steps for \( \lambda \) regular, and in exactly \( \lambda \) steps for \( \lambda \) singular.

**Example 6.2.14** [21]. (1) We present a set functor whose terminal-coalgebra chain converges in exactly \( \lambda + \lambda \) steps. Recall that a *filter* on a set \( X \) is a collection \( \mathcal{F} \) of nonempty subsets of \( X \) closed under finite intersection and upwards closed. The filter functor \( \mathcal{F} \) assigns to a set \( X \) the set \( \mathcal{F}X \) of all filters on \( X \), and for a map \( f: X \to Y \) we have

\[
\mathcal{F} f (F) = \{ M \subseteq Y : f^{-1}[M] \in \mathcal{F} \}.
\]

This functor \( \mathcal{F} \) has no fixed point, hence no terminal coalgebra. The \( \lambda \)-accessible coreflection \( \mathcal{F}_\lambda \) is the subfunctor of all \( \lambda \)-small filters; i.e. \( \mathcal{F}_\lambda X \) consists of those filters containing some member of size smaller than \( \lambda \). The convergence ordinal of the terminal-coalgebra chain of \( \mathcal{F}_\lambda \) is \( \lambda + \lambda \).

(2) We can also present a set functor whose terminal-coalgebra chain converges in exactly \( \lambda + \kappa \) steps for an arbitrary regular cardinal \( \kappa < \lambda \). A *base* of a filter \( \mathcal{F} \) is a family \( \mathcal{F}_0 \) of sets such that \( \mathcal{F} \) is the smallest filter with \( \mathcal{F}_0 \subseteq \mathcal{F} \). Let us call a filter \( \alpha \)-based if it has a base whose cardinality is less than \( \alpha \). The subfunctor \( \mathcal{F}_\lambda^\kappa \) of \( \mathcal{F}_\lambda \) consisting of all \( \lambda \)-small, \( \kappa \)-based filters has the terminal-coalgebra chain converging in exactly \( \lambda + \kappa \) steps.

(3) We present a modification \( \mathcal{P}_\lambda \) of \( \mathcal{P}_\lambda \) whose terminal-coalgebra chain converges in exactly \( \lambda \) steps. No modification is needed on objects, but for a morphism \( f: X \to Y \), given \( M \subseteq X \) with \( |M| < \lambda \), we put

\[
\mathcal{P}_\lambda (M) = \begin{cases} 
  f[M] \text{ if } f|_M \text{ is monic}, \\
  \emptyset \text{ otherwise}.
\end{cases}
\]

We have seen a necessary and sufficient condition for a set functor \( F \) to have an initial algebra: the existence of a fixed point (see Corollary 6.1.28). We shall see in Example 6.2.16 that a set functor with a fixed point need not have a terminal coalgebra.

Whenever a terminal coalgebra exists, it can be constructed by the terminal-coalgebra chain. This holds for sets, and more generally, for many-sorted sets:

**Theorem 6.2.15** [46]. *Whenever an endofunctor of \( \text{Set}^S \) has a terminal coalgebra, then its terminal-coalgebra chain converges.*

This generalizes the previous result for endofunctors on \( \text{Set} \) in Adámek and Koubek [20]. Both proofs heavily depend on the theory of algebraized chains developed by Jan Reiterman in his PhD thesis and summarized by Koubek and Reiterman [154].

The expected generalization of Theorem 6.2.15 to, say, all presheaf categories does not hold. Adámek and Trnková [46] constructed an endofunctor on the category of graphs that has a terminal coalgebra although the terminal-coalgebra chain does not converge.
Example 6.2.16 [20]. In contrast to what we saw in the Initial-Algebra Theorem 6.1.22, a set functor with a fixed point need not have a terminal coalgebra. Consider the set functor \( \mathcal{P}_{-\omega} \)

\[ \mathcal{P}_{-\omega} X = \text{the set of all finite or uncountable subsets of } X. \]

For a function \( f : X \to Y \) and \( A \in \mathcal{P}_{-\omega} X \),

\[ \mathcal{P}_{-\omega} f(A) = \begin{cases} f[A] & \text{if either } A \text{ is finite or } f|_{A} \text{ is injective,} \\ \emptyset & \text{otherwise.} \end{cases} \]

This functor has an initial algebra but not a terminal coalgebra. Obviously, \( \mathbb{N} \) is a fixed point of \( \mathcal{P}_{-\omega} \): the set \( \mathcal{P}_{-\omega} \mathbb{N} \) of all finite subsets is countable. Thus \( \mu_{\mathcal{P}_{-\omega}} \) exists by Corollary 6.1.28.

To verify that \( \nu \mathcal{P}_{-\omega} \) does not exist, note that \( \mathcal{P}_1 \) is a subfunctor of \( \mathcal{P}_{-\omega} \). If \( \nu \mathcal{P}_{-\omega} \) would exist, then by Proposition 6.3.8 below, it would be at least as large as \( \nu \mathcal{P}_1 \). Recall from Example 4.5.8 that \( \nu \mathcal{P}_1 \) is uncountable. To conclude the proof, we thus check that \( \mathcal{P}_{-\omega} \) has no uncountable fixed points. Let \( A \) be an uncountable set. Let \( C \) be the set of countably infinite subsets of \( A \), write \( D \) for \( \mathcal{P}_{-\omega} A \), and note that \(|C \cup D| = |\mathcal{P}(A)| > |A|\). The map \( X \mapsto A \setminus X \) is an injection \( \mathcal{P}A \mapsto \mathcal{P}A \). Since \( A \) is uncountable, it restricts to an injection \( C \mapsto D \). Therefore \(|C| \leq |D|\). Now \(|D| = \max(|C|, |D|) = |C| + |D| = |\mathcal{P}(A)|\).

In other words, \(|A| < |\mathcal{P}_{-\omega} A|\). In particular, \( A \) is not a fixed point.

Example 6.2.17 [45]. We now provide set functors \( F \) and \( G \) having initial algebras such that none of \( F \cdot G \), \( F + G \), or \( F \times G \) has any fixed points. Recall \( F X = \mathcal{P}_{-\omega} X \) from Example 6.2.16, and let \( G X = X + \mathbb{R} \) for the set \( \mathbb{R} \) of real numbers. Then \( F \) and \( G \) have initial algebras. We have seen in Example 6.2.16 that for every uncountable set \( A \), \(|A| < |F(A)|\). It follows that the same holds for \( F + G \), \( F \times G \), and \( G \cdot F \). Indeed, for \( A \) countable or finite, \((F + G)A \) is uncountable (easily), and the same for \( F \times G \) and \( G \cdot F \). As for \( F \cdot G \), note that \((F \cdot G)X = \mathcal{P}_{-\omega}(X + \mathbb{R})\). For all \( X \), \( X + \mathbb{R} \) is uncountable, and so \(|X| \leq |X + \mathbb{R}| < |(F \cdot G)X|\). Thus, these functors all fail to have fixed points.

We next present a sufficient condition for the existence of a terminal coalgebra for a set functor \( F \). The condition is the existence of a fixed point pair: this means that an infinite regular cardinal \( \lambda \) exists such that \( F \) has fixed points of cardinalities \( \lambda \) and \( \lambda^+ \), the successor cardinal of \( \lambda \). We prove in Theorem 6.2.18 that (assuming GCH) this existence of a fixed point pair implies that \( F \) has a terminal coalgebra.

However, a fixed point pair is not necessary for the existence of \( \nu F \). To see this consider the following functor

\[ F X = X + 1 + \{M \subseteq X : X \text{ is uncountable}\}. \]

On morphisms, \( F \) acts in the evident way, using direct images in the third summand. Then \( F \) has the same terminal-coalgebra chain as the functor \( X \mapsto X + 1 \) yielding a countable terminal coalgebra for both functors. Moreover, by an argument similar to what we saw in Example 6.2.16, we see that every fixed point of \( F \) has cardinality \( \aleph_0 \). So \( F \) does not have a fixed point pair.
6 Transfinite Iteration

In Theorem 6.2.18 we assume the Generalized Continuum Hypothesis (GCH): for every infinite cardinal $\lambda$, $\lambda^+ = 2^\lambda$. Under this hypothesis, for all cardinals $\kappa$ and all regular cardinals $\lambda$, we have $\lambda^\kappa = \lambda$; see [138, Thm. 5.15(iii)]. Even without GCH, we have $\lambda^\lambda = 2^\lambda$.

**Theorem 6.2.18** [20, Thm. 5]. Assuming GCH, every set functor with a fixed point pair has a terminal coalgebra.

**Proof.** Let $\lambda$ be a part of a fixed point pair of a set functor $F$. We prove that for every element $x \in V_{\lambda+1}$ there exists a subset $m: M \hookrightarrow V_{\lambda}$ such that $x$ lies in $Fm[Fm]$ and $|M| < \lambda$. This implies that $\nu_{\lambda+1, \lambda}$ is surjective. Hence it is an isomorphism, and $\nu F = V_{\lambda}$, as we have seen in the proof of Theorem 6.2.11. We can assume that $F_1 \neq \emptyset$, thus, $V_i \neq \emptyset$ for all ordinals $i$.

1. For every $j < \lambda$ we prove that $|V_j| \leq \lambda$ by transfinite induction. The base case is clear. If this holds for $V_j$, then there exists an injective map $m: V_j \hookrightarrow \lambda$. Since $m$ is a split monomorphism, so is $Fm$. Thus, $|FV_j| \leq |F\lambda| = \lambda$. This is the isolated step. Finally, let $j < \lambda$ be a limit ordinal, and let $\kappa < \lambda$ be $|j|$. Then $V_j = \lim_{i<j} V_i$ has cardinality at most that of $\prod_{i<j} V_i$ which, by the induction hypothesis is at most $\lambda^\kappa$. By GCH, $\lambda^\kappa = \lambda$.

2. Consequently, $|V_{\lambda}| \leq \lambda^+$, because

$$|V_{\lambda}| = |\prod_{i<\lambda} V_i| \leq \lambda^\lambda = 2^\lambda.$$ Since $2^\lambda = \lambda^+$ is a fixed point of $F$, it follows that $|V_{\lambda+1}| = |FV_{\lambda}| \leq \lambda^+$.

3. We now apply Lemma B.3.6 twice. First use the fixed point $\lambda^+$. Given $x \in V_{\lambda+1} = FV_{\lambda}$ there exists $m_0: M_0 \hookrightarrow V_{\lambda}$ with $x \in Fm_0[Fm_0]$ and $|M_0| < \lambda^+$. So $|M_0| \leq \lambda$. Choose $x_0 \in FM_0$ with $x = Fm_0(x_0)$. Now we use the fixed point $\lambda$: there exists $m_1: M_1 \hookrightarrow M_0$ with $x_0 \in FM_1[Fm_1]$ and $|M_1| < \lambda$. This concludes the proof as the subset $m = m_0 \cdot m_1: M_1 \hookrightarrow X$ has cardinality less than $\lambda$, and $x \in FM[Fm_1]$.

**Open Problem 6.2.19.** (1) For a set functor $F$ give a necessary and sufficient condition for the existence of a terminal coalgebra.

2. Characterize the convergence ordinals of terminal-coalgebra chains of set functors.

**Remark 6.2.20.** The following were proved by Barr [57].

1. The initial-algebra and terminal-coalgebra chain of a functor $F$ are connected by a unique collection of morphisms $h_i: W_i \rightarrow V_i$ ($i \in \text{Ord}$) satisfying $h_{i+1} = Fh_i$ for every ordinal $i$.

2. If some $h_i$ is an isomorphism and at least one algebra-to-coalgebra morphism exists, then $F$ has a canonical fixed point (Definition 5.0.1). In more detail, the hypothesis is that there is an algebra $\alpha: FA \rightarrow A$ and a coalgebra $\beta: B \rightarrow FB$ and some $f: A \rightarrow B$.
6.3 Subfunctors and quotient functors

such that the square below commutes:

\[
\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
Ff & \downarrow & \downarrow f \\
B & \xleftarrow{\beta} & FB
\end{array}
\]

Open Problem 6.2.21. Are there set functors with canonical fixed points, other than constant functors?

6.3 Subfunctors and quotient functors

This section presents results which transfer initial algebras and terminal coalgebras to subfunctors and quotient functors. So we are concerned with four situations. In some but not all of these, the results use the chain constructions which we have been studying in earlier parts of this chapter.

We first prove a technical result concerning the initial-algebra chains of pairs of functors related by a natural transformation. In the following we consider a pair of endofunctors \( F \) and \( F' \) with the initial-algebra chains \((W_i)\) for \( F \) and \((W'_i)\) for \( F' \). Note that a natural transformation between the corresponding chains is a collection of morphisms \( \psi_i : W_i \to W'_i \) \((i \in \text{Ord})\) making the following squares commutative for every pair \( i \leq j \) of ordinals:

\[
\begin{array}{ccc}
W_i & \xrightarrow{w_{i,j}} & W_j \\
\downarrow{\psi_i} & & \downarrow{\psi_j} \\
W'_i & \xrightarrow{w'_{i,j}} & W'_j
\end{array}
\]

Proposition 6.3.1. Every natural transformation \( \varphi : F \to F' \) between functors with initial-algebra chains \((W_i)\) and \((W'_i)\), respectively, induces a unique natural transformation \( \psi_i : W_i \to W'_i \) \((i \in \text{Ord})\) making the following squares commutative for every pair \( i \leq j \) of ordinals:

\[
\begin{array}{ccc}
W_i & \xrightarrow{w_{i,j}} & W_j \\
FW_i & \xrightarrow{\varphi W_i} & FW'_i \\
\downarrow{\psi_i} & & \downarrow{\psi_j} \\
W'_i & \xrightarrow{w'_{i,j}} & W'_j
\end{array}
\] (6.4)

Proof. (1) Uniqueness. Given any natural transformation \( \psi_i : W_i \to W'_i \) satisfying (6.5) we verify its that each \( \psi_i \) is uniquely determined by transfinite induction. The base case is clear since \( W_0 = W'_0 = 0 \). For the isolated step note that (6.5) determines \( \psi_{i+1} \) uniquely from \( \psi_i \). Now let \( j \) be a limit ordinal. Then we have \( W_j = \text{colim}_{i < j} W_i \) and \( W'_j = \text{colim}_{i < j} W'_i \). By hypothesis we know that \( (\psi_i)_{i < j} \) form a natural transformation of the corresponding \( j \)-chains. Since the morphisms \( w_{i,j} \cdot \psi_i : W_i \to W'_i \) clearly form a cocone of \( (W_i) \) with vertex \( W'_j \), we see that \( \psi_j : W_j \to W'_j \) is uniquely determined by the universal propery of the colimit \( W_j \) such that the naturality squares (6.4) commute for every \( i < j \).

(2) Existence. We proceed by transfinite recursion on \( j \) to define \( \psi_j \) and simultaneously prove the commutativity of (6.4) for every \( i < j \).
In the base case we put $\psi_0 = \text{id}_0$, and there is nothing to prove.

For the isolated step we define $\psi_{j+1} = \varphi_{W'_j} \cdot F\psi_j$ as in (6.5). We prove the commutativity of (6.4) for every $i < j + 1$ by transfinite induction on $i$. The base case $i = 0$ is clear since $W_0 = 0$ is the initial object. For the isolated step we know by the induction hypothesis on $j$ that (6.4) commutes. Then the corresponding square for the successors of $i$ and $j$ commutes, too:

\[
\begin{array}{c}
W_{i+1} \\
\downarrow \psi_{i+1} \\
W'_{i+1} \\
\downarrow F' \varphi_{W'_{i+1}} \\
F'w'_{i+1} \\
\downarrow F'w'_{i} \\
FW'_{i} \\
\downarrow F\psi_i \\
FW_i \\
\downarrow Fw_i \\
W_i \\
\end{array}
\]

Indeed, the upper and lower parts commute and by the definitions of $w_{i+1,j+1}$ and $w'_{i+1,j+1}$, respectively. The left- and right-hand squares commute by (6.5), the middle upper square commutes by the induction hypothesis, and the middle lower one by the naturality of $\varphi$.

Finally, suppose that $i$ is a limit ordinal. We use that $W_i = \text{colim}_{k < i} W_k$ and show that the desired square commutes when precomposed by every colimit injection $w_{k,i}$:

\[
\begin{array}{c}
W_k \\
\downarrow \psi_k \\
W'_k \\
\downarrow w'_{k,i} \\
W'_{i} \\
\downarrow \psi_i \\
W_i \\
\downarrow w_{k,i} \\
W_k \\
\end{array}
\]

The outside commutes by the induction hypothesis (for our induction on $i$), and the left-hand inner square does by the induction hypothesis (for the induction on $j$ because $i < j$). Since the upper and lower parts trivially commute, we obtain that the right-hand inner square commutes when precomposed by $w_{k,i}$ for all $k < i$. This proves that this square commutes since the colimit injections are jointly epic.

Finally, suppose that $j$ is a limit ordinal. By induction we know that for all $i \leq i' < j$ the squares (6.4) (with $i'$ in lieu of $j$) commute. Thus, we obtain $\psi_j: W_j \rightarrow W'_j$ uniquely such that (6.4) commutes for every $i < j$, similarly as in item (1), by the universal property of the colimit $W_j = \text{colim}_{i < j} W_i$. This completes the proof.
Remark 6.3.2. (1) If \( \varphi : F \to F' \) is a natural isomorphism then so is \( \psi_i : W_i \to W'_i \). Indeed, the natural transformation \( \varphi^{-1} : F \to F' \) induces a natural transformation \( \bar{\psi}_i : W'_i \to W_i \). It is easy to see that it is inverse to \( \psi_i \).

(2) In particular, the initial-algebra chain of a functor is unique up to natural isomorphism. To see this apply item (1) to \( \varphi = \text{id}_F \).

**Subfunctors and initial algebras**

Given a functor \( F : \mathcal{A} \to \mathcal{B} \), and a class \( \mathcal{M} \) of monomorphisms, an \( \mathcal{M} \)-subfunctor of \( F \) is represented by a functor \( F' : \mathcal{A} \to \mathcal{B} \) with a natural transformation \( m : F' \to F \) whose components \( m_X \) belong to \( \mathcal{M} \).

If an initial algebra of a functor \( F \) exists, do all \( \mathcal{M} \)-subfunctors have an initial algebra?

Not in general:

**Example 6.3.3.** Let \( \text{Gra} \) be the category of graphs and graph homomorphisms. Let \( F \) be the endofunctor mapping every graph \((X, \emptyset)\) without edges to the graph \((\mathcal{P}X, \{(\emptyset, \emptyset)\})\) on the power set that has just a loop at \( \emptyset \). All graphs with edges are mapped to the terminal graph \( 1 \). Homomorphisms \( f : (X, \emptyset) \to (Y, \emptyset) \) are mapped to \( \mathcal{P}f \). Then \( F \) has an initial algebra carried by the terminal graph:

\[ \mu F = 1. \]

But the subfunctor of \( F \) given by \((X, \emptyset) \mapsto (\mathcal{P}X, \emptyset) \) and else constantly 1 does not have an initial algebra, since 1 is its only fixed point, and there exist no algebra homomorphism from it to the algebras \( \alpha : (\mathcal{P}X, \emptyset) \mapsto (X, \emptyset) \).

**Corollary 6.3.4.** Let \( \mathcal{A} \) be a well-powered category, and let \( \mathcal{M} \) be a smooth class of monomorphisms. If an endofunctor \( F \) on \( \mathcal{A} \) has an initial algebra, then so does every \( \mathcal{M} \)-subfunctor \( F' \) of \( F \) preserving \( \mathcal{M} \)-monomorphisms. Moreover, \( \mu F \) is a subobject of \( \mu F' \).

**Proof.** Indeed, \( F' \) has the \( \mathcal{M} \)-pre-fixed point

\[ F'((\mu F)) \xrightarrow{m_{\mu F}} F(\mu F) \xrightarrow{\text{\text{structure}}} \mu F, \]

where the second morphism is the structure of the initial \( F \)-algebra. Now apply Theorem 6.1.22 to \( F' \) to see that it has an initial algebra. Since \( \mu F \) is a pre-fixed point of \( F' \), it follows that it is a subobject of \( \mu F' \).

Observe that the assumption that the subfunctor \( F' \) preserves \( \mathcal{M} \) whenever \( F \) does and \( \mathcal{M} \) is left cancellative; i.e. \( u \cdot v \in \mathcal{M} \) implies \( v \in \mathcal{M} \). (For example, the class of all monomorphisms is left cancellative.) Indeed, let \( m : F' \to F \) be a natural transformation with components in \( \mathcal{M} \). Then \( F' \) preserves \( \mathcal{M} \)-monomorphisms whenever \( F \) does: if \( u : X \to Y \) is an \( \mathcal{M} \)-monomorphism then \( F'u \) is one, too, since \( m_Y \cdot F'u = Fu \cdot m_X \) holds by the naturality of \( m \).

**Corollary 6.3.5.** Let \( \mathcal{A} \) be well-powered with a left-cancellative, smooth class \( \mathcal{M} \) of monomorphisms. If an endofunctor preserves \( \mathcal{M} \)-monomorphisms and has an initial algebra, then every \( \mathcal{M} \)-subfunctor has an initial algebra, too.
Remark 6.3.6. Even if \( F \) has no initial algebra the initial algebra of any \( M \)-subfunctor of \( F \) has a universal property with respect to all algebras for \( F \). Indeed, suppose that we have any natural transformation \( m: F' \to F \) and that the initial algebra \( (\mu F', \iota) \) exists. Then for every \( F \)-algebra \( (A, \alpha) \) there exists a unique morphism \( h: \mu F' \to A \) such that the diagram below commutes:

\[
\begin{array}{ccc}
F'(\mu F') & \xrightarrow{\iota} & \mu F' \\
m_{\mu F'} \downarrow & & \downarrow m \\
F(\mu F') & \xrightarrow{F'h} & F' A \\
mh \downarrow \quad \alpha \downarrow & & \downarrow \alpha \\
FA & \xrightarrow{\alpha - m A} & A
\end{array}
\]

Indeed, note that the left-hand triangle above commutes due to the naturality of \( m \). By the initiality of \( \mu F' \) there exists a unique \( h \) such that the right-hand part above commutes. Equivalently, the outside commutes.

Quotient functors and initial algebras Just as subfunctors inherit initial algebras, so do quotient functors.

Proposition 6.3.7. Let \( \mathcal{A} \) be a well-powered and co-well-powered category with a smooth class \( M \) of monomorphisms. Let \( F: \mathcal{A} \to \mathcal{A} \) preserve \( M \)-monomorphisms and epimorphisms. If \( F \) has an initial algebra, then so does every quotient functor \( q: F \to F' \), and \( \mu F' \) is a quotient object of \( \mu F \).

Proof. By Theorem 6.1.22, the initial-algebra chain \((W_i)\) of \( F \) converges. Thus, there exists an ordinal \( \lambda \) with \( \mu F = W_\lambda \). We prove that the initial-algebra chain \((W'_i)\) of \( F' \) converges, too.

(1) Applying Proposition 6.3.1, we obtain the natural transformation \( \psi_i: W_i \to W'_i \) induced by \( q \). We prove by transfinite induction that \( \psi_i \) is epic for every ordinal \( i \).

The base case is trivial since \( \psi_0 = \text{id}_0 \). For the isolated step use the induction hypothesis and that the components of \( q \) are epic and that epimorphisms compose and that \( F \) preserves epimorphisms to see that \( \psi_{i+1} \) is epic:

\[
\psi_{i+1} = (W_{i+1} = FW_i \xrightarrow{\psi_i} FW'_i \xrightarrow{q_{W'_i}} F'W'_i = W'_{i+1}).
\]

For a limit ordinal \( j \), observe that \( \psi_j: W_j \to W'_j \) was (necessarily) defined as the colimit of the chain of epimorphisms \((\psi_i)_{i < j}\) (in the category of \( \mathcal{A} \to \mathcal{A} \) of morphisms of \( \mathcal{A} \)). \( \psi_j = \text{colim}_{i < j} \psi_i \). It is easy to see that colimit of epimorphisms in \( \mathcal{A} \to \mathcal{A} \) are epimorphisms (cf. Remark 6.1.20).

(2) Since \( \mu F = W_\lambda \) we know that \( w_{\lambda,i} \) is an isomorphism for every \( i \geq \lambda \). Therefore the following composites are quotients of \( W_\lambda \) for every \( i \geq \lambda \):

\[
e_i = (W_\lambda \xrightarrow{w_{\lambda,i}} W_i \xrightarrow{\psi_i} W'_i).
\]
Due to the naturality of $\psi_i$ they form a chain of quotients of $W_\lambda$ with the factorizing morphisms $w'_{i,j}$ for $j \geq i \geq \lambda$:

$$
\begin{array}{ccc}
W_\lambda & \xrightarrow{w'_{i,j}} & W_j \\
W_i & \downarrow{w_{i,j}} & \downarrow{\psi_j} \\
W'_i & \xrightarrow{w'_{i,j}} & W'_j \\
\end{array}
$$

Since $W_\lambda$ has only a set of quotients, this chain of quotients converges, i.e. there exists an ordinal $i \geq \lambda$ such that $e_i$ and $e_{i+1}$ represent the same quotient of $W_\lambda$. In other words, $w'_{i,i+1}: W'_i \to W'_{i+1}$ is an isomorphism, as desired.

(3) The assertion on $\mu F$ and $\mu F'$ follows from the fact that $\psi_i: \mu F \to \mu F'$ is epic. \qed

**Proposition 6.3.8.** Let $\mathcal{A}$ be a well-powered category with limits of chains. Let $F': \mathcal{A} \to \mathcal{A}$ preserve monomorphisms, and assume that the terminal-coalgebra chain of $F'$ converges. Then so does the terminal coalgebra chain of every subfunctor $m: F \to F'$, and $\nu F$ is a subobject of $\nu F'$.

**Proof.** (1) Denote by $(V_i)$ and $(V'_i)$ the respective terminal-coalgebra chains of $F$ and $F'$. By the dual of Proposition 6.3.1 there exists a unique natural transformation $\psi_i: V_i \to V'_i$ ($i \in \text{Ord}$) such that for every ordinal $i$ we have

$$
\psi_{i+1} = (V_{i+1} = FV_i \xrightarrow{F\psi_i} FV'_i \xrightarrow{mV'_i} F'V'_i).
$$

By induction on $i$ one proves that every $\psi_i$ is monic. Indeed, the proof is almost identical to what we have seen for epimorphisms in item (1) of the proof of Proposition 6.3.7 except that one uses in the limit step that limits of monomorphisms (in $\mathcal{A}^{op}$) are monic.

(2) We know that $\nu F' = V'_\lambda$ for some ordinal $\lambda$. For every $i > \lambda$ we have the following subobject of $\nu F$:

$$
s_i = (V_i \xrightarrow{\psi_i} V'_i \xrightarrow{v'_{i,\lambda}} \nu F').
$$

We are using the fact that $v'_{i,\lambda}$ is an isomorphism. These subobjects form a decreasing chain of subobjects of $V'_\lambda$ with the factorizing morphisms $v_{j,i}$ for $j \geq i \geq \lambda$:

$$
\begin{array}{ccc}
V_i & \xleftarrow{v_{j,i}} & V_j \\
\downarrow{\psi_i} & & \downarrow{\psi_j} \\
V'_i & \xleftarrow{v'_{j,i}} & V'_j \\
\downarrow{v'_{i,\lambda}} & & \downarrow{v'_{j,\lambda}} \\
V'_\lambda & & \\
\end{array}
$$

171
Since $\nu F'$ has only a set of subjobjects, there is an ordinal $i \geq \lambda$ such that the subjobjects represented by $s_i$ and $s_{i+1}$ are the same. In other words, $\nu_{i+1}$ is an isomorphism. So as desired, $\nu F = V_i$.

(3) The assertion on the terminal coalgebras $\nu F$ and $\nu F'$ follows from the fact that $\psi_i$ is monic.

**Remark 6.3.9.** We will later see that a terminal coalgebra for $F'$ implies that all subfunctors of $F'$ have terminal coalgebras, without assuming that the terminal-coalgebra chain converges (see Proposition 11.2.2).

**Quotient functors and terminal coalgebras** For terminal coalgebras of quotient functors we do not have a result that would fit with the preceding ones. However, if the components of a natural transformations $\varepsilon: F \rightarrow F'$ are split epimorphisms, we have the following result due to Gumm and Schröder [122]:

**Observation 6.3.10.** (1) Let $\mathcal{A}$ be a complete and co-well-powered category. Suppose that $F$ and $G$ are endofunctors, and let $\varepsilon: F \rightarrow G$ be a natural transformation whose components are split epimorphisms. If $F$ has a terminal coalgebra, then so does $G$. Indeed, by Lemma 4.3.6, $G$ has a weakly terminal coalgebra. Now use Theorem 4.2.8 to the that $\nu G$ exists.

(2) Furthermore, we see that $\nu G$ is a canonical (split) quotient of $\nu F$ via the unique homomorphism shown below:

\[
\begin{array}{ccc}
\nu F & \xrightarrow{\tau F} & F(\nu F) \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon \nu F} \\
\nu G & \xrightarrow{\tau G} & G(\nu F)
\end{array}
\]

To see that $\varepsilon$ is a split epimorphism, we use that $\nu F$ is a weakly terminal $G$-coalgebra to obtain a homomorphism $\alpha: (\nu G, \tau G) \rightarrow (\nu F, \varepsilon \nu F \cdot \tau F)$ (cf. Lemma 4.3.6). Thus, $\varepsilon \cdot s = \text{id}_{\nu G}$ because it is a coalgebra homomorphism from $\nu G$ to itself.

**Corollary 6.3.11.** If a set functor has an initial algebra (or a terminal coalgebra), then so do all of its subfunctors and all of its quotients.

**Proof.** Let $F$ be a set functor, let $G$ be a subfunctor and $H$ a quotient. Assume that $F$ has an initial algebra, say $FI \cong I$. Then $|GI| \leq |FI| \leq |I|$. Likewise $|HI| \leq |I|$. By Corollary 6.1.28, $G$ and $H$ have initial algebras.

Assume that $F$ has a terminal coalgebra. Then by Theorem 6.2.15 the terminal-coalgebra chain of $F$ converges. We wish to apply Proposition 6.3.8 to see the same for $G$ even if $F$ does not preserve all monomorphisms. But note that in that result, we only need each morphism $\psi_i$ to be monic. If $G_1 = \emptyset$, then $G$ is constant with value $\emptyset$. Therefore $V_i = \emptyset$ for every $i > 0$, and we are done. Otherwise all $V_i$ are nonempty because there is (at least one) coalgebra $\alpha: 1 \rightarrow G_1$ and therefore we have the maps $\alpha_i: 1 \rightarrow V_i$ for every ordinal $i$. Using that nonempty injections are split monomorphisms in $\textbf{Set}$ and thus preserved by $G$, we see by transfinite induction that every $\psi_i$ is injective.
6.4 Canonical fixed points in CPO-enriched categories

We now return to the CPO-enriched categories and consider sufficient conditions for an endofunctor to have a canonical fixed point. We know from Theorem 5.1.23 that locally continuous endofunctors have canonical fixed points. We introduce a weaker condition, stability (which is, by Observation 6.4.5 below, more or less equivalent to preservation of embeddings) and prove that whenever a stable functor has a fixed point, it has a canonical one. Recall from Definition 5.1.20 that a category is called strict CPO-enriched if its hom-sets carry the structure of a cpo with \(\bot\), and composition is continuous and preserves \(\bot\) (in both variables).

**Definition 6.4.1.** A functor \(F\) between strict CPO-enriched categories is called **stable** if for every idempotent endomorphism \(f: X \to X\), \(f = f \cdot f\), we have:

\[
f \subseteq \text{id}_X \implies Ff \subseteq \text{id}_{FX}.
\]

**Examples 6.4.2.** (1) Every locally monotone functor (see Definition 5.1.15) is stable.
(2) A composite, product or coproduct of stable endofunctors on the category \(\text{CPO}_\bot\) (see Example 5.1.21) is stable. (Recall from Example 2.1.7(2) that coproducts in \(\text{CPO}_\bot\) are given by disjoint unions with least elements merged.)
(3) For every stable functor \(F: \text{CPO}_\bot \to \text{CPO}_\bot\) the lifting given by \(F\bot X = FX \cup \{\bot\}\) (\(\bot\) a new bottom element) is stable.
(4) It follows that for every signature \(\Sigma\), we have two stable endofunctors

\[
H_\Sigma X = \prod_{\sigma \in \Sigma} X^n \quad \text{and} \quad H'_\Sigma X = \prod_{\sigma \in \Sigma} X^n_{\bot} \quad n = \text{ar}(\sigma),
\]

**Remark 6.4.3.** Let \(\Sigma\) be a signature. The \(H_\Sigma\)-algebras in \(\text{CPO}_\bot\) are the **strict continuous \(\Sigma\)-algebras**: they are given by a cpo, \(A\), which is a \(\Sigma\)-algebra such that every operation \(\sigma_A: A^n \to A\) is continuous and strict: \(\sigma_A(\bot, \ldots, \bot) = \bot\).

The \(H'_\Sigma\)-algebras are \(\Sigma\)-algebras whose operations are continuous but not necessarily strict.

**Remark 6.4.4.** (1) A category is said to have **split idempotents** if every idempotent endomorphism, i.e. a morphism \(f: X \to X\) with \(f \cdot f = f\), has a factorization

\[
f = s \cdot u \quad \text{with} \quad u \cdot s = \text{id}_X.
\]

(2) Every category with equalizers has split idempotents: given \(f\) one takes the equalizer \(u\) of \(f\) and \(\text{id}_X\). Dually, a category with coequalizers has split idempotents.

(3) The above factorization is unique up to isomorphism, for suppose \(f = s' \cdot u'\) where \(u' \cdot s' = \text{id}_X\). Then \(i = u' \cdot s\) is an isomorphism with \(s' = s \cdot i^{-1}\) and \(u' = i \cdot u\).
Observation 6.4.5. (1) Stable functors $F$ preserve embeddings. To see this, let $e$ be an embedding with projection $\hat{e}$. Observe that $e \cdot \hat{e}$ is idempotent. Since $e \cdot \hat{e} \leq \text{id}$, we also have $F(e \cdot \hat{e}) \leq \text{id}$. Therefore $F\hat{e}$ is the projection for $Fe$.

(2) Let $\mathcal{A}$ be a CPO-enriched category with split idempotents. For every endofunctor $F: \mathcal{A} \to \mathcal{A}$ we have:

\[ F \text{ stable } \iff F \text{ preserves embedding-projection pairs.} \]

Indeed, let $F$ preserve these pairs and let $f \cdot f = f \subseteq \text{id}_X$. For the factorization in Remark 6.4.4 we have $s \cdot u = f \subseteq \text{id}_X$, thus, $s$ is an embedding with $u = \hat{s}$. Therefore $Fu = F\hat{s}$ and we conclude that $Ff = Fs \cdot F\hat{f} \subseteq \text{id}_{FX}$.

Lemma 6.4.6. Let $\mathcal{A}$ be a nonempty, strict CPO-enriched category with colimits of $\omega$-chains. The initial-algebra chain of every stable endofunctor consists of embeddings.

Proof. We prove that if $F$ is a stable endofunctor, then $w_{i,j}: W_i \to W_j$ is an embedding by transfinite induction analogous to that in Proposition 6.3.1. First, $w_{0,1}$ is an embedding since, by Lemma 5.1.22, we have a zero object $0 = 1$, and the unique morphism $F0 \to 0$ is $\hat{w}_{0,1}$ because we clearly have $\hat{w}_{0,1} \cdot w_{0,1} = \text{id}$.$0$. Since $w_{0,1} = \bot: 0 \to F0$, we conclude that $w_{0,1} \cdot \hat{w}_{0,1} = \bot \subseteq \text{id}_{F0}$.

If $w_{i,j}$ is an embedding, so is $w_{i+1,j+1} = Fw_{ij}$ by Observation 6.4.5.

For the limit steps, use the fact that a colimit of a chain of embeddings in formed by embeddings (see Basic Lemma 5.1.9 and Remark 5.1.11).

Remark 6.4.7. In the proof of Lemma 6.4.6, the projections $\hat{v}_{i,j}: V_j \to V_i$ form the terminal-coalgebra chain of $F$ (cf. Section 6.2). This is easily seen by transfinite induction.

Corollary 6.4.8. Let $\mathcal{A}$ be a strict CPO-enriched category with colimits of chains. Then every stable endofunctor $F$ with a fixed point has a canonical fixed point $\mu F = \nu F$.

This follows from Theorem 6.1.22 applied to the class $M$ of all embeddings. This is a smooth class, see Basic Lemma 5.1.9 and Remark 5.1.11. Due to Observation 6.4.5, $F$ preserves embeddings. Well-poweredness with respect to embeddings (which are split monics) follows from the observation that the number of split subobjects of an object $A$ is bounded by the number of endomorphisms of $A$. Indeed, given a subobject represented by $m: A' \to A$ with a splitting $e: A' \to A$, so that $e \cdot m = \text{id}$, the endomorphism $m \cdot e$ determines the subobject, since every endomorphism has, by Remark 6.4.4(3), up to isomorphism at most one such factorization.

Remark 6.4.9. Returning to CPO⊥-enriched categories, recall that locally continuous endofunctors have canonical fixed points (see Theorem 5.1.23). The concept of a locally continuous endofunctor $F$ can be generalized to that of a locally $\lambda$-continuous one, where $\lambda$ is a given infinite cardinal: This means that for every $\lambda$-chain $(f_i)_{i < \lambda}$ in $\mathcal{A}(X,Y)$ we have

\[ F(\bigsqcup_{i < \lambda} f_i) = \bigsqcup_{i < \lambda} Ff_i \quad \text{ in } \mathcal{A}(FX,FY). \]
6.5 Summary of this chapter

For these endofunctors we have, whenever $\mathcal{A}$ is strict CPO-enriched and has colimits of chains of cofinality at most $\lambda$, the following:

$$F \text{ has a canonical fixed point } \mu F = \nu F = \lim_{i<\lambda} V_i.$$  

This is completely analogous to Theorem 5.1.23.

**Example 6.4.10** [57]. The category of sets and partial one-to-one functions with homsets ordered by inclusion is CPO$_{\perp}$-enriched. An endomorphism $f$ is smaller than the identity if and only if it is idempotent. Thus, every endofunction is stable, and consequently, all endofunctors with a fixed point have a canonical one.

**Remark 6.4.11.** Corollary 6.4.8 stems from Barr [57], where the slightly stronger assumption that $F$ be order-preserving on homsets was made, and the proof was somewhat more technical. Barr also proves a result establishing canonical fixpoint for endofunctors on a (non-strict) CPO-enriched category (see [57, Theorem 5.1]).

**Example 6.4.12.** The functor $\text{Idl}: \text{DCPO}_{\perp} \to \text{DCPO}_{\perp}$ taking an object of DCPO$_{\perp}$ to the dcpo of its ideals (see Example 5.1.17) is stable, but it does not have any fixed point (see Corollary 15.2.7).

### 6.5 Summary of this chapter

Up to and including this chapter, the main method of constructing initial algebras and terminal coalgebras has been iteration. This was the only method in Chapter 2, where the iterations went on for precisely $\omega$ steps. This method worked on $\text{Set}$ and a host of other categories. For terminal coalgebras of finitary set functors, we extended the iteration to $\omega + \omega$ in Chapter 4. In the present chapter we take things to the limit, so to speak, allowing the iterations to go as long as needed. Indeed, the results in the chapter provide bounds for the convergence of both the initial-algebra chain and the terminal-coalgebra chain. We also have seen results which address the question of whether iteration is necessary for initial algebras and terminal coalgebras.

We want to emphasize the contrast between the results Chapter 2 and those in present chapter. We began with Theorem 6.1.1, and this chapter applied that result in key places. But the chapter was not mainly concerned with categorical generalizations. As noted in Remark 6.1.7, such generalizations do not exist. In their place, we have results like Theorem 6.1.22, the Initial Algebra Theorem.

Some highlights of this chapter: We presented a transfinite construction of initial algebras $\mu F$ and terminal coalgebras $\nu F$ for endofunctors on categories with (co)limits of (op-)chains. For a polynomial set functor $H_{\Sigma}$, the initial-algebra chain converges in $\lambda$ steps to $\mu H_{\Sigma}$, the algebra of all well-founded $\Sigma$-trees, provided that $\lambda$ is an infinite regular cardinal larger than all arities of the operation symbols in $\Sigma$. In contrast, the terminal-coalgebra chain converges already in $\omega$ steps to $\nu H_{\Sigma}$, the coalgebra of all $\Sigma$-trees. For every set functor $F$, an initial algebra exists iff $F$ has a (pre-)fixed point, and it is always constructed by the initial-algebra chain. In contrast, existence of a fixed point $X$ does not guarantee that a terminal coalgebra exists. A fixed point pair does guarantee
the existence of a terminal coalgebra, but this condition is not necessary. For a number of categories, given an endofunctor $F$ preserving monomorphisms, we proved that the existence of a fixed point implies that $\mu F$ exists.

All terminal coalgebras of endofunctor on $\text{Set}^S$, the category of $S$-sorted sets, that exist are obtained by the terminal-coalgebra construction. This does not hold for endofunctors on the (presheaf) category of graphs. We also discussed the precise length of the construction until convergence of the initial-algebra chain or the terminal coalgebra chain, respectively, happens.

For endofunctors $F$ on strict CPO-enriched categories the concept of stability was introduced. This is the (very weak) condition that if $f: X \to X$ is an idempotent morphism which $f \sqsubseteq \text{id}_X$, then $Ff \sqsubseteq \text{id}_{FX}$. For a stable endofunctor we proved that if it has a fixed point, then it has a canonical one, i.e. $\mu F = \nu F$.

The chapter also discussed $\lambda$-accessible set functors. This concept will be discussed further in Section 11.3.
7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras

A fixed point of a functor can be considered both as an algebra or as a coalgebra. For example, the terminal coalgebra can be viewed as an algebra (by inverting its structure map). The theme of this chapter are characterizations of the terminal coalgebra as the initial object in a category of algebras that allow one to interpret structured corecursive specifications. Dually, we are interested in characterizations of the initial algebra as a terminal object in a category of coalgebras that allow one to interpret specifications by structured recursion. The chapter is split into sections that present the various notions of (co)algebras. Each of those notions is interesting in its own right. In fact, they were originally introduced and studied for reasons not connected to our theme.

We begin in Section 7.1 with corecursive algebras, and we show that the terminal coalgebra is characterized as the initial corecursive algebra. In Section 7.2 we present the stronger notion of a completely iterative algebra and again characterize the terminal coalgebra as an initial completely iterative algebra. Dually, we then turn to recursive and parametrically recursive coalgebras, respectively, in Section 7.3, where the initial algebra is characterized as a terminal object.

Before we turn to Sections 7.1–7.3, let us informally motivate corecursive and completely iterative algebras with an example from general algebra. Consider the set functor $FX = X \times X + 1$ expressing the type of algebras with a binary operation $*$ and constant $c$. In this example, corecursive specifications take the form of systems of formal recursive equations, and we require that those are uniquely solvable. For example, the recursive equation $x \approx x * x$ has a unique solution in an algebra $A$ iff $A$ has a unique idempotent element $i \in A$. The following system of recursive equations

$$x \approx x * x \quad y \approx x * (y * c)$$

(7.1)

requires, in addition, that a unique element $a \in A$ with $a = i *^A (a *^A c^A)$. Moreover, we would like to have unique solutions of as many recursive equations as possible. It will not be possible to allow the right-hand sides to be bare variables because we do not expect equations like $x \approx x$ to have a unique solution. Similarly, “ungrounded” systems like

$$x_1 \approx x_2 \quad x_2 \approx x_3 \quad x_3 \approx x_4 \quad \cdots$$

are not expected to have unique solutions. So our formal definition of an equation is designed to forbid the right-hand sides of systems to be bare variables. In addition, our definition also includes an important simplification. Even if we wish to solve as many recursive equations as possible, it is sufficient to restrict attention to systems whose
right-hand sides are *flat terms*; i.e. terms containing precisely one operation symbol. Fortunately, every system of recursive equations where each right-hand side is a non-variable term (or even an infinite \( \Sigma \)-tree) can be flattened by introducing fresh auxiliary variables to represent subterms of non-flat terms. This implies that, as soon as we have solutions for all systems with flat right-hand sides, we also have solutions to systems where the right-hand sides are more complex. For example, for the above system \((7.1)\) we obtain the following flattened system (using fresh variables \(z\) and \(z'\)):

\[
x \approx x * x \quad y \approx x * z \quad z \approx y * z' \quad z' \approx c.
\]

Let \(X\) denote the set of recursion variables (i.e. the ones on the left-hand sides of the above formal equations). Then a flat system of formal recursive equations assigns to every element of \(X\) its right-hand side, viz. an element of \(FX = X \times X + 1\). Such assignments are simply coalgebras \(e: X \to FX\). Given an algebra \(\alpha: A \times A + 1 \to A\), a *solution* of \(e\) assigns to every recursion variable in \(X\) an element of \(A\), i.e. we obtain a morphism \(e^\dagger: X \to A\). The fact that \(e^\dagger\) solves \(e\) means precisely that the following square commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & A \\
\downarrow e & & \uparrow \alpha \\
FX & \xrightarrow{Fe^\dagger} & FA
\end{array}
\]

This means that a solution \(e^\dagger\) is just a coalgebra-to-algebra morphism. The \(F\)-algebra \((A, \alpha)\) is called *corecursive* if every coalgebra has a unique solution.

Although this notion is interesting, it is not exactly what we are after in this chapter. We are more interested in a stronger notion that allows a wider class of recursive specifications. For example, suppose that we again fix \((A, \alpha)\) and allow elements of \(A\) to appear as parameters on the right-hand sides of formal recursive equations such as \((7.1)\). This means that we require that all flat equation morphisms \(e: X \to FX + A\) have a unique solution \(e^\dagger\) in the sense that the following square commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & A \\
\downarrow e & & \uparrow \alpha, \text{id}_A \\
FX + A & \xrightarrow{Fe^\dagger + \text{id}_A} & FA + A
\end{array}
\]

Then we call the algebra \(A\) *completely iterative*, or a *cia*, for short. Again, the commutativity of the above square just expresses the fact that formal equations are turned into identities in \(A\). The main result in Section 7.2 is that a terminal coalgebra for \(F\) is precisely the same as an initial cia for \(F\).

The two dual notions are notions of coalgebras providing semantics for structured recursion. A coalgebra \((A, \alpha)\) is called *recursive* if for every morphism \(e: FX \to X\) there
7.1 Corecursive algebras

is a unique morphism $e^\dagger: A \to X$ such that the square below commutes:

\[
\begin{array}{c}
A \\ \alpha \downarrow \\ FA \\
\end{array}
\xrightarrow{e^\dagger} X
\quad
\begin{array}{c}
\downarrow \\ Fe^\dagger \\
FX \\
\end{array}
\]

The dual of complete iterativity is the notion of parametric recursiveness. A coalgebra $(A, \alpha)$ is parametrically recursive if for every morphism $e: FX \times A \to X$ there is a unique morphism $e^\dagger: A \to X$ so that the square below commutes:

\[
\begin{array}{c}
A \\ \langle \alpha, A \rangle \downarrow \\ FA \times A \\
\end{array}
\xrightarrow{e^\dagger} X
\quad
\begin{array}{c}
\downarrow \\ Fe^\dagger \times A \\
FX \times A \\
\end{array}
\]

Finally, we list the four definitions given above in Table 7.1.

<table>
<thead>
<tr>
<th>type of equation morphism</th>
<th>a _____ algebra has unique solutions to these coalgebra in the dual setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X \to FX$</td>
<td>corecursive</td>
</tr>
<tr>
<td>$X \to FX + A$</td>
<td>completely iterative</td>
</tr>
</tbody>
</table>

Table 7.1: Definitions studied in Sections 7.1–7.3.

7.1 Corecursive algebras

Corecursive algebras were first studied by Capretta et al. [81], and they compared that notion to a number of related concepts. Adámek et al. [17] investigated corecursive algebras further, and, in particular, they provided a description and construction of free corecursive algebras as well as a study of the ensuing monad of free corecursive algebras and its properties. Formally, a corecursive algebra is one that admits a unique coalgebra-to-algebra morphism for every given coalgebra. We use the terminology from the introduction of this chapter:

**Definition 7.1.1.** Let $F$ be an endofunctor on a category $\mathcal{A}$. By a solution of a coalgebra $e: X \to FX$ in an $F$-algebra $(A, \alpha)$ we mean a morphism $e^\dagger$ such that the following square commutes:

\[
\begin{array}{c}
X \\ \downarrow e \\
\end{array}
\xrightarrow{e^\dagger} A
\quad
\begin{array}{c}
\downarrow \alpha \\
FX \\
\end{array}
\xrightarrow{F e^\dagger} FA
\]

The algebra $(A, \alpha)$ is called corecursive if every coalgebra has a unique solution in it.
Remark 7.1.2. (1) Fields like general algebra and abstract data types study \( \Sigma \)-algebras, which are precisely the algebras for the functor \( H_\Sigma \) (see Example 2.1.5). Let \( X \) be a set of “recursion variables”. A coalgebra \( e : X \to H_\Sigma X \) can be understood as a system of formal mutually recursive equations

\[
x \approx \sigma(x_1, \ldots, x_n),
\]

one for every recursion variable \( x \in X \), where \( \sigma \) is an \( n \)-ary operation symbol in \( \Sigma \) and \( x_1, \ldots, x_n \in X \). Indeed, one just writes \( e(x) = \sigma(x_1, \ldots, x_n) \) in lieu of the above formal equation. For a \( \Sigma \)-algebra \( A \), a solution \( e^\dagger : X \to A \) assigns to every recursion variable \( x \) an element \( e^\dagger(x) \in A \) such that the formal equations in (7.3) become actual identities in \( A \) when recursion variables are substituted by their solutions, i.e. we have \( e^\dagger(x) = \sigma^A(e^\dagger(x_1), \ldots, e^\dagger(x_n)) \).

A concrete example of a corecursive algebra is the algebra \( T_\Sigma = \nu H_\Sigma \) of all \( \Sigma \)-trees, cf. Example 7.1.3(2) below.

(2) In Example 7.1.3, we will present further examples of corecursive \( \Sigma \)-algebras. However, let us observe that classical algebras (such as groups, lattices, or boolean algebras) are almost never corecursive. This is due to the uniqueness requirement for solutions, more precisely, it follows from the fact that a corecursive algebra with a binary operation has a unique idempotent element. Indeed, the formal recursive equation \( x \approx x \lor x \) has a unique solution in a lattice or boolean algebra \( A \) iff \( A \) is trivial. Similarly, the system \( x \approx x \cdot y, y \approx 1 \) has a unique solution in a group \( G \) iff \( G \) is trivial.

Example 7.1.3. (1) Capretta et al. [81] provide the following example of a corecursive algebra for the endofunctor \( FX = E \times X \times X \) on \( \text{Set} \). Consider the set \( E^\omega \) of all streams over \( E \) with the \( F \)-algebra structure \( \alpha : E \times E^\omega \times E^\omega \to E^\omega \) where \( \alpha(e, s, t) \) is the stream with head \( e \) and continuing by the merge of \( s \) and \( t \). This is a corecursive \( F \)-algebra.

(2) Every terminal coalgebra is a corecursive algebra. More precisely, by Lambek’s Lemma 2.2.5, the structure \( \tau : \nu F \to F(\nu F) \) of the terminal coalgebra is invertible, and hence \((\nu F, \tau^{-1})\) an \( F \)-algebra. This algebra is corecursive because for every \( e : X \to FX \) a coalgebra homomorphism from \((X, e)\) to \((\nu F, \tau)\) is the same thing as a solution of \( e \) in the algebra \((\nu F, \tau^{-1})\).

(3) The set functor \( FX = X \times X + 1 \) is the polynomial functor associated to the signature \( \Sigma \) with a binary operation symbol \( * \) and a constant symbol \( c \). By the previous point, the terminal coalgebra \( \nu F \) consisting of all finite and infinite binary trees (see Example 3.3.9) is corecursive. Let us consider inner nodes of binary trees as being labelled by \( * \) and leaves as labelled by \( c \). Let \( X = \{x_1, x_2\} \) and consider the following system of mutually recursive equations

\[
x_1 \approx c \ast x_2 \quad x_2 \approx x_1 \ast c
\]

Then its solution in \( \nu F \) consists of two infinite trees, \( x_1^\dagger \) and \( x_2^\dagger \), here written as infinite terms:

\[
x_1^\dagger = c \ast ((c \ast (\cdots \ast c)) \ast c) \quad \text{and} \quad x_2^\dagger = (c \ast ((c \ast (\cdots) \ast c)) \ast c.
\]

We invite the reader to draw these as pictures; cf. Example 7.2.6(1).
7.1 Corecursive algebras

(4) Here is a different example of a corecursive algebra which we shall revisit when we
discuss completely iterative algebras. It comes from Capretta et al. [81]. Let $FX = X \times X$
on $\mathsf{Set}$. Let $A = \{0, 1, 2\}$, and let $\alpha: A \times A \to A$ be the algebra defined by $\alpha(1, 2) = 1$,and otherwise $\alpha(i, j) = 0$. Then $A$ is corecursive: given any $e: X \to X \times X$, the unique
solution $e^\dagger: X \to A$ is the constant function on 0. Indeed, this is clearly a solution since
$\alpha(0, 0) = 0$. Moreover, since 2 cannot be written as $\alpha(i, j)$, we see that for all $x$, $e^\dagger(x) \neq 2$.
And we claim that the same is true for 1. For suppose that $e^\dagger(x) = 1$. Then $e(x)$ must
be $(y, z)$ for some $y$ and $z$ such that $e^\dagger(y) = 1$ and $e^\dagger(z) = 2$. However, this contradicts
what we have seen about 2. This proves that $e^\dagger$ is the constant function with value 0.

(5) In Section 15.4, we shall see a number of corecursive algebras for set functors
whose definitions involve elementary mathematics. Here is one example. Consider
$FX = \mathbb{N} \times X$. Then the half-open interval $(0, 1]$ is a corecursive algebra for $F$, via the
structure $n, r \mapsto (n + r)/(1 + n + r)$. See Theorem 15.4.9.

(6) All algebras $FV_i \to V_i$ (see Definition 6.2.1) in the terminal-coalgebra chain of $F$ are
corecursive. This is a consequence of Corollary 7.2.11 on completely iterative algebras.

Corecursive algebras for $F$ may be considered as a full subcategory of $\mathsf{Alg} F$ because of the following

Proposition 7.1.4 [17, Lem. 2.9]. Let $(A, \alpha)$ and $(B, \beta)$ be corecursive algebras. Every
homomorphism $h: (A, \alpha) \to (B, \beta)$ preserves solutions. This means that for every $e: X \to FX$ we have that

$X \xrightarrow{e^\dagger} A \xrightarrow{h} B$

is the unique solution of $e$ in $B$.

Proof. The following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & A \\
\downarrow{e} & & \downarrow{h} \\
FX & \xrightarrow{Fe^\dagger} & FA & \xrightarrow{Fh} FB \\
\end{array}
\]

\[\xrightarrow{F(h \cdot e^\dagger)}
\]

shows that $h \cdot e^\dagger$ is a solution of $e$ in $B$ and therefore is the unique one. \qed

In order to see that an initial corecursive $F$-algebra is a fixed point of $F$, a crucial step
is the following result by Capretta, Uustalu and Vene.

Proposition 7.1.5 [80, dual of Prop. 6]. If $(A, \alpha)$ is a corecursive algebra, then so is
$(FA, F\alpha)$.

Proof. Suppose that $(A, \alpha)$ is a corecursive algebra for $F$, let $e: X \to FX$ and denote by
$e^\dagger: X \to A$ its unique solution. We will show that

$e^\dagger = (X \xrightarrow{e} FX \xrightarrow{Fe^\dagger} FA)$

181
is the unique solution of $e$ in $FA$. First, the following diagram shows that it is a solution:

\[
\begin{array}{c}
X & \xrightarrow{e} & FX & \xrightarrow{Fe^\dagger} & FA \\
\downarrow & & \downarrow & & \downarrow \\
FX & \xrightarrow{F_e} & FFX & \xrightarrow{FFe^\dagger} & FFA \\
\end{array}
\]

Indeed, the upper and lower parts commute by the definition of $e^\dagger$, the right-hand square does since $e^\dagger$ solves $e$, and the left-hand square is trivial.

To see that $e^\dagger$ is the unique solution in $FA$, suppose that $s : X \to FA$ solves $e$. Then it follows that $\alpha \cdot s : X \to A$ is a solution of $e$ in $A$, since the following diagram commutes:

\[
\begin{array}{c}
X & \xrightarrow{s} & FA & \xrightarrow{\alpha} & A \\
\downarrow & & \downarrow & & \downarrow \\
FX & \xrightarrow{Fs} & FFA & \xrightarrow{F\alpha} & FA \\
\end{array}
\]

Indeed, the left-hand square commutes since $s$ solves $e$ in $FA$, and the remaining parts are trivial. Thus, we have $e^\dagger = \alpha \cdot s$, and we conclude that

\[e^\dagger = Fe^\dagger \cdot e = F\alpha \cdot Fs \cdot e = s,\]

where the last equation holds since, once again, $s$ solves $e$ in $FA$.

From this result one obtains Lambek’s Lemma for corecursive algebras.

**Corollary 7.1.6.** If an initial corecursive $F$-algebra exists, it is a fixed point of $F$.

Indeed, the proof is the same as the one for Lambek’s Lemma 2.2.5, using Proposition 7.1.5 to see that for an initial corecursive algebra $(I, \iota)$, $(FI, F\iota)$ is corecursive, too.

**Theorem 7.1.7** [80, dual of Prop. 7]. The initial corecursive algebra is precisely the same as the terminal coalgebra.

In more detail, let $F : \mathcal{A} \to \mathcal{A}$ be an endofunctor. Then we have:

1. If $(I, \iota)$ is an initial corecursive algebra, then $(I, \iota^{-1})$ is a terminal coalgebra.
2. If $(\nu F, \tau)$ is a terminal coalgebra, then $(\nu F, \tau^{-1})$ is an initial corecursive algebra.

**Proof.** (1) Suppose that $\iota : FI \to I$ is an initial corecursive algebra. By Corollary 7.1.6, we have a coalgebra $\iota^{-1} : I \to FI$. We need to verify that it is terminal. Let $(C, \gamma)$ be any coalgebra. The coalgebra homomorphisms from $(C, \gamma)$ to $(I, \iota^{-1})$ are the same as the solutions of $\gamma$ in $(I, \iota)$. So since $(I, \iota)$ is a corecursive algebra, $(I, \iota^{-1})$ is a terminal coalgebra.

182
(2) Suppose that \( \tau: \nu F \to F(\nu F) \) is the terminal coalgebra. We have seen in Example 7.1.3(2) that the algebra \((\nu F, \tau^{-1})\) is corecursive. It remains to verify its initiality. So let \((A, \alpha)\) be any corecursive algebra. There is a unique solution of \( \tau: \nu F \to F(\nu F) \) in \((A, \alpha)\), and this means that there is a unique algebra homomorphism \( h \) from \((\nu F, \tau^{-1})\) to \((A, \alpha)\).

**Remark 7.1.8.** The only fixed point among the corecursive algebras for a functor is the initial corecursive algebra. Indeed, notice that in part (1) of the above proof we used the initiality of the corecursive algebra \((I, \iota)\) only to see that the structure morphism is an isomorphism. Consequently, by the argument above, every corecursive algebra whose structure map is an isomorphism is a terminal coalgebra [80, dual of Prop. 7].

Theorem 7.1.7 connects to the theme of this chapter in that it expresses the terminal coalgebra of a functor as the initial object in an interesting category of algebras.

**Proposition 7.1.9.** Every limit of corecursive algebras is corecursive.

This is easy to prove, using that limits of algebras are formed on the level of the underlying category (cf. Proposition 4.1.5). We leave the details to the reader as we shall see a related proof in Lemma 7.2.10(2).

We next mention some results which allow for the propagation of the corecursive algebra structures. The first item is by Capretta et al. [80, dual of Prop. 3.9].

**Proposition 7.1.10.** Let \( n: F \to G \) be a natural transformation, and let \((A, \alpha)\) be an algebra for \( G \).

1. If \( \alpha \cdot n_A: FA \to A \) is corecursive for \( F \), then \((A, \alpha)\) is corecursive for \( G \).
2. If \((\nu G, \tau)\) is the terminal coalgebra, then \( \tau^{-1} \cdot n_{\nu G}: F(\nu G) \to \nu G \) is corecursive for \( F \).

**Proof.** (1) Given \( e: X \to FX \), \( e^\dagger: X \to A \) is a solution in the \( F \)-algebra \((A, \alpha \cdot n_A)\) iff it is a solution of \( n_X \cdot e: X \to GX \) in the \( G \)-algebra \((A, \alpha)\). To see this consider the diagram below:

\[
\begin{array}{ccc}
X & \xrightarrow{e} & FX \\
\downarrow & & \downarrow^{n_X} \\
FA & \xrightarrow{Fe^\dagger} & GX \\
\downarrow^{\alpha \cdot n_A} & \nearrow^{n_A} \\
A & & GA \\
\end{array}
\]

Since the right-hand square and the triangle clearly commute, we see that the outside commutes iff so does the left-hand square.
(2) Given $e: X \to FX$, we need $e^\dagger: X \to A$ so that the square on the left commutes:

The triangle commutes. The square on the right commutes by naturality, no matter what $e^\dagger$ is. We take $e^\dagger$ to be the unique $G$-coalgebra homomorphism from $(X, \eta_X \cdot e)$ to $\nu G$, so that the outside commutes. Thus, the square on the left commutes when followed by $\tau^{-1}$; thus it commutes outright. The uniqueness of $e^\dagger$ is easy to see.

At this point, we would like to briefly recall how free corecursive algebras are constructed [17]. Freeness is an analogous concept as that for ordinary $F$-algebras (cf. Remark 2.2.19): a free corecursive algebra on the object $Y$ is a corecursive algebra $\gamma_Y: FCY \to CY$ together with a universal morphism $\eta_Y: Y \to CY$. This means that for every corecursive algebra $(A, \alpha)$ and every morphism $f: Y \to A$ there exists a unique algebra homomorphism $f^\# : (CY, \gamma_Y) \to (A, \alpha)$ satisfying $f^\# \cdot \eta_Y = f$:

Theorem 7.1.11 [17]. Assuming that a terminal coalgebra $\nu F$ and a free $F$-algebra $\Phi X$ on $X$ exist, the free corecursive algebra is their coproduct $\nu F \oplus \Phi X$ in the category of algebras for $F$.

Example 7.1.12. For a polynomial endofunctor on $\text{Set}$, this means that the free corecursive algebra is formed by all $\Sigma$-trees that have only finitely many leaves labelled in $X$ (but possibly infinitely many leaves labelled by constant symbols from $\Sigma$).

Furthermore, op. cit. provides the following iterative construction of free corecursive algebras somewhat similar to the initial algebra chain.

Construction 7.1.13 (Free-Corecursive-Algebra Chain). Let $\mathcal{A}$ be a cocomplete category and let $F: \mathcal{A} \to \mathcal{A}$ have a terminal coalgebra. We define for every object $Y$ of $\mathcal{A}$ an essentially unique chain $U: \text{Ord} \to \mathcal{A}$ by the following transfinite recursion:

$$
U_0 = \nu F \\
U_{i+1} = FU_i + Y \\
U_j = \colim_{i<j} U_i 
$$

for limit ordinals $j$. 184
7.2 Completely Iterative Algebras

The connecting morphisms $u_{i,j} : U_i \to U_j$ are defined as follows:

\[
\begin{align*}
  u_{0,1} &= (\nu F \xrightarrow{\tau} F(\nu F)) \xrightarrow{\text{inl}} F(\nu F) + Y, \\
  u_{i+1,j+1} &= Fu_{i,j} + \text{id}_Y,
\end{align*}
\]

and for limit ordinals $j$, $(u_{i,j})_{i<j}$ is the colimit cocone.

**Theorem 7.1.14** [17, Thm. 4.6]. Let $\mathcal{A}$ be a locally presentable category. Assume that $F : \mathcal{A} \to \mathcal{A}$ is accessible and preserves monomorphisms. Then the free-corecursive-algebra chain converges to the free corecursive algebra on $Y$.

More precisely, there exists an ordinal $\lambda$ such that $u_{\lambda,\lambda+1} : U_\lambda \to FU_\lambda + Y$ is an isomorphism, and its inverse yields (by composition with the coproduct injections) the structure and universal morphism, respectively, of a free corecursive algebra on $Y$.

**Example 7.1.15** [17]. For a polynomial endofunctor $H_\Sigma$ on $\text{Set}$ the free corecursive algebra on a set $Y$ can be described as follows. Consider the $\Sigma$-algebra $T_\Sigma Y$ of all $\Sigma$-trees over $Y$, i.e., trees for the signature obtained by adding the elements of $Y$ as constant symbols to $\Sigma_0$, which means these trees are defined like $\Sigma$-trees except that the leaves are labelled by constant symbols in $\Sigma_0$ or generators in $Y$. The free corecursive $\Sigma$-algebra on $Y$ is the subalgebra $C_\Sigma Y$ of all $\Sigma$-trees over $Y$ having only finitely many leaves labelled by generators in $Y$ (and the remaining leaves labelled in $\Sigma_0$, the set of constants in the original signature).

7.2 Completely Iterative Algebras

We continue our study by looking at a natural strengthening of the defining property of corecursive algebras and verifying that again, the terminal coalgebra is, equivalently, the initial object in the relevant (smaller) subcategory of algebras.

The idea of algebras with unique solutions of recursive equations stems from work in general algebra by Evelyn Nelson [191] and Jerzy Tiuryn [230]. Nelson introduced iterative algebras for a signature as algebras with unique solutions of finite systems of mutually recursive equations such as (7.1). In these systems, parameters from $A$ are allowed to occur on right-hand sides of equations. Dropping the finiteness assumption one arrives at the notion of a completely iterative algebra, introduced by Milius [174]. Op. cit. also investigates the connection to terminal coalgebras.

Although the addition of parameters to formal recursive equations may seem like a small extension, the fact that each flat equation morphism has a unique solution is a strong property of an algebra and has many interesting consequences, as we shall explain in Remark 7.2.20.

**Assumption 7.2.1.** Throughout this section, we assume that $\mathcal{A}$ is a category with binary coproducts.

**Definition 7.2.2** [174]. Let $F$ be an endofunctor on $\mathcal{A}$. By a flat equation morphism in an object $A$ we mean a morphism $e : X \to FX + A$. A solution of $e$ in an algebra
(\(A, \alpha\)) is a morphism \(e^\dagger: X \to A\) such that the following square commutes:

\[
\begin{array}{c}
X \\
\downarrow e \\
FX + A \\
\downarrow F(e^\dagger + \text{id}_A) \\
\end{array} \\
\begin{array}{c}
A \\
\downarrow [\alpha, \text{id}_A] \\
FA + A \\
\end{array}
\]  

(7.5)

The algebra \((A, \alpha)\) is called completely iterative (or, a cia for short) provided that every flat equation morphism in \(A\) has a unique solution.

**Remark 7.2.3.** (1) Every cia \((A, \alpha)\) is, of course, a corecursive algebra. Indeed, for every coalgebra \(e: X \to FX\) one forms \(\bar{e} = (X \xrightarrow{e} FX \xrightarrow{\text{inl}} FX + A)\), and one readily verifies that the unique solution of \(\bar{e}\) in the present sense is a unique solution of \(e\) in \((A, \alpha)\) in the sense of Definition 7.1.1.

(2) Thus, classical algebras are seldom cias as they are not even corecursive, as explained in Remark 7.1.2(2).

We take cias to be a full subcategory of the category of all \(F\)-algebras. This choice is justified by the following result:

**Proposition 7.2.4** [174, Prop. 2.3]. Let \((A, \alpha)\) and \((B, \beta)\) be cias for \(F\). Then a morphism \(h: A \to B\) is an algebra homomorphism if and only if it preserves solutions, i.e. for every flat equation morphism \(e: X \to FX + A\) we have

\[
(X \xrightarrow{e} A \xrightarrow{h} B) = (X \xrightarrow{\bar{e}} FX + A \xrightarrow{FX+h} FX + B)^\dagger.
\]

**Sketch of Proof.** The ‘only if’ direction is very similar to what we have seen in Proposition 7.1.4: one proves with a straightforward diagram chase that \(h \cdot e^\dagger\) is a solution for \((FX + h) \cdot e\) in \(B\) and then obtains the desired result by the uniqueness of solutions in \(B\).

For the ‘if’ direction one first observes that the morphism \([\alpha, \text{id}_A]: FA + A \to A\) appears as the unique solution of the flat equation morphism \(e = F\text{inr} + \text{inr}: FA + A \to F(FA + A) + A\).

Similarly, one proves that \([\beta \cdot Fh, h]: FA + A \to B\) is the unique solution of \((F(FA + A) + h) \cdot e\). Since \(h\) preserves solutions, one then obtains

\[
h \cdot [\alpha, \text{id}_A] = h \cdot e^\dagger = (F(FA + A) + h) \cdot e^\dagger = [\beta \cdot Fh, h]: FA + A \to B,
\]

whose left-hand coproduct component shows that \(h\) is a homomorphism.

**Proposition 7.2.5** [174, Ex. 2.5]. For every endofunctor, the terminal coalgebra is a cia.
7.2 Completely Iterative Algebras

Proof. Let \( F : \mathcal{A} \to \mathcal{A} \) be an endofunctor for which \((\nu F, \tau)\) exists. Given a flat equation morphism \( e : X \to FX + \nu F \), form the following \( F \)-coalgebra

\[
\tau = (X + \nu F \xrightarrow{[e, \text{inr}]} FX + \nu F \xrightarrow{\text{can}} F(X + \nu F)),
\]

where \( \text{can} = [\text{Finl}, F\text{inr}] \). Let \( h : X + \nu F \to \nu F \) be the corresponding unique coalgebra homomorphism and define

\[
e^\dagger = (X \xrightarrow{\text{inl}} X + \nu F \xrightarrow{h} \nu F).
\]

One readily shows that \( \text{inr} : \nu F \to X + \nu F \) is a coalgebra homomorphism from \((\nu F, \tau)\) to \((X + \nu F, \tau)\); indeed, the following diagram clearly commutes:

Consequently \( h \cdot \text{inr} \) is a coalgebra homomorphism from \((\nu F, \tau)\) to itself, whence \( h \cdot \text{inr} = \text{id} \), which implies that \( h = [e^\dagger, \text{id}] \).

We now consider the following diagram:

Note that the left- and right-hand parts, as well as the middle one and the lower triangle clearly commute. For \( s = e^\dagger \), the left-hand component of \( h \), the outside of the diagram commutes. Thus, so does the upper part, which shows that \( e^\dagger \) is a solution of \( e \).

To show its uniqueness, suppose that \( s \) is any solution of \( e \). Then the upper square commutes, and therefore so does the outside. This implies that \( [s, \text{id}] \) is a coalgebra homomorphism from \((X + \nu F, \tau)\) to \((\nu F, \tau)\), and so we have \([s, \text{id}] = h\) by the universal property of the terminal coalgebra. We conclude that

\[
e^\dagger = h \cdot \text{inl} = [s, \text{id}] \cdot \text{inl} = s,
\]

which completes the proof. \( \square \)
Examples 7.2.6. (1) It follows from Proposition 7.2.5 that the collection of all infinite binary trees is a cia for $FX = X \times X + 1$ (cf. Example 7.1.3(3)). For example, fix an element $t \in \nu F$, and consider the following system of equations:

$$x_1 \approx x_2 \ast t \quad x_2 \approx x_1 \ast c$$

(7.6)

We emphasize that $t$ here is a fixed element of the terminal coalgebra (i.e. a finite or infinite binary tree). Then the solution of this equation in $\nu F$ consists of two infinite trees, $x_1^\dagger$ and $x_2^\dagger$. Here are two different depictions of these trees. First, we may write them as infinite $\Sigma$-terms:

$$x_1^\dagger = (\cdots (\cdots c \ast t \ast c) \ast t) \ast c$$

and

$$x_2^\dagger = (\cdots (\cdots t \ast c) \ast t) \ast c$$

Second, we may picture them as infinite $\Sigma$-trees. For example, here is $x_1^\dagger$ shown as a such a tree. The small triangles represent the infinite tree $t$.

(2) The collection of all finitely branching strongly extensional trees is a cia for $P_f$ (see Theorem 4.5.7). The collection of all unordered finitely branching trees is a terminal coalgebra for the bag functor $B$ (see Example 3.2.10.)

(3) The algebra of addition on the extended natural numbers $\tilde{\mathbb{N}} = \{1, 2, 3, \ldots\} \cup \{\infty\}$ is a cia for the functor $FX = X \times X$, see Adámek et al. [31].

(4) As shown by Milius [174] a unary algebra $\alpha : A \rightarrow A$ (here we consider $F = \text{Id}$ on Set) is a cia if and only if

(a) there exists a unique fixed point $x_0 \in A$ of $\alpha$, and

(b) if an infinite sequence $y_0, y_1, y_2, \ldots$ fulfils $\alpha(y_{n+1}) = y_n$ ($n < \omega$) then $y_n = x_0$ for all $n$.

(5) Recall the corecursive algebra $A$ from Example 7.1.3(4). This algebra $A$ is not a cia: the system $x \approx (x, y), y \approx 2$ has two solutions, corresponding to the facts that both $\alpha(1, 2) = 1$ and $\alpha(0, 2) = 0$. This point again comes from Capretta et al. [81].

(6) Free corecursive algebras often fail to be cias. For example, for every signature $\Sigma$ containing a binary operation symbol $\ast$ a free corecursive algebra on $Y = \{y\}$ (see Example 7.1.15) does not have a unique solution of recursive equation $x \approx x \ast y$.

We continue with examples of cias obtained from complete metric spaces. Recall from Notation 3.2.2 that CMS is the category of complete metric spaces with distances at most 1 and non-expanding maps. Recall locally contracting endofunctors from Definition 5.2.5.
Proposition 7.2.7 [33, Lem. 2.9]. Let $F : \text{CMS} \to \text{CMS}$ be a locally contracting endo-functor. Then every nonempty $F$-algebra is a cia.

Proof. Suppose that $F$ has a contraction factor $\varepsilon < 1$. Let $\alpha : FA \to A$ be a nonempty $F$-algebra and let $e : X \to FX + A$ be a flat equation morphism. Recall that the hom-set $\text{CMS}(X, A)$ is a complete metric space with the supremum metric $d_{X,A}$. Moreover, $\text{CMS}(X, A)$ is nonempty, since $A$ is nonempty. It is clear from Definition 7.2.2 that a solution of $e$ is equivalently a fixed point of the endomap $\Phi$ on $\text{CMS}(X, A)$ assigning to every $s : X \to A$ the map

$$\Phi(s) = (X \xrightarrow{e} FX + A \xrightarrow{Fs + A} FA + A \xrightarrow{[\alpha,A]} A).$$

We shall now prove that this function is a contraction on $\text{CMS}(X, A)$. Indeed, for two nonexpanding maps $s, t : X \to A$ we have

$$d_{X,A}(\Phi s, \Phi t) = d_{X,A}([\alpha,A] \cdot (Fs + A) \cdot e, [\alpha,A] \cdot (Ft + A) \cdot e) \leq d_{FX + A,FA + A}(Fs + A, Ft + A) \leq \varepsilon d_{X,A}(s, t) \quad (F \text{ is } \varepsilon\text{-contracting}).$$

By Banach’s Fixed Point Theorem 5.2.4, there exists a unique fixed point of $\Phi$. \qed

Remark 7.2.8. Note that Banach’s Fixed Point Theorem yields the unique solution $e^\dagger$ as the limit of a Cauchy sequence as follows. Choose some element $a \in A$ and define a sequence $(e_n^\dagger)_{n \in \mathbb{N}}$ in $\text{CMS}(X, A)$ inductively as follows: let $e_0^\dagger = \text{const}_a$, and given $e_n^\dagger$ define $e_{n+1}^\dagger$ by the commutativity of the following square

$$
\begin{array}{ccc}
X & \xrightarrow{e_{n+1}^\dagger} & A \\
\downarrow{e} & & \downarrow{[\alpha,\text{id}_A]} \\
FX + A & \xrightarrow{F e_{n+1}^\dagger + \text{id}_A} & FA + A \\
\end{array}
$$

Then $(e_n^\dagger)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\text{CMS}(X, A)$ whose limit is the unique solution of $e$:

$$e^\dagger = \lim_{n \to \infty} e_n^\dagger.$$

Examples 7.2.9. (1) Many set functors $F$ have a lifting to a locally contracting endofunctor $F'$ on CMS. We have seen this for polynomial endofunctors in Example 5.2.6. We call an $F$-algebra $\alpha : FA \to A$ completely metrizable if there exists a complete metric $d$ on $A$ such that the algebra structure is a non-expanding map $\alpha : F'(A, d) \to (A, d)$.

Every nonempty completely metrizable $F$-algebra is a cia. Indeed, to every equation morphism $e : X \to FX + A$ its unique solution is the unique solution of $e : (X, d_0) \to F'(X, d_0) + (A, d)$ in the $F'$-algebra $(A, \alpha)$, where $d_0$ is the discrete metric.
7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras

(2) Let $F': \text{CMS} \to \text{CMS}$ be given by

$$F'(X, d) = (X, \frac{1}{2}d) + (X, \frac{1}{2}d),$$

i.e. $F'$ takes a space into the disjoint union of two copies of itself, each equipped with a metric shrinking the distance by half, and with the distance between the copies set to 1. Let $A = [0, 1]$, and let $\alpha: F'A \to A$ be defined by $x \mapsto \frac{x}{2}$ on the left-hand copy, and $x \mapsto \frac{x}{2} + \frac{1}{2}$ on the right-hand one. Then $\alpha$ is clearly a non-expanding map, and hence $(A, \alpha)$ is a cia. Note that the functor $F'$ is a lifting of $FX = X + X$ on $\text{Set}$. By item (1), $[0, 1]$ with $\alpha$ considered as a $F$ algebra structure is completely metrizable and therefore a cia for $F$.

As an illustration, we show how to obtain every real number $r \in [0, 1]$ as $e^!(x)$ for some flat equation morphism $e: X \to FX + A$ and $x \in X$. We take the set $X = \{x_0, x_1, \ldots, x_i, \ldots\}$, and we start with $r$ in binary notation as $0.b_1b_2b_3 \cdots b_i \cdots$. Our flat equation morphism $e$ is given by the system

$$x_i \approx \frac{b_i}{2} + \frac{1}{2}x_{i+1} \quad (i \in \mathbb{N}).$$

It is not hard to see that $e!(x_i) = 0.b_ib_{i+1}b_{i+2} \cdots$ is the solution. So $e!(x_0)$ is the real number $r$ with which we started.

(3) Proposition 7.2.7 yields further interesting examples of cias, where solutions of recursive equations are fractals. The most basic of these examples is the following one by Milius and Moss [179].

Let $A$ be the set of nonempty closed subsets of the interval $[0, 1]$. Then $A$ is a complete metric space equipped with the Hausdorff metric:

$$d(S, T) = \max \{\sup_{x \in S} \inf_{y \in T} d(x, y), \sup_{y \in T} \inf_{x \in S} d(x, y)\} \quad \text{for closed subsets } S, T \subseteq [0, 1].$$

We also consider the functor $F(X, d) = (X \times X, \frac{1}{3}d_{\text{max}})$ sending the complete metric space $(X, d)$ to $X \times X$ equipped with the maximum metric scaled by $\frac{1}{3}$; this functor is clearly locally contracting. Then $A$ is an algebra for $F$ with structure $\alpha$ given by

$$\alpha(S, T) = \frac{1}{3}S \cup \left(\frac{1}{3}T + \frac{2}{3}\right)$$

with the obvious interpretation of addition and multiplication on the closed subsets $S, T \subseteq [0, 1]$, i.e. $\frac{1}{3}S = \{\frac{1}{3}s : s \in S\}$. Now let $X = \{\ast\}$ be the one-point space and $e: X \to FX + A$ be given by $e(\ast) = (\ast, \ast)$. Then $e^!(\ast)$ is the famous Cantor dust.

We now turn to constructions of cias as a preparation of the proof that initial cias and terminal coalgebras are the same.

**Lemma 7.2.10.** (1) If $(A, \alpha)$ is a cia, then so is $(FA, F\alpha)$.

(2) A limit of cias in $\text{Alg} F$ is a cia. In particular, if $\mathcal{A}$ has a terminal object $1$, the trivial (terminal) algebra $F1 \to 1$ is a cia.
7.2 Completely Iterative Algebras

Proof sketch. (1) Let \( e : X \to FX + FA \) be a flat equation morphism in \( FA \). Form the equation morphism
\[
\tau = (X \xrightarrow{e} FX + FA \xrightarrow{FX + \alpha} FX + A),
\]
and obtain \( \tau^\dagger : X \to A \). Using this, define
\[
e^\dagger = (X \xrightarrow{e} FX + FA \xrightarrow{[F\tau^\dagger,FA]} FA).
\]
It is not difficult to check that \( e^\dagger \) is the unique solution of \( e \) in \( FA \). For the details, see Milius [174, Proposition 2.6].

(2) Suppose we have a diagram \( D \) of cias \((A_i, \alpha_i)i \in I\), and let \((A, \alpha)\) be a limit in the category of algebras for \( F \) with limit projections \( p_i : (A, \alpha) \to (A_i, \alpha_i) \). Recall from Remark 4.1.4 that this limit is formed on the level of \( \mathcal{A} \). We need to show that \((A, \alpha)\) is a cia. Let \( e : X \to FX + A \) be a flat equation morphism. For every \( i \in I \) one forms the flat equation morphism
\[
e_i = (X \xrightarrow{e} FX + A \xrightarrow{FX + p_i} FX + A_i)
\]
and takes the unique solution \( e_i^\dagger : X \to A_i \). Then the \( e_i^\dagger \) \((i \in I)\) form a cone of \( D \). Indeed, for every connecting morphism \( h : (A_i, \alpha_i) \to (A_j, \alpha_j) \) we know from Proposition 7.2.4 that \( h \) preserves solutions, thus \( h \cdot e_i^\dagger = e_j^\dagger \). We define \( e^\dagger : X \to A \) to be the unique morphism with \( p_i \cdot e^\dagger = e_i^\dagger \) for every \( i \in I \). Then \( e^\dagger \) is a solution of \( e \): indeed, in order to see that (7.5) commutes, extend this by the limit projections \( p_i \) to obtain the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & A \\
\downarrow{e} & & \downarrow{p_i} \\
FX + A & \xrightarrow{[\alpha,A]} & FA + A \\
\downarrow{FX + p_i} & & \downarrow{[\alpha_i,A_i]} \\
FX + A_i & \xrightarrow{[\alpha_i,A_i]} & FA_i + A_i
\end{array}
\]

Its outside commutes since \( e_i^\dagger \) solves \((FX + p_i) \cdot e\), its right-hand part commutes because \( p_i \) is a homomorphism, and the lower and upper parts commute by the definition of \( e^\dagger \). Thus, the upper left-hand square commutes when extended by \( p_i \), as desired.

For the uniqueness of \( e^\dagger \) suppose that \( s : X \to A \) is any solution of \( e \) in \( A \). Then we see that for every \( i \in I \), \( p_i \cdot s \) is a solution of \((FX + p_i) \cdot e \) (to see this, repeat the reasoning in the ‘only if’ direction of the proof of Proposition 7.2.4). Thus \( p_i \cdot s = e_i^\dagger \), whence \( s = e^\dagger \), by the universal property of the limit \( A \).

Corollary 7.2.11. Suppose that \( \mathcal{A} \) is a complete category. Then in the terminal-coalgebra-chain of \( F \), all algebras \( FV_i \to V_i \) (see Definition 6.2.1) are cias.

191
Consequently, these algebras are corecursive by Remark 7.2.3. This holds even if the terminal coalgebra itself does not exist.

We also obtain a version of Lambek’s Lemma for cias:

**Corollary 7.2.12.** If an initial cia for \( F \) exists, then it is a fixed point of \( F \).

Indeed, the proof is the same as the one for Lambek’s Lemma 2.2.5 but using Lemma 7.2.10(1) to see that for an initial cia \((I, i)\), \((FI, Fi)\) is a cia, too.

**Theorem 7.2.13** [174, Thm. 2.8]. The initial cia is precisely the same as the terminal coalgebra.

More precisely, let \( F : \mathcal{A} \to \mathcal{A} \) be an endofunctor. Then \((I, i)\) is an initial cia for \( F \) iff \((I, i^{-1})\) is a terminal coalgebra. The proof is somewhat similar to what we saw in Theorem 7.1.7; we include it for the convenience of the reader.

**Proof.** (1) For a coalgebra \( \gamma : C \to FC \) and an algebra \( \alpha : FA \to A \) consider the following diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{h} & A \\
\downarrow\gamma & & \downarrow\alpha \\
FC & \xrightarrow{Fh} & FA \\
\downarrow\text{inl} & & \downarrow\text{inl} \\
FC + A & \xrightarrow{Fh + \text{id}} & FA + A \\
\end{array}
\]

Let us define \( e = \text{inl} \cdot \gamma \) so that the left-hand part of this diagram commutes. Notice also that the right-hand part and the lower square of the diagram obviously commute. Now the outside of the diagram commutes iff the upper square does; in other words, \( h \) is a solution of \( \gamma \) in the algebra \((A, \alpha)\) iff it is a solution of the flat equation morphism \( e \) in that algebra.

(2) Suppose that \( i : FI \to I \) is an initial cia. Then \( i \) is an isomorphism by Corollary 7.2.12. So we have the coalgebra \( i^{-1} : I \to FI \), and we need to verify that it is terminal. Indeed, for every coalgebra \((C, \gamma)\) replace \((A, \alpha)\) in Diagram (7.7) by the algebra \((I, i)\). Then since this algebra is a cia we have a unique coalgebra-to-algebra homomorphism, i.e. a unique coalgebra homomorphism from \((C, \gamma)\) to \((I, i^{-1})\).

(3) Conversely, suppose that \( \tau : \nu F \to F(\nu F) \) is a terminal coalgebra. Then it is a cia by Proposition 7.2.5, and it remains to verify its initiality. So let \((A, \alpha)\) be any cia and let \((C, \gamma)\) in Diagram (7.7) be \((\nu F, \tau)\). Since \( A \) is a cia, we have a unique solution of \( e = \text{inl} \cdot \tau \) in \( A \), equivalently a unique \( F \)-algebra homomorphism from the cia \((\nu F, \tau^{-1})\) to the cia \((A, \alpha)\).

**Remark 7.2.14.** The only fixed point among the cias for a functor is the initial one. This follows from Theorem 7.2.13 and Remark 7.1.8.

More generally, we can characterize free cias (cf. Remark 2.2.19). That is, the forgetful functor from the category of all cias for \( F \) to the base category \( \mathcal{A} \) has a left adjoint whenever all endofunctors \( F(-) + Y \) for object \( Y \) of \( \mathcal{A} \) have a terminal coalgebra:
Theorem 7.2.15 [174, Cor. 2.11]. A free cia for \( F \) on \( Y \) is precisely the same as a terminal coalgebra for \( F(\_)+Y \).

More precisely, given a terminal coalgebra \( \tau_Y : TY \to FTY + Y \), then extending \( \tau_Y^{-1} \) by the two coproduct injections yields an algebra structure \( \alpha_Y : FTY \to TY \) and a morphism \( \eta_Y : Y \to TY \) which are the structure and universal morphism, respectively, of a free cia on \( Y \). Conversely, given a free cia \( (TY, \alpha_Y) \) with universal morphism \( \eta_Y \) then \( [\alpha_Y, \eta_Y] : FTY + Y \to TY \) is an isomorphism whose inverse is the structure of a terminal coalgebra for \( F(\_)+Y \).

Remark 7.2.16. In general, corecursive algebras need not be cias, see Example 7.2.6(5) and (6). But are there interesting functors for which the two notions coincide? The answer is “yes” in the case where the base category \( \mathcal{C} \) has “very well-behaved” coproducts (this includes examples such as sets, posets, graphs, and presheaf categories) and the endofunctor has the form \( FX = A \times X + B \) for fixed objects \( A \) and \( B \) [26]. Moreover, among finitary set functors, there are essentially no other functors. In the appendix (see Remark B.6.4) we recall that that for every set functor \( F \) there exists a standard functor which agrees with \( F \) on all nonempty sets and functions. Standard means that the functor preserves finite intersections and inclusions.

Theorem 7.2.17 [26]. Let \( F \) be a standard finitary set functor. If every corecursive \( F \)-algebra is a cia, then \( F \) is naturally isomorphic to a functor given by \( X \mapsto A \times X + B \).

Remark 7.2.18. As observed by Paul Levy, a cia can, equivalently, be defined as an algebra \((A, \alpha)\) such that for every coalgebra \( e : X \to F(X+A) \) there exists a unique morphism \( e^\dagger : X \to A \) such that the square below commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & A \\
\downarrow{e} & & \downarrow{\alpha} \\
F(X+A) & \xrightarrow{F[e^\dagger, id_A]} & FA
\end{array}
\] (7.8)

Note that (7.8) is very closely related to primitive corecursion (see Theorem 2.6.4); in fact, primitive corecursion is precisely the unique existence of \( e^\dagger \) in (7.8) for the algebra \((\nu F, \tau^{-1})\).

However, one advantage of Definition 7.2.2 is that \((A, \alpha)\) is a cia iff \((A, [\alpha, id_A])\) is corecursive for \( F(\_)+A \). Moreover, observe that for a polynomial set functor, the flat equation morphism in Definition 7.2.2 regarded as systems of recursive equations have a slightly simpler syntactic format than systems arising maps \( X \to H_\Sigma(X+A) \) (cf. Remark 7.1.2(1)). Indeed, a flat equation morphism \( X : H_\Sigma X + A \) can be understood as a recursive system of equations whose right-hand sides to are either flat terms \( \sigma(x_1, \ldots, x_n) \) for an \( n \)-ary operation symbol \( \sigma \) and variables from \( X \) or an element of \( A \).

The theory of cias could be developed based on this alternative definition. However, we prefer to use the definition based on flat equation morphism from the literature.

Proposition 7.2.19. An algebra \((A, \alpha)\) is a cia iff for every \( e : X \to F(X+A) \) there exists a unique morphism \( e^\dagger : X \to A \) such that (7.8) commutes.
7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras

Proof. (1) Given a cia \((A, \alpha)\) and a morphism \(e: X \to F(X + A)\) form the flat equation morphism

\[
X + A \xrightarrow{e + \text{id}_A} F(X + A) + A.
\]

Its unique solution is easily seen to satisfy \((e + \text{id}_A)^\dagger \cdot \text{inr} = \text{id}_A\), and \((e + \text{id}_A)^\dagger \cdot \text{inl}: X \to A\) is then proved to be the desired unique morphism. We leave the details as an easy exercise for the reader.

(2) Conversely, suppose that \((A, \alpha)\) admits for every \(e: X \to F(X + A)\) a unique morphism \(e^\dagger\) such that (7.8) commutes. Given a flat equation morphism \(f: X \to FX + A\), form the morphism

\[
FX \xrightarrow{Ff} F(FX + A),
\]

and let \((Ff)^\dagger: FX \to A\) be the unique morphism such that (7.8) commutes (for \(e = Ff\)). Now let

\[
f^\dagger = (X \xrightarrow{f} FX + A \xrightarrow{(Ff)^\dagger \cdot \text{id}_A} A).
\]

One readily proves that this is a unique solution of \(f\). Again, we leave the details as an exercise for the reader.

Remark 7.2.20. To conclude this section, let us mention some further work on cia’s and related structures in the literature.

(1) As we mention in the introduction of this section, one of the reasons why cias are interesting is that one can uniquely solve much more general recursive equations than the flat ones in Definition 7.2.2 in a cia. For example, [4, 174] contain a solution theorem which, when instantiated for a polynomial set functor, states that mutually recursive systems

\[
x_i \approx t_i \quad (i \in I),
\]

where \(I\) is any set and the right-hand sides \(t_i\) are arbitrary (possibly infinite) non-variable \(\Sigma\)-trees have unique solutions.

(2) Assuming the existence of a free cia \(TX\) on every object \(X\) of a base category \(\mathcal{A}\) (see Theorem 7.2.15), it turns out that \(T\) is the object assignment of a monad on \(\mathcal{A}\). This monad is characterized by a universal property: it is the free completely iterative monad on \(F\). These results appear in [4, 174] generalizing work on (free) completely iterative theories by Elgot, Bloom and Tindell [94].

(3) In a number of settings, equation morphisms do have a solution albeit not a unique one. Then one is more interested in taking canonical solutions (or fixed points) than requiring their unique existence, e.g. least fixed points in complete partial orders. The idea of Bloom and Ésik’s iteration theories [69] is to study the equational properties that characterize least fixed points in complete partial orders. It turns out that similar ideas are important in connection with cias, and they lead to the notion of a complete Elgot algebra [33]. A complete Elgot algebra for a functor \(F\) is a triple \((A, \alpha, (\_)^\dagger)\) where \(\alpha: FA \to A\) is an algebra for \(F\) and \((\_)^\dagger\) is an operation taking a flat equation morphism \(e: X \to FX + A\) to a solution \(e^\dagger: X \to A\) such that two simple and well-motivated properties are satisfied.
Every cia is a complete Elgot algebra. Further examples are continuous algebras (i.e. algebras for locally continuous functors on CPO) or complete lattices, which are complete Elgot algebras for $FX = X \times X$ on Set.

Theorem 7.2.13 can be augmented to state that the terminal coalgebra is the same as the initial Elgot algebra (in addition, the terminal coalgebra for $F(-) + X$ is equivalently, the free complete Elgot algebra for $F$ on $X$). The main result [33, Thm. 5.8] is that complete Elgot algebras form precisely the category of Eilenberg-Moore algebras for the monad $T$ of item (2) above.

(4) Completely iterative algebras and complete Elgot algebras as in item (3) play a significant rôle in the category theoretic semantics of recursive program schemes as presented by Milius and Moss [179]. This approach to the semantics of recursive function definitions is based on terminal coalgebras for functors in lieu of concepts from general algebra like signatures and infinite trees. In the category theoretic setting, cias and Elgot algebras serve as those classes of algebras in which one interprets recursive program schemes – in those algebras one can canonically (or even uniquely, in the case of cias) solve recursive program schemes.

(5) Milius, Moss, and Schwencke [180] demonstrate how Turi and Plotkin’s abstract operational rules [202] give rise to new cia structures on the terminal coalgebra. The latter yield theorems that instantiate to several known unique solution theorems in the literature, for example, Milner’s solution theorem for CCS [185], and a unique solution theorem for stream circuits [209], as well as new ones for non-well founded sets (extending previous results of Barwise and Moss [60]) or formal languages. Besides, one obtains a modular framework for the specification of operations by abstract operational rules [180]; modularity here means that unique solution theorems are preserved when adding operations specified by abstract operational rules to a given cia structure on the terminal coalgebra.

7.3 Recursive coalgebras

After having looked at algebras with special properties, we now turn to coalgebras with special properties, and we will see the initial algebra appearing as a terminal object in certain categories of coalgebras. The first notion we mention is that of a recursive coalgebra, which is formally dual to the notion of a corecursive algebra, thus formulating a unique definition principle for recursive functions. Recursive coalgebras are closely connected to well-founded coalgebras, which we consider in Chapter 8 and which provide a categorical formulation of well-founded induction. In his work on categorical set theory, Osius [195] first studied the notions of well-founded and recursive coalgebras (for the power-set functor on sets and, more generally, the power-object functor on an elementary topos). He defined recursive coalgebras as those coalgebras $\alpha: A \to \mathcal{P}A$ which have a unique coalgebra-to-algebra homomorphism into every algebra (see Definition 7.3.1).

Taylor [229, 228] considered recursive coalgebras for a general endofunctor under the name ‘coalgebras obeying the recursion scheme’, and proved the General Recursion
Theorem that all well-founded coalgebras are recursive for more general endofunctors (see Chapter 8). Recursive coalgebras were also investigated by Eppendahl [95], who called them algebra-initial coalgebras.

Capretta, Uustalu, and Vene [80] studied recursive coalgebras, and they showed how to construct new ones from given ones by using comonads. They also explained nicely how recursive coalgebras allow for the semantic treatment of (functional) divide-and-conquer programs. More recently, Jeannin et al. [137] proved the general recursion theorem for polynomial functors on the category of many-sorted sets; they also provide many interesting examples of recursive coalgebras arising in programming.

In this section we will just recall a few basic results, which are dual to what we have seen in Section 7.1 and Section 7.2 on corecursive algebras, and some examples.

For the dual concept of a solution in a corecursive algebra we stick to the standard terminology of an algebra-to-coalgebra morphism:

**Definition 7.3.1.** A coalgebra \( \gamma : C \rightarrow FC \) is recursive if for every algebra \( \alpha : FA \rightarrow A \) there exists a unique coalgebra-to-algebra morphism \( \alpha^\dagger : C \rightarrow A \), i.e. a unique morphism such that the square below commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\gamma} & FC \\
\alpha^\dagger \downarrow & & \downarrow F\alpha^\dagger \\
A & \leftarrow & FA
\end{array}
\]

**Examples 7.3.2.** (1) The first examples of recursive coalgebras are well-founded relations.

Recall that a binary relation \( R \) on a set \( X \) is well-founded if there is no infinite descending sequence \( \cdots x_3 R x_2 R x_1 R x_0 \).

Now a binary relation \( R \subseteq X \times X \) is essentially a graph on \( X \), equivalently the coalgebra structure \( \alpha : X \rightarrow \mathcal{P}X \) with \( \alpha(x) = \{ y \mid y R x \} \) (cf. Example 1.3.2). Osius [195] showed that for every well-founded relation the associated \( \mathcal{P} \)-coalgebra is recursive. Shortly: a graph regarded as a coalgebra for \( \mathcal{P} \) is recursive iff it has no infinite path.

(2) If \( \mu F \) exists, then it is a recursive coalgebra. This is dual to Example 7.1.3(2).

(3) The initial coalgebra \( 0 \rightarrow F0 \) is recursive.

(4) If \( (C, \gamma) \) is recursive so is \( (FC, F\gamma) \). This is dual to Proposition 7.1.5.

(5) Every colimit of recursive coalgebras in \( \text{Coalg} F \) is recursive. This is easy to prove, using that colimits of coalgebras are formed on the level of the underlying category. The argument is the “parameter-less” form of the dual of the proof of Lemma 7.2.10(2).

(6) It follows from items (3)–(6) that in the initial-algebra chain \( (W_i \rightarrow FW_i)_{i \in \text{Ord}} \) (see Definition 6.1.4) all coalgebras are recursive.

Dually to Corollary 7.1.6, we see that a terminal recursive \( F \)-coalgebra is a fixed point of \( F \). Moreover, dually to Theorem 7.1.7 we have
Corollary 7.3.3 [80, Prop. 7], The initial algebra is precisely the same as the terminal recursive coalgebra.

The dual notion of a completely iterative algebra (see Definition 7.2.2) is called a parametrically recursive coalgebra by Capretta et al. [80] (cf. (7.2)). So the dual statement of Theorem 7.2.13 states that the initial algebra is, equivalently, the terminal parametrically recursive coalgebra. Of course, every parametrically recursive coalgebra is recursive. (To see this, form for a given \( e: FX \to X \) the morphism \( e' = e \cdot \pi \), where \( \pi: FX \times A \to FX \) is the projection.) However, in general the converse fails:

Example 7.3.4 [7]. Let \( R: \text{Set} \to \text{Set} \) be the functor defined by \( RX = \{(x,y) \in X \times X : x \neq y\} + \{d\} \) for sets \( X \), and for a function \( f: X \to Y \) put \( Rf(d) = d \) and \( Rf(x,y) = \begin{cases} d & \text{if } f(x) = f(y) \\ (f(x), f(y)) & \text{else.} \end{cases} \)

Now let \( C = \{0, 1\} \), and define \( \gamma: C \to RC \) by \( \gamma(0) = \gamma(1) = (0, 1) \). Then \((C, \gamma)\) is a recursive coalgebra. Indeed, for every algebra \( \alpha: RA \to A \) the constant map \( h: C \to A \) with \( h(0) = h(1) = \alpha(d) \) is the unique coalgebra-to-algebra morphism.

However, \((C, \gamma)\) is not parametrically recursive. To see this, consider any morphism \( e: RX \times \{0, 1\} \to X \) such that \( RX \) contains more than one pair \((x_0, x_1)\), \( x_0 \neq x_1 \) with \( e((x_0, x_1), i) = x_i \) for \( i = 0, 1 \). Then each such pair yields \( h: C \to X \) with \( h(i) = x_i \) making (7.2) commutative. Thus, \((C, \gamma)\) is not parametrically recursive.

The situation in Example 7.3.4 is relatively rare and artificial because for functors preserving inverse images, recursive and parametrically recursive coalgebras coincide (see Corollary 8.7.2 and Corollary 8.7.10).

Proposition 7.3.5. If a coalgebra \((C, \gamma)\) has the property that all of its subcoalgebras are recursive, then \((FC, F\gamma)\) has that property, too, provided that

(1) \( F \) preserves inverse images, or
(2) \( F \) preserves finite intersections and \( \gamma \) is monic.

Proof. Let \( s: (S, \sigma) \to (FC, F\gamma) \) be a subcoalgebra and \( \alpha: FA \to A \) be any algebra.

(1) Existence of a coalgebra-to-algebra morphism from \((S, \sigma)\) to \((A, \alpha)\) follows from the recursivity of \((FC, F\gamma)\) (see Example 7.3.2(4)): we have a coalgebra-to-algebra morphism \( h: C \to A \), and therefore \( h \cdot s: S \to A \) is a coalgebra-to-algebra morphism.

(2) In order to show the uniqueness, suppose that \( k: S \to A \) is any coalgebra-to-algebra morphism. Now form the pullback of \( s \) and \( \gamma \):

\[
\begin{array}{ccc}
P & \xrightarrow{q} & S \\
p \downarrow & & \downarrow s \\
C & \xrightarrow{\gamma} & FC \\
\end{array}
\]

Note that \( \gamma: (C, \gamma) \to (FC, F\gamma) \) clearly is a coalgebra homomorphism. Thus, since \( F \) preserves finite intersections, there exists a unique coalgebra structure \( \pi: P \to FP \) such
7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras

that \( p: (P, \pi) \to (C, \gamma) \) and \( q: (P, \pi) \to (S, \sigma) \) are homomorphisms (cf. Remark 4.1.4). Thus, \( k \cdot q \) is a coalgebra-to-algebra morphism from \( (P, \pi) \) to \( (A, \alpha) \), and so is \( h \cdot s \cdot q \). By hypothesis, \( (P, \pi) \) being a subcoalgebra of \( (C, \gamma) \), is recursive. Hence we have

\[
k \cdot q = h \cdot s \cdot q. \tag{7.9}
\]

Furthermore, since the inside square in the diagram below is a pullback by assumption (1) or (2) and its outside clearly commutes, we obtain the morphism \( \pi_0 \) such that the two triangles commute:

\[
\begin{array}{ccc}
S & \xrightarrow{\pi_0} & FS \\
\downarrow{s} & & \downarrow{F\gamma} \\
FP & \xrightarrow{Fq} & FS \\
& \xrightarrow{Fp} & FC \\
& & \xrightarrow{Fs} FFC
\end{array}
\tag{7.10}
\]

We conclude that

\[
k = \alpha \cdot Fk \cdot \sigma \quad \text{(since \( k \) is a coalgebra-to-algebra morphism)}
\]

\[
= \alpha \cdot Fk \cdot Fq \cdot \pi_0 \quad \text{(by the upper triangle in (7.10))}
\]

\[
= \alpha \cdot F(k \cdot q) \cdot \pi_0 
\]

\[
= \alpha \cdot F(h \cdot s \cdot q) \cdot \pi_0 \quad \text{(by (7.9))}
\]

\[
= \alpha \cdot F(h \cdot s) \cdot Fq \cdot \pi_0 
\]

\[
= \alpha \cdot F(h \cdot s) \cdot \sigma \quad \text{(by the upper triangle in (7.10))}
\]

\[
= h \cdot s 
\quad \text{(since \( h \cdot s \) is a coalgebra-to-algebra morphism).}
\]

In the following corollary we use the notion of a \textit{simple} initial object 0. This means that the unique morphisms \( 0 \to X \) are monomorphisms. This result is essentially due to Urbat and Schröder [236, Prop. A.8].

**Corollary 7.3.6.** Every subcoalgebra of a coalgebra in the initial-algebra \( \omega \)-chain is recursive, provided that

(1) \( F \) preserves inverse images, or

(2) \( \mathcal{A} \) has a simple initial object and \( F: \mathcal{A} \to \mathcal{A} \) preserves finite intersections.

**Proof.** Let \( w_{n,n+1}: W_n \to FW_n, n \in \mathbb{N}, \) be the coalgebras in the initial-algebra \( \omega \)-chain (cf. Definition 6.1.4).

The desired result follows easily by induction on \( n \). For \( n = 0 \), the statement clearly holds since every subcoalgebra of \( 0 \to F0 \) is carried by an initial object. Indeed, for every subobject \( s: S \to 0 \) we have \( s \cdot u_S = \text{id}_0 \) for the unique morphism \( u_A: 0 \to S \), which shows that \( s \) is a split epimorphism whence an isomorphism.

For the induction step, use Proposition 7.3.5. In the case of assumption (2) we know that since \( F \) preserves finite intersections it preserves monomorphisms. Hence, using that 0 is simple, an easy induction shows that all coalgebra structures \( w_{n,n+1} \) are monomorphisms.

\[198\]
Remark 7.3.7. (1) The same proof as in Proposition 7.3.5 shows that for a coalgebra $(C,\gamma)$ such that all coalgebras having a homomorphism to $(C,\gamma)$ are recursive, the coalgebra $(FC,F\gamma)$ has that property, too, provided that
(a) $F$ preserves pullbacks, or
(b) $F$ preserves inverse images and $\gamma$ is monic.

(2) Similarly as in Corollary 7.3.6 one then obtains that for a category $\mathcal{A}$ having a simple initial object and a functor $F$ preserving inverse images, every coalgebra with a homomorphism into a coalgebra $F^n0 \to F(F^n0)$ in the initial-algebra-$\omega$-chain is recursive.

(3) It is easy to extend the result in Corollary 7.3.6 to all subcoalgebras of coalgebras $(W_i \to FW_i)_{i\in\text{Ord}}$ in the initial-algebra chain provided that $\mathcal{A}$ has universally smooth monomorphisms (see Definition 6.1.17 and Definition 8.5.10).

We conclude this section with a few examples explaining how recursive coalgebras capture familiar recursive function definitions as well as functional divide-and-conquer programs.

Examples 7.3.8. (1) The set functor $FX = X + 1$ has unary algebras with a constant as algebras, and coalgebras for $F$ may be identified with partial unary algebras. As we know from Example 2.2.7(3), the initial algebra is the set of natural numbers $\mathbb{N}$ with the successor function and the constant $0$. The inverse of the initial $F$-algebra is the coalgebra given by the partial unary operation $n \mapsto n - 1$ (defined iff $n > 0$). This coalgebra is parametrically recursive. Hence every function $e = [u,v] : FX \times \mathbb{N} \cong X \times \mathbb{N} + \mathbb{N} \to X$ defines a unique sequence $e^\dagger : \mathbb{N} \to X$, $e^\dagger(n) = x_n$ such that (7.2) commutes. This means that $x_0 = v(0)$ and $x_{n+1} = u(x_n, n+1)$. For example, the factorial function is then given by the choice $X = \mathbb{N}$; $u(n,m) = n \cdot m$ and $v(0) = 1$.

(2) For the set functor $F$ given by $FX = X \times X + 1$, coalgebras $\gamma : C \times C + 1$ are deterministic systems with a state set $C$, a binary input and with halting states (expressed by $\gamma^{-1}(1)$).

The coalgebra $\mathbb{N}$ of natural numbers with halting states $0$ and $1$ and input structure $\gamma : n \mapsto (n - 1, n - 2)$ for $n \geq 2$ is parametrically recursive (see Example 8.6.6).

For example, to define the Fibonacci sequence starting with $a_0, a_1 \in \mathbb{N}$, consider the morphism $e : F\mathbb{N} \times \mathbb{N} \cong \mathbb{N}^3 + \mathbb{N} \to \mathbb{N}$ with

$$e(i,j,k) = i + j \quad \text{and} \quad e(n) = \begin{cases} a_0 & \text{if } n = 0 \\ a_1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

We know that there is a unique sequence $e^\dagger : \mathbb{N} \to \mathbb{N}$ such that the diagram (7.2) commutes, which means $e^\dagger(0) = a_0$, $e^\dagger(1) = a_1$ and $e^\dagger(n+2) = e^\dagger(n+1) + e^\dagger(n)$.

(3) Capretta et al. [81] showed how to obtain Quicksort using parametric recursivity. Let $A$ be any linearly ordered set (of data elements). Then Quicksort is usually defined
as the recursive function \( q : A^* \to A^* \) given by

\[
q(\varepsilon) = \varepsilon \quad \text{and} \quad q(aw) = q(w\leq a) \ast (aq(w > a)),
\]

where \( A^* \) is the set of all lists on \( A \), \( \varepsilon \) is the empty list, \( \ast \) is the concatenation of lists and \( w\leq a \) denotes the list of those elements of \( w \) which are less than or equal to \( a \); analogously for \( w > a \).

Now consider the functor \( FX = A \times X \times X + 1 \) on \( \text{Set} \), where 1 = \( \{\bullet\} \), and form the coalgebra \( s : A^* \to A \times A^* \times A^* + 1 \) given by

\[
s(\varepsilon) = \bullet \quad \text{and} \quad s(aw) = (a, w\leq a, w > a) \quad \text{for} \ a \in A \ \text{and} \ w \in A^* . \quad (7.11)
\]

Again, we shall see that this coalgebra is recursive in Example 8.6.6. Thus, for the \( F \)-algebra \( m : A \times A^* \times A^* + 1 \to A^* \) given by

\[
m(\bullet) = \varepsilon \quad \text{and} \quad m(a, w, v) = w \ast (av)
\]

there exists a unique function \( q \) on \( A^* \) such that \( q = m \cdot Fq \cdot s \). Notice that the last equation reflects the idea that Quicksort is a divide-and-conquer algorithm. The coalgebra structure \( s \) divides a list into two parts \( w\leq a \) and \( w > a \). Then \( Fq \) sorts these two smaller lists, and finally in the combine- (or conquer-)step, the algebra structure \( m \) merges the two sorted parts to obtain the desired whole sorted list.

Similarly, functions defined by parametric recursivity (cf. Diagram (7.2)), can be understood as divide-and-conquer algorithms, where the combine-step is allowed to access the original parameter additionally. For instance, in the current example the divide-step \( \langle s, \text{id}_{A^*} \rangle \) produces the pair consisting of \( (a, w\leq a, w > a) \) and the original parameter \( aw \), and the combine-step, which is given by an algebra \( FX \times A^* \to X \) will, by the commutativity of (7.2), get \( aw \) as its right-hand input.

Jeannin et al. [137, Sec. 4] provide a number of recursive functions arising in programming that are determined by recursivity of a coalgebra, e.g. the \( \text{gcd} \) of integers, the Ackermann function, and the Towers of Hanoi.

We conclude this chapter with a new proof of Theorem 6.1.22 using the topic of this section, recursive coalgebras.

We need several preliminary results which are also used in later chapters, especially Chapter 8. We begin with an induction principle. It is often called Scott induction, especially in the special case when \( f \) is a continuous function on a cpo \( P \). It easily follows from the proof of Theorem 6.1.1.

**Corollary 7.3.9.** Let \( P \) be a chain-complete poset, and let \( S \) be a subset which is closed under joins of chains. If a monotone function \( f : P \to P \) preserves \( S \) (in symbols: \( f[S] \subseteq S \)), then \( \mu f \in S \).

Indeed, \( S \) contains \( \bot \), the join of the empty chain. By transfinite induction we have \( f^i(\bot) \in S \) for every ordinal \( i \).

**Notation 7.3.10.** We write \( \text{Sub}(A) \) for the collection of subobjects of \( A \) ordered by factorization (cf. Remark 6.1.21). This is a poset in every well-powered category. The top of this poset is represented by \( \text{id}_A \).
7.3 Recursive coalgebras

Recall the notion of smooth monomorphisms from Definition 6.1.17. When monomorphisms are smooth, we have a representation of joins in terms of colimits in the base category \( \mathcal{A} \), as we show in Proposition 7.3.13. Let us first clarify what we mean by colimits representing joins.

**Definition 7.3.11.** We say that an object \( A \) has *smooth subobjects* provided that \( \text{Sub}(A) \) is chain-complete with joins given by colimits of the corresponding chains of subobjects.

**Remark 7.3.12.** In more detail, let \( \lambda \) be an ordinal, and let \( m_i : A_i \rightarrow A \) \((i < \lambda)\) be a \( \lambda \)-chain of subobjects of \( A \). Then the join \( m : C \rightarrow A \) exists. Moreover, consider the \( \lambda \)-chain of objects \((A_i)_{i<\lambda}\) with connecting morphisms \( a_{i,j} : A_i \rightarrow A_j \) for \( i \leq j < \lambda \) given by the unique factorizations witnessing \( m_i \leq m_j \):

![Diagram](image)

Then for every \( i < \lambda \) we have monomorphisms \( c_i : A_i \rightarrow C \) with \( m \cdot c_i = m_i \) since \( m_i \leq m \), and these monomorphisms form a colimit cocone.

The following equivalent formulation, which is easier to verify in concrete examples, is completely parallel to Definition 6.1.17.

**Proposition 7.3.13.** The object \( A \) has smooth subobjects if and only if for every \( \lambda \)-chain \((A_i)_{i<\lambda}\) in \( \mathcal{A} \) for some ordinal \( \lambda \) which comes equipped with a cocone \( m_i : A_i \rightarrow A \) of monomorphisms, the following holds:

1. a colimit exists, and
2. the factorization morphism of the cocone \((m_i)\) is again a monomorphism.

**Proof.** The ‘only if’ direction is obvious. For the ‘if’ direction, suppose we are given a \( \lambda \)-chain \( m_i : A_i \rightarrow A \) of subobjects \((i < \lambda)\) as in Remark 7.3.12. By (1), this chain has a colimit \( C \), and we will prove that this yields the join \( \bigvee_{i<\lambda} m_i \). We denote the colimit injections by \( c_i : A_i \rightarrow C \) for all \( i < \lambda \). By (2), we have a unique monomorphism \( m : C \rightarrow A \) such that \( m \cdot c_i = m_i \) for all \( i < \lambda \).

Now let \( s : S \rightarrow A \) be any subobject with \( m_i \leq s \) for all \( i < \lambda \). That is, we have monomorphisms \( s_i : A_i \rightarrow S \) with \( s \cdot s_i = m_i \) for all \( i < \lambda \). Then the \( s_i \) form a cocone because for every \( a_{i,j} : A_i \rightarrow A_j \) we have

\[
s \cdot a_{i,j} \cdot s_i = m_j \cdot a_{i,j} = m_i = s \cdot s_i,
\]

whence \( s \cdot a_{i,j} = s_i \) since \( s \) is monic. We therefore obtain a unique \( t : C \rightarrow S \) with \( t \cdot c_i = s_i \) for all \( i < \lambda \). Consequently, we have

\[
s \cdot t \cdot c_i = s \cdot s_i = m_i = m \cdot c_i \quad \text{for all } i < \lambda.
\]

Since the colimit injections \( c_i \) form an epic family, we conclude that \( s \cdot t = m \), which means that \( m \leq t \) in \( \text{Sub}(A) \) as desired.
Example 7.3.14. (1) Clearly, a category with smooth monomorphisms has smooth subobjects of each of its objects.

(2) In the category of finite sets, every object has smooth subobjects, yet monomorphisms are not smooth because not all colimits of chains of monomorphisms exist.

In Theorem 6.1.22 we proved that, under hypotheses, the existence of a pre-fixed point implies the existence of an initial algebra. We now present an alternative proof that does not use transfinite recursion. Instead, it uses fixed points and recursive coalgebras. Moreover, the assumption that monomorphisms are smooth in that theorem can be slightly weakened as follows.

Theorem 7.3.15. Let $A$ be a prefixed point of a functor $F$ preserving monomorphisms. If $A$ has smooth subobjects, then $F$ has an initial algebra.

Proof. Let $m: FA \to A$ be a monomorphism. We have the following endomap $f: \text{Sub}(A) \to \text{Sub}(A)$, taking a given subobject $u: B \to A$ to

$$f(u) = FB \hookrightarrow FA \hookrightarrow A.$$  

This map is clearly monotone. We are going to apply Corollary 7.3.9 to it. We take the subset $S \subseteq \text{Sub}(A)$ of all $u: B \to A$ such that $u \leq fu$ via some recursive coalgebra. More precisely,

$$S = \{ u: B \to A : u = m \cdot Fu \cdot \beta \text{ where } \beta: B \to FB \text{ is recursive} \}.$$  

Note that if $\beta$ exists for $u$, then it is unique. Moreover, $u \in S$ is a coalgebra-to-algebra morphism from $(B, \beta)$ to $(A, m)$.

The set $S$ is closed under $f$ since $(FB, F\beta)$ is a recursive coalgebra by Example 7.3.2(4) and since for $u \in S$ we have

$$f(u) = m \cdot Fu = m \cdot F(m \cdot Fu \cdot \beta) = m \cdot F(f(u)) \cdot F\beta.$$  

We now prove that $S$ is closed under joins of chains. Given an ordinal $\lambda$ and a chain $u_i: B_i \to A (i < \lambda)$ in $S$ with recursive coalgebras $\beta_i: B_i \to FB_i$ witnessing $u_i \leq f(u_i)$, we show that these recursive coalgebras form a chain. Indeed, suppose that $v_{i,j}: B_i \to B_j$ witnesses $u_i \leq u_j$; that is, we have $u_j \cdot v_{i,j} = u_i$. Then we have the following diagram:

Since the outside, the lower square and the left-hand and right-hand parts commute, we see that the upper square commutes when extended by the monomorphism $m \cdot Fu_j$. Thus the upper square commutes, proving that $v_{i,j}$ is a coalgebra homomorphism. 

202
Denote by \( u: B \to A \) the join of the chain \((u_i)\). Since \( A \) has smooth subobjects, it is given by the colimit of the chain of the objects \( B_i \) and morphisms \( v_{i,j} \) in \( \mathcal{A} \) (see Definition 7.3.11). By Example 7.3.2(5), we obtain a canonical structure \( \beta: B \to FB \) of a recursive coalgebra. Moreover, since \( u_i \in S \) we know that \( u_i \) is (the unique) coalgebra-to-algebra morphism from \((B_i, \beta_i)\) to \((A, m)\), and we know that \( u \) is the unique morphism induced by the cocone formed by the \( u_i \). Thus, \( u \) is the unique coalgebra-to-algebra morphism from \((B, \beta)\) to \((A, m)\) (cf. the dual of the proof of Lemma 7.2.10(2)). In symbols, we have \( u = m \cdot Fu \cdot \beta \), which states that \( u \) lies in \( S \), as desired.

By Theorem 6.1.1, \( f \) has a least fixed point, and by Corollary 7.3.9, \( \mu f \in S \). We write this subobject as \( u: I \to A \). We know that \( u \) and \( f(u) = m \cdot Fu \) represent the same subobject of \( A \). Since \( u \in S \), this is witnessed by a recursive coalgebra structure \( \iota: I \to FI \), which is an isomorphism. Thus \((I, \iota^{-1})\) is an initial algebra, by Remark 7.1.8.

**Remark 7.3.16.** In the proof of Theorem 6.1.22 we could have worked, more generally, with a class \( M \) of monomorphisms. Theorem 7.3.15 also holds in that setting: in lieu of \( \text{Sub}(A) \) we work with the collection \( \text{Sub}_M(A) \) of all subobject of \( A \) represented by monomorphisms in \( M \). Definition 7.3.11 needs to be adjusted to speak about \( M \)-subobjects. We conclude that if \( A \) is an \( M \)-prefixed point, \( F \) preserves \( M \) and \( \text{Sub}_M(A) \) is chain-complete with joins given by colimits, then \( F \) has an initial algebra.

We shall meet recursive coalgebras again Section 8.6, where we see that the condition of well-foundedness of coalgebras studied in the next chapter implies parametric corecursivity, and under mild assumptions, the parametric corecursivity coincides with well-foundedness.

### 7.4 Summary of this Chapter

Our theme in this chapter were characterizations of the initial algebra as a terminal object in a category of coalgebras and of the terminal coalgebra as an initial object in a category of algebras.

We have presented corecursive and completely iterative algebras (cias), which allow for the unique solution of recursive equation systems, in other words, they allow for the interpretation of definitions by structured corecursion. The terminal coalgebra turns out to be the initial corecursive algebra as well as the initial cia. Dually, the initial algebra is the initial (parametrically) recursive coalgebra. The latter notions allow for the unique interpretation of definitions by structured recursion.
8 Well-Founded Coalgebras

In the last chapter we have seen recursive coalgebras, which have the property that functions out of them may be specified by structured recursion. This is a desirable property, but it may not be straightforward to establish in general. In this chapter we consider well-foundedness of coalgebras, which captures well-founded induction and is usually much easier to establish for a given coalgebra. As mentioned at the beginning of Section 7.3, recursive and well-founded coalgebras were first studied by Osius [195] in connection with categorical set theory. He studied them for the power-object functor on an elementary topos.

Taylor [229, 228] took Osius’ ideas much further. He introduced well-founded coalgebras for a general endofunctor, capturing the notion of a well-founded relation categorically, and considered recursive coalgebras under the name ‘coalgebras obeying the recursion scheme’. He then proved the General Recursion Theorem that all well-founded coalgebras are (parametrically) recursive for every endofunctor on sets (and on more general categories) preserving inverse images.

We will present a proof of Taylor’s General Recursion Theorem for endofunctors preserving monomorphisms on a complete and well-powered category having smooth monomorphisms (see Definition 6.1.17). For the category of sets, this implies that “well-founded ⇒ recursive” holds for all endofunctors, strengthening Taylor’s result. We then discuss the converse: is every recursive coalgebra well-founded? Here the assumption that the endofunctor preserves inverse images cannot be lifted, and one needs additional assumptions. In fact, we present two proofs: one assumes the functor has a pre-fixed point and universally smooth monomorphisms (see Theorem 8.7.1). Under these assumptions we also give a new equivalent characterization of recursiveness and well-foundedness of a coalgebra, namely that the coalgebra admits a coalgebra-to-algebra morphism into the initial algebra (which exists under our assumptions), see Corollary 8.7.2. This characterization was previously established for finitary functors on sets [24]. The other proof of the above implication is due to Taylor [228], and it uses the concept of a subobject classifier (Theorem 8.7.8). It implies that ‘recursive’ and ‘well-founded’ are equivalent concepts for all set functors preserving inverse images. We also prove that a similar result holds for the category of vector spaces over a fixed field (Corollary 8.7.14).

Finally, we shall see that for every set functor, the initial algebra is characterized as the terminal well-founded coalgebra. This is connected to the theme of the previous chapter, where we have seen in Corollary 7.3.3 that the initial algebra is the terminal (parametric) recursive coalgebra.

In Chapter 9 we will see that well-founded coalgebras lead to a surprising description of the initial algebra for a set functor preserving intersections.

On a more technical note, we shall once again make use of transfinite iteration that
we already studied in Chapter 6. Here we make use of the induction principle in Corollary 7.3.9, when we prove the General Recursion Theorem 8.6.3.

8.1 Well-Founded Coalgebras and Well-Founded Graphs

The concept of well-foundedness is well-known for directed graphs: it means that the graph has no infinite directed paths. Similarly for relations: for example, the elementhood relation $\in$ of set theory is well-founded; this is precisely the Foundation Axiom.

Taylor [229, Def. 6.2.3] gave a more general category theoretic formulation of well-foundedness. His definition can be presented based on the concept of subcoalgebra; recall that this means a coalgebra homomorphism into the given coalgebra carried by a monomorphism in the base category (see Remark 2.4.13(1)). For example, consider a graph as a coalgebra $\alpha: A \to P^A$ for the power-set functor (see Example 1.3.2). A subcoalgebra is a subset $m: B \hookrightarrow A$ such that with every vertex it contains all the neighbors. The coalgebra structure $\beta: B \to P^B$ is then the domain-codomain restriction of $\alpha$. We call a subcoalgebra $m: B \hookrightarrow A$ cartesian if the square below expressing that $m$ is a homomorphism is a pullback:

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & P^B \\
m \downarrow & & \downarrow P^m \\
A & \xrightarrow{\alpha} & P^A
\end{array}
\]

(Being a pullback is indicated by the “corner” symbol.) This means that whenever a vertex of $A$ has all neighbors in $B$, it also lies in $B$. (8.1)

This implies that a graph is well-founded iff no proper subcoalgebra is cartesian. That is, whenever the above square is a pullback, $m$ is an isomorphism.

Definition 8.1.1. Let $\alpha: A \to FA$ be a coalgebra.

(1) A subcoalgebra $m: (B, \beta) \hookrightarrow (A, \alpha)$ is cartesian provided that the square below is a pullback.

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
m \downarrow & & \downarrow Fm \\
A & \xrightarrow{\alpha} & FA
\end{array}
\]

(8.2)

(2) $(A, \alpha)$ is called well-founded if whenever $m: (B, \beta) \hookrightarrow (A, \alpha)$ is a cartesian subcoalgebra, then $m$ is an isomorphism.

Lemma 8.1.2 [229, Example 6.3.3]. The well-founded coalgebras for the power-set functor are precisely the graphs without infinite paths.

Proof. Let $\alpha: A \to P^A$ be a graph without any infinite paths. Given a cartesian subcoalgebra $m: (B, \beta) \hookrightarrow (A, \alpha)$, we prove that $B = A$ by contradiction: every node
8.1 Well-Founded Coalgebras and Well-Founded Graphs

$\alpha_0 \in A \setminus B$ has, by (8.1), a neighbor $\alpha_1 \in A \setminus B$. By the same argument, $\alpha_1$ has neighbor $\alpha_2 \in A \setminus B$, etc. This yields an infinite path $\alpha_0, \alpha_1, \alpha_2, \ldots$, which is a contradiction.

Conversely, if $(A, \alpha)$ is a well-founded coalgebra, then there is no infinite path. Indeed, the subset $B \subseteq A$ of all vertices that do not lie on an infinite path satisfies (8.1). Thus, $B$ is a cartesian subcoalgebra of $A$, whence $B = A$. □

Remark 8.1.3. Recall from Example 7.3.2(1) that a binary relation $R \subseteq X \times X$ can be represented as a coalgebra for $\mathcal{P}$ by defining $\alpha : X \to \mathcal{P}X$ by $\alpha(x) = \{ y \in X \mid y R x \}$. This is the coalgebra representing the graph $(X, R^\mathcal{P})$. It is well-founded iff it is well-founded as a relation, which means that there is no sequence $x_0, x_2, x_2, \ldots$ of elements of $A$ such that $x_{n+1} R x_n$ for all $n \in \mathbb{N}$.

Examples 8.1.4. (1) If an initial algebra $\mu F$ exists, then (considered as a coalgebra) it is well-founded. Indeed, in every pullback (8.2), since $\alpha$ is invertible, so is $\beta$. The unique algebra homomorphism from $\mu F$ to the algebra $\beta^{-1} : FB \to B$ is clearly inverse to $m$.

(2) If a set functor $F$ fulfils $F\emptyset = \emptyset$, then the only well-founded coalgebra is the empty one. Indeed, this follows from the fact that the empty coalgebra is a cartesian subcoalgebra of every coalgebra for $F$.

For example, a deterministic automaton, as a coalgebra for $FX = \{0, 1\} \times X^\Sigma$ (see Example 2.4.2(3)), is well-founded iff it is empty.

(3) Non-deterministic automata were discussed in Example 5.1.27 as coalgebras for an endofunctor on $\text{Rel}$. But they can also be considered as coalgebras for the set functor $FX = \{0, 1\} \times (\mathcal{P}X)^\Sigma$. Recall that the state transition graph of an automaton has the states as its nodes, and an edge leads from every state to every of its successor states. An $F$-coalgebra is well-founded iff the state transition graph of the corresponding non-deterministic automaton is well-founded (i.e. has no infinite path). This follows immediately from Corollary 8.3.9 below.

(4) Well-founded linear weighted automata. We have introduced weighted automata in Example 2.4.5 as coalgebras for endofunctors $FX = S \times X^\Sigma$, where $S$ is a semiring and the underlying category is the category of $S$-semimodules. For the case that $S$ is a field $K$, this category $K\text{-Vec}$ is the category of vector spaces over $K$ and linear functions.

A linear weighted automaton is a coalgebra $(A, \alpha)$ for $FX = K \times X^\Sigma$. We characterize the well-founded coalgebras of this functor in Proposition 8.4.8: a coalgebra is well-founded iff every path in its state transition graph eventually leads to 0. This means that every path starting in a state $s \in A$ leads to the state 0 after finitely many steps (where it stays).

(5) Let $F$ be the set functor defined by $F\emptyset = 1$ and $FX = 1 + 1$ for all nonempty sets $X$ and $Ff = \text{inl}$ for all maps $f : \emptyset \to X$ with $X$ nonempty.

The empty coalgebra is well-founded. The coalgebra $\text{inr} : 1 \to F1$ is not well-founded because its empty subcoalgebra is cartesian. On the other hand, consider $\text{inl} : 1 \to F1$. It and all its subcoalgebras are well-founded: these are itself and the empty coalgebra (not cartesian). Applying $F$ to the last coalgebra gives $\text{id} : 1 + 1 \to 1 + 1$. This coalgebra is the initial algebra and is thus well-founded, but it has a subcoalgebra which is not, namely $\text{inr} : 1 \to F1$ via the embedding $\text{inr}$.
These example show that a well-founded coalgebra for a set functor can have non-well-founded subcoalgebras, and also that even if all subcoalgebras of a given coalgebra are well-founded, the same need not hold after applying the functor. But both of these are very rare; see Lemma 8.1.5 just below.

**Lemma 8.1.5.** Let $\mathcal{A}$ have and $F : \mathcal{A} \to \mathcal{A}$ preserve inverse images, and let $\alpha : A \to FA$ be a coalgebra.

1. If $(A, \alpha)$ is a well-founded, then so is $(FA, F\alpha)$.
2. If every subcoalgebra of $(A, \alpha)$ is well-founded, then $(FA, F\alpha)$ has the same property.

**Proof.** (1) Let $s : (S, \sigma) \to (FA, F\alpha)$ be a cartesian subcoalgebra; we prove that $s$ is invertible. Form its inverse image under $\alpha$:

\[
\begin{array}{ccc}
T & \xrightarrow{q} & S \\
\downarrow s & & \downarrow \sigma \\
A & \xrightarrow{\alpha} & FA \\
\end{array}
\]

Since $F$ preserves inverse images, we see that $Ft$ is the inverse image of $Fs$ under $F\alpha$. In other words, the pullback square on the right above can be chosen as

\[
\begin{array}{ccc}
FT & \xrightarrow{Fq} & FS \\
\downarrow Ft & & \downarrow Fs \\
FA & \xrightarrow{F\alpha} & FF\!\!A
\end{array}
\]

This means that $s = Ft$ as subobjects of $FA$. Hence $T$ carries a cartesian subcoalgebra of $(A, \alpha)$. Due to the well-foundedness of $(A, \alpha)$ we know that $t$ is an isomorphism, thus so is $Ft = s$, as required.

(2) Suppose that every subcoalgebra of $(A, \alpha)$ is well-founded. Let $m : (B, \beta) \to (FA, F\alpha)$ be a subcoalgebra; we prove that $(B, \beta)$ is well-founded, too. Form the inverse image of $m$ along $\alpha$. Using that $F$ preserves inverse images we then obtain the following commutative diagram, where we obtain the dashed arrow $\beta'$ using the universal property of the right-hand pullback:

\[
\begin{array}{ccc}
P & \xrightarrow{q} & B \\
\downarrow p & & \downarrow \beta' \\
A & \xrightarrow{\alpha} & FA \\
\end{array}
\]

\[
\begin{array}{ccc}
P & \xrightarrow{q} & FP \\
\downarrow p & & \downarrow F\!\!\beta' \\
A & \xrightarrow{\alpha} & FA \\
\downarrow Fp & & \downarrow F\alpha \\
B & \xrightarrow{\beta} & FB \\
\end{array}
\]

\[
\begin{array}{ccc}
P & \xrightarrow{q} & FP \\
\downarrow p & & \downarrow Fp \\
A & \xrightarrow{\alpha} & FA \\
\downarrow Fm & & \downarrow F\!\!\alpha \\
B & \xrightarrow{\beta'} & FB \\
\end{array}
\]

We see that $(P, \beta' \cdot q)$ is a subcoalgebra of $(A, \alpha)$ and is therefore well-founded by hypothesis.
Now consider a cartesian subcoalgebra $s: (S, \sigma) \hookrightarrow (B, \beta)$. We prove that $s$ is an isomorphism. Take its inverse image under $q: P \to A$:

$$
\begin{array}{ccc}
T & \xrightarrow{r} & S \\
\downarrow{t} & & \downarrow{s} \\
P & \xrightarrow{q} & B
\end{array}
$$

Note that since $\beta = Fq \cdot \beta'$ we can obtain $s$ as the pullback of $Fs$ first along $\beta'$ and further along $Fq$. Using that $F$ preserves inverse images, we thus obtain the following commutative diagram (with a uniquely determined morphism $\sigma'$):

$$
\begin{array}{ccc}
T & \xrightarrow{r} & S & \xrightarrow{\sigma'} & FT & \xrightarrow{Fr} & FS \\
\downarrow{t} & & \downarrow{s} & & \downarrow{Fr} & & \downarrow{Fs} \\
P & \xrightarrow{q} & B & \xrightarrow{\beta'} & FP & \xrightarrow{Fq} & FB \\
\end{array}
$$

This shows that $(T, \sigma' \cdot r)$ is a cartesian subcoalgebra of $(P, \beta' \cdot q)$. Thus, $t$ is an isomorphism by well-foundedness. Therefore so are $Ft$ and its inverse image $s$ under $\beta'$. This completes the proof. \hfill \Box

**Remark 8.1.6.** (1) Observe that when $\alpha: A \to FA$ in Lemma 8.1.5 is a monomorphism we need only assume that $\mathcal{A}$ has and $F$ preserves finite intersections since all the pullback squares are then intersections, i.e. pullbacks of two monomorphisms.

(2) We obtain a version of Corollary 7.3.6 for well-founded coalgebras as follows. Suppose that, in addition to the these assumptions, $\mathcal{A}$ has a simple initial object, i.e. the unique morphisms $0 \to X$ are monomorphisms. Then it follows from Lemma 8.1.5 that every subcoalgebra of a coalgebra $F^00 \to F(F^00)$ in the initial-algebra $\omega$-chain for $F$ is well-founded. This is easy to see by induction. Indeed, for the unique coalgebra $0 \to F0$ this clearly holds since every subcoalgebra yields a subobject $m: S \to 0$, and we have $m \cdot u = \text{id}_0$ for the unique morphism $u: 0 \to S$. Thus, $m$ is a split epimorphism, whence an isomorphism. For the induction step use Lemma 8.1.5 and the observation in point (1).

We next show that to every coalgebra for a set functor $F$ one may associate a graph, in a canonical way. Moreover, if $F$ preserves intersections, then a coalgebra is well-founded if and only if so is its canonical graph.

**Notation 8.1.7.** Given a set functor $F$, we define for every set $X$ the map $\tau_X: FX \to \mathcal{P}X$ assigning to every element $x \in FX$ the intersection of all subsets $m: M \hookrightarrow X$ such that $x$ lies in the image of $Fm$:

$$
\tau_X(x) = \bigcap\{m \mid m: M \hookrightarrow X \text{ satisfies } x \in Fm[FM]\}. \hspace{1cm} (8.3)
$$
Definition 8.1.8. Let $F$ be a set functor. For every coalgebra $\alpha : A \to FA$ its canonical graph is the following coalgebra for $P$:

$$A \xrightarrow{\alpha} FA \xrightarrow{\tau_A} PA.$$ 

Examples 8.1.9. (1) Given a graph as a coalgebra $\alpha : A \to PA$, the condition $\alpha(x) \in Pm[PM]$ states precisely that all successors of $x$ lie in the set $M$. The least such set is $\alpha(x)$. Therefore, the canonical graph of $(A,\alpha)$ is itself (see [229, Example 6.3.3]).

(2) For the type functor of $FX = \{0,1\} \times X^{\Sigma}$ of deterministic automata, we have

$$\tau_X(i,t) = \{t(s) : s \in \Sigma\} \quad \text{for } i = 0,1 \text{ and } t : \Sigma \to X.$$ 

Thus, the canonical graph of a deterministic automaton $A$ is precisely its state transition graph (forgetting the labels of transitions and the finality of states), i.e. we have an edge $(a,a')$ iff $a' = \delta(a,s)$ for some $s \in \Sigma$, where $\delta$ is the next state function of $A$.

Similarly, for the type functor $FX = \{0,1\} \times (P X)^{\Sigma}$ of non-deterministic automata we have

$$\tau_X(i,g) = \bigcup_{s \in \Sigma} t(s) \quad \text{for } i = 0,1 \text{ and } t : \Sigma \to PX,$$

and we again obtain the state transition relation, forgetting the labels.

(3) For the functor $FX = P(\Sigma \times X)$ whose coalgebras are labeled transition systems we have

$$\tau_X = (P(\Sigma \times X) \xrightarrow{P\pi_X} PX),$$

where $\pi_X : \Sigma \times X \to X$ is the projection. Again, the canonical graph of a labelled transition system is its state transition graph. Thus $(a,a')$ is an edge iff some action leads from state $a$ to $a'$.

Recall that a functor preserves intersections if it preserves (wide) pullbacks of families of monomorphisms. Gumm [124, Theorem 7.3] observed that for a set functor preserving intersections, the maps $\tau_X : FX \to PX$ in (8.3) form a “subnatural” transformation from $F$ to the power-set functor $P$. Subnaturality means that (although these maps do not form a natural transformation in general) for every monomorphism $i : X \to Y$ we have a commutative square:

$$\begin{array}{c}
FX \downarrow \tau_X \downarrow \quad PX \\
Fi \downarrow \quad \downarrow \pi_i \\
FY \xrightarrow{\tau_Y} PY
\end{array}$$

For such set functors this is even a pullback square:

Theorem 8.1.10 [124, Thm. 7.4] and [228, Prop. 7.5]. A set functor $F$ preserves intersections iff the above squares (8.4) are pullbacks.

Remark 8.1.11. The condition that $F$ preserves intersections is an extremely mild one for set functors. We discuss this further in Example B.2.3; see especially Figure B.1.
on page 423. Furthermore, the collection of set functors which preserve intersections is closed under products, coproducts, and compositions. A subfunctor $m: G \rightarrow F$ of an intersection preserving functor $F$ preserves intersections whenever $m$ is a cartesian natural transformation, i.e. all naturality squares are pullbacks:

$$
\begin{array}{c}
GX \\ Gf
\end{array}
\begin{array}{c}
\downarrow m_X \\
\downarrow Ff
\end{array}
\begin{array}{c}
FX \\ Ff
\end{array}
\begin{array}{c}
GY \\ m_Y
\end{array}
\begin{array}{c}
FY
\end{array}
$$

Observe that all the functors in Example 8.1.9 preserve intersections.

As we show next, “almost all” finitary set functors preserve intersections. This calls on an important construction due to Trnková [233]. We have seen in Proposition 4.4.1 that every set functor preserves finite nonempty intersections. By changing a set functor only at the empty set one can ensure preservation of finite intersections:

**Proposition 8.1.12** [233]. For every set functor $F$ there exists an essentially unique set functor $\bar{F}$ which coincides with $F$ on non-empty sets and functions and preserves finite intersections (whence monomorphisms).

For the proof, see Theorem B.4.2. The original proof may be found in Trnková [233, Propositions III.5 and II.4]; for a more direct proof see Adámek and Trnková [45, Theorem III.4.5].

**Definition 8.1.13.** We call the functor $\bar{F}$ in Proposition 8.1.12 the Trnková hull of $F$.

Note that it is the reflection of $F$ into the full subcategory of endofunctors on sets preserving finite intersections (see Section B.4 for details).

**Examples 8.1.14.** (1) Let $F$ be the functor in Example 8.1.4(5). Its Trnková hull is constant functor with value $1 + 1$.

(2) Let $\mathcal{P}'$ be the subfunctor of the power set functor $\mathcal{P}$ consisting of all non-empty subsets. Its Trnková hull is $\mathcal{P}$.

**Remark 8.1.15.** Note that the Trnková hull $\bar{F}$ has the same coalgebras as $F$. More precisely, the categories $\text{Coalg} F$ and $\text{Coalg} \bar{F}$ are isomorphic. Note that the initial coalgebra $0 \rightarrow F0$ of the former category is related to $0 \rightarrow \bar{F}0$ in the latter category. Similary, the Trnková hull has the same nonempty algebras as $F$.

**Remark 8.1.16.** If $F$ is a finitary set functor, then so is its Trnková hull $\bar{F}$. This follows immediately from the characterization of finitary set functor in terms of finite boundedness in Proposition 4.3.4.

**Corollary 8.1.17.** The Trnková hull of a finitary set functor preserves all intersections. The proof is a slightly simpler version of the proof of Proposition 4.4.3. In fact, in lieu of Proposition 4.4.1 one uses that $\bar{F}$ preserves finite intersections so that one need not worry about intersections which might be nonempty.

**Theorem 8.1.18** [124, Thm. 8.1]. Let $F$ be a set functor which preserves inverse images and intersections. Then $\tau: F \rightarrow \mathcal{P}$ is a natural transformation.
Example 8.1.19. To see that $\tau$ is not a natural transformation in general, we use the set functor $R$ from Example 7.3.4. Let $X = \{0, 1\}$, $Y = \{0\}$, and $f : X \to Y$ the evident function. Then $(0, 1) \in FX$, and $\tau_X(0, 1) = X$. Further, $\mathcal{P}f(X) = Y$. But $Rf(0, 1) = d$, and $\tau_Y(d) = \emptyset$.

Remark 8.1.20. Taylor [229, Rem. 6.3.4] proved that, for functors preserving intersections and inverse images, a coalgebra is well-founded iff its canonical graph is so. We shall see in Corollary 8.3.9 that the proof only needs preservation of intersections.

Examples 8.1.21. (1) A coalgebra for the identity functor $FX = X$ on Set is a set $A$ equipped with a function $\alpha : A \to A$. The canonical graph of $(A, \alpha)$ is the graph of the function $\alpha$, i.e. the graph with edges $(a, \alpha(a))$ for all $a \in A$. Hence, $(A, \alpha)$ is well-founded iff it is empty (see Example 8.1.4(2)).

(2) For $FX = X + 1$ coalgebras are sets $A$ equipped with a partial function $\alpha : A \to A$, and the canonical graph is the graph of $\alpha$. This functor has many nonempty well-founded coalgebras. For example, the initial $F$-algebra, considered as the coalgebra on $\mathbb{N}$ with the structure given by the partial function $n \mapsto n - 1$, for $n > 0$ (cf. Example 7.3.8(1)), is well-founded since its canonical graph is so.

(3) Consider the functor $FX = X \times X + 1$ and a coalgebra $\alpha : A \to A \times A + 1$. The edges in its canonical graph are all of the pairs $(a, a_1)$ and $(a, a_2)$ such that $a \in A$ and $\alpha(a) = (a_1, a_2)$. For example, the coalgebra $(\mathbb{N}, \gamma)$ from Example 7.3.8(2) has the canonical graph with edge set $\{(n, n - 1), (n, n - 2) : n \geq 2\}$, which is clearly well-founded, and therefore so is the coalgebra.

Similarly, for the functor $FX = A \times X \times X + 1$, the coalgebra $(A^*, s)$ in Example 7.3.8(3) is easily seen to be well-founded via its canonical graph. Indeed, this graph has for every list $a$ one outgoing edge to the list $w_0 a$ and one to $w_0 a$, for every $a \in A$. Hence, this is a well-founded graph, and therefore $(A^*, s)$ is a well-founded coalgebra.

(4) More generally, for a polynomial functor $H_{\Sigma} : \mathsf{Set} \to \mathsf{Set}$ associated to a finitary signature $\Sigma$, a coalgebra $\alpha : A \to \coprod_{n \in \mathbb{N}} \Sigma_n \times A^n$ has the canonical graph where a vertex $a \in A$ has an outgoing edge $(a, a')$ iff $a' \in A$ occurs in some tuple $\alpha(a) \in \Sigma_n \times A^n$ for some $n < \omega$.

Thus, the coalgebra $(A, \alpha)$ is well-founded iff for every $a \in A$ its tree-unfolding, i.e. its image under the unique homomorphism $h : A \to \nu F$, is a finite $\Sigma$-tree.

In particular, if the signature $\Sigma$ does not contain any constant symbols, then the only well-founded $H_{\Sigma}$-coalgebra is $A = \emptyset$.

For further use we now compare well-founded and recursive coalgebras for a given set functor $F$ with those of its Trnková hull $\bar{F}$ (see Proposition 8.1.12). Since empty coalgebras are (trivially) well-founded and recursive, we can restrict ourselves to the nonempty ones. We observe in Remark 8.1.15 that $\mathsf{Coalg} F$ and $\mathsf{Coalg} \bar{F}$ are isomorphic.

Lemma 8.1.22. Let $(A, \alpha)$ be a nonempty coalgebra for a set functor $F$. If it is well-founded or (parametrically) recursive, then it also has those properties as a coalgebra for the Trnková hull $\bar{F}$.

Proof. (1) Let $(A, \alpha)$ be well-founded for $F$. Nonempty subcoalgebras of $(A, \alpha)$ for $F$
and for $\bar{F}$ coincide. Thus, we only need to show that the left-hand square below, where $r_X : \emptyset \to X$ denotes the empty map, is not a pullback:

$$
\begin{array}{ccc}
\emptyset & \xrightarrow{r_F} & F\emptyset \\
\downarrow{r_A} & & \downarrow{\bar{F}r_A} \\
A & \xrightarrow{\alpha} & FA = FA
\end{array}
\quad \begin{array}{ccc}
\emptyset & \xrightarrow{r_{F\emptyset}} & F\emptyset \\
\downarrow{r_A} & & \downarrow{Fr_A} \\
A & \xrightarrow{\alpha} & FA
\end{array}
$$

Since $(A, \alpha)$ is well-founded, we know that the right-hand square is not a pullback. Thus, there exist $a \in A$ and $x \in F\emptyset$ with $\alpha(a) = Fr_A(x)$. In particular, $F\emptyset \neq \emptyset$. But then the two squares above are the same, so the one on the left is not a pullback, either.

(2) Let $(A, \alpha)$ be a nonempty recursive coalgebra for $F$. Given an algebra $e : \bar{F}X \to X$ we know that $X \neq \emptyset$, for otherwise the existence of a unique coalgebra-to-algebra morphism $A \to X$ would force $A$ to be empty. But then the unique coalgebra-to-algebra morphism from $(A, \alpha)$ to $(X, e)$ w.r.t. $F$ is also one for $\bar{F}$.

\section*{8.2 Factorization of Coalgebra Homomorphisms}

Before we continue our study of well-founded coalgebras, we shortly digress and consider factorizations of coalgebra homomorphisms inherited from the factorizations in the base category.

**Assumption 8.2.1.** Throughout the remainder of this chapter we work with a category $\mathcal{A}$ which is complete and well-powered. We also assume that $F : \mathcal{A} \to \mathcal{A}$ preserves monomorphisms. (See Remark 8.2.6 for a weakening of the last assumption.)

**Remark 8.2.2.** (1) Recall that an epimorphism $e : A \to B$ is called strong if it satisfies the following diagonal fill-in property: given a monomorphism $m : C \to D$ and morphisms $f : A \to C$ and $g : B \to D$ such that $m \cdot f = g \cdot e$ (i.e. the outside of the square below commutes) then there exists a unique $d : B \to C$ such that the diagram below commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xleftarrow{d} & D
\end{array}
$$

(8.5)

(2) A very basic fact is that if a morphism is both a monomorphism and a strong epimorphism, then it is an isomorphism. For this, take $f$ and $g$ to be identities and $m = e$, and observe that $d$ is the inverse of $m = e$.

(3) Every complete and well-powered category has factorizations of morphisms $f$ as $f = m \cdot e$, where $e$ is a strong epimorphism and $m$ is a monomorphism [74, Prop. 4.4.3]. We call the subobject $m$ the image of $f$.

(4) We indicate monomorphisms by $\hookrightarrow$ and strong epimorphisms by $\twoheadrightarrow$.

It follows from a result in Kurz’ thesis [161, Prop. 1.3.6] that factorizations of morphisms lift to coalgebras:
Proposition 8.2.3 (Coalg\(F\) inherits factorizations from \(\mathcal{A}\)). Since \(F\) preserves monomorphisms, the category Coalg\(F\) has factorizations of homomorphisms \(f\) as \(f = m \cdot e\), where \(e\) is carried by a strong epimorphism and \(m\) by a monomorphism in \(\mathcal{A}\).

Proof. Let \(f : (A, \alpha) \to (B, \beta)\) be a coalgebra homomorphism and take its (strong epi, mono)-factorization \(f = m \cdot e\) in \(\mathcal{A}\). Now consider the diagram below:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow e & & \downarrow F_e \\
C & \xrightarrow{\gamma} & FC \\
\downarrow m & & \downarrow F_m \\
B & \xrightarrow{\beta} & FB
\end{array}
\]  

(8.6)

Since \(Fm\) is monic we can use the diagonal fill-in property to obtain a unique coalgebra structure \(\gamma\) such that \(e\) and \(m\) are coalgebra homomorphisms.

Remark 8.2.4. Following Remark 8.2.2(3) we see that \(C\) in (8.6) above is the image of \(f\). We therefore call \((C, \gamma)\) the image of \((A, \alpha)\) under the homomorphism \(f\).

Remark 8.2.5. (1) Recall from Corollary 2.4.12 and Remark 2.4.13(1) that coalgebra homomorphisms carried by monomorphisms (or epimorphisms) in \(\mathcal{A}\) have the same property in Coalg\(F\). Since \(F\) preserves monomorphisms, the diagonal fill-in property in Remark 8.2.2(1) lifts to coalgebra homomorphisms. This shows that we obtain a factorization system on Coalg\(F\). In fact, if \(m\) and \(g\) in (8.5) are coalgebra homomorphisms, then so is \(d\). To see this we use that \(F\) preserves monomorphisms and consider the diagram below:

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow d & & \downarrow F_d \\
C & \xrightarrow{\gamma} & FC \\
\downarrow m & & \downarrow F_m \\
D & \xrightarrow{\delta} & FD
\end{array}
\]

Its outside, left-hand, and right-hand parts commute. Thus so does the upper square when extended by the monomorphism \(Fm\), which implies that it commutes.

(2) Note that Proposition 8.2.3 generalizes to an arbitrary factorization system \((\mathcal{E}, \mathcal{M})\) on the base category and endofunctors \(F\) such that \(m \in \mathcal{M}\) implies \(Fm \in \mathcal{M}\). However, we are not going to work in this generality.

Remark 8.2.6. (1) Note that for \(\mathcal{A} = \text{Set}\), Proposition 8.2.3 holds even without the assumption that \(F\) preserve monomorphisms. In fact, every set functor preserves epimorphisms and nonempty monomorphisms, and if \(m\) in (8.6) is empty, we have \(C = \emptyset\) and therefore \(A = \emptyset\). Hence, we can take \(\gamma\) to be the empty map in this case.
8.3 The Next Time Operator on Coalgebras

A compact characterization of well-foundedness can be given by using an operator that generalizes the semantics of the ‘next time’ operator of temporal logics for nondeterministic (or even probabilistic) automata and transitions systems (see e.g. Manna and Pnueli [171]). It is also strongly related to the algebraic semantics of modal logic, where one passes from a graph $G$ to a function on $\mathcal{P}G$. Jacobs [135] defined and studied the ‘next time’ operator on coalgebras for Kripke polynomial set functors. We generalize his definition to arbitrary functors, obtaining an operator $\boxdot$ on the subobjects of the carrier of a coalgebra. In the next section, we will see that a coalgebra is well-founded iff it has no proper subcoalgebra as a fixed point of $\boxdot$ (Proposition 8.4.9). Recall from Notation 7.3.10 that for every object $A$ we have a poset $\text{Sub}(A)$ of subobjects of $A$.

**Remark 8.3.1.** Since $\mathcal{A}$ is complete and well-powered we have that $\text{Sub}(A)$ is a complete lattice: it is small since $\mathcal{A}$ is well-powered, and a meet of subobjects $m_i: A_i \rightarrow A$, $i \in I$, is their intersection, obtained by forming their wide pullback. For example, $\bot_A$ is the intersection of all subobjects of $A$.

It follows that $\text{Sub}(A)$ has all joins as well.

We shall need that forming inverse images, i.e. pulling back along a morphism, is a right adjoint.

**Notation 8.3.2.** For every morphism $f: B \rightarrow A$ we have two operators:

1. The inverse image operator

$$\mathcal{f} : \text{Sub}(A) \rightarrow \text{Sub}(B),$$

assigning to every subobject $s: S \hookrightarrow A$ its inverse image under $f$ obtained by the following pullback

$$
\begin{array}{ccc}
P & \rightarrow & S \\
\downarrow & & \downarrow s \\
B & \mathcal{f} & \rightarrow & A
\end{array}
$$

(2) Milius et al. [182, Lemma 2.5] show that this argument can easily be generalized. Recall [83] that an initial object $0$ is called strict if every morphism $I \rightarrow 0$ is an isomorphism. A nonempty monomorphism in a category $\mathcal{A}$ is a monomorphism whose domain is not a strict initial object. Proposition 8.2.3 (and in fact all the results of this section) then hold for endofunctors $F$ on $\mathcal{A}$ preserving nonempty monomorphisms. Note that in categories not having (strict) initial objects, e.g. the categories of groups or vector spaces over a field, all monomorphisms are nonempty. In such categories, preservation of nonempty monomorphisms is the same as mono-preservation. However, in every category of algebras containing the empty algebra, all functors lifted from $\text{Set}$ preserve nonempty monomorphisms.
The (direct) image operator
\[ \overrightarrow{f} : \text{Sub}(B) \to \text{Sub}(A), \]
assigning to every subobject \( t : T \to B \) the image of \( f \cdot t \) (see Remark 8.2.2(3)).

**Remark 8.3.3.** (1) Let \( X \) and \( Y \) be posets, considered as categories. A monotone map \( r : X \to Y \) is a right adjoint iff there exists a monotone map \( \ell : Y \to X \) such that
\[ \ell(y) \leq x \quad \text{iff} \quad y \leq r(x) \quad \text{for every} \ x \in X \ \text{and} \ y \in Y. \]

(2) If in addition \( X \) is a complete lattice, then \( r \) is a right adjoint iff it preserves meets. This in particular holds for \( r : \text{Sub}(B) \to \text{Sub}(A) \). Indeed, the necessity follows since right adjoints preserve limits. For the sufficiency, suppose that \( r \) preserves intersections, and define \( \ell : Y \to X \) by
\[ \ell(y) = \bigwedge_{y \leq r(x)} x \quad \text{for every} \ y \in Y. \]

Then \( \ell \) is clearly monotone, and for every \( x \in X, \ y \in Y \) we have
\[ \ell(x) \leq y \quad \text{iff} \quad x \leq r(y). \]

Thus, \( \ell \) is the desired left adjoint of \( r \).

**Proposition 8.3.4.** For every morphism \( f : B \to A \) we have an adjoint situation:
\[ \text{Sub}(A) \quad \overleftarrow{\overrightarrow{f}} \quad \text{Sub}(B). \]

In other words: \( \overrightarrow{f}(t) \leq s \) iff \( t \leq \overleftarrow{f}(s) \) for all subobjects \( s : S \to A \) and \( t : T \to B \).

**Proof.** In order to see this, we consider the following diagram:

By the universal property of the lower middle pullback square and the diagonal fill-in property, we have the dashed morphism on the left iff we have the one on the right. Thus, \( t \leq \overleftarrow{f}(s) \) iff \( \overrightarrow{f}(t) \leq s \), as desired. \( \square \)
The Next Time Operator on Coalgebras

We now come to the main definition in this section. It was introduced by Jacobs [135] for Kripke polynomial functors on sets.

**Definition 8.3.5.** Every coalgebra \( \alpha : A \to FA \) induces an endofunction \( \Box \) on \( \text{Sub}(A) \), called the *next time operator*:

\[
\Box : \text{Sub}(A) \to \text{Sub}(A), \quad \Box(s) = \Box(\alpha(s)) \text{ for } s \in \text{Sub}(A).
\]

In more detail: for a subobject \( s : S \hookrightarrow A \), we define \( \Box s \) and \( \alpha(s) \) by the following pullback:

\[
\begin{array}{c}
\Box S \xrightarrow{\alpha(s)} FS \\
\downarrow \quad \downarrow Fs \\
A \xrightarrow{\alpha} FA
\end{array}
\]  

(8.7)

Since \( Fs \) is a monomorphism, \( \Box s \) is a monomorphism and \( \alpha(s) \) is (for every representation \( \Box s \) of that subobject of \( A \)) uniquely determined.

**Example 8.3.6.** (1) Let \( A \) be a graph, considered as a coalgebra for \( \mathcal{P} : \text{Set} \to \text{Set} \). If \( S \subseteq A \) is a set of vertices, then \( \Box S \) is the set of those vertices all of whose successors belong to \( S \).

(2) For the set functor \( FX = \mathcal{P}(\Sigma \times X) \) expressing labelled transition systems the operator \( \Box \) for a coalgebra \( \alpha : A \to FA \) is the semantic counterpart of the next time operator of classical linear temporal logic, see e.g. Manna and Pnueli [171]. In fact, for a labelled transition system \( \alpha : A \to \mathcal{P}(\Sigma \times A) \) and a subset \( S \hookrightarrow A \) we have that \( \Box S \) consists of those states whose next states lie in \( S \), in symbols:

\[
\Box S = \{ x \in A \mid (s, y) \in \alpha(x) \text{ implies } y \in S, \text{ for all } s \in \Sigma \}.
\]

(3) For the endofunctor \( FX = K \times X^\Sigma \) on \( K\text{-Vec} \) whose coalgebras are linear weighted automata, given a subspace \( S \) of a linear automaton, \( \Box S \) consists of all states whose successors all lie in \( S \).

**Lemma 8.3.7.** The next time operator \( \Box \) is monotone: if \( m \leq n \), then \( \Box m \leq \Box n \).

**Proof.** Suppose that \( m : B \to A \) and \( n : C \to A \) are subobjects such that \( m \leq n \), i.e. \( n \cdot x = m \) for some \( s : B \to C \). Then we obtain the dashed arrow in the diagram below using that its lower square is a pullback:

\[
\begin{array}{ccc}
B & \xrightarrow{\alpha(m)} & FB \\
\downarrow w & & \downarrow Fs \\
\Box C & \xrightarrow{\alpha(n)} & FC \\
\downarrow \Box & & \downarrow Fn \\
A & \xrightarrow{\alpha} & FA
\end{array}
\]  

(8.8)

This shows that \( \Box m \leq \Box n \). \( \square \)
Lemma 8.3.8. For every set functor $F$ preserving intersections, the next time operator of a coalgebra $(A, \alpha)$ coincides with that of its canonical graph (see Definition 8.1.8).

Proof. In the diagram below the outside is a pullback if and only if so is the left-hand square:

\[
\begin{array}{ccccccc}
\bigcirc A' & \xrightarrow{\alpha(m)} & FB & \xrightarrow{\tau_B} & \mathcal{P}B \\
\downarrow A' \xleftarrow{\alpha(m)} & & \downarrow Fm & & \downarrow \mathcal{P}m \\
A & \xrightarrow{\alpha} & \mathcal{P}A & \xrightarrow{\tau_A} & \mathcal{P}A \\
\end{array}
\]

\[\square\]

Corollary 8.3.9 [229, Rem. 6.3.4]. A coalgebra for a set functor preserving intersections is well-founded iff its canonical graph is a well-founded graph.

The following lemma will be useful when we establish the universal property of the well-founded part of a coalgebra in the next section.

Lemma 8.3.10. For every coalgebra homomorphism $f: (B, \beta) \to (A, \alpha)$ we have

\[\bigcirc \beta \cdot \mathcal{T}f \leq \mathcal{T}f \cdot \bigcirc \alpha,\]

where $\bigcirc \alpha$ and $\bigcirc \beta$ denote the next time operators of the coalgebras $(A, \alpha)$ and $(B, \beta)$, respectively.

Proof. Let $s: S \to A$ be a subobject. We see that $\mathcal{T}f(\bigcirc \alpha s)$ is obtained by pasting two pullback squares as shown below:

\[
\begin{array}{cccc}
T & \xrightarrow{f} & \bigcirc \alpha S & \xrightarrow{\alpha(s)} \xrightarrow{FS} \\
\downarrow \mathcal{T}(\bigcirc \alpha s) & & \downarrow \bigcirc \alpha s & & \bigcirc \alpha s \\
B & \xrightarrow{f} & A & \xrightarrow{\alpha} \xrightarrow{FA} \\
\end{array}
\]

(8.9)

In order to show that $\bigcirc \beta(\mathcal{T}(s)) \leq \mathcal{T}(\bigcirc \alpha s)$, we consider the following diagram:

\[
\begin{array}{ccccccccc}
\bigcirc \beta U & \xrightarrow{\beta(\mathcal{T}(s))} & FU & \xrightarrow{Fu} & FS \\
\downarrow \bigcirc \beta(\mathcal{T}(s)) & & \downarrow F(\mathcal{T}(s)) & & \downarrow Fs \\
B & \xrightarrow{\beta} & FB & \xrightarrow{Ff} & F \mathcal{T}(s) \\
\downarrow f & & \downarrow \mathcal{T}(s) & & \downarrow F \mathcal{T}(s) \\
A & \xrightarrow{\alpha} & FA & & & & \\
\end{array}
\]

(8.10)

The left-hand part is the pullback square defining $\bigcirc \beta(\mathcal{T}(s))$, and the right-hand one is that defining $\mathcal{T}(\bigcirc \alpha s)$, with $F$ applied. On the bottom, we use that $f$ is a coalgebra homomorphism. Thus, the outside of the diagram commutes. Since the outside of the
8.3 The Next Time Operator on Coalgebras

diagram in (8.9) is a pullback, we have some $g: \Box \beta U \to T$ such that $\Box \beta(f(s)) = \Box \beta(U) \cdot g$, which proves the desired inequality.

**Corollary 8.3.11.** For every coalgebra homomorphism $f: (B, \beta) \to (A, \alpha)$ we have

(1) $f$ is a monomorphism in $\mathcal{A}$ and $F$ preserves finite intersections, or

(2) $F$ preserves inverse images.

**Proof.** In case (1), the right-hand part in Diagram (8.10) is clearly a pullback. In case (2), this holds because $F(\Box f(s)) = F(f(Fs))$. Either way, pasting this part with the pullback in the upper left of (8.10) and using that the lower part commutes, we see that $\Box \beta(f(s))$ is obtained by pulling back $Fs$ along $f \cdot \alpha$. This implies the desired equality since this is how $\Box \beta(U) \cdot g$ is obtained (see (8.9)).

**Lemma 8.3.12.** Let $\alpha: A \to FA$ be a coalgebra and $m: B \hookrightarrow A$ be a monomorphism.

(1) There is a coalgebra structure $\beta: B \to FB$ for which $m$ gives a subcoalgebra of $(A, \alpha)$ iff $m \leq \Box m$.

(2) There is a coalgebra structure $\beta: B \to FB$ for which $m$ gives a cartesian subcoalgebra of $(A, \alpha)$ iff $m = \Box m$.

**Proof.** We prove the left-to-right directions of both assertions first, and then the right-to-left ones.

Suppose first that there exists $\beta: B \to FB$ such that $m: (B, \beta) \hookrightarrow (A, \alpha)$ is a homomorphism. Then the fact that $\Box B$ is given by a pullback yields the unique morphism $x$ such that the diagram below commutes:

\[
\begin{array}{ccc}
B & \xrightarrow{x} & \Box B \\
\downarrow m & & \downarrow \Box m \\
A & \xrightarrow{\alpha} & FA \\
\end{array}
\]

\[\beta\]

\[\Box \alpha (m) \]

\[FB\]

\[\Box \alpha (m) \cdot x = m \text{ yields } m \leq \Box m.\] If $(B, \beta)$ is a cartesian subcoalgebra, then the outside of the above diagram is a pullback square. This implies that $x$ is an isomorphism, thus $m = \Box m$ in Sub$(A)$.

Conversely, suppose that $m \leq \Box m$ via $x: B \hookrightarrow \Box B$. Then (8.11) shows that $\beta = \alpha(m) \cdot x: B \to FB$ is a coalgebra, and $m: B \hookrightarrow A$ is a homomorphism. If in addition $m = \Box m$, i.e. $x$ is an isomorphism, we see that $m$ is a cartesian subcoalgebra.

**Remark 8.3.13.** The next time operator of a coalgebra restricts to its subcoalgebras. Given a subcoalgebra $m: (B, \beta) \hookrightarrow (A, \alpha)$, we use the morphism $x$ from (8.11) to obtain
a (unique) coalgebra structure on \( \bigcirc \) turning it into a subcoalgebra of \((A, \alpha)\):

\[
\begin{array}{c}
\bigcirc B \\
\downarrow \circ m \quad \downarrow \alpha (m) \quad \downarrow Fm \quad \downarrow F(\bigcirc m) \quad \downarrow F(\bigcirc B) \\
A \\
\uparrow \alpha \\
FA
\end{array}
\]

(8.12)

From Remark 2.4.13(3) and Lemma 8.3.7 we conclude that \( \bigcirc \) is a monotone operator on the poset of subcoalgebras of \((A, \alpha)\).

Recall from Definition 4.3.1 that a functor is called finitary if it preserves directed colimits.

**Proposition 8.3.14.** Suppose that \( \mathcal{A} \) has smooth monomorphisms and that \( F \) preserves monomorphisms and is finitary. Then the next time operator of a coalgebra \((A, \alpha)\) is continuous, i.e. it preserves joins of \( \omega \)-chains, whenever \( \overset{\alpha}{\rightarrow} : \text{Sub}(FA) \to \text{Sub}(A) \) preserves them.

**Proof.** For every set \( A \) we have a function

\[
\text{Sub}(A) \to \text{Sub}(FA) \quad (s : S \hookrightarrow A) \mapsto (Fs : FS \hookrightarrow FA)
\]

using that \( F \) preserves monomorphisms (see Assumption 8.2.1). This function is continuous because for a given \( \omega \)-chain in \( \text{Sub}(A) \)

\[
S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S
\]

its join \( s : S \hookrightarrow A \) is, equivalently, a colimit of the above \( \omega \)-chain \((t_i)\) (cf. Proposition 7.3.13). Since \( F \) preserves this colimit, we see that \( Fs \) is the join of all \( Fs_i : FS_i \hookrightarrow FS, i \in \mathbb{N} \).

Now note that the next time operator of a coalgebra \( \alpha : A \to FA \) is the composite of the function in (8.13) and \( \overset{\alpha}{\rightarrow} : \text{Sub}(FA) \to \text{Sub}(A) \). The latter is continuous by assumption.

**Remark 8.3.15.** (1) The result in Proposition 8.3.14 holds for every coalgebra for a finitary set functor. Indeed, we know from Example 6.1.19(1) that monomorphisms are smooth, and taking inverse images clearly commutes with unions of \( \omega \)-chains (of subobjects). The assumption that \( F \) preserves monomorphisms may be dropped by working with the Trnková hull (cf. Remark 8.1.15).

(2) Similarly, Proposition 8.3.14 holds for every coalgebra for a finitary functor on \( K \cdot \text{Vec} \), e.g. the functor \( FX = K \times X^F \) whose coalgebras are linear weighted automata.

(3) More generally, for readers familiar with these concepts we would like to point out that the result in Proposition 8.3.14 holds for every coalgebra for a finitary endofunctor preserving monomorphisms on a locally finitely presentable category (see Definition 10.1.6).
Examples of such categories are, besides Set, $K$-$\text{Vec}$, every finitary variety of algebras, and the categories of posets and graphs.

Indeed, finitary functors clearly preserve $\omega$-colimits, a locally finitely presentable category has smooth monomorphisms [43, Prop. 1.62], and that $\overline{\alpha}$ is continuous follows since joins are given by colimits (cf. Proposition 7.3.13) and the fact the colimit of $\omega$-chains are universal (cf. Example 8.5.13(5)).

We close this section with a characterization result: $F$ preserves intersections if and only if the following “generalized next time” operators are right adjoints. Given a morphism $f: A \to FB$, we have the operator $\bigcirc f: \text{Sub}(B) \to \text{Sub}(A)$ that maps $m: B' \hookrightarrow B$ to the pullback of $Fm$ along $f$:

\[
\begin{array}{ccc}
\bigcirc f A' & \xrightarrow{f(m)} & FB' \\
\downarrow \bigcirc fm & & \downarrow Fm \\
A & \xrightarrow{f} & FB
\end{array}
\]

**Proposition 8.3.16** [242]. An endofunctor $F$ preserves intersections if and only if every generalized next time operator $\bigcirc f$ is a right adjoint.

**Proof.** For the “if”-direction, choose $f = \text{id}_{FY}$. Then $\bigcirc \text{id}_{FY} : m \mapsto Fm$ is a right adjoint and so preserves all meets, i.e. $F$ preserves intersections.

The converse follows from the easily-established fact that intersections are stable under inverse image, i.e. for every morphism $f: X \to Y$ and every family $m_i: S_i \hookrightarrow Y$ of subobjects, the intersection $m: P \hookrightarrow X$ of the inverse images of the $m_i$ under $f$ yields a pullback

\[
\begin{array}{ccc}
P & \xrightarrow{m} & \bigcap S_i \\
\downarrow f & & \downarrow \bigcap m_i \\
X & \xrightarrow{f} & Y
\end{array}
\]

Hence, if $F$ preserves intersections, then so does every operator $\bigcirc f$. Equivalently, $\bigcirc f$ is a right adjoint. \qed

We will mention another equivalent characterization of intersection preservation in Remark 9.2.28.

### 8.4 The Well-Founded Part of a Coalgebra

We introduced well-founded coalgebras in Section 8.1. We now discuss the well-founded part of a coalgebra, i.e. its largest well-founded subcoalgebra. We shall prove that the well-founded part of a coalgebra always exists (Proposition 8.4.6) and is the least fixed point of the next time operator. We will also see that a coalgebra is well-founded iff it is its own well-founded part. Then we prove that the well-founded part is the coreflection of a coalgebra in the category of well-founded coalgebras (Proposition 8.4.10).
8 Well-Founded Coalgebras

Definition 8.4.1. The well-founded part of a coalgebra is its largest well-founded subcoalgebra.

Example 8.4.2. The well-founded part of a graph (as a coalgebra for $\mathcal{P}$) is the set of points which do not lie on any infinite path.

Construction 8.4.3. Let $\alpha : A \to FA$ be a coalgebra. We know that $\text{Sub}(A)$ is a complete lattice and that the next time operator $\bigcirc$ is monotone (see Lemma 8.3.7). Hence, by Theorem 6.1.1, $\bigcirc$ has a least fixed point, which we denote by $a^* : A^* \to A$.

Moreover, by Lemma 8.3.12(2), we know that there is a coalgebra structure $\alpha^* : A^* \to FA^*$ so that $a^* : (A^*, \alpha^*) \to (A, \alpha)$ is the smallest cartesian subcoalgebra of $(A, \alpha)$.

Remark 8.4.4. (1) From the proof of Theorem 6.1.1 we know how to construct $a^*$, the least fixed point of $\bigcirc$, as the join of the following transfinite chain of subobjects $a_i : A_i \to A$, $i \in \text{Ord}$. First, put $a_0 = \bot_A$, the least subobject of $A$. Given $a_i : A_i \to A$, put $a_{i+1} = \bigcirc a_i : A_{i+1} = \bigcirc A_i \to A$. For every limit ordinal $j$, put $a_j = \bigvee_{i<j} a_i$. There exists an ordinal $i$ such that $a_i = a^* : A^* \to A$.

(2) Under the conditions in Proposition 8.3.14 the least ordinal $i$ with $a^* = a_i$ is at most $\omega$. Indeed, since $\bigcirc$ is continuous we have $a^* = \bigvee_{i\in\mathbb{N}} a_i$ by Kleene’s fixed point theorem.

In particular, by Remark 8.3.15, this holds for every coalgebra for a finitary functor on $\text{Set}$ or $\text{K-Vec}$, and more generally, for any finitary functor on a locally finitely presentable category (see Definition 10.1.6).

Example 8.4.5. For a graph $\alpha : A \to \mathcal{P}A$ we have $a_0 : \emptyset \to A$, $a_1 : A_1 \to A$ is the set of nodes with no successors, $a_2 : A_2 \to A$ contains the nodes from $A_1$ plus those nodes all of whose successors have no successors, etc. Hence, $a_i : A_i \to A$ is the sets of nodes $x$ such that every path from $x$ has length at most $i$. The union $a^* : A^* \to A$ is the well-founded part.

Proposition 8.4.6. For every coalgebra $(A, \alpha)$, the coalgebra $(A^*, \alpha^*)$ is well-founded.

Proof. Let $m : (B, \beta) \to (A^*, \alpha^*)$ be a cartesian subcoalgebra. By Lemma 8.3.12, $a^* \cdot m : B \to A$ is a fixed point of $\bigcirc$. Since $a^*$ is the least fixed point, we have $a^* \leq a^* \cdot m$, i.e. $a^* = a^* \cdot m \cdot x$ for some $x : A^* \to B$. Since $a^*$ is monic, we thus obtain $m \cdot x = \text{id}_{A^*}$. So $m$ is a monomorphism and a split epimorphism, whence an isomorphism. 

Example 8.4.7. Consider the coalgebra $G$ for $\mathcal{P}$ depicted as the following graph:

```
\begin{array}{c}
a \rightarrow b \\
\text{ } \\
\text{ } \\
\text{ } \\
d \leftarrow c
\end{array}
```

We list all subcoalgebras below (the structures are the obvious ones given by the picture of $G$). Those are $\emptyset$, $\{b\}$, $\{a, b\}$, $\{c, d\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$. Of these, the cartesian subcoalgebras of $G$ are $\{a, b\}$, and $\{a, b, c, d\}$. The well-founded part of $G$ is the least cartesian subcoalgebra, namely $\{a, b\}$.

Let us come back to a loose end from Example 8.1.4(4):
Proposition 8.4.8. A linear weighted automaton is well-founded iff every path in its state transition graph eventually leads to 0.

Proof. Given a linear weighted automaton, i.e. a coalgebra \((A, \alpha)\) for \(FX = K \times X^\Sigma\) on \(K\text{-Vec}\), denote by \(a^* : A^* \to A\) the subset of all states with the property in the statement of the proposition. Clearly, \(A^*\) is a subspace of \(A\). Furthermore, we obtain from Remark 8.4.4(2) that the least fixed point of \(\bigcirc\) is \(\bigvee_{n \in \mathbb{N}} \bigcirc^n(\bot_A)\). We also know that \(\bot_A\) is the 0-subspace, and for every subspace \(s : S \to A\), \(\bigcirc s\) is the space of all nodes whose successors are in \(S\) (see Example 8.3.6(3)). Therefore \(\bigcirc^n(\bot_A)\) consists of precisely those states from which every path reaches 0 in at most \(n\) steps. Since \(A^* = \bigvee_{n \in \mathbb{N}} \bigcirc^n(\bot_A)\), and since \((A, \alpha)\) is well-founded iff \(A = A^*\), our result follows. \(\square\)

For the next result cf. Taylor [229, Exercise VI.17] or Adámek et al. [28, Corollary 2.19]; the formulation (3) is Taylor’s definition of a well-founded coalgebra [229, Def. 6.3.2]:

Proposition 8.4.9. For every coalgebra \((A, \alpha)\), the following are equivalent:

1. \((A, \alpha)\) is well-founded,
2. \(\text{id}_A\) is the only fixed point of \(\bigcirc\),
3. \(\text{id}_A\) is the only pre-fixed point of \(\bigcirc\).

Proof. Since \(\bigcirc\) is monotone, (2) and (3) state the same fact, see Remark 6.1.3.

1 \((\Rightarrow)\) 2. Let \((A, \alpha)\) be well-founded. Since \(A\) is a cartesian subcoalgebra of itself via \(\text{id}_A\), we have \(\text{id}_A = \bigcirc \text{id}_A\) by Lemma 8.3.12. Consider an arbitrary subobject \(m : A' \to A\) such that \(m = \bigcirc m\). By Lemma 8.3.12, there is a coalgebra structure \(\alpha : A' \to FA'\) giving a cartesian subcoalgebra. By well-foundedness, \(m\) is an isomorphism. Thus it represents the same subobject as \(\text{id}_A\).

2 \((\Rightarrow)\) 1. We know that the least fixed point \(a^*\) in Construction 8.4.3 is \(\text{id}_A\). Hence, \((A, \alpha)\) is isomorphic to \((A^*, \alpha^*)\) and therefore well-founded by Proposition 8.4.6. \(\square\)

We know from Proposition 8.4.6 that for every coalgebra \((A, \alpha)\) its subcoalgebra represented by \(a^* : A^* \to A\) is well-founded. We now prove that, categorically, this subcoalgebra is characterized uniquely up to isomorphism by the following universal property: every homomorphism from a well-founded coalgebra into \((A, \alpha)\) factorizes uniquely through \(a^*\). In particular, this implies that \(a^* : A^* \to A\) is the largest well-founded subcoalgebra of \(A\), viz. the well-founded part of \(A\).

Proposition 8.4.10. The full subcategory of \(\text{Coalg}\ F\) given by well-founded coalgebras is coreflective. In fact, the well-founded coreflection of a coalgebra \((A, \alpha)\) is its well-founded part \((A^*, \alpha^*)\).

Proof. We are to prove that for every coalgebra homomorphism \(f : (B, \beta) \to (A, \alpha)\), where \((B, \beta)\) is well-founded, there exists a coalgebra homomorphism \(f^\sharp : (B, \beta) \to (A^*, \alpha^*)\) such that \(a^* \cdot f^\sharp = f\). It is unique since \(a^* : A^* \to A\) is a monomorphism. It then follows that \(a^* : (A^*, \alpha^*) \to (A, \alpha)\) is the largest well-founded subcoalgebra.
For the existence of \( f^\sharp \), we first observe that \( f^\rho(a^\ast) \) is a pre-fixed point of \( \oplus_\beta \): indeed, using Lemma 8.3.10 we get

\[
\oplus_\beta(f^\rho(a^\ast)) \leq f^\rho(\oplus_\alpha(a^\ast)) = f^\rho(a^\ast).
\]

By Proposition 8.4.9, we therefore have \( id_B = b^\ast \leq f^\rho(a^\ast) \) in \( \text{Sub}(B) \). Using the adjunction in Proposition 8.3.4, we obtain \( f^\rho(id_B) \leq a^\ast \) in \( \text{Sub}(A) \). Now let

\[
f = (B \xrightarrow{e} C \xrightarrow{m} A)
\]

be the factorization of \( f \) as in Remark 8.2.2(3). This implies that \( f^\rho(id_B) = m \). Thus we obtain

\[
m = f^\rho(id_B) \leq a^\ast,
\]

i.e. there exists a morphism \( h: C \rightarrow A^\ast \) such that \( a^\ast \cdot h = m \). Thus, \( f^\sharp = h \cdot e: B \rightarrow A^\ast \) is a morphism satisfying

\[
a^\ast \cdot f^\sharp = a^\ast \cdot h \cdot e = m \cdot e = f.
\]

It follows that \( f^\sharp \) is a coalgebra homomorphism from \((B, \beta)\) to \((A^\ast, \alpha^\ast)\) since \( f \) and \( a^\ast \) are and \( F \) preserves monomorphisms (cf. Remark 8.2.5(1)).

### 8.5 Closure Properties of Well-Founded Coalgebras

In this section we will see that strong quotients and subcoalgebras of well-founded coalgebras are well-founded. For subcoalgebras we need to assume more about \( \mathcal{A} \) and \( F \) than what we mentioned in Assumption 8.2.1. We present two variants in Proposition 8.5.7 and Corollary 8.5.15.

**Remark 8.5.1.** Recall from Remark 4.2.4 that quotient coalgebras of \((A, \alpha)\) are represented by homomorphisms with domain \((A, \alpha)\) carried by epimorphisms in \( \mathcal{A} \). In this chapter we will consider a stronger notion of quotient coalgebra.

**Definition 8.5.2.** By a strong quotient of a coalgebra \((A, \alpha)\) is meant a quotient coalgebra of \((A, \alpha)\) represented by a strong epimorphism in \( \mathcal{A} \) (see Remark 8.2.2(1)).

Later, in Section 9.3, we will make use of the following corollary to Proposition 8.4.10. For endofunctors on sets preserving inverse images this was stated by Taylor [229, Exercise VI.16]:

**Corollary 8.5.3.** The subcategory of \( \text{Coalg} F \) formed by all well-founded coalgebras is closed under strong quotients and coproducts in \( \text{Coalg} F \).

This follows from a general result on coreflective subcategories: the category \( \text{Coalg} F \) has the factorization system of Proposition 8.2.3 (cf. Remark 8.2.5(1)), and its full subcategory of well-founded coalgebras is coreflective with monic coreflections (see Proposition 8.4.10). Consequently, it is closed under strong quotients and colimits.
Remark 8.5.4. Let \( \mathcal{A} \) have a simple initial object and colimits of chains. If \( F \) preserves finite intersections, then every coalgebra \( W_i \to FW_i, i \in \text{Ord} \), in the initial-algebra-chain (see Definition 6.1.4) is well-founded. This is shown by transfinite induction. The base case and isolated step are shown as in Remark 8.1.6(2) using Lemma 8.1.5(1), and for the limit step one uses that well-founded coalgebras are closed under colimits.

Remark 8.5.5. We prove next that, for an endofunctor preserving finite intersections, well-founded coalgebras are closed under subcoalgebras provided that \( \text{Sub}(A) \) forms a frame. Recall that this means that for every subobject \( m: B \to A \) and every family \( m_i (i \in I) \) of subobjects of \( A \) we have

\[
m \wedge \bigvee_{i \in I} m_i = \bigvee_{i \in I} (m \wedge m_i).
\]

Equivalently, the inverse image operator \( \mathcal{m}: \text{Sub}(A) \to \text{Sub}(B) \) from Notation 8.3.2 has a right adjoint \( \mathcal{m}_*: \text{Sub}(B) \to \text{Sub}(A) \) (use the dual of Remark 8.3.3).

Examples 8.5.6. (1) Set has the property that all \( \text{Sub}(A) \) are frames. In fact, given subsets \( S \) and \( S_i (i \in I) \) of \( A \) the equality \( S \cap (\bigcup_{i \in I} S_i) = \bigcup_{i \in I} (S \cap S_i) \) clearly holds.

(2) This property is shared by categories such as posets and monotone maps, graphs and homomorphisms, unary algebras and homomorphisms, and presheaf categories \( \text{Set}^{\mathcal{C}^{\text{op}}} \), with \( \mathcal{C} \) small. This follows from the fact that joins and meets of subobjects of an object \( A \) are formed on the level of subsets of the underlying set of \( A \).

(3) The category \( \text{CPO} \) does not have the above property: for the cpo \( A = \mathbb{N}^\top \) of natural numbers with a top element \( \top \) (linearly ordered) the lattice \( \text{Sub}(A) \) is not a frame. Consider the subobjects given by inclusion maps \( m_i: \{0,\ldots,i\} \to \mathbb{N}^\top \) for \( i \in \mathbb{N} \), with domains linearly ordered. It is easy to see that \( \bigvee_{i \in \mathbb{N}} m_i = \text{id}_A \). For the inclusion map \( m: \{\top\} \to \mathbb{N}^\top \) we have \( m \wedge m_i = 0 (i \in I) \), the empty subobject. Thus, \( \bigvee_{i \in \mathbb{N}} (m \wedge m_i) = 0 \neq m = m \wedge \bigvee_{i \in I} m_i \).

(4) For every Grothendieck topos, the posets \( \text{Sub}(A) \) are frames. In fact, it is sufficient for a topos to have all coproducts or intersections to satisfy this requirement.

Proposition 8.5.7. Suppose that \( F \) preserves finite intersections, and let \( (A, \alpha) \) be a well-founded coalgebra such that \( \text{Sub}(A) \) is a frame. Then every subcoalgebra of \( (A, \alpha) \) is well-founded.

Proof. Let \( m: (B, \beta) \to (A, \alpha) \) be a subcoalgebra. We will show that the only pre-fixed point of \( \bigcup_\beta \) is \( \text{id}_B \) (cf. Proposition 8.4.9). Suppose \( s: S \to B \) fulfils \( \bigcup_\beta(s) \leq s \). Since \( F \) preserves finite intersections, we have

\[
\mathcal{m} \cdot \bigcup_\alpha = \bigcup_\beta \cdot \mathcal{m}.
\]

by Corollary 8.3.11(1). The counit of the adjunction \( \mathcal{m} \dashv m_* \) yields \( \mathcal{m}(m_*(s)) \leq s \), so that we obtain

\[
\mathcal{m}(\bigcup_\alpha(m_*(s))) = \bigcup_\beta(\mathcal{m}(m_*(s))) \leq \bigcup_\beta(s) \leq s.
\]

Using again the adjunction \( \mathcal{m} \dashv m_* \), we have equivalently that \( \bigcup_\alpha(m_*(s)) \leq m_*(s) \), i.e. \( m_*(s) \) is a pre-fixed point of \( \bigcup_\alpha \). Since \( (A, \alpha) \) is well-founded, Corollary 8.3.11(1)
implies that $m_*(s) = \text{id}_A$. Since $\hat{m}$ is also a right adjoint (Proposition 8.3.4) and therefore preserves the top element of $\text{Sub}(B)$, we thus obtain

$$\text{id}_B = \hat{m}(\text{id}_A) = \hat{m}(m_*(s)) \leq s,$$

which completes the proof. \(\square\)

**Remark 8.5.8.** For a set functor $F$ preserving inverse images, a much better result was proved by Taylor [229, Cor. 6.3.6]: for every coalgebra homomorphism $f: (B, \beta) \to (A, \alpha)$ with $(A, \alpha)$ well-founded so is $(B, \beta)$. In fact, our proof above is essentially Taylor’s who (implicitly) uses Corollary 8.3.11(2) instead.

**Corollary 8.5.9.** If a set functor preserves finite intersections, then subcoalgebras of well-founded coalgebras are well-founded.

We know from Trnková’s result (Proposition 4.4.1) that every set functor preserves all nonempty finite intersections. However, this does not suffice for Corollary 8.5.9, as we saw in Example 8.1.4(5).

The fact that subcoalgebras of a well-founded coalgebra are well-founded does not necessarily need the assumption that $\text{Sub}(A)$ is a frame. Using the construction of the least fixed point $a^*$ of $\bigcup$ provided by the (proof of the) Knaster-Tarski fixed point theorem, it is essentially sufficient that $\hat{m}$ in the proof of Proposition 8.5.7 preserves joins of unions of chains in $\text{Sub}(A)$. We now discuss this more in detail.

Recall the notion of smooth monomorphisms from Definition 6.1.17.

**Definition 8.5.10.** Let $\mathcal{A}$ be a category with pullbacks. We say that $\mathcal{A}$ has universally smooth monomorphisms if it has smooth monomorphisms, and colimits of chains of monomorphisms are universal, i.e. for every morphism $f: X \to Y$, the functor $\mathcal{A}/Y \to \mathcal{A}/X$ forming pullbacks along $f$ preserves those colimits.

**Remark 8.5.11.** The property of having universally smooth monomorphisms is easily seen to be equivalent to the following property: Given a chain $(A_i)_{i < k}$ of monomorphisms in $\mathcal{A}$ with colimit cocone $a_i: A_i \to A$ and a morphism $f: X \to A$, form the pullback of $a_i$ along $f$:

$$X_i \xleftarrow{f_i} A_i \xrightarrow{a_i} A$$

for every $i < k$.

Then $(X_i)_{i < k}$ forms a chain, using the universal property of pullbacks, and the $x_i: X_i \to X$ form its colimit cocone.

**Remark 8.5.12.** Let $\mathcal{A}$ have universally smooth monomorphisms.

(1) The initial object $0$ is strict (see Remark 8.2.6(2)). Indeed, consider the empty chain in Definition 8.5.10.

(2) For every morphism $f: A \to B$, the operator $\hat{f}: \text{Sub}(B) \to \text{Sub}(A)$ preserves unions of chains. Indeed, this follows from universal smoothness using that monomorphisms are
stable under pullback and that unions of chains are given by the corresponding colimits (cf. Proposition 7.3.13).

**Example 8.5.13.** (1) Set has universally smooth monomorphisms.

(2) K-Vec has smooth monomorphisms, but not universally so because the initial object is not strict (cf. Remark 8.6.2(1)).

(3) CPO does not have universally smooth monomorphisms. In fact, they are not even smooth (see Example 6.1.19(4)).

(4) Categories in which colimits of chains and pullbacks are formed “set-like” have universally smooth monomorphisms. These include the categories of posets, graphs, topological spaces, presheaf categories, and many varieties, such as monoids, graphs, and unary algebras.

(5) Every locally finitely presentable category $\mathcal{A}$ with a strict initial object (see Remark 8.5.12) has universally smooth monomorphisms. To see this use [43, Prop. 1.62] and the fact that filtered colimits (whence colimits of nonempty chains) are universal as we now prove. Indeed, suppose that $c_i: C_i \to C$, $i \in I$ is a filtered colimit, and let $f: B \to C$ be a morphism. Form the pullback of every $c_i$ along $f$:

$$
\begin{array}{ccc}
B_i & \xrightarrow{f_i} & C_i \\
\downarrow{b_i} & & \downarrow{c_i} \\
B & \xrightarrow{f} & C
\end{array}
$$

Then $b_i: B_i \to B$ is a colimit cocone. Indeed, in the category of commutative squares in $\mathcal{A}$, the filtered diagram formed by the above pullbacks squares has as a colimit a pullback square

$$
\begin{array}{ccc}
\colim B_i & \xrightarrow{\colim f_i} & \colim C_i = C \\
\downarrow{\cong} & & \downarrow{c_i} \\
B & \xrightarrow{f} & C
\end{array}
$$

Unfortunately, the example of rings demonstrates that the assumption of strictness of 0 cannot be lifted. In fact, the collections of monomorphisms is not smooth in this category (see Example 6.1.19(6)).

Our proof in Theorem 8.7.1 of the converse of the General Recursion Theorem hinges on Proposition 8.5.14, which shows that well-foundedness is reflected by coalgebra homomorphisms. For categories in which $\text{Sub}(A)$ is a frame this has been proved by Taylor [229, Cor. 6.3.6] and [228, Prop. 7.3].

Our proof makes use of the induction principle which we saw in Corollary 7.3.9.

**Proposition 8.5.14.** Let $\mathcal{A}$ be a complete and well-powered category with universally smooth monomorphisms, and suppose that $F: \mathcal{A} \to \mathcal{A}$ preserves inverse images. Then for every coalgebra homomorphism $f: (B, \beta) \to (A, \alpha)$ with $(A, \alpha)$ well-founded, $(B, \beta)$ is well-founded, too.
8 Well-Founded Coalgebras

Proof. Let $s: S \twoheadrightarrow B$ be a pre-fixed point of $\bigcirc_\beta$, i.e. $\bigcirc_\beta(s) \leq s$. It is our task to prove that $id_B \leq s$ (see Proposition 8.4.9(3)).

We apply the induction principle in Corollary 7.3.9 to the subset of $\text{Sub}(A)$ given by

$$S = \{t: T \twoheadrightarrow A \mid \text{f}(t) \leq s\}.$$  

We observe that $\bigcirc_\alpha$ preserves $S$: given $t: T \twoheadrightarrow A$ in $S$ we have

$$\text{f}(\bigcirc_\alpha(t)) = \bigcirc_\beta(\text{f}(t)) \leq (s) \leq s,$$

where the left-hand equation holds by Corollary 8.3.11(2). In addition, $S$ is closed of chains. For suppose that $(t_i: T_i \twoheadrightarrow A), i \in I$, is a chain in $S$. Then we have

$$\text{f}(\bigvee_{i \in I} t_i) = \bigvee_{i \in I} \text{f}(t_i) \leq \bigvee_{i \in I} s = s,$$

where the first equation holds due to Remark 8.5.12(2). Thus, by Corollary 7.3.9 we obtain that the least fixed point of $\bigcirc_\alpha$ lies in $S$, in symbols $id_A \in S$, since $(A, \alpha)$ is well-founded. Consequently, we obtain

$$id_B = \text{f}(id_A) \leq s,$$

which completes the proof. □

Corollary 8.5.15. Suppose that $\mathcal{A}$ satisfies the assumptions in Proposition 8.5.14 and that $F$ preserves finite intersections. Then every subcoalgebra of a well-founded coalgebra is well-founded.

Indeed, for a monomorphism $f: (B, \beta) \twoheadrightarrow (A, \alpha)$ we can use Corollary 8.3.11(1) in the above proof of Proposition 8.5.14.

We close this section with a result concerning subcoalgebras of the coalgebras in the initial-algebra chain (see Definition 6.1.4).

Example 8.5.16. Suppose that $\mathcal{A}$ has colimits of chains, has a simple initial object, $F: \mathcal{A} \rightarrow \mathcal{A}$ preserves finite intersections, and $\text{Sub}(W_i)$ is a frame for all objects $W_i$ in the initial-algebra chain. Then by Remark 8.5.4 and Proposition 8.5.7 we have that every subcoalgebra of $W_i \rightarrow FW_i$, $i \in \text{Ord}$, is well-founded. In lieu of all $\text{Sub}(W_i)$ being frames and 0 being simple, one may also assume that $\mathcal{A}$ is complete and well-powered with universally smooth monomorphisms (cf. Corollary 8.5.15).

8.6 The General Recursion Theorem

The main consequence of well-foundedness is parametric recursivity. This is Taylor’s General Recursion Theorem [229, Theorem 6.3.13]. Taylor assumed that $F$ preserves inverse images. We present a new proof for which it is sufficient that $F$ preserves monomorphisms, assuming those are smooth. In the next section, we discuss the converse of the General Recursion Theorem in Theorems 8.7.1 and 8.7.8.
Assumption 8.6.1. For the rest of this section, we assume that in our complete and well-powered category $\mathcal{A}$ the class of all monomorphisms is smooth (see Definition 6.1.17).

Remark 8.6.2. (1) In this setting, the bottom element of each poset $\text{Sub}(A)$ is $\perp_A: 0 \to A$, where 0 is the initial object of $\mathcal{A}$ (cf. Remark 6.1.18(1)).

(2) Furthermore, a join of a chain in $\text{Sub}(A)$ is obtained by forming the colimit of the corresponding chain (Proposition 7.3.13).

Theorem 8.6.3. Let $\mathcal{A}$ be a complete and well-powered category with smooth monomorphisms. For $F: \mathcal{A} \to \mathcal{A}$ preserving monomorphisms, every well-founded coalgebra is parametrically recursive.

Proof. (1) Let $(A, \alpha)$ be well-founded. We first prove that it is recursive. Recall from Remark 2.4.13(3) that the subcoalgebras of $(A, \alpha)$ form a subposet of $\text{Sub}(A)$. Let $S$ be the subset of the poset $\text{Sub}(A)$ given by the recursive subcoalgebras of $(A, \alpha)$. It follows from Example 7.3.2(5) and Remark 8.6.2(2) that $S$ is closed under joins of chains in $\text{Sub}(A)$. We prove below that the next time operator $\circlearrowleft$ preserves $S$. Using Corollary 7.3.9, we conclude that the least fixed point of $\circlearrowleft$ is contained in $S$, and since $(A, \alpha)$ is well-founded, this fixed point is $\text{id}_A$ (see Proposition 8.4.9). In other words, $(A, \alpha)$ is recursive.

To complete the proof, we now show that for every recursive subcoalgebra $(B, \beta)$ the subcoalgebra on $\circlearrowleft B$ in Remark 8.3.13 is recursive, too.

First, we know from Example 7.3.2(4) that $(F B, F \beta)$ is recursive. Given an algebra $e: F X \to X$ we have the coalgebra-to-algebra morphism $h$ so that the lower square of the following diagram commutes:

Moreover, in the upper square the left-hand and middle triangles commute trivially, and the right-hand triangle does by the definition of $x$ (see Remark 8.3.13). Thus $h \cdot \alpha(m)$ is a coalgebra-to-algebra morphism.

For the uniqueness suppose that $g: \circlearrowleft B \to X$ is a coalgebra-to-algebra morphism:

\begin{equation}
\begin{array}{ccc}
\circlearrowleft B & \xrightarrow{\alpha(m)} & FB \\
\downarrow{\alpha(m)} & & \downarrow{F \beta} \\
FB & \xrightarrow{F \beta} & FFB \\
\downarrow{h} & & \downarrow{F h} \\
X & \xleftarrow{e} & FX
\end{array}
\end{equation}

(8.14)
Then \( e \cdot Fg \cdot Fx \) is a coalgebra-to-algebra morphism from \((FB,F\beta)\) to \((X,e)\):

\[
\begin{array}{ccc}
FB & \xrightarrow{F\beta} & FFB \\
Fx \downarrow & & \downarrow FFX \\
F(\circ B) & \xrightarrow{F(\circ m)} & FFB \\
Fg \downarrow & & \downarrow FFg \\
FX & \xrightarrow{Fe} & FFX \\
\downarrow e & & \downarrow Fe \\
X & \xrightarrow{e} & FX
\end{array}
\]

The left-hand triangle commutes by the definition of \( x \), the middle and right-hand triangles are trivial, the middle square is the \( F \)-image of (8.14), and the lower part is trivial again.

Thus, we conclude that \( h = e \cdot Fg \cdot Fx \) and therefore

\[
g = e \cdot Fg \cdot Fx \cdot \alpha(m) = h \cdot \alpha(m),
\]

as desired.

(2) Finally, we prove that the coalgebra \((A,\alpha)\) is parametrically recursive.

Consider the coalgebra \( \langle \alpha, \text{id}_A \rangle : A \to FA \times A \) for \( F(-) \times A \). This functor preserves monomorphisms since \( F \) does and monomorphisms are closed under products. The next time operator \( \circ \) on \( \text{Sub}(A) \) is the same for both coalgebras, since the square (8.7) is a pullback if and only if the square below is one:

\[
\begin{array}{ccc}
\circ A' & \xrightarrow{(\alpha(m), \circ(m))} & FA' \times A \\
\circ m \downarrow & & \downarrow Fm \times A \\
A & \xrightarrow{(\alpha, A)} & FA \times A
\end{array}
\]

Since \( \text{id}_A \) is the unique fixed point of \( \circ \) w.r.t. \( F \) (see Proposition 8.4.9), it is also the unique fixed point of \( \circ \) w.r.t. \( F(-) \times A \). Thus \((A, \langle \alpha, A \rangle)\) is a well-founded coalgebra for \( F(-) \times A \). By point (1), it is thus recursive for \( F(-) \times A \). This states equivalently that \((A, \alpha)\) is a parametrically recursive coalgebra for \( F \).

\[ \square \]

**Remark 8.6.4.** In the category \( K-\text{Vec} \) of vector spaces over the field \( K \) and linear maps every monomorphism splits. Given a subspace \( s : V \rightarrow W \), choose a base \((v_i)_{i \in I}\) of \( V \) and extend it to a basis of \( W \) using the vectors \((w_j)_{j \in J}\). The linear map \( e : W \rightarrow V \) determined by \( e(v_i) = v_i \) and \( e(w_j) = 0 \) fulfills \( e \cdot s = \text{id}_V \).

Analogously, all epimorphisms split in \( K-\text{Vec} \). Given a surjective linear map \( e : W \rightarrow V \), let \((v_i)_{i \in I}\) be a base of \( V \) and choose, for every \( i \in I \), an element \( w_i \in W \) with \( e(w_i) = v_i \). The linear map \( s : V \rightarrow W \) determined by \( s(v_i) = w_i \) for \( i \in I \) fulfills \( e \cdot s = \text{id}_V \).

**Corollary 8.6.5.** For every endofunctor on \( \text{Set} \) or \( K-\text{Vec} \), every well-founded coalgebra is parametrically recursive.
8.6 The General Recursion Theorem

Proof. For Set, we apply Theorem 8.6.3 to the Trnková hull $\bar{F}$ (see Proposition 8.1.12), noting that $F$ and $\bar{F}$ have the same (nonempty) coalgebras. By Lemma 8.1.22 the desired result follows. For $K$-Vec, use that monomorphisms split and are therefore preserved by every endofunctor $F$.

Example 8.6.6. For the set functor $FX = X \times X + 1$ the coalgebra $(\mathbb{N}, \gamma)$ from Example 8.1.21(3) is well-founded. Hence it is parametrically recursive.

Similarly, we saw that for $FX = A \times X \times X + 1$ the coalgebra $(A, s)$ from Example 7.3.8(3) is well-founded, and therefore it is (parametrically) recursive.

Example 8.6.7. Well-founded coalgebras need not be recursive when $F$ does not preserve monomorphisms. We take $\mathcal{A}$ to be the category $\text{Pred}$ of sets with a predicate. Objects are pairs $(X, A)$, where $A \subseteq X$. Morphisms $f: (X, A) \to (Y, B)$ are functions that satisfy $f[A] \subseteq B$. $\text{Pred}$ clearly fulfills the assumptions in Remark 8.6.2. Denote by $\mathbb{1}$ the terminal object $(1, 1)$. We define an endofunctor $F$ by $F(X, \emptyset) = (X + 1, \emptyset)$, and for $A \neq \emptyset$, $F(X, A) = \mathbb{1}$. For a morphism $f: (X, A) \to (Y, B)$, put $F = f + \text{id}$ if $A = \emptyset$; if $A \neq \emptyset$, then also $B \neq \emptyset$ and $Ff$ is $\text{id}: \mathbb{1} \to \mathbb{1}$.

The terminal coalgebra is $\text{id}: \mathbb{1} \to \mathbb{1}$, and it is easy to see that it is well-founded. But it is not recursive: there are no coalgebra-to-algebra morphisms into an algebra of the form $F(X, \emptyset) = (X, \emptyset)$.

We know that the terminal recursive coalgebra is precisely the initial algebra (Theorem 7.1.7). Under our current assumptions the same holds for the terminal well-founded coalgebra:

Corollary 8.6.8. The terminal well-founded coalgebra is precisely the initial algebra.

Proof. If $(A, \alpha)$ is a terminal well-founded coalgebra, then $\alpha$ is an isomorphism: this is proved precisely as Lambek’s Lemma 2.2.5, using Lemma 8.1.5(1).

Conversely, if $(A, \alpha^{-1})$ is an initial algebra, then $(A, \alpha)$ is well-founded (Example 8.1.4(1)). By Theorem 7.1.7, $(A, \alpha)$ is a terminal recursive coalgebra. Hence, Theorem 8.6.3 implies that it is a terminal well-founded coalgebra, for if $(B, \beta)$ is any well-founded coalgebra it is recursive, whence there exists a unique coalgebra homomorphism from $(B, \beta)$ to $(A, \alpha)$.

For $\mathcal{A} = \text{Set}$ the previous result even holds for all endofunctors (even those not preserving monomorphisms):

Remark 8.6.9 [28, Thm. 2.46]. For every set functor, a terminal well-founded coalgebra is precisely an initial algebra.

Indeed, combine Corollaries 8.6.5 and Corollaries 8.6.8.

The fact that no assumptions on $F$ are needed seems very special to Set. Without any assumptions on $F$, Remark 8.6.9 does not even generalize from Set to the category $\text{Gra}$ or other presheaf categories, as the following example shows.

Example 8.6.10. Let $\text{Gra}$ be the category of graphs, i.e. the category of presheaves over the category $\{\bullet \Rightarrow \bullet\}$ given by two parallel morphisms. Here is a simple endofunctor $F$ whose initial algebra is infinite and whose terminal well-founded coalgebra is a singleton.

231
graph: On objects $A$ put $FA = 1$ (the terminal graph) if $A$ has edges. For a graph $A$ without edges, let $FA$ be the graph without edges whose vertices are those of $A$ plus an additional vertex. The definition of $F$ on morphisms $h: A \to B$ is as expected: $Fh$ maps the additional vertex of $A$ to that of $B$ in the case where $B$ has no edges. Then $\mu F$ is the graph of natural numbers without edges. However, the terminal well-founded coalgebra is $F1 \bisim 1$.

We close this section with a general fact on well-founded parts of fixed points of functors:

**Theorem 8.6.11.** Let $\mathcal{A}$ be a complete and well-powered category with smooth monomorphisms. For $F$ preserving monomorphisms, the well-founded part of every fixed point is the initial algebra. In particular, the only well-founded fixed point is the initial algebra.

**Proof.** Let $\alpha: A \to FA$ be a fixed point of $F$. By Theorem 6.1.22 we know that the initial algebra $(\mu F, \iota)$ exists. Now let $\alpha^*: (A^*, \alpha^*) \to (A, \alpha)$ be the well-founded part of $A$ given in Proposition 8.4.6. This is a cartesian subcoalgebra, i.e. we have a pullback square

$$
\begin{array}{ccc}
A^* & \xrightarrow{\alpha^*} & FA^* \\
\alpha & \downarrow & \downarrow F\alpha^* \\
A & \xrightarrow{\alpha} & FA
\end{array}
$$

Since $\alpha$ is an isomorphism, so is $\alpha^*$. By Theorem 8.6.3, $(A^*, \alpha^*)$ is recursive. Finally, by the dual of Remark 7.1.8, $(A^*, \alpha^*)$ is the initial algebra. \qed

**Corollary 8.6.12.** If $F$ in Theorem 8.6.11 has a terminal coalgebra $\nu F$, it also has an initial algebra which is the well-founded part of $\nu F$.

**Remark 8.6.13.** For set functors, the assumption that $F$ preserves monomorphisms may be dropped in Theorem 8.6.11 and Corollary 8.6.12. For Theorem 8.6.11, we modify the proof above, replacing Theorem 6.1.22 by Corollary 6.1.28, and Theorem 8.6.3 by Corollary 8.6.5.

**Example 8.6.14.** We illustrate that for a set functor $F$, the well-founded part of the terminal coalgebra is the initial algebra.

1. Consider $FX = A \times X + 1$. The terminal coalgebra is the set $A^\infty \cup A^*$ of finite and infinite sequences from the set $A$. The initial algebra is $A^*$. It is easy to check that $A^*$ is the well-founded part of $A^\infty \cup A^*$.

2. For a polynomial set functor $H_\Sigma$ the final coalgebra $\nu H_\Sigma$ is formed by all $\Sigma$-trees (see Example 6.2.6). Its well-founded part is given by all well-founded $\Sigma$-trees (see Example 6.1.15(2)).

**Example 8.6.15.** When the functor $F$ does not preserve monomorphisms, the well-founded part of the terminal coalgebra need not be the initial algebra. Indeed, for the functor $F$ in Example 8.6.7, the initial algebra is carried by $(\mathbb{N}, \emptyset)$, which is not even a subcoalgebra of 1.
8.7 The Converse of the General Recursion Theorem

Our last topic is in this chapter is a converse to Theorem 8.6.3, under various hypotheses. This is based on Taylor [228, 229]. It also generalizes results by Adámek et al. [7] and Jeannin et al. [137]. In order to prove the implication

\[
\text{recursive} \implies \text{well-founded},
\]

one needs to assume more than preservation of finite intersections. In fact, we will assume that \( F \) preserves inverse images. But even this is not enough. We additionally assume that either

1. the underlying category \( \mathcal{A} \) has universally smooth monomorphisms and the endofunctor \( F \) has a pre-fixed point, i.e. an object \( A \) with a monomorphism \( \alpha: FA \to A \).
2. the underlying category \( \mathcal{A} \) has a subobject classifier.

The first of these possible assumptions leads to Theorem 8.7.1, the second is a theorem derived from Taylor’s preprint [228]. Finally, at the end of this section we prove the above converse implication for every functor on vector spaces preserving inverse images (see Theorem 8.7.13). This last result is not covered by the previous two, since \( K\text{-Vec} \) neither has universally constructive monomorphisms nor a subobject classifier. We have not found a unifying proof that would cover the cases we consider.

**Theorem 8.7.1.** Let \( \mathcal{A} \) be a complete and well-powered category with universally smooth monomorphisms, and suppose that \( F: \mathcal{A} \to \mathcal{A} \) preserves inverse images and has a pre-fixed point. Then every recursive \( F \)-coalgebra is well-founded.

**Proof.** By Theorem 6.1.22, we know that the initial algebra \( \mu F \) exists. Let \((A, \alpha)\) be a recursive coalgebra, then we have a unique coalgebra-to-algebra morphism \( f \) from \((A, \alpha)\) to \((\mu F, \iota)\), i.e. a unique coalgebra homomorphism \( f: (A, \alpha) \to (\mu F, \iota^{-1}) \). By Example 8.1.4(1), the coalgebra \((\mu F, \iota^{-1})\) is well-founded. Hence, so is \((A, \alpha)\) by Proposition 8.5.14.

**Corollary 8.7.2.** Let \( \mathcal{A} \) and \( F \) satisfy the assumptions of Theorem 8.7.1. Then the following properties of a coalgebra are equivalent:

1. well-foundedness,
2. parametric recursiveness,
3. recursiveness,
4. existence of a homomorphism into \((\mu F, \iota^{-1})\),
5. existence of a homomorphism into a well-founded coalgebra.

**Proof.** We already know \((1) \implies (2) \implies (3)\) by Theorem 8.6.3. We have also seen \((3) \implies (4) \implies (1)\) in the proof of Theorem 8.7.1. The implication \((4) \implies (5)\) holds since \((\mu F, \iota^{-1})\) is well-founded (see Example 8.1.4(1)). Finally, by Corollary 8.6.8, \((\mu F, \iota^{-1})\) is a terminal well-founded coalgebra. Thus, \((5) \implies (4)\), which completes the proof.
**Remark 8.7.3.** Observe that one can replace universally smooth monomorphisms in Corollary 8.7.2 by a universally smooth class $M$ of monomorphisms. Taking as $M$ the class of all strong monomorphisms, this yields examples such as posets, graphs, and topological spaces (where the strong monomorphisms represent subposets, subgraphs, and subspaces, respectively).

**Example 8.7.4.** (1) The category of many-sorted sets satisfies the assumptions of Theorem 8.7.1, and polynomial endofunctors on that category preserve inverse images. Thus, we obtain Jeannin et al.'s result [137, Thm. 3.3] that conditions (1)–(4) in Corollary 8.7.2 are equivalent as a special instance.

(2) We will see in Theorem 8.7.13 that for the category $K$-$Vec$ recursive coalgebras are well-founded whenever $F$ preserves inverse images. However, this is not a consequence of Corollary 8.7.2 since vector spaces fail to have universally smooth monomorphisms (see Example 8.5.13(2)). In contrast, the implication (4) ⇒ (3) in Corollary 8.7.2 does not even hold for the identity functor on $K$-$Vec$. To see this, note that we have $\mu \text{id} = (0, \text{id})$. Hence, every coalgebra has a homomorphism into $\mu \text{id}$. However, not every coalgebra is recursive, e.g. the coalgebra $(K, \text{id})$ admits many coalgebra-to-algebra morphisms to the algebra $(K, \text{id})$. Similarly, the implication (4) ⇒ (1) does not hold. In fact, a coalgebra $\alpha: A \to A$ is well-founded if and only if for every $x \in A$ there exists a natural number $n$ with $\alpha^n(x) = 0$ (cf. Example 8.1.4(4)). Clearly, not every coalgebra satisfies this property.

We mention in passing a consequence of the general recursion theorem which uses universally smooth monomorphisms but not preservation of inverse images.

**Corollary 8.7.5.** Let $\mathcal{A}$ satisfy the assumptions of Theorem 8.7.1, let $F: \mathcal{A} \to \mathcal{A}$ preserve intersections and have an initial algebra. Then every subcoalgebra of $\mu F$ is recursive.

Indeed, every subcoalgebra of $\mu F$ is well-founded by Proposition 8.5.7 and Corollary 8.5.15 and therefore recursive by Theorem 8.6.3.

In Theorem 8.7.1, we assumed that the endofunctor has a pre-fixed point. For set functors, this assumption may be lifted. Indeed, whenever a category has a subobject classifier, then every recursive coalgebra is well-founded, as shown by Taylor [228, Rem. 3.8]. We present this in all details, beginning with a recollection of the following definition from topos theory.

**Definition 8.7.6.** (1) A subobject classifier is an object $\Omega$ with a subobject $t: 1 \to \Omega$ such that for every subobject $b: B \to A$ there is a unique $\hat{b}: A \to \Omega$ such that the square below is a pullback:

$$
\begin{array}{ccc}
B & \to & 1 \\
\downarrow & & \downarrow 1 \\
B & \to & \Omega \\
\downarrow & & \downarrow \hat{t} \\
A & \to & \Omega
\end{array}
$$

(8.15)

**Examples 8.7.7.** (1) Set has a subobject classifier given by $\Omega = \{\bot, \top\}$ with the evident morphism $t: 1 = \{\top\} \to \Omega$. Indeed, subsets $b: B \to A$ are in one-to-one correspondence with characteristic maps $\hat{b}: B \to \{\bot, \top\}$. 

234
Every elementary topos has a subobject classifier; for example, every category $\text{Set}^\xi$ with $\xi$ small.

Our standing assumption that $\mathcal{A}$ is a complete and well-powered category is not needed for the next result: finite limits are sufficient. The following proof is based on Taylor’s work [228, Rem. 3.8].

**Theorem 8.7.8** [228]. Let $F$ be an endofunctor on a finitely complete category with a subobject classifier. If $F$ preserves inverse images, then every recursive coalgebra is well-founded.

**Proof.** Let $(A, \alpha)$ be a recursive coalgebra. Clearly, $\text{id}_A$ is a fixed point of $\varnothing$, and we prove below that it is the unique one. Thus, $(A, \alpha)$ is well-founded.

Let $b: B \to A$ be any fixed point of $\varnothing$. Consider the following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\alpha(b)} & FB \\
\downarrow b & & \downarrow Fb \\
A & \xrightarrow{\alpha} & FA
\end{array}
\quad
\begin{array}{ccc}
F1 & \xrightarrow{!} & 1 \\
\downarrow Ft & & \downarrow t \\
F\Omega & \xrightarrow{\hat{F}_t} & \Omega
\end{array}
\]

The square on the left is a pullback because $b = \varnothing b$. The middle square is $F$ applied to the pullback square (8.15). The square on the right is the pullback square (8.15) for $b$ chosen as $Ft: F1 \to F\Omega$. The upper composite morphism is $!: B \to 1$, and so the lower one is $\hat{b}$. Thus the outside rectangle is again a pullback. In particular,

$$\hat{b} = \hat{F}t \cdot \hat{F}b \cdot \alpha.$$ 

So $\hat{b}$ is a coalgebra-to-algebra morphism

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow \hat{b} & & \downarrow F\hat{b} \\
\Omega & \xrightarrow{\hat{F}_t} & F\Omega
\end{array}
\]

Since $(A, \alpha)$ is recursive, this means that $\hat{b}$ is uniquely determined from $\alpha$, independent of which fixed point $b$ of $\varnothing$ was used in our argument. Thus $\hat{b} = \text{id}_A$, which implies $b = \text{id}_A$, as desired.

**Corollary 8.7.9.** For every set functor preserving inverse images, recursive coalgebras are well-founded.

**Corollary 8.7.10.** Let $\mathcal{A}$ and $F$ satisfy the assumptions of Theorem 8.7.8. Then the following properties of a coalgebra are equivalent:

- well-foundedness $\iff$ recursiveness $\iff$ parametric recursiveness.
Example 8.7.11. The hypothesis in Theorems 8.7.1 and 8.7.8 that $F$ preserves inverse images cannot be lifted. In order to see this, we consider the functor $R: \text{Set} \to \text{Set}$ of Example 7.3.4. It preserves monomorphisms but not inverse images (see Example B.2.3(2)). The coalgebra $A = \{0, 1\}$ with the structure $\alpha$ constant to $(0, 1)$ is recursive: given an algebra $\beta: RB \to B$, the unique coalgebra-to-algebra homomorphism $h: \{0, 1\} \to B$ is given by

$$h(0) = h(1) = \beta(d).$$

But $A$ is not well-founded: $\emptyset$ is a cartesian subcoalgebra.

As the last result of this section we prove the implication “recursive $\Rightarrow$ well-founded” for endofunctors on the category $K\text{-Vec}$ preserving inverse images. This follows neither from either Theorem 8.7.1 (since monomorphism are not universally smooth in $K\text{-Vec}$) nor from Theorem 8.7.8 (since $K\text{-Vec}$ does not have a subobject classifier).

Remark 8.7.12. (1) For any vector space $X$, we denote by $z: 0 \to X$ the zero map. We also use this notation for composites $z: X \to 0 \to Y$.

(2) Recall that the kernel of a linear map $f: X \to Y$ is the equalizer of $f$ and the zero map $z: X \to 0 \to Y$ represented by the subspace $\ker f = \{x \in X : f(x) = 0\}$.

(3) A functor $F: K\text{-Vec} \to K\text{-Vec}$ preserves kernels if for every linear map $f: X \to Y$ its kernel $s: \ker f \hookrightarrow X$ is mapped to the kernel of $Ff$, shortly $Fs = \ker Ff$.

(4) Observe that for every linear map $f: X \to Y$ its kernel $s: \ker f \hookrightarrow X$ is the inverse image of the zero map $z: 0 \hookrightarrow Y$.

If $F: K\text{-Vec} \to K\text{-Vec}$ preserves inverse images and $F0 = 0$, then it preserves kernels. Indeed, $Fs$ is then the inverse image of $Fz$ under $Ff$, and $Fz: 0 = F0 \hookrightarrow FY$ is the zero map. Thus $Fs$ is the kernel of $Ff$ as desired.

(5) Conversely, if $F$ preserves kernels, then $F0 = 0$ (the terminal object) and $F$ preserves inverse images. In fact, $F$ preserves finite limits: by [105, Thm. 3.12], a functor preserving kernels is additive, and for an additive functor preservation of kernels is equivalent to preservation of finite limits (see [75, Prop. 1.11.2]).

(6) Every subspace $s: S \hookrightarrow X$ induces a quotient space $X/S$ with vectors $x + S$ for $x \in X$. We denote the corresponding canonical quotient map $x \mapsto x + S$ by $\coker s: S \to X/S$. Its kernel is $S$.

(7) Every linear map $f: X \to Y$ induces an isomorphism $X \cong \ker f + f[X]$, where $f[X]$ denotes the image of $f$ in $Y$.

(8) For a linear map $f: X \to Y$ and a subspace $s: S \hookrightarrow Y$ let $t: T = f^{-1}[S] \hookrightarrow Y$. Then by the universal property of $\coker t$, there exists a unique monomorphism $u: X/T \hookrightarrow Y/S$. 

236
such that the following diagram commutes:

\[
\begin{array}{ccccccc}
T & \longrightarrow & S \\
\downarrow t & & \downarrow s \\
X & \longrightarrow & Y \\
\coker t & \downarrow & \coker s \\
X/T & \overset{u}{\longrightarrow} & Y/S
\end{array}
\]

Moreover, we see that \( u \) is injective: if \( x + T \) satisfies \( u(x + T) = 0 \), i.e. \( f(x) + S = 0 \), then we have \( f(x) \in S \), thus \( x \in T \).

**Theorem 8.7.13.** Let \( F \) be an endofunctor on \( K\text{-Vec} \) preserving inverse images. Then every recursive coalgebra is well-founded.

**Proof.** Let \( \alpha: A \to FA \) be a recursive coalgebra and let \( a^*: (A^*, \alpha^*) \to (A, \alpha) \) be its well-founded part.

(1) Assume first that \( F0 = 0 \). Then \( F \) preserves zero maps and kernels by Remark 8.7.12(4). Therefore \( Fa^* \) is the kernel of \( F(\coker a^*) \) as shown in the following diagram:

\[
\begin{array}{ccccccc}
A^* & \longrightarrow & FA^* \\
\downarrow \alpha^* & & \downarrow Fa^* \\
A & \longrightarrow & FA \\
\coker a^* & \downarrow \alpha & \downarrow F(\coker a^*) \\
A/A^* & \overset{u}{\longrightarrow} & F(A/A^*)
\end{array}
\]

Since \( \coker a^* \) is epic, so is \( F(\coker a^*) \) because epimorphisms split in \( K\text{-Vec} \) (see Remark 8.6.4). Thus, we have \( F(\coker a^*) = \coker(Fa^*) \), and by Remark 8.7.12(8) we obtain the unique monomorphism \( u: A/A^* \to F(A/A^*) \) such that the diagram above commutes. Choose a splitting \( e: F(A/A^*) \to A/A^* \), i.e. \( e \cdot u = \text{id} \). It follows that \( q = \coker a^* \) is a coalgebra-to-algebra morphism from \( (A, \alpha) \) to \( (A/A^*, e) \). Indeed, we obtain

\[
e \cdot Fq \cdot \alpha = e \cdot u \cdot q = q.
\]

Since \( F \) preserves zero morphisms, the zero morphism \( z: A \to A/A^* \) is also a coalgebra-to-algebra morphism. Consequently, \( q = z \), which is equivalent to \( a^* \) being an isomorphism \( A \cong A^* \) as desired.

(2) Let \( F \) be arbitrary, and put \( R = F0 \). We verify that there is an endofunctor \( G \) on \( K\text{-Vec} \) with \( G0 = 0 \) and preserving inverse images such that \( FX = R \times GX \). Indeed, for every vector space \( X \), let \( t_X: X \to 0 \) denote the zero map, and let \( k_X: GX \to FX \) be the kernel of \( Ft_X \). For every linear map \( f: X \to Y \) the equality \( t_X = t_Y \cdot f \) implies that
$Ff$ yields a linear map $Gf$ making the following square commutative:

\[
\begin{array}{ccc}
GX & \xrightarrow{k_X} & FX \\
\downarrow{Gf} & & \downarrow{Ff} \\
GY & \xrightarrow{k_Y} & FY
\end{array}
\]

It is easy to verify that this defines an endofunctor $G$ and $k: G \to F$ is a natural transformation. Observe that $t_X$ is a split epimorphism (whose splitting is the zero morphism $z_X: 0 \to X$), whence $Ft_X$ is a split epimorphism with splitting $Fz_X: R \to FX$. Using Remark 8.7.12(8), this implies that $FX \cong R + GX$ with coproduct injections $Fs_X$ and $k_X$. Since $+$ is also product, we obtain $FX \cong R \times GX$ as desired.

(3) We prove that $G$ preserves kernels. By Remark 8.7.12(5), $G$ then preserves finite limits, whence inverse images. Suppose that $s = \ker f$ so that we have the pullback on the left below

\[
\begin{array}{ccc}
S & \xrightarrow{s} & 0 \\
\downarrow{f} & & \downarrow{Gf} \\
X & \xrightarrow{Gs} & Y
\end{array}
\]

\[
\begin{array}{ccc}
GS & \xrightarrow{0} & 0 \\
\downarrow{Gf} & & \downarrow{Gf} \\
GX & \xrightarrow{Gf} & FY
\end{array}
\]

It is our task to prove that the square on the right above is a pullback. Since $F$ preserves inverse images, applying it to left-hand square yields the pullback square below: note here that $G0 = 0$ implies that $F(S \to 0)$ is the product projection $\pi: R \times GS \to R$ and $F(0 \to Y)$ the coproduct injection $i: R \to R \times GY$:

\[
\begin{array}{ccc}
R \times GS & \xrightarrow{\pi} & R \\
\downarrow{R \times Gs} & & \downarrow{R \times Gf} \\
R \times GX & \xrightarrow{R \times Gf} & R \times GY
\end{array}
\]

Now suppose we have $g: Z \to GX$ with $Gf \cdot g = z$, where $z: Z \to 0 \to GY$ is the zero morphism. We prove that $g$ factorizes through $Gs$. For the zero morphism $z': Z \to 0 \to R$ we clearly have

\[(R \times Gf) \cdot (z', g) = (z', z) = i \cdot z',\]

since the latter two are both the zero morphism $Z \to R \times GY$. Therefore, there is a unique morphism $h: Z \to R \times GS$ with $(R \times Gs) \cdot h = (z', g)$ and $\pi \cdot h = z'$. This implies that $h = (z', h')$ for a unique morphism $h': Z \to GS$ such that $Gs \cdot h' = g$, which proves the claim.

(4) Our recursive coalgebra $\alpha = (\alpha_1, \alpha_2): A \to R \times GA$ for $G$ yields a coalgebra $\alpha_2: A \to GA$, and we prove that it is recursive, too. Indeed, given any algebra $\beta: GB \to B$, we use the zero morphism $z: R \to GB$ to get an algebra

\[
R \times GB \cong R + GB \xrightarrow{[z, \beta]} GB
\]

238
for $F$. Now observe that a morphism $h: A \to B$ is a coalgebra-to-algebra morphism for $F$ iff it is a coalgebra-to-algebra morphism from $(A, \alpha_2)$ to $(B, \beta)$ for $G$. Since the former exists uniquely, so does the latter. This proves that $(A, \alpha_2)$ is recursive.

Since $G0 = 0$, we use item (1) to see that the coalgebra $(A, \alpha_2)$ is well-founded for $G$. Its next time operator $\bigcirc$ is the same as that of the $F$-coalgebra $(A, \alpha)$ because in the diagram below the outside is a pullback iff the left-hand square is:

\[
\begin{array}{c}
\bigcirc S \xrightarrow{\alpha(s)} R \times GS \\
\bigcirc S \xrightarrow{s} R \times GS \\
At \xrightarrow{\alpha} R \times GA \xrightarrow{\pi_r} GA
\end{array}
\]

Since $\text{id}_A$ is the unique fixed point of $\bigcirc$ w.r.t. $G$, it is also the unique fixed point w.r.t. $F$. Thus $(A, \alpha)$ is well-founded for $F$, as desired.

\[\square\]

**Corollary 8.7.14.** For every functor on $K\text{-Vec}$ preserving inverse images, the following properties of a coalgebra are equivalent:

\[
\text{well-foundedness} \iff \text{parametric recursiveness} \iff \text{recursiveness}.
\]

**8.8 Summary of this Chapter**

We have presented well-founded coalgebras, thereby capturing the concept of well-founded induction on an abstract level. We have also provided a new proof of Taylor’s General Recursion Theorem stating that every well-founded coalgebra is parametrically recursive. This holds for functors preserving monomorphisms on a complete and well-founded category with smooth monomorphisms. In the category of sets, this even holds for every endofunctor, and the converse holds for endofunctors preserving inverse images. Moreover, for every set functor the initial algebra is, equivalently, the terminal well-founded coalgebra.

We have also provided an iterative construction of the well-founded part of a given coalgebra. It is carried by the least fixed point of Jacobs’ next time operator. In addition, the well-founded part yields the coreflection of a coalgebra in the category of well-founded coalgebras.

Finally, let us remark that a dual notion to well-foundedness has been studied by Capretta, Uustalu, and Vene [81].

In the next chapter we will meet well-founded coalgebras again in a different role. There we will describe the initial algebra of set a functor as the algebra consisting of all well-founded, well-pointed coalgebras up to isomorphism.
9 State Minimality and Well-Pointed Coalgebras

The notions and constructions we present in this chapter are inspired by the well-known notion of *minimality* for deterministic automata. Recall that a minimal deterministic automaton is one that is *reachable* and *observable*: every state is reachable from the initial state and no two states accept the same language. Both notions turn out to have equivalent element-free characterizations: a deterministic automaton is observable iff it has no proper quotient (via a coalgebra homomorphism, see Example 2.4.2(3)), and it is reachable iff it has no proper subautomaton containing the initial state. Both of these element-free characterizations can be formulated, more generally, for coalgebras. These formulations are the main topic of this chapter.

We first turn to the study of *simple* coalgebras, i.e. those having no proper quotient coalgebra, in Section 9.1. Under rather mild assumptions, we show how to construct for every coalgebra its simple quotient.

In Section 9.2 we study reachable coalgebras. The coalgebraic formulation of that property requires us to consider *pointed* coalgebras. These are coalgebras equipped with a *point*, a morphism from the terminal object, modelling an initial state. Again under mild assumptions, every pointed coalgebra has what we call a *reachable part*. Moreover, we construct this reachable part in a way parallel to what we have seen for the well-founded part in Chapter 8.

Finally, in Section 9.3, we use the foregoing work to provide new descriptions of initial algebras and terminal coalgebras for set functors preserving intersections. We treat *well-pointed* coalgebras which are defined to be both simple and reachable and thus model state minimality of coalgebras. We present a construction of the well-pointed modification of a given coalgebra much in the spirit of the well-known minimization of deterministic automata. Moreover, we shall see that the set of all well-pointed $F$-coalgebras, considered up to isomorphism, carries the terminal coalgebra $\nu F$, and the subset given by well-founded and well-pointed $F$-coalgebras carries the initial algebra $\mu F$.

9.1 Simple Coalgebras

In this section we study coalgebras having no proper quotient. This property generalizes the notion of an *observable* deterministic automaton, one where distinct states accept distinct languages.
**Assumption 9.1.1.** Throughout this section, $\mathcal{A}$ denotes a cocomplete, well-powered and co-well-powered category.

**Remark 9.1.2.** (1) For every endofunctor $F$ on $\mathcal{A}$, $\text{Coalg } F$ is cocomplete with colimits formed on the level of $\mathcal{A}$. This follows from Proposition 4.1.1, since the forgetful functor of $\text{Coalg } F$ creates all colimits.

(2) As in Remark 8.2.2(3), we work with factorizations of morphisms $f$ as $f = m \cdot e$, where $e$ is a strong epimorphism and $m$ is a monomorphism. It follows from Adámek et al. [18, dual of Theorem 14.19 and Exercise 14C(e)] that every cocomplete and co-well-powered category has such factorizations. Furthermore, recall from Proposition 8.2.3 that these factorizations lift to homomorphisms in $\text{Coalg } F$ for every endofunctor $F$ preserving monomorphisms (a condition that can be dropped for set functors, see Remark 8.2.6(1)).

(3) In this chapter we work with strong quotient coalgebras (see Definition 8.5.2), i.e. quotients represented by strong epimorphisms in the base category.

(4) We recall some basic facts about strong epimorphisms:

(a) Recall from Remark 8.2.2(2) that a morphism which is both a strong epimorphism and a monomorphism is an isomorphism.

(b) Strong epimorphisms are stable under pushout [74, Prop. 4.3.8]. That is, in the diagram below $e'$ is a strong epimorphism if $e$ is one:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
e| & & \downarrow e' \\
C & \longrightarrow & D
\end{array}
$$

Analogously, a wide pushout of a collections $e_i : A \rightarrow B_i$ ($i \in I$) of strong epimorphisms is formed by strong epimorphisms.

(c) Strong epimorphisms are extremal, i.e. if a strong epimorphism $e : A \twoheadrightarrow B$ factorizes through a monomorphism $m : S \rightarrow B$, then $m$ is an isomorphism [74, Prop. 4.3.6].

(d) Every coequalizer is a strong epimorphism [74, Prop. 4.3.6].

(5) We call a wide pushout of a span of strong epimorphisms a cointersection. Thus, our assumptions imply that all cointersections exist in $\mathcal{A}$. Moreover, for every endofunctor $F$ all cointersections exist in $\text{Coalg } F$, and they are formed on the level of $\mathcal{A}$.

**Definition 9.1.3.** A coalgebra $(A, \alpha)$ is called simple if it has no proper strong quotient coalgebra: every strong quotient coalgebra of $(A, \alpha)$ is an isomorphism.

**Examples 9.1.4.** (1) If $F$ has a terminal coalgebra, then it is simple. Indeed, given any strong quotient $e : \nu F \twoheadrightarrow A$ then the unique homomorphism $f : A \rightarrow \nu F$ satisfies $f \cdot e = \text{id}_{\nu F}$. Thus, $e$ is a split monomorphism, and since it is also an epimorphism, it is an isomorphism.

(2) A deterministic automaton, considered as a coalgebra for $FX = \{0, 1\} \times X^\Sigma$ is simple iff it is observable. To see this, let $(A, \alpha)$ be simple and consider the unique coalgebra homomorphism $f : A \rightarrow \nu F$ assigning to every state the formal language it accepts (see.
9.1 Simple Coalgebras

Example 2.5.5. Take its (strong epi, mono)-factorization \( f = m \cdot e \) to obtain its image \( C \) in \( \nu F \). Then \( C \) is a quotient of \( A \) via \( e \), and therefore \( e \) is an isomorphism by simplicity. It follows that two states \( x \) and \( y \) in \( A \) accept the same language (i.e. \( f(x) = f(y) \)) iff \( e(x) = e(y) \), and equivalently, \( x = y \).

(3) A graph considered as a coalgebra for \( \mathcal{P} \) is simple iff no distinct nodes are bisimilar. Indeed, for every graph \( (A, \alpha) \), the largest bisimulation \( \sim \) on it is a congruence (see Remark 4.2.6), i.e. the quotient \( A/\sim \) carries the structure of a \( \mathcal{P} \)-coalgebra, such that the canonical quotient map \( e: A \to A/\sim \) is a coalgebra homomorphism. Thus, \( (A, \alpha) \) is simple iff \( e \) is an isomorphism, which is equivalent to saying that for every bisimilar pair \( x, y \) in \( A \) we have \( x = y \).

Proposition 9.1.5 [133]. Every coalgebra has a unique simple quotient represented by the cointersection

\[ e_{(A,\alpha)}: (A, \alpha) \to (\bar{A}, \bar{\alpha}) \]

of all strong quotient coalgebras of \( (A, \alpha) \). This is the reflection of \( (A, \alpha) \) in the full subcategory of \( \text{Coalg}_F \) given by all simple coalgebras.

Proof. (1) Take the cointersection \( e_{(A,\alpha)}: (A, \alpha) \to (\bar{A}, \bar{\alpha}) \) of all strong quotient coalgebras of \( (A, \alpha) \) in \( \text{Coalg}_F \). Then \( e_{(A,\alpha)} \) is a strong quotient coalgebra by Remark 9.1.2(4).

To see that \( (\bar{A}, \bar{\alpha}) \) is simple, consider any strong quotient coalgebra \( q: (\bar{A}, \bar{\alpha}) \to (B, \beta) \). Then \( q \cdot e_{(A,\alpha)}: (A, \alpha) \to (B, \beta) \) is a strong quotient coalgebra. Hence, by the construction of \( e_{(A,\alpha)} \) we have a homomorphism \( f: (B, \beta) \to (\bar{A}, \bar{\alpha}) \) such that \( f \cdot q \cdot e_{(A,\alpha)} = e_{(A,\alpha)} \).

Since \( e_{(A,\alpha)} \) is epic, we obtain \( f \cdot q = \text{id}_B \). Thus, \( q \) is a strong epimorphism which is also monic, hence an isomorphism.

(2) In order to prove that \( e_{(A,\alpha)} \) is a reflection, let \( h: (A, \alpha) \to (B, \beta) \) be any coalgebra homomorphism with \( (B, \beta) \) simple. Form a pushout in \( \text{Coalg}_F \) as shown below:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{e_{(A,\alpha)}} & & \downarrow{e} \\
A & \xrightarrow{\bar{h}} & B \\
\end{array}
\]

By the stability of strong epimorphisms under pushouts, we obtain that \( \bar{e} \) is strongly epic in the base category. Since \( (B, \beta) \) is simple, we thus know that \( \bar{e} \) is an isomorphism and therefore \( k = \bar{e}^{-1} \cdot \bar{h} \) is the desired coalgebra homomorphism satisfying \( k \cdot e_{(A,\alpha)} = h \).

Uniqueness of \( k \) is clear, since \( e_{(A,\alpha)} \) is epic.

Observation 9.1.6. (1) If \( F \) preserves monomorphisms, then a coalgebra \( (A, \alpha) \) is simple iff every homomorphism with domain \( (A, \alpha) \) is monic. Indeed, given a coalgebra homomorphism \( f: (A, \alpha) \to (B, \beta) \) with \( (A, \alpha) \) simple, we obtain its image by taking the (strong epi, mono) factorization \( f = m \cdot e \) (see Remark 8.2.4). Then \( (A, \alpha) \) must be isomorphic to its image via \( e \), which implies that \( f \) is monic.

Conversely, suppose that every coalgebra homomorphism with domain \( (A, \alpha) \) is monic. Then, in particular the simple quotient \( e_{(A,\alpha)}: (A, \alpha) \to (\bar{A}, \bar{\alpha}) \) from Proposition 9.1.5 is monic. Hence it is an isomorphism.

243
(2) For every pair of coalgebra homomorphisms \( f_1, f_2 : (A, \alpha) \to (B, \beta) \) with \((B, \beta)\) simple we have \( f_1 = f_2 \). Indeed, take the coequalizer \( e \) of \( f_1, f_2 \) in \( \text{Coalg} F \). Then, \( e \) is also a coequalizer in the base category, thus it is an isomorphism by simplicity (since every coequalizer is strongly epic by Remark 9.1.2(4)). It follows that \( f_1 = f_2 \).

**Remark 9.1.7.** If the terminal coalgebra \( \nu F \) exists, then the simple reflection of a coalgebra \( A \) is given by the image of the unique coalgebra homomorphism \( A \to \nu F \). In other words, an \( F \)-coalgebra \( A \) is simple iff the unique homomorphism \( h : A \to \nu F \) is a monomorphism. We leave the easy proof to the reader.

**Theorem 9.1.8.** If \( F \) preserves monomorphisms, then the following are equivalent:

(1) \( F \) has a terminal coalgebra, and

(2) \( F \) has only a set of simple coalgebras up to isomorphism (of \( \text{Coalg} F \)).

**Proof.** Suppose that a terminal coalgebra \( \nu F \) exists. Since \( \mathcal{A} \) is well-powered, \( \nu F \) has only a set of subobjects in \( \mathcal{A} \). Every simple coalgebra \( A \) gives such a subobject, since the coalgebra homomorphism \( m_A : A \to \nu F \) is, by Observation 9.1.6(1), monic. Moreover, if \( A \) and \( B \) determine the same subobject, then \( m_A \) and \( m_B \) factor through each other and are thus isomorphic.

Conversely, if \((A_i, \alpha_i) \ (i \in I)\) is a representative set of all simple coalgebras for \( F \), then we prove that their coproduct 

\[
(B, \beta) = \coprod_{i \in I} (A_i, \alpha_i)
\]

is weakly terminal. The existence of \( \nu F \) then follows from Theorem 4.2.8. Let \((A, \alpha)\) be a coalgebra. Since \((\bar{A}, \bar{\alpha})\) in Proposition 9.1.5 is simple, we have an isomorphism \( u : (\bar{A}, \bar{\alpha}) \to (A_i, \alpha_i) \) for some \( i \in I \). Denote by \( v_i : (A_i, \alpha_i) \to (B, \beta) \) the coproduct injection. Then \( v_i \cdot u \cdot e_{(A, \alpha)} : (A, \alpha) \to (B, \beta) \) is the desired homomorphism. \( \square \)

**Remark 9.1.9.** Note that for \( \mathcal{A} = \text{Set} \) we may drop the assumption in Observation 9.1.6 and Theorem 9.1.8 that \( F \) preserves monomorphisms. This follows from Remark 8.2.6(1).

**Proposition 9.1.10.** Every subcoalgebra of a simple coalgebra is simple.

**Proof.** Suppose that \((A, \alpha)\) is simple, and let \( m : (B, \beta) \to (A, \alpha) \) represent a subcoalgebra. Given a quotient \( e : (B, \beta) \to (C, \gamma) \), we prove that \( e \) is an isomorphism. Form the following pushout in \( \text{Coalg} F \):

\[
\begin{array}{ccc}
(B, \beta) & \xleftarrow{m} & (A, \alpha) \\
\downarrow{e} & & \downarrow{\bar{e}} \\
(C, \gamma) & \xrightarrow{\bar{m}} & (C, \bar{\gamma})
\end{array}
\]

Since \( e \) is a strong epi in \( \mathcal{A} \), so is \( \bar{e} \) by Remark 9.1.2(4), i.e. \( \bar{e} \) represents a strong quotient coalgebra in \( \text{Coalg} F \). Hence, \( \bar{e} \) is an isomorphism by simplicity. Thus, \( \bar{m} \cdot e = \bar{e} \cdot m \) is monic, whence \( e \) is monic. Since \( e \) is a strong epi which is also a mono, it is an isomorphism, and we are done. \( \square \)
9.2 Pointed and Reachable Coalgebras

In this section we study reachability on the level of coalgebras. For automata, this is the second ingredient of minimality, besides observability.

Reachability has a simple formulation: a coalgebra with a given distinguished point (thought of as an initial state) is reachable if it does not have any proper pointed subcoalgebras. We will also see an iterative construction of the reachable part of a given pointed coalgebra, which is reminiscent of the usual breadth-first search algorithm for computing the reachable part of a pointed graph. This construction appears in work by Barlocco et al. [55], and independently by Wißmann et al. [242].

Assumption 9.2.1. Throughout this section $\mathcal{A}$ denotes a complete and well-powered category, and $F$ an endofunctor preserving intersections.

We denote by $1$ the terminal object of $\mathcal{A}$. Morphisms $1 \to A$ are called points (a.k.a. global elements) of the object $A$.

Definition 9.2.2 (Adámek et al. [28]). (1) By a pointed coalgebra is meant a triple $(A, \alpha, x)$ consisting of a coalgebra $\alpha: A \to FA$ and a point $x: 1 \to A$. The category $\text{Coalg}_F$ of pointed coalgebras has as morphisms from $(A, \alpha, x)$ to $(B, \beta, y)$ those coalgebra homomorphisms $f: (A, \alpha) \to (B, \beta)$ which preserve the point: $f \cdot x = y$.

(2) A pointed coalgebra $(A, \alpha, x)$ is called reachable if it has no proper pointed subcoalgebra. That is, every homomorphism $m: (A', \alpha', x') \to (A, \alpha, x)$ of pointed coalgebras, where $m$ is a monomorphism of $\mathcal{A}$, is an isomorphism.

Examples 9.2.3. (1) A deterministic automaton with a given initial state is a pointed coalgebra for the set functor $FX = \{0, 1\} \times X^\Sigma$. Reachability means that every state can be reached (in finitely many steps) from the initial state.

(2) For the power-set functor the pointed coalgebras are the pointed directed graphs. Reachability here means that every vertex can be reached by a directed path from the distinguished vertex. This property of a pointed graph is sometimes called accessibility [3].

(3) A labelled transition systems with a given initial state is a pointed coalgebra for $FX = \mathcal{P}(\Sigma \times X)$ on Set. It is reachable iff every state can be reached from the initial one by a finite sequence of transitions.

Definition 9.2.4. By the reachable part of a pointed coalgebra we mean its smallest pointed subcoalgebra.

Example 9.2.5. (1) For a deterministic automaton as a pointed coalgebra for $FX = \{0, 1\} \times X^\Sigma$ the reachable part is the subautomaton given by all states reachable (by input words from $\Sigma$) from the initial state.

(2) Similarly, a labelled transition system as a pointed coalgebra for $\mathcal{P}(\Sigma \times A)$ has as reachable part the states reachable (by some sequence of actions from $\Sigma$) from the initial state.
Proposition 9.2.6. Every pointed coalgebra has a reachable part. It is its unique reachable subcoalgebra.

Proof. (1) Recall from Proposition 4.1.5 that the forgetful functor \( \text{Coalg} F \to \mathcal{A} \) creates all limits that \( F \) preserves. Since \( F \) is assumed to preserve intersections, it follows that intersections (of coalgebra homomorphisms carried by monomorphisms) exist in \( \text{Coalg} F \) and are formed on the level of \( \mathcal{A} \).

Fix a pointed \( F \)-coalgebra \((A, \alpha, x)\). Since \( \mathcal{A} \) is well-powered we have a set \( I \) indexing all pointed subcoalgebras

\[
m_i: (A_i, \alpha_i, x_i) \to (A, \alpha, x),
\]

where each \( m_i \) is a monomorphism in \( \mathcal{A} \). Let \( m_0 \) be the intersection of all \( m_i \), for \( i \in I \).

The morphisms \( x_i: 1 \to A_i \) form a cone over the evident diagram, and so by the universal property of the wide pullback we have \( x_0: 1 \to A_0 \) such that for all \( i \in I \), \( m_i \cdot x_0 = x \).

We thus obtain a pointed subcoalgebra

\[
m_0: (A_0, \alpha_0, x_0) \to (A, \alpha, x).
\]

(2) We check that \((A_0, \alpha_0, x_0)\) is reachable. Let \( i \in I \) and \( m' \) be such that \( m': (A_i, \alpha_i, x_i) \to (A_0, \alpha_0, x_0) \) is a pointed subcoalgebra. By definition of \( m_0 \), there is a monomorphism \( n: A_0 \to A' \) such that \( m_0 = m_0 \cdot m' \cdot n \).

Since \( m_0 \) and \( m_0 \cdot m' \) are monic, \( m' \) and \( n \) are inverses. In particular, \( m' \) is an isomorphism, as desired.

(3) For the uniqueness, suppose that \((A_1, \alpha_1, x_1)\) is a reachable pointed subcoalgebra of \((A, \alpha, x)\). Let \((B, \beta, y)\) be the intersection of \((A_0, \alpha_0, x_0)\) and \((A_1, \alpha_1, x_1)\) in \( \text{Coalg}_F F \). Then as above \( B \) is isomorphic to \( A_0 \) and \( A_1 \), and hence \( A_0 \) and \( A_1 \) are themselves isomorphic. \( \square \)

Remark 9.2.7. Note that for \( \mathcal{A} = \text{Set} \) it suffices for Proposition 9.2.6 that \( F \) preserves nonempty intersections. The following result is the converse:

Proposition 9.2.8. Let \( F \) be a set functor such that every pointed coalgebra has a reachable part. Then \( F \) preserves nonempty intersections.

Proof. Let \( m_i: B_i \hookrightarrow A \) (\( i \in I \)) have intersection \( m: B \hookrightarrow A \) with \( B \neq \emptyset \). Since \( Fm \) and \( Fm_i \) are (split) monomorphisms, it is sufficient to take a collection \( b_i \in FB_i \) so that all \( Fm_i(b_i) \) are equal, say to \( a \in FA \), and to find \( b \in FB \) so that \( Fm(b) = a \).

We assume without loss of generality that \( B_i \subseteq B \subseteq A \) for all \( i \in I \) and that \( m \) and \( m_i \) are the inclusion maps.

Form the pointed coalgebra \((A, \alpha, x)\), where \( x \in B \) is arbitrary and \( \alpha \) is the constant function of value \( a \). We have a reachable subcoalgebra \( n: (C, \gamma, x) \hookrightarrow (A, \alpha, x) \).

(1) The first case is when \( m_i \cap n \) is nonempty for all \( i \). Then \( C \neq \emptyset \) as well, and it follows that \( Fn \) is a monomorphism. Therefore, we know that the composite \( Fn \cdot \gamma = \alpha \cdot n \) is constant with value \( Fn \cdot \gamma(x) = a \). Hence, \( \gamma \) is constant with value \( c = \gamma(x) \). For every \( i \),
9.2 Pointed and Reachable Coalgebras

the intersection of $m_i$ and $n$ is formed by the inclusion maps of $B_i^* = B_i \cap C$ as follows:

\[
\begin{align*}
B_i^* & \xrightarrow{m_i^*} C \\
& \downarrow n_i \downarrow \quad \downarrow n \\
B_i & \xrightarrow{m_i} A
\end{align*}
\]

Clearly, $B_i^*$ contains the point $x$. By Proposition 4.4.1, $F$ preserves this intersection. Consequently, from $Fm_i(b_i) = a = Fn(\gamma(x)) = Fn(c)$ we conclude that there exists $b_i^* \in FB_i^*$ with $Fm_i(b_i^*) = c$. Let $\beta_i : B_i \rightarrow FB_i$ be constant with value $b_i^*$. Since $\gamma$ is constant with value $c = Fm_i(b_i^*)$ we obtain a pointed subcoalgebra

\[m_i^* : (B_i^*, \beta_i, x) \hookrightarrow (C, \gamma, x).\]

From the reachability of $(C, \gamma, x)$ we conclude that $m_i^*$ is an isomorphism, whence $n \subseteq m_i$. Since this holds for all $i$, we conclude that $n \subseteq m$, which means that we have $f : C \hookrightarrow B$ with $n = m \cdot f$. The desired element is therefore $b = Ff(c)$: we clearly have $Fm(b) = Fm(Ff(c)) = Fn(c) = a$.

(2) The second case is when $m_i \cap n$ is empty for some $i$. For every nonempty set $X$ choose some $f : A \rightarrow X$, and set $a_X = Ff(a)$. We claim that $a_X$ is independent of the choice of $f$. To see this, let $g : A \rightarrow X$ be given. Since $m_i \cap n = \emptyset$, we can find $h : A \rightarrow X$ be which agrees with $f$ on $B_i$ and with $g$ on $C$:

\[h \cdot m_i = f \cdot m_i \quad \text{and} \quad h \cdot n = g \cdot n. \quad (9.1)\]

This implies $Ff(a) = Fg(a)$ because using $Fn(c) = a = Fm_i(b_i)$ and (9.1) we have \[Ff(a) = Ff(Fm_i(b_i)) = Fh(Fm_i(b_i)) = Fh(a) = Fh(Fn(c)) = Fg(Fn(c)) = Fg(a).\]

We conclude that for all functions $p : X \rightarrow Y$ with $X \neq \emptyset$ we have $Fp(a_X) = a_Y$. Now take $p = m$ and also $p = \text{id}_A$. We see that $a_B$ is the desired point with $Fm(a_B) = F\text{id}_A(a_A) = a$. \hfill $\square$

Remark 9.2.9. Note that Proposition 9.2.8 does not hold with intersections in lieu of nonempty ones. In fact, the functor $C_{01}$ with $C_{01}\emptyset = \emptyset$ and mapping nonempty sets to 1 is a counterexample. Every pointed coalgebra $(A, \alpha, x)$ carries a reachable subcoalgebra carried by $\{x\}$. However, $C_{01}$ does not preserve the intersection of a pair of disjoint nonempty subsets.

Remark 9.2.10. (1) Recall from Remark 8.3.1 that for every object $A$ of $\mathbb{A}$ the poset $\text{Sub}(A)$ of subobjects of $A$ is a complete lattice with meets obtained by forming intersections.

(2) Since $F$ preserves intersections, we know that for every $F$-coalgebra $(A, \alpha)$ the next time operator $\bigcirc$ has a left adjoint by Proposition 8.3.16.

We are ready to proceed to an iterative construction of the reachable part of a coalgebra.
Definition 9.2.11. For every $F$-coalgebra $(A, \alpha)$ the previous time operator is the left adjoint of the next time operator. We denote it by

$$\ominus : \text{Sub}(A) \to \text{Sub}(A).$$

Remark 9.2.12. (1) More explicitly, $\ominus$ assigns to every subobject $s : S \to A$ in $\mathcal{A}$ the intersection of all subobjects $m$ with $s \leq \ominus m$. This follows from the formula for the left adjoint given in Remark 8.3.3:

$$\ominus s = \bigwedge_{s \leq \ominus m} m.$$

(2) Thus, for the least subobject $\bot$ of $A$, we have $\ominus \bot = \bot$.

Proposition 9.2.13. Let $(A, \alpha)$ be a coalgebra. The previous time operator assigns to every subobject $s$ of $A$ the least subobject $m$ of $A$ for which $\alpha \cdot s$ factorizes through $Fm$:

$$\begin{array}{ccc}
S & \xrightarrow{=} & F(\ominus S) \\
\downarrow s & & \downarrow F(\ominus s) \\
A & \xrightarrow{\alpha} & FA
\end{array}$$

Proof. (1) We prove that $\alpha \cdot s$ factorizes through $F(\ominus s)$. Indeed, since $F$ preserves intersections, $F(\ominus s) = \bigwedge_{s \leq \ominus m} Fm$. Thus, we just need to verify that for every $m : M \to A$ with $s \leq \ominus m$, $\alpha \cdot s$ factorizes through $Fm$:

$$\begin{array}{ccc}
S & \xrightarrow{=} & \ominus M \\
\downarrow s & & \downarrow Fm \\
A & \xrightarrow{\alpha} & FA
\end{array}$$

(2) Conversely, given a subobject $m : M \to A$ with such a factorization, i.e. we have $g : S \to FM$ with $\alpha \cdot s = Fm \cdot g$, it is our task to prove that $\ominus s \leq m$. One uses the universal property of the pullback to obtain the dashed arrow in the diagram below:

$$\begin{array}{ccc}
S & \xrightarrow{=} & \ominus M \\
\downarrow s & & \downarrow Fm \\
A & \xrightarrow{\alpha} & FA
\end{array}$$

This proves that $s \leq \ominus m$, and equivalently, $\ominus s \leq m$. \qed

The name of the operator $\ominus$ comes from the fact that it is a generalized semantic counterpart of the previous time operator of classical linear temporal logic (see e.g. Manna and Pnueli [171]), as we will now illustrate.
9.2 Pointed and Reachable Coalgebras

Examples 9.2.14. (1) For the functor $FX = \mathcal{P}(\Sigma \times X)$ on $\text{Set}$, consider a coalgebra $\alpha: A \to \mathcal{P}(\Sigma \times A)$, i.e. a labelled transition system. Then for every subset $S \subseteq A$, the set $\bigcirc S$ consists of those states which are reachable from $S$ by a single transition:

$$\bigcirc S = \{y \in A \mid y \in \alpha(s, x) \text{ for some } s \in \Sigma \text{ and } x \in S\}.$$  

Cf. Example 8.3.6(2) on the “next time” operator $\heartsuit$.

(2) Analogously for graphs as coalgebras for $\mathcal{P}$: given a set $S$ of vertices, $\bigcirc S$ consists of all successor vertices of $S$.

(3) Finally, deterministic automata $A$ considered as coalgebras for $FX = \{0, 1\} \times X^\Sigma$ have an analogous description of the previous time operator: for a set $S \subset A$ of states, $\bigcirc S$ are the states reachable from a state in $S$ by a single transition.

In the case where $\mathcal{A} = \text{Set}$, the canonical graph (see Definition 8.1.8) of a given pointed $F$-coalgebra is a pointed graph. Moreover, the operator $\bigcirc$ can be computed on the canonical graph:

**Corollary 9.2.15.** The previous time operator of a coalgebra and its canonical pointed graph are the same.

Indeed, this follows from Lemma 8.3.8 and the fact that left adjoints are unique.

**Corollary 9.2.16.** A pointed coalgebra for an intersection preserving set functor is reachable iff so is its canonical pointed graph.

This result can be used to prove a co-algebraic version of the well-known consequence of König’s Lemma [148] that every finitely branching, well-founded and reachable graph (in the sense of Definition 9.2.2(2)) is finite. (Finitely branching means that every node has only finitely many neighbours.) This statement holds more generally for every finitary endofunctor on sets.

**Proposition 9.2.17.** Every well-founded, reachable pointed coalgebra for a finitary set functor is finite.

Note that in this proposition we do not assume that the given set functor preserves intersections.

**Proof.** Let $(A, \alpha, x)$ be a well-founded, reachable coalgebra for the finitary set functor $F$. Suppose first that $F$ preserves intersections. Then we construct the canonical graph $G = (A, \tau_A \cdot \alpha)$. We see that $G$ is finitely branching using that $F$ is finitely bounded (see Definition 4.3.3). Indeed, for every $x \in A$ we have a finite subset $s: S \to A$ and $a \in FS$ such that $Fs(a) = \alpha(x)$. Using that $\tau: F \to \mathcal{P}$ is “subnatural” (see (8.4)), we obtain that $\tau_A(\alpha(x))$ lies in $P S \subset PA$, i.e. the set of successor nodes of $x$ is finite.

Moreover, $G$ is well-founded by Corollary 8.3.9 and reachable (from the given point $x \in A$) by Corollary 9.2.16. It follows from König’s Lemma that $G$ is finite.

In general, a finitary functor $F$ need not preserve intersections. In this case, consider the Trnková hull $\bar{F}$ (see Proposition 8.1.12), whose category of coalgebras is clearly isomorphic to that of $F$-coalgebras. From Corollary 8.1.17 we know that $\bar{F}$ preserves intersections.

249
Thus, the only fact we need to prove is that the given well-founded and reachable coalgebra \((A, \alpha, x)\) is well-founded and reachable as an \(\bar{F}\)-coalgebra. We have \(FA = \bar{FA}\) since \(A\) is nonempty because \(x \in A\) is given.

For reachability this is clear because pointed subcoalgebras of \((A, \alpha, x)\) are nonempty, and therefore they are the same for \(F\) and \(\bar{F}\). For well-foundedness, see Lemma 8.1.22. □

Barlocco et al. [55] proved the following fact. Here we obtain it as a corollary of Lemma 8.3.12(1).

**Corollary 9.2.18.** Let \((A, \alpha)\) be a coalgebra. A subobject \(m: A' \rightarrow A\) carries a subcoalgebra of \((A, \alpha)\) if and only if \(m\) is a pre-fixed point of \(\ominus\), i.e. \(\ominus m \leq m\).

Indeed, the above inequality is equivalent to \(m \leq \ominus m\).

**Remark 9.2.19.** For every pointed coalgebra \((A, \alpha, x)\) we have, besides the previous time operator \(\ominus\) the constant endomap on \(\text{Sub}(A)\) with value \(x: 1 \rightarrow A\). Indeed, since 1 is a terminal object, \(x\) is a monomorphism. We can then form the (pointwise) join

\[ x \vee \ominus(-): \text{Sub}(A) \rightarrow \text{Sub}(A), \]

which assigns to every subobject \(m: S \rightarrow A\) the join of \(x\) and \(\ominus m\) in the complete lattice \(\text{Sub}(A)\).

**Theorem 9.2.20** [55]. A pointed coalgebra \((A, \alpha, x)\) is reachable if and only if the operator \(x \vee \ominus(-)\) on \(\text{Sub}(A)\) has the unique fixed point \(\top = \text{id}_A\).

**Proof.** The operator \(x \vee \ominus(-)\) is monotone since \(\ominus\) is. Thus it has a least fixed point by Theorem 6.1.1. Suppose that \((A, \alpha, x)\) is reachable, and let \(r: R \rightarrow A\) be the least fixed point of \(x \vee \ominus(-)\). Then we have \(\ominus r \leq x \vee \ominus r = r\). Hence, by Corollary 9.2.18, \(R\) carries a subcoalgebra of \((A, \alpha)\) via \(r\). Moreover, we have \(x \leq x \vee \ominus r = r\). So there exists a morphism \(x_0: 1 \rightarrow R\) such that \(r \cdot x_0 = x\). Thus \(R\) is a pointed subcoalgebra of \(A\) via \(r\), which implies that \(r\) is an isomorphism.

Conversely, suppose that \(\text{id}_A\) is the only fixed point of \(x \vee \ominus(-)\). Let \(m: (A', \alpha', x') \rightarrow (A, \alpha, x)\) be any pointed subcoalgebra. By Corollary 9.2.18, we know that \(\ominus m \leq m\). Since \(m \cdot x' = x\), we also know that \(x \leq m\). Thus \(x \vee \ominus m \leq m\); i.e. \(m\) is a pre-fixed point of \(x \vee \ominus(-)\). By the proof of Theorem 6.1.1, the least fixed point is also the least pre-fixed point. Thus, we obtain \(\text{id}_A \leq m\), which implies that \(m\) is an isomorphism. □

**Corollary 9.2.21.** For every pointed coalgebra \((A, \alpha, x)\) its reachable part is the least fixed point of \(x \vee \ominus(-)\).

A related construction of the reachable part was given independently by Wißmann et al. [242]. It can be obtained as a consequence of Corollary 9.2.21 using the following general fact from lattice theory.

**Remark 9.2.22.** For every join-preserving map \(\varphi: L \rightarrow L\) on a complete lattice \(L\), and every \(\ell \in L\), the least fixed point of \(\ell \vee \varphi(-)\) is given by the following join

\[ \bigvee_{i<\omega} \varphi^i(\ell). \quad (9.2) \]
To see this, note that \( \ell \lor \varphi(-) \) preserves joins. Hence, by the proof of Kleene’s Fixed Point Theorem 3.1.1, the least fixed point of \( \ell \lor \varphi(-) \) is the join of the \( \omega \)-chain given by \( x_0 = \bot \), the least element of \( L \), and \( x_{n+1} = \ell \lor \varphi(x_n) \). Since \( \varphi \) preserves joins, we know that \( \varphi(\bot) = \bot \) and furthermore that this \( \omega \)-chain is the following one:

\[
\bot, \ \ell \lor \varphi(\bot) = \ell, \ \ell \lor \varphi(\ell) \lor \varphi^2(\ell), \ \ldots,
\]

whose join clearly is the one in (9.2).

Since the left adjoint \( \ominus \) is a join-preserving map on the complete lattice \( \text{Sub}(A) \), we obtain:

**Corollary 9.2.23** [242]. For every pointed coalgebra \( (A, \alpha, x) \) its reachable part is carried by the following join of subobjects of \( A \):

\[
\bigvee_{i<\omega} \ominus^i x = x \lor \ominus x \lor (\ominus x) \lor \cdots .
\]  

(9.3)

More explicitly, one defines subobjects \( m_i: A_i \hookrightarrow A \) as follows: \( m_0 = x \), and given \( m_i: A_i \hookrightarrow A \), \( m_{i+1} = \ominus m_i: A_{i+1} = \ominus A_i \hookrightarrow A \) is the least subobject such that \( \alpha \cdot m_i \) factorizes through \( Fm_{i+1} \):

\[
\begin{array}{ccc}
A_i & \longrightarrow & A_{i+1} = \ominus A_i \\
\downarrow m_i & & \downarrow m_{i+1} = \ominus m_i \\
A & \xrightarrow{\alpha} & FA
\end{array}
\]

The reachable part is then the union of all \( m_i: A_i \hookrightarrow A \) for \( i < \omega \).

**Remark 9.2.24.** Note that the above subobjects \( m_i \) yield a precise connection to standard algorithms for computing reachability. Indeed, we may compute the subsets \( m_i: A_i \hookrightarrow A \) as subgraphs of the canonical graph of \( (A, \alpha, x) \). This follows by an easy induction from Corollary 9.2.15. Furthermore, observe that in this case the subset \( m_i: A_i \hookrightarrow A \) consists of precisely those states of \( A \) that are reachable by a directed path of length precisely \( i \) from the initial state \( x \). Consequently, one can compute the reachable part of a given pointed coalgebra by a standard graph algorithm such as breadth-first search.

**Theorem 9.2.25.** Suppose that \( F \) preserves inverse images. Then the full subcategory of \( \text{Coalg}_F \) given by all reachable coalgebras is coreflective in \( \text{Coalg}_F \).

**Proof.** Let \( (A, \alpha, x) \) be a pointed coalgebra and \( m: (A_0, \alpha_0, x_0) \to (A, \alpha, x) \) its reachable part. We will show that this is a coreflection.

Given a homomorphism \( h: (B, \beta, y) \to (A, \alpha, x) \) where \( (B, \beta, y) \) is reachable, we need to prove that \( h \) factorizes uniquely through \( m \). Uniqueness is clear since \( m \) is monic. For the existence, we form the inverse image of \( m \) under \( h \):

\[
\begin{array}{ccc}
P & \xrightarrow{h'} & R \\
m' \downarrow & & \downarrow m \\
B & \xrightarrow{h} & A
\end{array}
\]

(9.4)
Since $m \cdot x_0 = x = h \cdot y$, we obtain a point $z : 1 \to P$, and since $F$ preserves inverse images we also obtain a coalgebra structure $\pi$:

This defines a pointed coalgebra $(P, \pi, z)$ making $m'$ and $h'$ pointed coalgebra homomorphisms. Since $(B, \beta, y)$ is reachable and $m'$ is monic, the latter must be an isomorphism. Thus, $h' \cdot (m')^{-1}$ is the desired factorization of $h$ through $m$, cf. (9.4).

**Corollary 9.2.26.** If $F$ preserves inverse images, then all strong quotients of a reachable coalgebra are reachable.

**Proof.** Suppose we have a reachable coalgebra $(A, \alpha, x)$ and a quotient $e : (A, \alpha, x) \to (B, \beta, y)$ in $\text{Coalg}_F$. Denote by $m : (B_0, \beta_0, y_0) \to (B, \beta, y)$ the reachable part of $(B, \beta, y)$. By Theorem 9.2.25, $e$ factorizes through $m$:

Thus, $m$ is an isomorphism by Remark 9.1.2(4).

**Example 9.2.27.** For functors not preserving inverse images, reachable coalgebras need not be closed under quotients. For example, recall the functor $R : \text{Set} \to \text{Set}$ from Example 7.3.4, which preserves intersections but not inverse images. Consider the coalgebras $\gamma : C \to RC$ with $C = \{x, y, z\}$ and $\gamma(x) = (y, z)$ and $\gamma(y) = \gamma(z) = d$ and $\delta : D \to RD$ with $D = \{x', y'\}$ and $\delta(x') = \delta(y') = d$. Then $(D, \delta)$ is a quotient of $(C, \gamma)$ via the coalgebra homomorphism $q$ with $q(x') = x'$ and $q(y') = q(z') = y'$. However, $(C, \gamma, x)$ is reachable, whereas $(D, \delta, x')$ is not.

Note that, in light of the proof of Corollary 9.2.26, this example also shows that reachable coalgebras need not form a coreflective subcategory if $F$ does not preserve inverse images.

We have seen that intersection preservation by $F$ plays an important role in the results above on the reachable part of a pointed coalgebra. Let us remark that this property has an equivalent characterization in terms of the least subobjects that the previous time operator $\ominus$ delivers (cf. Proposition 9.2.13):

252
Remark 9.2.28. (1) Let \( F: \mathcal{A} \to A \) preserve monomorphisms. Then \( F \) preserves intersections if and only if

for every morphism \( f: X \to FY \) there is a least subobject \( m: Z \to Y \) such that \( f \) factorizes through \( Fm \).

This means that there exists some \( g: X \to FZ \) with

\[
\begin{array}{c}
X \\
g \\
\downarrow \downarrow \\
FY \\
\end{array} \quad \begin{array}{c}
f \\
\downarrow \\
Fm \\
\end{array} \quad \begin{array}{c}
FY \\
FZ \\
\end{array}
\]

and satisfying the following universal property: For every subobject \( m': Z' \to Y \) and every morphism \( g': X \to FZ' \) with \( Fm' \cdot g' = f \) there exists a (necessarily unique) \( h: Z \to Z' \) with \( m' \cdot h = m \).

For \( \mathcal{A} = \text{Set} \), the above equivalence was established by Gumm [124, Corollary 4.8]. For a proof for the current setting of a complete and well-powered category see Wißmann at al. [242, Proposition 5.9].

(2) For a morphism \( f: X \to FY \), the above factor \( g \) provides the part of \( Y \) used by \( f \). As an example, consider a polynomial functor \( F = H\Sigma \) on \( \text{Set} \). Then for every \( f: X \to H\Sigma Y \) the set \( Z \) above consists of all those elements occurring in the flat terms \( f(x), x \in X \).

We finish this section with a few remarks on extensions of the results on reachability presented here.

Remark 9.2.29. (1) The results presented in this section hold more generally in a category \( \mathcal{A} \) equipped with a class \( M \) of monomorphisms. One then works with \( \text{Sub}_M(A) \), the class of subobjects of \( A \) represented by \( M \)-morphisms \( m: S \to A \) (cf. Remark 6.1.21). One requires that \( \text{Sub}_M(A) \) is a complete lattice for every object \( A \), that inverse images (i.e. pullbacks along monomorphisms in \( M \)) exist, and that for every morphism \( f: A \to B \) the map \( f^* : \text{Sub}(B) \to \text{Sub}(A) \) given by forming inverse images is a right adjoint.

(2) Of course, these requirements hold in our present setting (see Assumption 9.2.1) where \( M \) is the class of all monomorphisms. Another class of examples are well-powered categories \( \mathcal{A} \) having coproducts and an \((E, M)\)-factorization system where \( M \) is a class of monomorphisms. It is then easy to see that inverse images exist (see Wißmann et al. [242, Remark 5.2(3)]). Moreover, for every \( f: X \to Y \), the left adjoint to \( f^* \) is \( f_*: \text{Sub}(X) \to \text{Sub}(Y) \), mapping an \( M \)-subobject \( s: S \to X \) to the image \( m \) of \( f \cdot s \), i.e. \( m \) is given by taking an \((E, M)\) factorization \( m \cdot e \) of \( f \cdot s \).

As explained in loc. cit., this setting allows one to construct the reachable part of coalgebras over categories which are not complete, such as \( \text{Rel} \), the category of sets and relations (viz. the Kleisli category for the power-set monad \( \mathcal{P} \)), and other Kleisli categories, e.g. that for the distribution monad.

(3) In some applications, and notably in the setting outlined in point (2), it is desirable to replace the terminal object in a point \( 1 \to A \) by a different object \( I \). This leads to the notion of an \( I \)-pointed \( F \)-coalgebra, i.e. an \( F \)-coalgebra \((A, \alpha)\) equipped with a morphism
9 State Minimality and Well-Pointed Coalgebras

In fact, Wißmann et al. [242] formulated and proved the results of this section for \( I \)-pointed \( F \)-coalgebras over a category satisfying the assumptions in (2) above.

Finally, let us come back to Kleisli categories for a monad. They are used as base categories when one wants to obtain the finite trace or language semantics of state-based systems modelled as coalgebras (cf. Example 5.1.27, and for further related examples see Subsection 11.5.5).

Wißmann et al. [242, Section 6] show how to obtain the reachable part of a coalgebra over a Kleisli category for a monad, generalizing point (2), even if this category is not equipped with an \((\mathcal{E}, \mathcal{M})\)-factorization system. One works under an additional assumptions on the base category \( \mathcal{A} \). In addition one assumes that the monad \( T \) on \( \mathcal{A} \) as well as the coalgebraic type functor \( F: \mathcal{A} \to \mathcal{A} \) preserve intersections, and that \( F \) has an extension \( \overline{F} \) to the Kleisli category \( \mathcal{A}_T \). One then obtains that the reachable part of a given pointed \( \overline{F} \)-coalgebra is the reachable part of a related coalgebra for \( F \) in \( \mathcal{A} \).

9.3 Well-pointed Coalgebras

We now present a notion that best captures the notion of minimality from automata theory, well-pointed coalgebras. We also present a new description of the initial algebra and the terminal coalgebra for set functors preserving intersections: we describe \( \nu F \) as the set of all well-pointed \( F \)-coalgebras up to isomorphism, and \( \mu F \) as the set of all well-pointed, well-founded \( F \)-coalgebras.

Although we restrict our presentation of these descriptions to endofunctors on \( \text{Set} \), the same description holds for endofunctors on concrete categories \( \mathcal{A} \) satisfying a number of natural assumptions (see Adámek et al. [28]). In particular, the results hold for endofunctors on varieties, as we mention at the end of this section.

Assumption 9.3.1. Throughout this section \( F \) denotes a set functor preserving intersections.

In particular, \( F \) preserves monomorphisms, and we know that \( \text{Set} \) is complete, cocomplete, well-powered, and co-well-powered. Thus, all of the results in Sections 9.1 and 9.2 apply.

Definition 9.3.2 [28]. A pointed coalgebra is called well-pointed if it is reachable and simple (i.e. it has no proper strong quotient coalgebra and no proper pointed subcoalgebra).

Examples 9.3.3. We combine Example 9.1.4 and Example 9.2.3:

(1) A deterministic automaton with a given initial state is a well-pointed coalgebra for \( FX = \{0, 1\} \times X^\Sigma \) iff it is reachable and observable (= simple), i.e. iff it is a minimal automaton.

(2) A well-pointed \( \mathcal{P} \)-coalgebra is a pointed graph which is reachable in the usual sense (see Example 9.2.3(2)) and no distinct nodes are bisimilar (see Example 9.1.4(3)).

(3) A labelled transition systems with a given initial state is a well-pointed coalgebra for \( FX = (\mathcal{P}X)^\Sigma \) iff it is reachable in the usual sense (see Example 9.2.3(3)) and no distinct states are bisimilar.
Notation 9.3.4. There is a canonical construction for turning an arbitrary pointed coalgebra \((A, \alpha, x)\) into a well-pointed one: first form the simple quotient \((\bar{A}, \bar{\alpha})\) (see Proposition 9.1.5) pointed by \(e((A, \alpha))\) \((x) \in \bar{A}\), and then form the reachable part of \((\bar{A}, \bar{\alpha}, e((A, \alpha))\) \((x))\) (see Proposition 9.2.6).

\[
\begin{align*}
x_0 & \xrightarrow{e((A, \alpha))} \bar{A}_0 \\
1 & \xrightarrow{x} A \xrightarrow{e((A, \alpha))} \bar{A}
\end{align*}
\]

We have \(m: (\bar{A}_0, \bar{\alpha}_0, x_0) \rightarrow (\bar{A}, \bar{\alpha}, e((A, \alpha))\) \((x))\). Then \((\bar{A}_0, \bar{\alpha}_0, x_0)\) is well-pointed, and we denote it by \(wp(A, \alpha, x)\).

We call it the \textit{well-pointed modification} of \((A, \alpha, x)\).

Remark 9.3.5. As in the case of deterministic automata, well-pointed coalgebras are rather special in general. In fact, for many functors \(F\) there exists, up to isomorphism of \(\text{Coalg}_F\), only a set of well-pointed coalgebras. This is the case for all bounded set functors.

Notation 9.3.6. Let \(F\) be a set functor for which there is a representative set \(T\) of well-pointed coalgebras. In other words, every well-pointed coalgebra for \(F\) is isomorphic in \(\text{Coalg}_F\) to precisely one coalgebra in \(T\). Then we can assume, without loss of generality, that all well-pointed modifications are chosen to be elements of \(T\). For every coalgebra \(\alpha: A \rightarrow FA\) we have a function

\[
\alpha^+: A \rightarrow T \quad \text{defined by} \quad \alpha^+(x) = wp(A, \alpha, x).
\]

Lemma 9.3.7 [28]. For every coalgebra homomorphism \(h: (A, \alpha) \rightarrow (B, \beta)\) the following triangle commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\alpha^+ & \downarrow & \beta^+ \\
T & \xleftarrow{\alpha} & B
\end{array}
\]

Proof. (1) Assume that \((A, \alpha)\) and \((B, \beta)\) are simple. In particular, \(h\) is a monomorphism in \(\text{Coalg} F\), by simplicity of \((A, \alpha)\); see Observation 9.1.6(1). Fix \(x \in A\). By simplicity again, we know that \(\alpha^+(x)\) is the reachable part \(m: (A_0, \alpha_0, x_0) \rightarrow (A, \alpha, x)\). Analogously, \(\beta^+(h(x))\) is the reachable part \(n: (B_0, \beta_0, y_0) \rightarrow (B, \beta, h(x))\). We have a monomorphism \(h \cdot m: (A_0, \alpha_0, x_0) \rightarrow (B, \beta, h(x))\). By Proposition 9.2.6, \((A_0, \alpha_0, x_0)\) is isomorphic to \((B_0, \beta_0, y_0)\). Now \(T\) contains just one representative of every well-pointed coalgebra up to isomorphism, consequently, \(\beta^+(h(x)) = \alpha^+(x)\).

(2) If the two coalgebras are arbitrary, form their simple reflections \((\bar{A}, \bar{\alpha})\) and \((\bar{B}, \bar{\beta})\). By Proposition 9.1.5, there is a unique homomorphism \(\bar{h}\) such that the diagram below
commutes:
\[
\begin{array}{ccl}
(A, \alpha) & \xrightarrow{h} & (B, \beta) \\
\downarrow e_{(A,\alpha)} & & \downarrow e_{(B,\beta)} \\
(A, \bar{\alpha}) & \xrightarrow{\bar{h}} & (B, \bar{\beta})
\end{array}
\]

Let \( x \in A \). We have that \( \alpha^+(x) \) is the reachable part of \((\bar{A}, \bar{\alpha}, \bar{x})\), where \( \bar{x} = e_{(A,\alpha)}(x) \). Thus \( \alpha^+(x) = \bar{\alpha}^+(\bar{x}) \); analogously for \( \beta^+(h(x)) \). By applying (1) to \( \bar{h} \) in lieu of \( h \) we conclude \( \alpha^+(x) = \bar{\alpha}^+(\bar{x}) = \bar{\beta}^+(\bar{h}(\bar{x})) = \beta^+(h(x)) \).

**Theorem 9.3.8.** A set functor \( F \) preserving intersections has a terminal coalgebra iff it has only a set \( T \) of well-pointed coalgebras up to isomorphism. Moreover, \( T \) carries the terminal coalgebra with the structure \( \tau: T \rightarrow FT \) which assigns to every element \((A, \alpha, x)\) of \( T \) the following element of \( FT \):

\[
\tau(A, \alpha, x) = F\alpha^+ \cdot \alpha(x)
\] (9.6)

For every coalgebra \((A, \alpha)\), the unique homomorphism to \( T \) is \( \alpha^+ : A \rightarrow T \).

**Proof.** Necessity follows from Theorem 9.1.8: since \( \nu F \) exists, there is only a set of simple coalgebras up to isomorphism. For each simple coalgebra \((A, \alpha)\) there is only a set of (subcoalgebras of) pointed coalgebras \((A, \alpha, x)\). Thus, the set \( T \) exists.

Let us prove sufficiency.

(1) We first show that the coalgebra \((T, \tau)\) is weakly terminal. Let \((A, \alpha)\) be a coalgebra. Since there is a homomorphism of \( A \) into a simple coalgebra (Proposition 9.1.5), we can assume that is \((A, \alpha)\) is simple. Fix \( x \in A \). Write \( wp(A, \alpha, x) \) as \((A_0, \alpha_0, x_0)\). By the simplicity of \((A, \alpha)\) we know that \((A_0, \alpha_0, x_0)\) is the reachable pointed subcoalgebra of \((A, \alpha, x)\). Thus, we have a monic coalgebra homomorphism \( m: (A_0, \alpha_0) \rightarrow (A, \alpha) \) with \( m(x_0) = x \).

We first observe the equation

\[
F\alpha^+ \cdot \alpha(x) = F\alpha_0^+ \cdot \alpha_0(x_0).
\] (9.7)

Indeed, \( \alpha^+ \cdot m = \alpha_0^+ \) by Lemma 9.3.7. Since \( m \) is a coalgebra homomorphism, we have \( Fm \cdot \alpha_0 = \alpha \cdot m \). Thus

\[
F\alpha_0^+ \cdot \alpha_0 = F\alpha^+ \cdot Fm \cdot \alpha_0 = F\alpha^+ \cdot \alpha \cdot m.
\]

Apply both sides to \( x_0 \) to obtain (9.7). Using this, we prove that \( \alpha^+ \) is a homomorphism, i.e. we have \( \tau \cdot \alpha^+ = F\alpha^+ \cdot \alpha \). Indeed, for every \( x \in A \) we have

\[
\tau \cdot \alpha^+(x) = \tau(wp(A, \alpha, x)) = \tau(A_0, \alpha_0, x_0) = F\alpha_0^+ \cdot \alpha_0(x_0) = F\alpha^+ \cdot \alpha(x).
\]

This verifies that \( \alpha^+: (A, \alpha) \rightarrow (T, \tau) \) is a homomorphism.
(2) Having shown \((T, \tau)\) to be weakly terminal, we conclude by proving that for every homomorphism \(h: (A, \alpha) \rightarrow (T, \tau)\) we have \(\alpha^+ = h\). By Lemma 9.3.7, \(\tau^+ \cdot h = \alpha^+\). We need only show that \(\tau^+ = \text{id}_T\). Every element \((A, \alpha, x)\) of \(T\) is well-pointed, and so we have \(\alpha^+(x) = (A, \alpha, x)\). By Lemma 9.3.7 again, \(\tau^+ \cdot \alpha^+ = \alpha^+\). This proves that 
\[
\tau^+(A, \alpha, x) = \tau^+ \cdot \alpha^+(x) = \alpha^+(x) = (A, \alpha, x).
\]

**Remark 9.3.9.** We have seen that if \(T\) is a set which contains representatives of all well-pointed coalgebras, then \(T\) is a terminal coalgebra. Even if \(T\) is a proper class (of representatives), it is a terminal coalgebra for a functor related to \(F\). More precisely, consider \(\text{Set}\) as the full subcategory of the category \(\text{Class}\) of all classes and functions, then \(F\) has an canonical extension to an endofunctor \(\hat{F}\) on the latter (see Adámek et al. [32]). Moreover, \(T\) is a terminal coalgebra for \(\hat{F}\). (The proof is completely analogous to that of Theorem 9.3.8.)

**Remark 9.3.10.** An analogous description of \(\nu F\) is possible for all endofunctors on a variety \(\mathcal{A}\) preserving intersections. Here \(\mathcal{A}\) is a category of algebras specified by a (finitary) signature \(\Sigma\) and a set of equations. The concept of a pointed and a well-pointed coalgebra is defined analogously to the case \(\mathcal{A} = \text{Set}\): a pointed coalgebra is \((A, \alpha, x)\) where \(x\) is an element of the coalgebra \((A, \alpha)\). Thus, we work here with the generalization mentioned in Remark 9.2.29(3): we choose \(I\) to be the free algebra on one generator in the variety \(\mathcal{A}\). A pointed coalgebra \((A, \alpha, x)\) is well-pointed if \((A, \alpha)\) is simple and has no proper subcoalgebra containing \(x\). Since \(F\) preserves intersections, it preserves monomorphisms. Thus, by the proof of Theorem 9.1.8, \(\nu F\) exists iff \(F\) has only a set of simple coalgebras (up to isomorphism). Now let \(T\) be a representative set of all well-pointed \(F\)-coalgebras. Then \(T\) carries a canonical structure of an object of \(\mathcal{A}\) such that \(\tau: T \rightarrow FT\) is a morphism of \(\mathcal{A}\). Moreover, \((T, \tau)\) is the terminal coalgebra.

**Initial algebras and well-founded well-pointed coalgebras** We have seen in Theorem 9.3.8 that the terminal coalgebra \(\nu F\) for a set functor preserving intersections is carried by a representative set of all well-pointed coalgebras. We now show that the initial algebra \(\mu F\) can be described similarly as the algebra of all well-founded, well-pointed coalgebras (up to isomorphism). For this result we recall from Remark 8.6.9 that \((\mu F, \iota^{-1})\) is the terminal well-founded coalgebra.

**Remark 9.3.11.** The collection of well-founded coalgebras is closed under subcoalgebras and strong quotients by Corollary 8.5.9 and Corollary 8.5.3. Hence, if a coalgebra is well-founded, then so is its well-pointed modification.

**Notation 9.3.12.** Suppose that \(F\) has only a set of well-founded, well-pointed coalgebras up to isomorphism, and choose a set \(W\) of representatives. Analogously to Notation 9.3.6, we have a function 
\[
\alpha^+: A \rightarrow W \quad \text{defined by} \quad \alpha^+(x) = \text{wp}(A, \alpha, x).
\]

We obtain a coalgebra structure \(\omega: W \rightarrow FW\) analogously to \(\tau\) in Theorem 9.3.8: it assigns to \((A, \alpha, x)\) in \(W\) the element \(F\alpha^+ \cdot \alpha(x) \in FW\).
Theorem 9.3.13 [28, Thm. 3.48]. A set functor preserving intersections has an initial algebra iff it has only a set $W$ of well-founded, well-pointed coalgebras up to isomorphism. Moreover, $W$ carries the initial algebra with the structure $\omega^{-1}: FW \to W$.

Proof. For the necessity, suppose that $\mu F$ exists. By Remark 8.6.9, $(\mu F, \iota^{-1})$ is the terminal well-founded coalgebra. For every well-founded, well-pointed coalgebra $(A, \alpha, x)$ the unique homomorphism $f: (A, \alpha) \to (\mu F, \iota^{-1})$ is injective (see Observation 9.1.6(1)). Since $\mu F$ has only a set of subsets, this implies that there is only a set of well-founded coalgebras up to isomorphism. Thus, the same holds for well-founded, well-pointed coalgebras.

For the sufficiency, suppose that the set $W$ is given. By Remark 8.6.9, we only need to prove that $(W, \omega)$ is a terminal well-founded coalgebra. This is done in a number of steps, and the order diverges from what we saw in Theorem 9.3.8.

We first prove a version of Lemma 9.3.7 saying that if $h: (A, \alpha) \to (B, \beta)$ is a homomorphism of well-founded coalgebras, then $\beta^+ \cdot h = \alpha^+$. This is as in Lemma 9.3.7; recall that a strong quotient of a well-founded coalgebra is again well-founded (Corollary 8.5.3).

Second, we observe that for every well-founded coalgebra $(A, \alpha)$, the function $\alpha^+$ is a coalgebra homomorphism $\alpha^+: (A, \alpha) \to (W, \omega)$. This is entirely as in the part of the proof of Theorem 9.3.8 where we showed $(T, \tau)$ to be weakly terminal.

Third, we prove that $(W, \omega)$ is a well-founded coalgebra. To this end notice that for every well-pointed, well-founded coalgebra $(A, \alpha, x)$ in $I$ we have that $\alpha^+(x) = (A, \alpha, x)$.

Now take the coproduct (in $\text{Coalg} F$) of all $(A, \alpha)$ for which there is an $x \in A$ such that $(A, \alpha, x)$ lies in $W$. This coproduct is a well-founded coalgebra by Corollary 8.5.3. Moreover, as we have just seen, the unique induced homomorphism from the coproduct into $(W, \omega)$ is surjective, whence $(W, \omega)$ is a strong quotient coalgebra of the coproduct. Thus, another application of Corollary 8.5.3 shows that $(W, \omega)$ is a well-founded coalgebra as desired.

At this point, we know that $(W, \omega)$ is well-founded, and indeed it is a weakly terminal well-founded coalgebra. We conclude as in the proof of Theorem 9.3.8. Let $(A, \alpha)$ be any well-founded coalgebra. Consider a homomorphism $h: (A, \alpha) \to (W, \omega)$. By the version of Lemma 9.3.7 noted above, $\alpha^+ = \omega^+ \cdot h$ and $\alpha^+ = \omega^+ \cdot \alpha^+$. As before, $\omega^+ = \text{id}_W$. So $h = \alpha^+$.

Remark 9.3.14. Note that for a finitary set functor $F$, the elements of $\mu F \cong W$ in Theorem 9.3.13 are finite coalgebras. This follows from Proposition 9.2.17.

We conclude this section with a number of examples of concrete descriptions of initial algebras and terminal coalgebras we obtain from Theorems 9.3.8 and 9.3.13.

Examples 9.3.15. (1) For the functor $FX = \{0, 1\} \times X^\Sigma$ the terminal coalgebra $T$ is carried by the set of (isomorphism classes of) all minimal deterministic automata with initial states. The coalgebra structure $\tau: T \to FT$, interpreted as an automaton on $T$, has as accepting states those minimal automata $A$ whose initial state is accepting in $A$.
and the next-state function maps a pair consisting of a minimal automaton \((A, \delta, a_0, F)\) and an input symbol \(s \in \Sigma\) to the minimization of the automaton \((A, \delta, \delta(a_0, s), F)\), i.e. one shifts the initial state along its \(s\)-transition and then minimizes the result.

The unique coalgebra homomorphism \(\alpha^+: A \to T\) assigns to a state \(a\) of the automaton \(A\) the minimization of \(A\) with that initial state.

As there are no well-founded coalgebras for \(F\), the initial algebra \(W\) in Theorem 9.3.13 is empty.

(2) For the power-set functor \(\mathcal{P}\), \(T\) is the collection of pointed, reachable and simple graphs (cf. Example 9.3.3(2)), which is not a set. However, for the finite power-set functor \(\mathcal{P}_f\) the well-pointed coalgebras are the pointed, reachable and simple finitely branching graphs. Here a set \(T\) can be chosen since every such graph is countable, which can be seen using that it is finitely branching and every vertex is reachable from the chosen vertex. Thus we obtain

\[
\nu\mathcal{P}_f \cong \text{all pointed, reachable and simple finitely branching graphs,}
\]

\[
\mu\mathcal{P}_f \cong \text{all pointed, reachable, simple and well-founded finitely branching graphs}
\]

(up to isomorphism).

(3) We next consider the set functor \(FX = X \times X + 1\). Here we view coalgebras for \(F\) (differently than earlier in this chapter) as directed graphs with edges labelled in \(\{l, r\}\) such that every node either has exactly one successor node reachable by an \(l\)-labelled edge and one by an \(r\)-labelled edge, respectively, or no successor node. We call pointed such graphs \textit{binary term graphs}. Here are two examples:

\[
\begin{align*}
\quad & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
From Theorem 9.3.8 and Theorem 9.3.13 we obtain:

\[ \nu F \cong \text{all minimal binary term graphs}, \]
\[ \mu F \cong \text{all well-founded, minimal binary term graphs} \]

(up to isomorphism). Indeed, the well-founded coalgebras are precisely those yielding a finite tree expansion.

(4) \(\Sigma\)-term graphs. More generally, consider a polynomial functor \(H_\Sigma\) (see Example 2.1.5). Analogously to the previous example, well-pointed \(H_\Sigma\)-coalgebras may be identified with minimal \(\Sigma\)-term graphs. A \(\Sigma\)-term graph is a pointed directed graph such that every node with \(n\) successors is labelled by some \(\sigma \in \Sigma_n\), and moreover, the successor nodes are reachable by an edge labelled in \(\{0, \ldots, n-1\}\) so that each edge label appears precisely once. The previous example is the special case where \(\Sigma\) is a signature containing a binary operations symbol \(\ast\) and a constant symbol \(c\), and the two binary term graphs in (9.8) above can be presented as \(\Sigma\)-term graphs as follows:

![Diagram 1]

Minimality of a \(\Sigma\)-term graph means that every node is reachable from the given point and that the \(\Sigma\)-tree expansions of distinct nodes are always different. We thus obtain

\[ \nu H_\Sigma \cong \text{all minimal } \Sigma\text{-term graphs}, \]
\[ \mu H_\Sigma \cong \text{all well-founded minimal } \Sigma\text{-term graphs} \]

(up to isomorphism).

(5) Binary decision diagram (BDD). Given a set \(V\) (of boolean variables), we have seen in Example 3.2.8(2) that all binary decision trees form the initial algebra for \(FX = \{0, 1\} + V \times X \times X\). Now a BDD is given by a pointed directed acyclic graph whose nodes are labelled by variables or by 0 or 1, and every node labelled by a variable has a 0-successor and a 1-successor, whereas nodes labelled by 0 or 1 are leaves (i.e. they do not have successors). Then BDDs are precisely the finite, pointed, well-founded \(F\)-coalgebras. In practice, one is usually more interested in reduced BDDs, i.e. reachable BDDs that are obtained as the result of a reduction process applying the following two rules:

- merge any two isomorphic subgraphs;
- eliminate any node whose 0- and 1-successors yield isomorphic subgraphs.

Reduced BDDs are well-founded, well-pointed \(F\)-coalgebras, but not conversely; for example, the following well-founded well-pointed \(F\)-coalgebra is not a reduced BDD:

![Diagram 2]
9.4 Summary of this chapter

We have presented simple and reachable coalgebras, and we have seen constructions of the simple quotient of a coalgebra and the reachable part of a pointed one. In addition, well-pointed coalgebras are the ones which are both reachable and simple, and those yield a coalgebraic formulation of minimality of state-based systems.

For set functors preserving intersections, we also saw new descriptions of the initial algebra and the terminal coalgebra. In fact, the terminal coalgebra is formed by all well-pointed coalgebras (considered up to isomorphism), and the initial algebra is formed by all well-founded, well-pointed coalgebras.
10 Fixed Points Determined by Finite Behaviour

In this chapter we study a fixed point of an endofunctor that in general is different from its initial algebra and its terminal coalgebra. This rational fixed point, as it is called, arises in the study of the final semantics of coalgebras. It collects precisely the behaviours of all ‘finite’ coalgebras of a given endofunctor, which, in the case of state-based systems, are those with finitely many states. Thus, the rational fixed point provides a semantic domain for finite-state behaviour. Accordingly, examples of rational fixed points include regular languages, eventually periodic and rational streams, regular trees for a signature, and many other types of finite-state behaviours.

Of course, for categories more general than \( \text{Set} \), ‘finite’ needs to be replaced by a suitable category theoretic concept. We use the concept of a finitely presentable object and work in a category that has enough such objects. More precisely, we work in a locally finitely presentable category, a notion originally introduced by Gabriel and Ulmer [107]. In addition, the endofunctor under consideration needs to be determined by its action on the finitely presentable objects, which means that it is finitary. We briefly review these concepts in Section 10.1. We shall see that for \( \text{Set} \) and many other categories the rational fixed point is fully abstract with respect to final semantics; i.e. it is a subcoalgebra of the terminal coalgebra.

Moreover, similar to what we have seen in the previous chapters for the initial algebra and the terminal coalgebra, the rational fixed point has two equivalent characterizations by a universal property: as a coalgebra it is the terminal locally finitely presentable coalgebra (Section 10.2), and as an algebra it is the initial iterative algebra (Section 10.3). For rational fixed points in \( \text{Set} \) we present two explicit descriptions (Section 10.4): as the coalgebra formed by all finite well-pointed coalgebras, and another one as a quotient of the coalgebra of rational \( \Sigma \)-trees modulo basic equations for a given presentation of the functor. Then we treat full abstractness of the rational fixed point (Section 10.5). We conclude the chapter with a brief discussion of further work based on the rational fixed point (Section 10.6). In particular, we mention monads arising from iterative algebras which are closely related to Elgot’s iterative theories [93].

10.1 Locally Finitely Presentable Categories

We need an appropriate notion of a ‘finite’ object for our study of the rational fixed point. We use the notion of a finitely presentable object. The name stems from general algebra, where an algebra (e.g., a group, ring, or module) is finitely presentable if it is presented by finitely many generators and finitely many relations. The categorical generalization
of this concepts is formulated with the help of filtered colimits. This then leads to the
concepts of finitary functors and locally finitely presentable categories. We now briefly
recall all these notions. A more comprehensive introduction may be found in [43].

**Remark 10.1.1.** (1) A **filtered category** is a category $\mathcal{D}$ such that every finite subcate-
gory has a cocone in it. Equivalently, the following three conditions hold:

(a) $\mathcal{D}$ is nonempty,

(b) for every pair $A_1, A_2$ of object of $\mathcal{D}$ there exists a cocone $A_1 \rightarrow B \leftarrow A_2$, and

(c) for every parallel pair of morphisms $f, g: A_1 \Rightarrow A_2$ in $\mathcal{D}$ there exists a morphism $h: A_2 \rightarrow A_3$ with $h \cdot f = h \cdot g$.

Note that every finitely cocomplete category is filtered.

(2) A **filtered diagram** in the category $\mathcal{A}$ is a diagram $\mathcal{D} \rightarrow \mathcal{A}$ whose scheme $\mathcal{D}$ is a
filtered category, and a **filtered colimit** is a colimit of a filtered diagram.

Important examples are chains (see Remark 6.0.4(2)), i.e. where $\mathcal{D}$ is an ordinal
(considered as the category given by a linearly ordered poset).

**Definition 10.1.2.** An object $X$ of a category $\mathcal{A}$ is called **finitely presentable** if its
covariant hom functor $\mathcal{A}(X, -): \mathcal{A} \rightarrow \text{Set}$ preserves filtered colimits.

**Example 10.1.3.** A set is finitely presentable in $\text{Set}$ iff it is finite. Analogously, in
the categories of posets and of graphs the finitely presentable objects are precisely the
finite posets and graphs, respectively. In the category of $K$-Vec of vector spaces over the
field $K$ the finitely presentable objects are precisely the finite-dimensional vector spaces,
i.e. those isomorphic to $K^n$ for some $n \in \mathbb{N}$.

**Remark 10.1.4.** More explicitly, an object $X$ is finitely presentable iff every morphism
$f$ from $X$ to a colimit $C = \text{colim}_{i \in I} C_i$ of a filtered diagram $D$ factorizes essentially
uniquely through one of the colimit injections $c_i: C_i \rightarrow C$. This means that

(1) there exists an $i \in I$ and a morphism $f': X \rightarrow C_i$ such that $c_i \cdot f' = f$, and

(2) given two such factorizations $c_i \cdot f' = c_i \cdot f''$ of $f$ there exists a connecting morphism $c_{i,j}: C_i \rightarrow C_j$ of $D$ such that $c_{i,j} \cdot f' = c_{i,j} \cdot f''$.

**Example 10.1.5.** No nonempty cpo is finitely presentable in $\text{CPO}$. To see this, consider
the $\omega$-chain of inclusion maps

$$0^\top \hookrightarrow 1^\top \hookrightarrow 2^\top \hookrightarrow \ldots,$$

where $n^\top = \{0, \ldots, n\} \cup \{\top\}$ is linearly ordered with the top element $\top$. Its colimit in $\text{CPO}$ is the cpo $C = \mathbb{N} \cup \{v, \top\}$ with colimit injections given by the obvious inclusion maps, where $v = \bigvee_{n \in \mathbb{N}} n$ and $v < \top$. Then for every nonempty cpo $X$ the continuous
map $f: X \rightarrow C$ which is constantly $v$ does not factorize through any of the colimit
injections $n^\top \hookrightarrow C$.

**Definition 10.1.6.** A category $\mathcal{A}$ is called **locally finitely presentable** if

(1) it is cocomplete,

(2) it has only a set of finitely presentable objects (up to isomorphism), and
every object in $\mathcal{A}$ is a filtered colimit of finitely presentable objects.

**Examples 10.1.7.** We list a number of locally finitely presentable categories:

1. **Set:** every set $X$ is a filtered colimit of its finite subsets. More precisely, $X$ is the colimit of the diagram by all its finite subsets and all inclusion maps between them. Note that this diagram is clearly filtered.

2. **Pos:** every poset if a filtered colimits of its finite subposets. Similarly for graphs.

3. **$K$-Vec:** every vector space $X$ is a filtered colimits if its finite-dimensional subspaces. This follows from the fact that every finite subset of $X$ generates a finite-dimensional subspace.

4. More generally, every finitary variety of algebras (i.e. every category of algebras specified by operations of finite arity and equations) is locally finitely presentable. The finitely presentable objects are precisely those algebras which can be presented by finitely many generators and relations [43, Theorem 3.12]. In other words, finitely presentable algebras are precisely the quotients of free algebras on finitely many generators modulo a congruence generated by finitely many pairs.

Concrete examples are the categories of monoids, groups, rings, vector spaces, modules over a (semi-)ring etc.

5. Interesting special cases are **locally finite** varieties, i.e. varieties where finitely presentable algebras are precisely the finite ones. Concrete examples are the categories of Boolean algebras, join-semilattices, distributive lattices, and vector spaces over a finite field.

6. The category $\mathbf{Nom}$ of nominal sets and equivariant maps (see Example 2.1.9) is locally finitely presentable (see e.g. Pitts [201, Rem. 5.17]). As shown by Petrişan [200, Prop. 2.3.7], the finitely presentable nominal sets are precisely the orbit-finite ones (cf. Example 2.4.7).

7. Presheaf categories $\mathbf{Set}^{C^{\text{op}}}$, where $C$ is a small category, are locally finitely presentable. One particular example of interest is the category of sets in context, i.e. the presheaf category $\mathbf{Set}^{\mathcal{F}}$, where $\mathcal{F}$ is the category of all finite sets and all maps between them. This category has been used by Fiore et al. [100] to capture structural induction on terms with variable binding operators (e.g. $\lambda$-terms) by initial algebras (see Examples 2.1.8 and 2.2.18(2)). The finitely presentable objects in $\mathbf{Set}^{\mathcal{F}}$ are precisely the super-finitary presheaves [39], where a presheaf $P: \mathcal{F} \to \mathbf{Set}$ is super-finitary if $P(\Gamma)$ is finite for every context $\Gamma$ and there is a context $\Gamma_0$ generating $P$ in the sense that for every context $\Gamma$ we have

$$P(\Gamma) = \bigcup_{\gamma: \Gamma_0 \to \Gamma} P\gamma[P(\Gamma_0)].$$

**Remark 10.1.8.** It is well-known that colimits commute with other colimits. In a locally finitely presentable category filtered colimits commute with finite products. This means that for any pair of filtered diagrams $D_i: \mathcal{D}_i \to \mathcal{A}$, $i = 1, 2$, the colimit of the diagram $\mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{A}$ taking $(d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$ to $D_1d_1 \times D_2d_2$ is given by $\text{colim} \ D_1 \times \text{colim} \ D_2$. Similarly, filtered colimits commutes with all finite limits.
Finitary endofunctors on $\mathbf{Set}$ were already discussed in Section 4.3. Recall from Definition 4.3.1 and Remark 4.3.2 that a functor is called \textit{finitary} if it preserves filtered colimits.

\textbf{Examples 10.1.9.} (1) A hom-functor $\mathcal{A}(X, -): \mathcal{A} \to \mathbf{Set}$ is finitary iff $X$ is a finitely presentable object of $\mathcal{A}$.

(2) As already mentioned for $\mathcal{A} = \mathbf{Set}$ in Example 4.3.5, constant functors and the identity functor are clearly finitary. Furthermore, finitary functors are closed under taking coproducts and colimits, composition, and finite products (see Remark 10.1.8), in general.

(3) It follows that polynomial set functors $H_\Sigma$ for (finitary) signatures $\Sigma$ are finitary.

(4) The finite power-set functor $\mathcal{P}$ is finitary, but the full power-set functor $\mathcal{P}$ is not (cf. Example 4.3.13).

(5) On the category $\mathbf{Nom}$ of nominal sets and equivariant maps, the functors $FX = A + X \times X + [A]X$ (see Example 2.1.8), $FX = \{0, 1\} \times X^k$ (see Example 2.4.7(2)) and $FX = \{0, 1\} \times X^k \times [A]X$ (see Example 2.4.7(2)) are finitary. Similarly, every functor on $\mathbf{Nom}$ arising from a binding signature in the sense of Fiore et al. [100] is finitary.

The finitely supported power-set functor $\mathcal{P}$ for $\mathbf{Nom}$ is not finitary.

(6) On the category $\mathbf{Set}^\mathcal{F}$ of sets in context the functor $FX = V + X \times X + \delta X$ (see Example 2.1.8) is finitary. Similarly, every functor on $\mathbf{Set}^\mathcal{F}$ arising from a binding signature is finitary.

\textbf{Remark 10.1.10.} The category of finitary set functors is equivalent to the category $\mathbf{Set}^\mathcal{F}$ of sets in context. In fact, the equivalence restricts a given set functor $F: \mathbf{Set} \to \mathbf{Set}$ by precomposing it with the inclusion functor $J: \mathcal{F} \hookrightarrow \mathbf{Set}$, and in the opposite direction the equivalence assigns to a given presheaf $P: \mathcal{F} \to \mathbf{Set}$ its left Kan extension along $J$.

Under this equivalence, the super finitary presheaves in Example 10.1.7(7) correspond precisely to the quotients of polynomial functors $H_\Sigma$, where $\Sigma$ contains only finitely many operation symbols [30, Corollary 3.34].

\textbf{Notation 10.1.11.} For a locally finitely presentable category $\mathcal{A}$ we denote by $\mathcal{A}_{fp}$ the (essentially small) full subcategory given by all finitely presentable objects.

\textbf{Remark 10.1.12.} Let us collect a few useful properties of locally finitely presentable categories that we shall subsequently use.

(1) Every locally finitely presentable category is complete [43, Rem. 1.56].

(2) A finite colimit of finitely presentable objects is finitely presentable again. Thus, the category $\mathcal{A}_{fp}$ from Notation 10.1.11 is finitely cocomplete.

(3) By [43, Prop. 1.57], every object $X$ of a locally finitely presentable category $\mathcal{A}$ is the filtered colimit of the \textit{canonical filtered diagram} given by all morphisms with codomain $X$ having a finitely presentable domain. In other words, for the following diagram

$$\mathcal{A}_{fp}/X \xrightarrow{DX} \mathcal{A}, \quad DX(p: P \to X) = P$$
the colimit is \(X\) with the colimit injections given by all the morphisms \(p\).

(4) Let \(D : \mathcal{D} \to \mathcal{A}\) be a filtered diagram in a locally finitely presentable category. A cocone \(c_i : Di \to C, \ i \in \mathcal{D}\), is a colimit cocone iff every morphism \(p : P \to C\) with \(P\) finitely presentable factorizes essentially uniquely through one of the colimit injections \(c_i : C_i \to C\). This means that

(a) there exist \(i \in \mathcal{D}\) and \(p'\) such that \(p = c_i \cdot p'\), and

(b) given two factorizations \(p = c_i \cdot p_t (t = 1, 2)\) there exists a morphism \(h : i \to j\) in \(\mathcal{D}\) such that \(Dh \cdot p_1 = Dh \cdot p_2\).

Indeed, this states precisely that the hom-functors \(\mathcal{A}(P, -)\) of finitely presentable objects \(P\) collectively reflect filtered colimits \([30, \text{Lemma 2.5}]\).

(5) Every locally finitely presentable category \(\mathcal{A}\) has (strong epi, mono)-factorizations \([43, \text{Prop. 1.61}]\) (cf. Remark 8.2.2).

Furthermore, let \(F\) be an endofunctor on \(\mathcal{A}\) preserving monomorphisms. By Proposition 8.2.3, \(\text{Coalg } F\) inherits these factorizations. Note that for a set functor \(F\) we may drop the assumption that \(F\) preserves monomorphisms (see Remark 8.2.6(1)).

**Lemma 10.1.13.** Let \(\mathcal{A}\) be a category with finite colimits. Then every diagram in \(D : \mathcal{D} \to \mathcal{A}\) can be extended to a filtered diagram with the same colimit (in fact, essentially the same cocones).

**Proof.** Let \(J : \mathcal{D} \to \hat{\mathcal{D}}\) be the free completion of \(\mathcal{D}\) under finite colimits. That means that precomposition with \(J\) yields an equivalence between the functor category \([\mathcal{D}, \mathcal{A}]\) and the category of all functors in \([\hat{\mathcal{D}}, \mathcal{A}]\) preserving finite colimits. A construction of \(\hat{\mathcal{D}}\) can be found e.g. in Kelly’s work \([146]\).

Thus, \(D\) extends to the functor \(\hat{D} : \hat{\mathcal{D}} \to \mathcal{A}\). (Note that this extension is given by left Kan extension along \(J\), i.e. \(\hat{D} = \text{Lan}_J D\).) Furthermore, cocones of \(\hat{D}\) with vertex \(X\), which are natural transformations from \(\hat{D}\) to \(\Delta_X : \hat{\mathcal{D}} \to \mathcal{A}\), the constant functor with value \(X\), bijectively correspond to natural transformations from \(D\) to \(\Delta_X \cdot J\) via the bijection

\[
\text{Nat}(\hat{D}, \Delta_X) \cong \text{Nat}(D, \Delta_X \cdot J)
\]

(10.1)

that maps a cocone \(d : \hat{D} \to \Delta_X\) to its restriction \(dJ : D \cong \hat{D} \cdot J \to \Delta_X \cdot J\). On the right \(\Delta_X \cdot J\) is also the constant functor with value \(X\) with domain \(\mathcal{D}\). Thus the right-hand side is precisely the cocones of \(D\) with vertex \(X\). This implies that \(D\) and \(\hat{D}\) have the same colimit; indeed, a cocone \(c : D \to \Delta_C\) is a colimit of \(\hat{D}\) iff we have a bijection \(\mathcal{C}(C, X) \cong \text{Nat}(\hat{D}, \Delta_X)\) given by precomposition with (the components of) \(c\). Now compose that last bijection with the one in (10.1) to see that \(C\) with the restricted cocone \(cJ\) is, equivalently, a colimit of \(D\). \(\square\)

### 10.2 The Rational Fixed Point

We now turn to the coalgebraic description of the rational fixed point as the terminal locally finitely presentable coalgebra.
Assumption 10.2.1. Throughout the rest of this chapter we assume that $\mathcal{A}$ is a locally finitely presentable category and that $F: \mathcal{A} \to \mathcal{A}$ is a finitary functor.

Remark 10.2.2. Note that these assumptions ensure that $F$ has both an initial algebra $\mu F$ (by Corollary 6.1.13) and a terminal coalgebra $\nu F$ (this follows from Theorem 11.3.12).

Notation 10.2.3. We denote by $\mathsf{Coalg}_f F$ the full subcategory of $\mathsf{Coalg} F$ formed by all coalgebras on finitely presentable carriers.

Remark 10.2.4. The category $\mathsf{Coalg}_f F$ is finitely cocomplete. This follows from the fact that $\mathcal{A}_{fp}$ is finitely cocomplete (see Remark 10.1.12(2)) and that the forgetful functor $U: \mathsf{Coalg}_f F \to \mathcal{A}$ creates colimits (Proposition 4.1.1). Thus $\mathsf{Coalg}_f F$ is an essentially small filtered category (see Remark 10.1.1(1)). Indeed, for essential smallness note that there is (up to isomorphism) only a set of finitely presentable objects $A$ each having a set of coalgebra structures $\alpha: A \to FA$.

Definition 10.2.5. An $F$-coalgebra $(A, \alpha)$ is called locally finitely presentable (or lfp, for short) if it is the colimit of a diagram of coalgebras in $\mathsf{Coalg}_f F$.

Remark 10.2.6. (1) Every lfp coalgebra is in fact a filtered colimit of coalgebras in $\mathsf{Coalg}_f F$. This follows from the fact that $\mathsf{Coalg}_f F$ is finitely cocomplete (see Remark 10.2.4).

(2) Note that a coalgebra $(A, \alpha)$ is lfp if it is the colimit of the filtered diagram of all homomorphisms from coalgebras of $\mathsf{Coalg}_f F$ to $(A, \alpha)$. More precisely, $(A, \alpha)$ is the canonical colimit of the following diagram $D: \mathsf{Coalg}_f F/(A, \alpha) \to \mathsf{Coalg} F$, $D(m: (B, \beta) \to (A, \alpha)) = (B, \beta)$.

Equivalently, the following canonical forgetful functor is final: $\mathsf{Coalg}_f F/(A, \alpha) \to \mathcal{A}_{fp}/A$, which is how the notion was originally introduced [175, Def. 3.7].

(3) Suppose that $F$ is a finitary set functor, every element $x \in A$ generates a subcoalgebra $s: (A', \alpha') \to (A, \alpha)$. That means that $(A', \alpha')$ is the smallest subcoalgebra of $(A, \alpha)$ containing $x$. Indeed, by Remark 8.1.16 and Corollary 8.1.17, we may assume that $F$ preserves all intersections. Therefore, the intersections of all subcoalgebras containing $x$ is a subcoalgebra by Proposition 4.1.5.

(4) Note that Proposition 8.2.3 can be slightly sharpened for set functors: the image of a coalgebra homomorphism $f: (A, \alpha) \to (B, \beta)$ may be taken to be a subcoalgebra $(C, \gamma) \to (B, \beta)$. Indeed, in the proof of Proposition 8.2.3 choose $C = f[A]$.

Recall (e.g. Silva et al. [222]) that a coalgebra for a set functor is called locally finite if each of its elements generates a finite subcoalgebra.

Proposition 10.2.7. Let $F$ be a finitary set functor. Then a coalgebra is lfp iff it is locally finite.

We will obtain this as an easy consequence of a much more general categorical result in Section 10.5. Here we provide a set theoretic proof.
Proof. For the “only if” direction, let \((A, \alpha)\) be an lfp coalgebra. Then it is a filtered colimit of finite coalgebras with limit injections \(c_i: (A_i, \alpha_i) \to (A, \alpha)\), say. Every element \(x \in A\) lies in \(c_i[A_i]\) for some \(i\), and we know that \(c_i[A_i]\) carries a subcoalgebra of \((A, \alpha)\) by Remark 10.2.6(4).

For the “if” direction, form the filtered diagram of all finite subcoalgebras \(s_i: (A_i, \alpha_i) \to (A, \alpha)\). Then \(A\) is the union of all the \(A_i\). Therefore, for the underlying diagram in \(\text{Set}\), the cocone \(s_i: A_i \to A\) is a filtered colimit. By Remark 4.1.7, this cocone is a filtered colimit in \(\text{Coalg}\ F\), which proves that \((A, \alpha)\) is an lfp coalgebra.

Corollary 10.2.8. For a finitary set functor every a coalgebra is lfp iff it is the union of all of its finite subcoalgebras.

Example 10.2.9. A deterministic automaton, considered as a coalgebra for \(FX = \{0, 1\} \times X^\Sigma\) (see Example 2.4.2(4)), is lfp iff from every of its states only finitely many states are reachable (by inputs from \(\Sigma^*\)). Indeed, the subcoalgebra generated by a state \(x\) consists precisely of those states reachable from \(x\).

The terminal coalgebra \(\nu F\) formed by all formal languages on \(\Sigma\) (see Example 2.5.5) is not an lfp coalgebra. Indeed, the states reachable from a language \(L\) in \(\nu F\) are its left derivatives \(w^{-1}L\) for \(w \in \Sigma^*\). It is well-known that a language has finitely many left derivatives iff it is regular. Thus, every non-regular language demonstrates that \(\nu F\) is not lfp.

In contrast, the set of all regular languages over \(\Sigma\) carries a locally finite subcoalgebra of \(\nu F\).

Notation 10.2.10. From now on we denote the full subcategory of \(\text{Coalg}\ F\) given by all lfp coalgebras by \(\text{Coalg}_{\text{lfp}} F\).

Remark 10.2.11. (1) Every category \(C\) has a free completion \(\text{Ind} C\) under filtered colimits (called the \(\text{Ind}\)-completion of \(C\)). This means that \(\text{Ind} C\) contains \(C\) as a full subcategory, it has filtered colimits, and the following universal property: every functor \(F: C \to \mathcal{A}\) where \(\mathcal{A}\) has filtered colimits has a finitary extension \(\hat{F}: \text{Ind} C \to \mathcal{A}\) which is unique up to natural equivalence.

(2) Recall [169] that the \(\text{Ind}\)-completion of a category \(\mathcal{C}\) with finite colimits is the category of all functors from \(\mathcal{C}^{\text{op}}\) to \(\text{Set}\) preserving finite limits. The category \(\mathcal{C}\) is considered as a full subcategory of \(\text{Ind} \mathcal{C}\) via the contravariant hom-functors \(\mathcal{C}(-, C): \mathcal{C}^{\text{op}} \to \text{Set}\). In this case \(\text{Ind} \mathcal{C}\) is a locally finitely presentable category.

Theorem 10.2.12 [178, Thm. 2.7]. The category \(\text{Coalg}_{\text{lfp}} F\) is the \(\text{Ind}\)-completion of \(\text{Coalg}_l F\).

Proof. We use a result from Johnstone’s book [139, Sect. VI.1.8]: if (a) the category \(\mathcal{C}\) has finite colimits, and (b) \(I: \mathcal{C} \to \mathcal{E}\) is a full embedding into a cocomplete category \(\mathcal{E}\) whose image consists of finitely presentable objects in \(\mathcal{E}\), then the unique finitary extension \(I^*: \text{Ind} \mathcal{C} \to \mathcal{E}\) is also a full embedding.

We apply this to \(\mathcal{E} = \text{Coalg}_l F\) and \(\mathcal{C} = \text{Coalg}_{\text{lfp}} F\). The former is cocomplete since so is \(\mathcal{A}\) and the forgetful functor \(U: \text{Coalg}_l F \to \mathcal{A}\) creates colimits. Similiary, \(\mathcal{C}\) is finitely cocomplete (see Remark 10.2.4).

269
Furthermore, we know [41] that for every finitary functor $F$ on a locally finitely presentable category, every coalgebra with a finitely presentable carrier is a finitely presentable object in $\text{Coalg}(F)$.

Then we can apply Johnstone’s theorem: the unique (finitary) extension of the full embedding $\text{Coalg}_f F \hookrightarrow \text{Coalg} F$ is itself a full embedding $\text{Ind}(\text{Coalg}_f F) \to \text{Coalg} F$. The definition of this extension is that it takes formal filtered diagrams of objects in $\text{Coalg}_f F$ and constructs their colimit in $\text{Coalg} F$. Therefore, according to Definition 10.2.5, its image is precisely $\text{Coalg}_{lfp}(F)$, so by restricting we obtain an equivalence $\text{Ind}(\text{Coalg}_f F) \cong \text{Coalg}_{lfp} F$ as desired.

**Corollary 10.2.13.** The category $\text{Coalg}_{lfp} F$ is locally finitely presentable.

In particular, it follows that $\text{Coalg}_{lfp} F$ is complete (see Remark 10.1.12(1) and therefore has a terminal object.

**Definition 10.2.14.** The rational fixed point of $F$ is the terminal lfp coalgebra. We denote it by $\varrho F$.

We shall see in Theorem 10.2.20 that the coalgebra structure of $\varrho F$ is indeed an isomorphism.

As an immediate consequence we obtain the following coalgebraic construction of the rational fixed point due to [34, Section 3]:

**Corollary 10.2.15.** The rational fixed point is the colimit of all coalgebras on finitely presentable carriers:

$$\varrho F = \text{colim}(\text{Coalg}_f F \hookrightarrow \text{Coalg} F).$$

Indeed, in every locally finitely presentable category the terminal object is the colimit of all finitely presentable objects.

For a finitary set functor, we obtain the following construction of the rational fixed point. It shows that the rational fixed point is fully abstract w.r.t. final semantics: every of its elements is the semantics of a state in a finite coalgebra (“no junk”) and no two behaviourally equivalent states are separated (“no confusion”). Shortly, $\varrho F$ is a subcoalgebra of $\nu F$. We shall see a more general result in Proposition 10.5.6 where we present sufficient conditions on a general locally finitely presentable category $\mathcal{A}$ and $F$ ensuring full abstractness. Here we present a proof for the special case of set functors for the convenience of the reader.

**Proposition 10.2.16.** The rational fixed point $\varrho F$ of a finitary set functor $F$ is the subcoalgebra of $\nu F$ on all elements generating a finite subcoalgebra of $\nu F$.

**Proof.** It suffices to prove that $\varrho F$ is (isomorphic to) the union of all finite subcoalgebras of $\nu F$. Let us denote this union by $R$ and denote by $m: \varrho F \to \nu F$ the unique coalgebra homomorphism.

We first prove that $m$ is injective. We know from Corollary 10.2.8 that $\varrho F$ is the union of all its finite subcoalgebras. Thus, given a pair $x, y \in \varrho F$ with $m(x) = m(y)$, we can choose a subcoalgebra $(A, \alpha)$ of $\varrho F$ containing $x$ and $y$. The unique coalgebra homomorphism $h: (A, \alpha) \to (\nu F, \tau)$ merges $x$ and $y$. We factorize $h$ into a strong
epimorphism \( e : (A, \alpha) \to (C, \gamma) \) followed by an inclusion \( s : (C, \gamma) \hookrightarrow (\nu F, \tau) \). Thus we have \( e(x) = e(y) \). We also have a unique coalgebra homomorphism \( g \) from \((C, \gamma)\) to \(qF\). By the universal property of \( qF \), we obtain \( g \cdot e = i \), where \( i \) is the inclusion of the subcoalgebra \((A, \alpha)\) in \( qF \). We thus have

\[
x = i(x) = g \cdot e(x) = g \cdot e(y) = i(y) = y.
\]

We proceed to prove that \( R = qF \). We clearly have an injection \( qF \hookrightarrow R \) since \( qF \) is the union of its finite subcoalgebras, and these may be identified with finite subcoalgebras of \( \nu F \) via extending the inclusions by \( m \).

Conversely, let \( s : (A, \alpha) \hookrightarrow (\nu F, \tau) \) be any finite subcoalgebra. Then \( s = m \cdot h \) where \( h : A \to qF \) is the unique coalgebra homomorphism into the terminal lfp coalgebra. Since \( s \) is injective, it follows that \( h \) is injective, too. Thus we obtain an injection \( R \hookrightarrow qF \), which completes the proof.

**Examples 10.2.17.**

1. Let \( FX = \{0, 1\} \times X^\Sigma \) be the functor whose coalgebras are deterministic automata and whose terminal coalgebra is formed by all formal languages over \( \Sigma \) (see Example 2.5.5). The rational fixed point is thus the subcoalgebra formed by all regular languages over \( \Sigma \).

2. For a polynomial endofunctor \( H_\Sigma \) on \( \text{Set} \), an initial algebra is carried by the set of all finite \( \Sigma \)-trees (see Example 2.2.11) and a terminal coalgebra by the set of all (finite and infinite) \( \Sigma \)-trees (see Theorem 2.5.9). The unique coalgebra homomorphism from a coalgebra \((A, \alpha)\) to the terminal coalgebra assigns to every element its tree expansion (see Remark 2.5.10).

Let us call a \( \Sigma \)-tree *rational* if it has, up to isomorphism, only finitely many subtrees (Ginari [112] has shown that these are precisely Courcelle’s regular \( \Sigma \)-trees [84]). We are going to show that \( qH_\Sigma \) is formed by all rational \( \Sigma \)-trees.

As a concrete example, consider \( FX = X \times X + 1 \) on \( \text{Set} \), which is the polynomial functor for the signature \( \Sigma \) with one binary operation symbol and one constant. We saw in Example 2.5.11(4) that its terminal coalgebra consists of all binary trees. For example, all finite trees and the following infinite one

![Binary Tree Example](image-url)

are rational, but the tree represented by

\[
(t_0 * (t_1 * (t_2 * (\cdots))))
\]

where \( t_i \) is the complete binary tree of height \( i \), is not rational.

Using Proposition 10.2.7, we derive that the rational fixed point \( qH_\Sigma \) is the subcoalgebra of \( \nu H_\Sigma \) formed by all rational \( \Sigma \)-trees. Indeed, recall from Example 2.5.8 that the coalgebra structure of \( \nu H_\Sigma \) takes a \( \Sigma \)-tree \( t \) whose root is labelled by an \( n \)-ary operation
symbol $\sigma$ to the pair $(\sigma,(t_0,\ldots,t_{n-1}))$ where the $t_i$ are the maximum proper subtrees of $t$. Consequently, every tree $t$ generates the subcoalgebra of all subtrees of $t$ up to isomorphism.

(3) As an illustration, let $\Sigma$ be an alphabet meaning finite nonempty set; everybody knows that and we do not need to say "of $n$ elements" because we do not need the number $n$. Let $\mathcal{P}_\sigma$ be the polynomial functor for the signature of two $|\Sigma|$-ary operation symbols. Trees for that signature are clearly in bijective correspondence with functions $\Sigma^* \to \{0,1\}$, i.e. formal languages over $\Sigma$ (cf. Remark 2.2.13). Under this correspondence, rational trees are precisely the regular languages over $\Sigma$.

(4) Similarly, all rational finitely branching strongly extensional trees form the rational fixed point of $\mathcal{P}$ (cf. Example 2.2.7(4) and Theorem 4.5.7).

(5) For the bag functor $\mathcal{B}: \mathbf{Set} \to \mathbf{Set}$ the initial algebra consists of all finite unordered trees (see Example 3.2.10) and the terminal coalgebra of all unordered trees (see Example 4.3.30(4)). Thus the rational fixed point is formed by all rational unordered trees.

(6) Let $FX = \Sigma \times X$ for a set $\Sigma$, then we have $\nu F = \Sigma^\omega$, the coalgebra of all streams in $\Sigma$ (see Example 2.5.3(3)). Since the coalgebra structure on $\nu F$ is given by the head and tail functions we see that the subcoalgebra generated by a stream $(\sigma_n)_{n \in \mathbb{N}}$ consists of all suffixes $(\sigma_{n+i})_{n \in \mathbb{N}}$. Clearly, a stream has finitely many different suffixes iff it is eventually periodic, i.e. it is of the form $u\omega = uvv\cdots$ for finite words $u, v \in \Sigma^*$. Thus, the rational fixed point is formed precisely of all eventually periodic streams in $\Sigma$.

(7) Automatic sequences [49] are binary streams computed by Moore automata. More precisely, a Moore automaton $(Q,\{0,1\},\delta,s_0,\lambda)$ over the input alphabet $\{0,1\}$ and output alphabet $A$ computes the stream $(s_n)_{n \in \mathbb{N}}$ with $s_n = \lambda(\delta^*(q_0,(n)_2))$, where $(n)_2$ denotes the binary representation of $n$, and $\delta^*$ is the extension of the next state function $\delta: Q \times \Sigma \to Q$ to words. A stream $s$ is called 2-automatic if it is computed by some Moore automaton. In Example 2.4.2(6) and 2.5.11(8) we have seen that Moore automata are coalgebras for $FX = A \times X \times X$ and that the set of streams $A^\omega$ carries a terminal $F$-coalgebra. Grabmeyer et al. [118, Thm. 52] proved that a stream $s$ is 2-automatic if and only if the smallest subcoalgebra of $A^\omega$ containing it is finite. Hence, it follows from Proposition 10.2.16 that $\rho F$ consists precisely of the 2-automatic streams. Analogously, $k$-automatic sequences in $A^\omega$ are computed by Moore automata with input alphabet $\Sigma = \{1,\ldots,k\}$, which are coalgebras for the functor $FX = A \times X^k$, $k \in \mathbb{N}$. We mentioned in Example 2.5.11(8) that $A^\omega$ carries a terminal $F$-coalgebra, and it follows from the result in op. cit. that $\rho F$ consists precisely of the $k$-automatic sequences.

**Remark 10.2.18.** For the set functor $FX = \mathbb{N} \times X$ we shall see, in addition to the description of $\nu F$ and $\rho F$ as (eventually periodic) streams, an interesting description of these two fixed points in Section 15.4: $\nu F$ is carried by the set $\mathbb{R}_{\geq 0}$ of non-negative real numbers, and $\rho F$ is then given by all quadratic irrationals, i.e. the roots of irreducible polynomials with rational coefficients.

The key ingredient for proving that $\rho F$ is a fixed point of $F$ is the following
Proposition 10.2.19. If \((A, \alpha)\) is an lfp coalgebra, then so is \((FA, F\alpha)\).

Proof. It is sufficient to prove the result for coalgebras with a finitely presentable carrier. The general case then follows: given \((A, \alpha)\) in \(\text{Coalg}_{\text{fip}} F\), expressed as a filtered colimit of coalgebras \(\alpha_i: A_i \to FA_i\) with \(A_i\) finitely presentable, then \(F\alpha: FA \to FFA\) is a filtered colimit of the coalgebras \((FA_i, F\alpha_i)\) since \(F\) is finitary. Thus \((FA, F\alpha)\) is lfp since each \((FA_i, F\alpha_i)\) is lfp and \(\text{Coalg}_{\text{fip}} F\) is closed under filtered colimits.

Now suppose that \((A, \alpha)\) is a coalgebra with \(A\) finitely presentable. We prove that \((FA, F\alpha)\) is a colimit of the diagram \(D: \emptyset \to \text{Coalg}_F \hookrightarrow \text{Coalg} F\) that we now describe. The diagram scheme \(\emptyset\) has as objects morphisms \(p: P \to FA\) with \(P\) finitely presentable, and \(D\) assigns to it the following coalgebra in \(\text{Coalg}_F\):

\[
\bar{p} = (P + A \xrightarrow{[p, \alpha]} FA \xrightarrow{\text{F\text{inr}}} F(P + A)).
\]

A morphism in \(\emptyset\) from \(p: P \to FA\) to \(q: Q \to FA\) is a coalgebra homomorphism \(h: (P + A, \bar{p}) \to (Q + A, \bar{q})\) which fulfills \([p, \alpha] = [q, \alpha] \cdot h:\)

\[
\begin{array}{ccc}
P + A & \xrightarrow{\bar{p}} & F(P + A) \\
\downarrow{h} & & \downarrow{Fh} \\
Q + A & \xrightarrow{\bar{q}} & F(Q + A)
\end{array}
\]

By proving that \((FA, F\alpha)\) is the colimit of \(D\) we conclude that this coalgebra is lfp: while \(D\) is itself not filtered, its closure under finite colimits in \(\text{Coalg}_F\) is and has the same colimit by Lemma 10.1.13. We first prove some preliminary facts:

(1) The unique morphism \(u: 0 \to FC\), where 0 is the initial object, yields \(\bar{u} = (\alpha: A \to FA)\). Moreover, \(\alpha: A \to FA\) itself yields the object \((A + A, \bar{\alpha})\) for which the codiagonal \(\nabla: A + A \to A\) is a connecting morphism of \(D\) from \((A + A, \bar{\alpha})\) to \((A, \bar{u})\) as shown by the commutative diagram below:

\[
\begin{array}{ccc}
A + A & \xrightarrow{[\alpha, \alpha]} & FA \\
\downarrow{\nabla} & & \downarrow{F\nabla} \\
FA & \xrightarrow{\alpha} & FA
\end{array}
\]

(2) Every morphism \(m\) from \(p: P \to FA\) to \(q: Q \to FA\) in the slice category \(\text{Coalg}_F/FA\), i.e. with \(q \cdot m = p\), yields the connecting morphism \(m + A\) of \(D\) as shown by the
10 Fixed Points Determined by Finite Behaviour

commutative diagram below:

![Diagram](image)

(3) The diagram $D$ has the following cocone in $\text{Coalg} F$:

$$[p, \alpha]: (P + A, \bar{p}) \to (FA, F\alpha),$$

where $p$ ranges over $\mathcal{A}_p/FA$. Indeed, the fact that each $[p, \alpha]$ is a homomorphism is clear:

$$[p, \alpha]: P + A \to FA$$

Compatibility follows from the left-hand triangle in (10.2).

(4) We are ready to prove that the cocone in point (3) is a colimit of $D$. In order to prove the universal property we use that the forgetful functor $U: \text{Coalg} F \to \mathcal{A}$ creates all colimits (see Proposition 4.1.1). Hence, it suffices to prove that $[p, \alpha]: P + A \to FA$, for all $p$ in $\mathcal{A}_p/FA$, form a colimit of $UD$. So suppose that

$$c_p: P + A \to C \quad (p \text{ in } \mathcal{A}_p/FA)$$

is a cocone of $UD$ in $\mathcal{A}$. We shall show that there exists a unique $c: FA \to C$ such that $c_p = c \cdot [p, \alpha]$ for all $p$.

First, it is easy to see that the morphisms $c_p \cdot \text{inl}: P \to C$ form a cocone of the canonical diagram $\mathcal{A}_p/FA \to \mathcal{A}$ whose colimit is $FA$: in fact, given a morphism $m$ as in part (2), the given cocone fulfils $c_q \cdot (m + A) = c_p$, hence, $c_q \cdot \text{inl} \cdot m = c_p \cdot \text{inl}$. Therefore there exists a unique $c: FA \to C$ such that $c_p \cdot \text{inl} = c \cdot p$ for all $p$ in $\mathcal{A}_p/FA$. Thus, to conclude $c_p = c \cdot [p, \alpha]$, it only remains to prove that $c_p \cdot \text{inr} = c \cdot \alpha$ holds for all $p$. The connecting morphism $\nabla$ in part (1) yields

$$c_u \cdot \nabla = c_\alpha: A + A \to C.$$

This implies that

$$c_u = c_\alpha \cdot \text{inl} = c \cdot \alpha \quad (10.3)$$

by the definition of $c$ for $p = \alpha$. Moreover, the unique morphism $u: 0 \to P$ is a morphism in $\mathcal{A}_p/FA$ from the unique $0 \to FA$ to $p: P \to FA$ and therefore yields the connecting morphism $u + A = \text{inr}: (A, \bar{u}) \to (P + A, \bar{p})$ using part (2). Thus, we conclude

$$c_p \cdot (u + A) = c_p \cdot \text{inr} = c_u = c \cdot \alpha,$$

where the last step uses (10.3).
The following result was first proved in [34, Thm. 3.3]. We now obtain a short proof by virtue of Proposition 10.2.19.

**Theorem 10.2.20.** The coalgebra structure of \( \varrho F \) is an isomorphism, i.e. \( \varrho F \) is a fixed point of \( F \).

**Proof.** We argue as in Lambek’s lemma: denote by \( \omega : \varrho F \to F(\varrho F) \) be the coalgebra structure. By Proposition 10.2.19 we know \((F(\varrho F), F\omega)\) is an lfp coalgebra. Therefore we have a homomorphism \( s : (F(\varrho F), F\omega) \to (\varrho F, \omega) \). Since \( \omega : (\varrho F, \omega) \to (F(\varrho F), F\omega) \) is clearly a homomorphism, we have that \( s \cdot \omega \) is a homomorphism from \((\varrho F, \omega)\) to itself. Thus, \( s \cdot \omega = \text{id} \) by the universal property of \((\varrho F, \omega)\). It follows that

\[
\omega \cdot s = Fs \cdot F\omega = F(s \cdot \omega) = F\text{id} = \text{id}.
\]

10.3 Iterative Algebras

Recall from Theorem 7.2.13 that the terminal coalgebra \( \nu F \) is the initial cia for \( F \). A very similar characterization holds for the rational fixed point \( \varrho F \): it is the initial iterative algebra for \( F \). This notion is weaker than the notion of a completely iterative algebra (see Definition 7.2.2) in that one only requires unique solutions of flat equation morphisms whose domain is finitely presentable. (In \( \text{Set} \) this means that only finite systems (7.3) are required to have unique solutions.)

Iterative algebras for polynomial set functors were introduced by Nelson [191] (see Tiuryn [230] for a related concept) in connection with Elgot’s iterative theories [93]. Much later this was generalized from \( \text{Set} \) to arbitrary locally finitely presentable categories [31, 34]. In this section we only mention the most important results on iterative algebras.

Recall from Definition 7.2.2 the notions of a flat equation morphism and its solution in an \( F \)-algebra.

**Definition 10.3.1** [34, Def. 2.5]. A flat equation morphism \( e : X \to FX + A \) is called finitary if \( X \) is a finitely presentable object of \( A \).

An algebra \((A, \alpha)\) is called iterative if every finitary flat equation morphism has a unique solution \( \hat{e} \) in \( A \).

**Example 10.3.2.** (1) Every cia is, of course, an iterative algebra.

For example, we saw in Example 7.2.9(2) that the interval \([0, 1]\) is a cia for the functor \( FX = X + X \) on \( \text{Set} \), and so it is iterative. It is interesting to note that the solutions of the finitary flat equation morphisms are precisely the rational numbers in the unit interval.

(2) Unary algebras \( \alpha : A \to A \) in \( \text{Set} \) are the algebras for the identity functor. Iterativity of \((A, \alpha)\) means that \( \alpha \) has a unique cycle, and this is a fixed point (see [34, Example 2.13] and cf. Example 7.2.6(4)).

An example of an iterative algebra that is not a cia is the unary algebra \( A = \mathbb{N} \) with the algebra structure given by \( \alpha(0) = 0 \) and \( \alpha(n + 1) = n \). Indeed, its only cycle is the fixed point 0. However, for \( X = \{x_n : n \in \mathbb{N}\} \) the flat equation morphism \( e : X \to X + A \)
10 Fixed Points Determined by Finite Behaviour

with \( e(x_n) = x_{n+1} \) has two solutions in \( A \): the constant function with value 0 and the function \( x_n \mapsto n \).

(3) The algebra of addition on the extended real numbers \( \bar{\mathbb{R}} = \mathbb{R} \cup \{ \infty \} \) is an iterative algebra for the functor \( FX = X \times X \) [34]. Note that this is not a completely iterative algebra: the system of equations \( x_0 \approx x_1 + 1, x_1 \approx x_2 + 1, \ldots \) has more than one solution (e.g. \( x_n^\dagger = \infty \) or \( x_n^\dagger = -n \)).

**Remark 10.3.3.** Note that in an iterative algebra \( (A, \alpha) \) we have a unique solution for every lfp coalgebra \( e: X \to FX + A \). Indeed, write \( (X, e) \) as a filtered colimit of a diagram \( D \) of coalgebras \( (X_i, e_i) \) in \( \text{Coalg}_f(F(-) + A) \). Then the latter coalgebras are finitary flat equation morphisms. Their unique solutions \( e_i^\dagger: X_i \to A \) form a cocone of \( D \) and therefore induce a unique cocone morphism \( e^\dagger: X \to A \). A routine verification then shows that \( e^\dagger \) is the unique solution of \( e \) as desired.

Iterative algebras constitute a full subcategory of \( \text{Alg} F \). The proof is a variation of the corresponding proof for cias (cf. Proposition 7.2.4). Since the reasoning in the second part of the proof is quite typical for an argument in a locally finitely presentable category we fully spell out the details:

**Proposition 10.3.4** [34, Prop. 2.18]. Let \( (A, \alpha) \) and \( (B, \beta) \) be iterative algebras for \( F \). Then a morphism \( h: A \to B \) is an algebra homomorphism if and only if it preserves solutions.

**Proof.** For the ‘only if’ direction one proceeds as in the proofs of Propositions 7.2.4 and 7.1.4, i.e. given a homomorphism \( h: (A, \alpha) \to (B, \beta) \) one shows for every finitary flat equation morphism \( e: X \to FX + A \) that \( h \cdot e^\dagger \) is a solution of \( (FX + h) \cdot e \) and concludes using uniqueness of solutions.

For the ‘if’ direction, suppose that \( h: A \to B \) preserves solutions. By Remark 10.1.12(3), \( A \) is the colimit of the canonical diagram \( D_A \) of all morphisms \( p: P \to A \) with \( P \) finitely presentable. Similarly \( FA = \text{colim} D_{FA} \). Hence, in order to prove that \( h \) is a homomorphism it suffices to prove

\[
\tag{10.4}
h \cdot \alpha \cdot p = \beta \cdot Fh \cdot p \quad \text{for all } p: P \to FA \text{ with } P \text{ finitely presentable.}
\]

Since \( F \) is finitary, we have \( FA = \text{colim} FD_A \). Because \( P \) is finitely presentable we know that the hom-functor \( \mathcal{A}(P, -) \) preserves that colimit. Therefore, we obtain morphisms \( q \) in \( \mathcal{A}_{fp}/A \) and \( p': P \to FQ \) such that the triangle below commutes:

\[
\begin{array}{ccc}
FQ & \xrightarrow{\psi'} & Fq \\
\downarrow & & \downarrow \\
P & \xrightarrow{p} & FA
\end{array}
\]

We then form the following finitary flat equation morphism

\[
e = (P + Q \xrightarrow{\psi+Q} FQ + Q \xrightarrow{\text{inr}+q} F(P + Q) + A).
\]

276
The solution \( e^\dagger \) yields the commutative square below:

\[
\begin{array}{ccc}
P + Q & \xrightarrow{e^\dagger} & A \\
\downarrow{p' + Q} & & \uparrow{[\alpha, A]} \\
FQ + Q & \xrightarrow{F[\alpha, p] + h} & FA + B \\
\downarrow{F[\alpha, p] + h} & & \uparrow{[\beta, B]} \\
F(P + Q) + A & \xrightarrow{F[\alpha, p] + h} & FB + B \\
\end{array}
\]

This shows that \( e^\dagger \cdot \text{inr} = q \) and therefore

\[
e^\dagger \cdot \text{inl} = \alpha \cdot F(e^\dagger \cdot \text{inr}) \cdot p' = \alpha \cdot Fq \cdot p' = \alpha \cdot p.
\]

Since \( h \) preserves the solution of \( e \) we have \( h \cdot e^\dagger = f^\dagger \) for

\[
f = (X \xrightarrow{\epsilon} F(P + Q) + A \xrightarrow{F(P + Q) + h} F(P + Q) + B).
\]

Thus we have the following equations:

\[
f^\dagger = h \cdot e^\dagger = h \cdot [\alpha \cdot p, q] = [h \cdot \alpha \cdot p, h \cdot q]. \tag{10.5}
\]

Analysing the solution of \( f \) further we obtain the following commutative diagram:

\[
\begin{array}{ccc}
P + Q & \xrightarrow{f^\dagger} & B \\
\downarrow{p' + Q} & & \uparrow{[\beta, B]} \\
FQ + Q & \xrightarrow{F[\alpha, p] + h} & FA + B \\
\downarrow{F[\alpha, p] + h} & & \uparrow{[\beta, B]} \\
F(P + Q) + A & \xrightarrow{F[\alpha, p] + h} & FB + B \\
\end{array}
\]

The outside commutes because \( f^\dagger \) is a solution of \( f \), for the lowest triangle use (10.5), and the remaining triangles are trivial. Thus the upper right-hand part commutes:

\[
f^\dagger = [\beta \cdot Fh \cdot p, h \cdot q]. \tag{10.6}
\]

The left-hand components of (10.5) and (10.6) establish the desired equality (10.4).

**Lemma 10.3.5.** If \( (A, \alpha) \) is an iterative algebra, then so is \( (FA, F\alpha) \).

The proof is the same as for cias in Lemma 7.2.10(1).

**Proposition 10.3.6** [34, Thm. 2.20]. The category of iterative algebras is closed under limits and filtered colimits in the category \( \text{Alg}_F \) of \( F \)-algebras. Thus, it is a reflective subcategory of \( \text{Alg}_F \) with limits and filtered colimits constructed on the level of the base category \( \mathcal{A} \).
Proof. (1) For limits the argument is identical to that in Lemma 7.2.10(2) but using the ‘only if’ part of Proposition 10.3.4 in lieu of Proposition 7.2.4.

(2) We now prove closedness under filtered colimits. Suppose we have a filtered diagram $D$ of iterative algebras $(A_i, \alpha_i)$, $i \in I$, and let $(A, \alpha)$ be its colimit in $\mathcal{A}$ with colimit injections $c_i : (A_i, \alpha_i) \to (A, \alpha)$. Since $F$ is finitary, filtered colimits of $F$-algebras are formed on the level of $\mathcal{A}$; more precisely, the forgetful functor $\text{Alg} F \to \mathcal{A}$ creates filtered colimits (cf. Remark 4.1.7). Let $e : X \to FX + A$ be a finitary flat equation morphism.

Since $X$ is finitely presentable and $FX + A = \text{colim}_{i \in I}(FX + A_i)$, there exist an $i$ and a morphism $e_i : X \to FX + A_i$ such that the triangle below commutes:

$$
\begin{array}{ccc}
FX + A_i & \rightarrow & FX + c_i \\
\downarrow & & \downarrow \\
X & \rightarrow & FX + A.
\end{array}
$$

By Proposition 10.3.4, we know that $c_i \cdot e_i^\dagger : X \to A$ is a solution of $e$ in $A$.

In order to prove uniqueness, we show for any solution $s : X \to A$ of $e$ in $A$ that $s = c_i \cdot e_i^\dagger$. First, use that $X$ is finitely presentable to obtain some $j \in I$ and $s'$ such that the following triangle commutes:

$$
\begin{array}{ccc}
A_j & \rightarrow & A \\
\downarrow & & \downarrow \\
X & \rightarrow & A
\end{array}
$$

Since $D$ is filtered, we can assume that for $j$ above we have a homomorphism $h : (A_i, \alpha_i) \to (A_j, \alpha_j)$ in the diagram, i.e. $c_j \cdot h = c_i$. Furthermore, we show that $j$ can be chosen such that $s'$ is a solution of $e_j = (FX + h) \cdot e_i$. To see this we consider the following diagram:

$$
\begin{array}{ccc}
X & \rightarrow & A_j & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
FX + A_i & \rightarrow & (A_j, \alpha_j) & \rightarrow & [\alpha, A] \\
\downarrow & & \downarrow & & \downarrow \\
FX + h & \rightarrow & FA_j + A_j & \rightarrow & FA + A \\
\downarrow & & \downarrow & & \downarrow \\
FX + c_i & \rightarrow & FX + c_j & \rightarrow & FA + A
\end{array}
$$

Note that all its inner parts except perhaps part $(\ast)$ commute. Thus, this part commutes when postcomposed by $c_j$. By filteredness we can therefore assume that $(\ast)$ commutes (for otherwise we can choose a homomorphism $g : (A_j, \alpha_j) \to (A_k, \alpha_k)$ in the diagram $D$ equating the sides of $(\ast)$ and work with $k$ in lieu of $j$). Thus, by the uniqueness of
10.3 Iterative Algebras

solutions in \( A_j \), we have \( s' = e_j^\dagger \). Moreover, by Proposition 10.3.4, we obtain \( e_j^\dagger = h \cdot e_i^\dagger \) and we conclude

\[
s = c_j \cdot s' = c_j \cdot e_j^\dagger = c_j \cdot h \cdot e_i^\dagger = c_i \cdot e_i^\dagger.
\]

(3) Reflectivity now follows from the fact that the category \( \text{Alg} F \) of all algebras for \( F \) is locally finitely presentable (cf. [43, Cor. 2.75]). Moreover, every full subcategory of a locally finitely presentable category closed under limits and filtered colimits is reflective [43, Thm. 2.48].

\[
\square
\]

Corollary 10.3.7. Every object of \( \mathcal{A} \) generates a free iterative algebra.

Indeed, it follows from Proposition 2.2.20 and Theorem 3.1.7 that on every object of \( \mathcal{A} \) a free algebra \( (A, \alpha) \) exists. The reflection of \( (A, \alpha) \) in the category of iterative algebras is the free iterative algebra on \( A \). (This follows from the fact that left adjoint functors compose.)

In particular, the reflection of the initial algebra \( \mu F \) in the category of iterative algebras is the initial iterative \( F \)-algebra. Furthermore, we next prove that the colimit in Corollary 10.2.15 yields the initial iterative \( F \)-algebra.

Remark 10.3.8. Recall [164, Sec. IX.3] that a functor \( P: \mathcal{D} \to \mathcal{C} \) is final if for every object \( X \) of \( \mathcal{D} \) the comma category \( X/F \) is nonempty and connected. In the case where \( \mathcal{C} \) and \( \mathcal{D} \) are filtered this is equivalent to the following two conditions:

1. for every object \( C \) of \( \mathcal{C} \) there exists a morphism \( C \to PX \) for some object \( X \) of \( \mathcal{D} \),
2. for every parallel pair \( f, g: C \to PX \) in \( \mathcal{C} \) there exists \( d: X \to X' \) in \( \mathcal{D} \) with \( Pd \) merging \( f \) and \( g \), i.e. \( Pd \cdot f = Pd \cdot g \).

It follows that for every diagram \( D: \mathcal{C} \to \mathcal{A} \) the colimits of \( D \) and \( D \cdot P \) coincide [164, Thm. IX.3.1].

Theorem 10.3.9 [34, Thm. 3.3]. The rational fixed point \( \varrho F \) is the initial iterative algebra.

Proof. Denote by \( \omega: \varrho F \to F(\varrho F) \) the coalgebra structure. We prove in part (2) below that the algebra \( (\varrho F, \omega^{-1}) \) is iterative. Its initiality is then easy to establish.

1. Initiality of \( \varrho F \). Let \( (A, \alpha) \) be an iterative algebra for \( F \). First observe that for every coalgebra \( e: X \to FX \) in \( \text{Coalg}_F \) we have a unique coalgebra-to-algebra morphism \( e^\dagger: X \to A \). The proof is identical to what we saw in Remark 7.2.3(1). It follows that for every lfp coalgebra \( e: X \to FX \) in \( \text{Coalg}_F \) we have a unique coalgebra-to-algebra morphism \( e^\dagger: X \to A \). Indeed, write \( (X, e) \) as a colimit of coalgebras \( (X_i, e_i), \ i \in I \), in \( \text{Coalg}_F \).

Then the \( e_i^\dagger: X_i \to A \) form a cocone and the induced morphism \( e^\dagger \) with \( e^\dagger \cdot e_i^\dagger = e_i^\dagger \) is the desired unique coalgebra-to-algebra morphism. Applying this to the lfp coalgebra \( (\varrho F, \omega) \) we see that there exists a unique coalgebra-to-algebra morphism from \( (\varrho F, \omega) \) to \( (A, \alpha) \), i.e. a unique algebra homomorphism from \( (\varrho F, \omega^{-1}) \) to \( (A, \alpha) \), as desired.

2. \( \varrho F \) is an iterative algebra. The proof follows the same reasoning as that in Proposition 7.2.5. Here we need to establish that the construction performed there yields an lfp
coalgebra. More precisely, given a finitary flat equation morphism \( e: X \to FX + \varrho F \), we form the coalgebra

\[
\bar{e} = (X + \varrho F \xrightarrow{\text{can}} FX + \varrho F \xrightarrow{FX + \omega} FX + F(\varrho F) \xrightarrow{\text{can}} F(X + \varrho F)),
\]

where \( \text{can} = [\text{Finl}, \text{Finr}] \). We prove below that this is an lfp coalgebra.

The remainder of the proof then proceeds precisely as the one of Proposition 7.2.5. In fact, we obtain a unique homomorphism \( \bar{e}^\sharp: (X + \varrho F, \bar{e}) \to (\varrho F, \omega) \) using that \((\varrho F, \omega)\) is the terminal lfp coalgebra. One then proves that \( e^\dagger = \bar{e}^\sharp \cdot \text{inl}: X \to \varrho F \) is the desired unique solution of \( e \).

We now turn to the proof that the coalgebra \((X + \varrho F, \bar{e})\) is lfp. Thus, our proof will be finished once we present a diagram \( D: \mathcal{D} \to \text{Coalg}_F \) whose colimit is \((X + \varrho F, \bar{e})\).

(1a) The diagram \( D \). The diagram scheme \( \mathcal{D} \) has as objects pairs consisting of a coalgebra \((A, \alpha) \in \text{Coalg}_F\) and a morphism \( f: X \to FX + A \) (where \( X \) is the above domain of \( e \)) such that for the unique homomorphism \( \alpha^\sharp: (A, \alpha) \to (\varrho F, \omega) \) the following triangle commutes:

\[
\begin{array}{ccc}
FX + A & \xrightarrow{\alpha^\sharp} & FX + \varrho F \\
X & \xrightarrow{e} & FX + \varrho F,
\end{array}
\]

(10.7)

A morphisms in \( \mathcal{D} \) from \(((A, \alpha), f)\) to \(((B, \beta), g)\) is a homomorphism \( h: (A, \alpha) \to (B, \beta) \) such that \((FX + h) \cdot f = g\).

For every object \(((A, \alpha), f)\) of \( \mathcal{D} \) we have a coalgebra in as follows:

\[
\tilde{f} = (X + A \xrightarrow{[f, \text{inr}]} FX + A \xrightarrow{FX + \alpha} FX + FA \xrightarrow{\text{can}} F(X + A)),
\]

Since \( X \) and \( A \) are finitely presentable, so is \( X + A \), and therefore \((X + A, \tilde{f})\) lies in \( \text{Coalg}_F \). This defines the diagram \( D \) on objects, and for a morphism \( h \) of \( \mathcal{D} \) we define \( Dh = X + h \). We need to verify that \( Dh \) is a homomorphism. This follows from the commutativity of the following diagram:

\[
\begin{array}{ccc}
X + A & \xrightarrow{[f, \text{inr}]} & FX + A \\
X + h & \xrightarrow{[g, \text{inr}]} & FX + h \\
X + B & \xrightarrow{g} & FX + B \xrightarrow{FX + \beta} FX + FB \xrightarrow{\text{can}} F(X + B)
\end{array}
\]

(10.8)

(1b) Our diagram \( D \) is clearly essentially small, and we now prove that it is filtered by verifying (a)–(c) in Remark 10.1.1(1).

Observe first that, since the forgetful functor \( U: \text{Coalg}_F \to \mathcal{A} \) creates colimits (see Proposition 4.1.1), we know that the unique homomorphisms \( \alpha^\dagger: (A, \alpha) \to (\varrho F, \omega) \), for \((A, \alpha)\) in \( \text{Coalg}_F \), form a colimit cocone in \( \mathcal{A} \). Furthermore, since the functor \( FX + (-): \mathcal{A} \to \mathcal{A} \) clearly is finitary (see Example 10.1.9(2)) we see that the morphisms
Iterative Algebras

\[ FX + \alpha^\sharp : FX + A \to FX + \varrho F \]
form a colimit cocone, too. Since \( X \) is finitely presentable, the morphism \( e : X \to FX + \varrho F \) admits some factorization as in (10.7), which proves that \( \mathcal{D} \) is nonempty, i.e. we have established condition (a).

For condition (b), suppose we are given two objects \(((A, \alpha), f)\) and \(((B, \beta), f')\) of \( \mathcal{D} \). Then we first use that \( \text{Coalg}_F F \) is filtered to obtain homomorphisms \( h : (A, \alpha) \to (C, \gamma) \) and \( h' : (B, \beta) \to (C, \gamma) \). This implies that

\[
\gamma^\sharp \cdot h = \alpha^\sharp \quad \text{and} \quad \gamma^\sharp \cdot h' = \beta^\sharp ,
\]

and furthermore we see that \( FX + \gamma^\sharp \) merges \((FX + h) \cdot f\) and \((FX + h') \cdot f'\):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & FX + A \\
\downarrow{f'} & & \downarrow{FX + h} \\
FX + B & \xrightarrow{e} & FX + C \\
\downarrow{FX + h'} & & \downarrow{FX + \alpha^\sharp} \\
FX + C & \xrightarrow{FX + \gamma^\sharp} & FX + \varrho F
\end{array}
\]

Since \( FX + \gamma^\sharp \) is the colimit injection of a filtered colimit, we thus obtain a homomorphism \( k : (C, \gamma) \to (D, \delta) \) in \( \text{Coalg}_F F \) such that \( FX + k \) merges \((FX + h) \cdot f\) and \((FX + h') \cdot f'\). Defining

\[
f'' = (FX + k \cdot h) \cdot f = (FX + k \cdot h') \cdot f',
\]

we see that \( k \cdot h : ((A, \alpha), f) \to ((D, \delta), f'') \) and \( k \cdot h' : ((B, \beta), f') \to ((D, \delta), f'') \) are morphisms in \( \mathcal{D} \), as desired.

Concerning condition (c), given a parallel pair of morphisms \( h, k : ((A, \alpha), f) \rightrightarrows ((B, \beta), f') \) in \( \mathcal{D} \), their coequalizer \( c : (B, \beta) \to (C, \gamma) \) in \( \text{Coalg}_F F \) yields a morphism from \(((B, \beta), f')\) to \(((C, \gamma), f'')\) in \( \mathcal{D} \), where \( f'' = (FX + c) \cdot f' \). Indeed, we have

\[
FX + \beta^\sharp \cdot f'' = (FX + \gamma^\sharp) \cdot (FX + c) \cdot f' = (FX + \varrho F) \cdot f' = e.
\]

(1c) We prove that the forgetful functor

\[ P : \mathcal{D} \to \text{Coalg}_F F \]

given by \(((A, \alpha), f) \mapsto (A, \alpha)\) is final. For condition (1) in Remark 10.3.8, suppose we are given \((B, \beta)\) in \( \text{Coalg}_F F \). Due to the nonemptiness of \( \mathcal{D} \) we have some \(((A, \alpha), f)\) in \( \mathcal{D} \). Since \( \text{Coalg}_F F \) is filtered we obtain \((C, \gamma)\) with \( C \in \mathcal{A}_f \) and homomorphisms \( h : (A, \alpha) \to (C, \gamma) \) and \( k : (B, \beta) \to (C, \gamma) \). Then \(((C, \gamma), g)\) with

\[
g = (X \xrightarrow{f} FX + A \xrightarrow{FX + h} FX + C)
\]
is the desired object of \( \mathcal{D} \) because the diagram below clearly commutes:
10 Fixed Points Determined by Finite Behaviour

The desired morphism of $\text{Coalg}_f F$ is $k: (B, \beta) \to P((C, \gamma), g)$.

Condition (2) is verified similarly. Given a parallel pair $h, k: (B, \beta) \to P((C, \gamma), g)$ of homomorphisms, their coequalizer $c: (C, \gamma) \to (C', \gamma')$ in $\text{Coalg}_f F$ is a morphism in $\mathcal{D}$ whose image under $P$ merges $h$ and $k$; this is shown by a similar argument as in item (1b) above.

(1d) We are ready to prove that $(X + \varrho F, \bar{e})$ is the colimit of $D$. For every object $((A, \alpha), f)$ in $\mathcal{D}$ we have a homomorphism

$$X + \alpha^\sharp: (X + A, \bar{f}) \to (X + \varrho F, \bar{e}),$$

which follows from the condition (10.7) using a diagram similar to (10.8). Given a morphism $h: ((A, \alpha), f) \to ((B, \beta), g)$ of $\mathcal{D}$ we obtain from $\alpha^\sharp \cdot h = \beta^\sharp$ a commutative triangle

$$\begin{array}{c}
X + A \\
\downarrow X + h \\
X + \alpha^\sharp \\
\downarrow X + \beta^\sharp \\
X + \varrho F
\end{array}$$

Thus, the homomorphisms $X + \alpha^\sharp$ form a cocone on $D$.

In order to prove that this is a colimit cocone, it suffices to show that the morphisms $X + \alpha^\sharp: X + A \to X + \varrho F$ form a colimit of $UD$ in $\mathcal{A}$, since the forgetful functor $U: \text{Coalg}_F \to \mathcal{A}$ creates colimits. Observe that $UD$ can be decomposed as follows:

$$UD = (\mathcal{D} \xrightarrow{P} \text{Coalg}_f F \xrightarrow{I} \text{Coalg}_F \xrightarrow{U} \mathcal{A} \xrightarrow{X+(-)} \mathcal{A}),$$

where $I$ is the inclusion. Since $X + (-)$ is finitary (see Example 10.1.9(2)) and $\varrho F = \text{colim} UI$ we are done because $P$ is final by point (1c).

**Corollary 10.3.10.** For every finitary functor on a locally finitely presentable category, the initial iterative algebra is precisely the terminal lfp coalgebra.

**Remark 10.3.11.** Similar to the alternative definition of cias in Remark 7.2.18, an iterative algebra can, equivalently, be defined as an algebra $(A, \alpha)$ such that for every coalgebra $e: X \to F(X + A)$ with $X$ finitely presentable there exists a unique morphism $e^!: X \to A$ such that the square below commutes:

$$\begin{array}{c}
X \\
\downarrow e \\
F(X + A)
\end{array} \xrightarrow{F[e^!, \text{id}_A]} \begin{array}{c}
A \\
\uparrow \alpha
\end{array}$$

(10.9)

As before for cias, the theory of iterative algebras could be developed based on this alternative definition.

The proof of the equivalence is considerably more involved than for cias. We only give a rough proof sketch.
Proposition 10.3.12. An algebra \((A, \alpha)\) is iterative iff for every \(e: X \to F(X + A)\) with \(X\) finitely presentable there exists a unique morphism \(e^1: X \to A\) such that (10.9) commutes.

Sketch of proof. (1) Suppose that \((A, \alpha)\) is iterative, and let \(e: X \to F(X + A)\) with \(X\) finitely presentable be given. One forms

\[
X + A \xrightarrow{e+A} F(X + A) + A.
\]

We indicate below how one proves that this is an lfp coalgebra for \(F(-) + A\). (Note that the full details are somewhat similar to the proof of part (2) in Theorem 10.3.9). By Remark 10.3.3, we obtain a unique solution \((e + \text{id}_A)^\dagger: X + A \to A\). It is easy to show that \((e + \text{id}_A)^\dagger \cdot \text{inr} = \text{id}_A\), and \((e + \text{id}_A)^\dagger \cdot \text{inl}: X \to A\) is then proved to be the desired unique morphism. As in the proof of Proposition 7.2.19, we leave the details as an easy exercise for the reader.

To see that \((X + A, e + \text{id}_A)\) is an lfp coalgebra one defines a diagram \(D: \mathcal{D} \to \text{Coalg}\ f(F(-) + A)\) as follows. The category \(\mathcal{D}\) consists of triples \((P, p, e_p)\), where \(p: P \to A\) lies in \(A_{fp}/A\) and \(e_p\) witnesses that \(e\) factorizes through \(F(X + P)\):

\[
\begin{array}{ccc}
X & \xrightarrow{e} & F(X + A) \\
& \xleftarrow{e_p} & \downarrow \text{F(X+p)} \\
& & F(X + P)
\end{array}
\]

The morphisms of \(\mathcal{D}, h: (P, p, e_p) \to (Q, q, e_q)\) are morphisms \(h: (P, p) \to (Q, q)\) in \(A_{fp}/A\) such that the triangle below commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{e_p} & F(X + P) \\
& \xleftarrow{e_q} & \downarrow \text{F(X+h)} \\
& & F(X + Q)
\end{array}
\]

The functor \(D\) maps \((P, p, e_p)\) to the coalgebra

\[
D(P, p, e_p) = (X + P \xrightarrow{e_p} F(X + P) + A),
\]

and it is easy to prove that \(Dh = X + h\) is a homomorphism for every morphism \(h\) in \(\mathcal{D}\).

We obtain a cocone of \(D\) with the cocone morphisms

\[
X + p: (X + P, e_p + p) \to (X + A, e + A)
\]

for every object \((P, p, e_p)\) in \(\mathcal{D}\). Indeed, it is easy to see that these are homomorphisms and form a compatible cocone.

The category \(\mathcal{D}\) is clearly nonempty because from the fact that \(X\) is finitely presentable we conclude that there exists \(p: P \to A\) in \(A_{fp}/A\) and \(e_p\) such that (10.10) commutes. Furthermore, one can show that \(\mathcal{D}\) is filtered.
10 Fixed Points Determined by Finite Behaviour

Next one proves that the projection functor \( \Pr: \mathcal{D} \to \mathcal{A}_{fp}/A \) given by \( (P, p, e_p) \mapsto (P, p) \) is final.

Finally, to see that \( (X + A, e + \text{id}_A) = \text{colim} D \) one uses that the forgetful functor \( U: \text{Coalg}(F(-) + A) \to \mathcal{A} \) creates colimit and obtains \( X + A = \text{colim} UD \) using the finality of \( \Pr \) and the following equation

\[
UD = (\mathcal{D} \xrightarrow{\Pr} \mathcal{A}_{fp}/A \xrightarrow{DA} \mathcal{A} \xrightarrow{X+(-)} \mathcal{A}),
\]

where \( DA \) is the functor from Remark 10.1.12(3).

(2) Conversely, suppose that \( (A, \alpha) \) admits for every \( e: X \to F(X + A) \) with \( X \) finitely presentable a unique morphism \( e^\#: X \to A \) such that (10.9) commutes.

First, a similar argument as in Remark 10.3.3 shows that the same holds for every lfp coalgebra \( e: X \to F(X + A) \).

Given a flat equation morphism \( f: X \to FX + A \), we indicate below how one proves that \( Ff: FX \to F(X + A) \) is an lfp coalgebra. Then there exists a unique \( (Ff)^\#: FX \to A \) such that such that (10.9) commutes (for \( e = Ff \)). Now let

\[
f^\# = (X \xrightarrow{f} FX + A \xrightarrow{(Ff)^\# \cdot \text{id}_A} A).
\]

One readily proves that this is a unique solution of \( f \). Again, we leave the details as an exercise for the reader.

To see that \( (FX, Ff) \) is an lfp coalgebra for \( F((-) + A) \) one forms the following diagram

\[
D: \mathcal{D} \to \text{Coalg}_f F((-) + A). \quad \text{The objects of the diagram scheme } \mathcal{D} \text{ are triples } (P, p, e_p) \text{ where } (P, p) \text{ is an object of } \mathcal{A}_{fp}/FX \text{ and } e_p: X \to P + A \text{ witnesses a factorization of } e \text{ through } p + A:
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & FX + A \\
\downarrow{e_p} & & \downarrow{p + A} \\
P + A & \xleftarrow{p + A} & Q + A
\end{array}
\]

Morphisms \( h: (P, p, e_p) \to (Q, q, e_q) \) of \( \mathcal{D} \) are morphisms \( h: (P, p) \to (Q, q) \) of \( \mathcal{A}_{fp}/FX \) such that the triangle below commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{e_p} & P + A \\
\downarrow{e_q} & & \downarrow{h + A} \\
Q + A & \xleftarrow{h + A} & Q + A
\end{array}
\]

The functor \( D \) maps an object \( (P, p, e_p) \) to the coalgebra

\[
D(P, p, e_p) = (P \xrightarrow{p} FX \xrightarrow{Fe_p} F(P + A)),
\]

and it is easy to prove that \( Dh = h \) is coalgebra homomorphism for every morphism \( h \) in \( \mathcal{D} \).

284
One obtains a cocone of $D$ cocone morphisms $p: (P, Fe_p, p) \rightarrow (FX, Fe)$ for every object $(P, p, e_p)$ of $\mathcal{D}$. Indeed, it is easy to prove that $p$ is a homomorphism as indicated and that these form a compatible cocone.

The category $\mathcal{D}$ is clearly nonempty because from the fact that $X$ is finitely presentable we conclude that there exists $p: P \rightarrow FX$ in $\mathcal{A}_{fp}/A$ and $e_p$ such that (10.11) commutes. Furthermore, one can show that $\mathcal{D}$ is filtered.

Now one proves that the projection functor $\text{Pr}: \mathcal{D} \rightarrow \mathcal{A}_{fp}/FX$ given by $\text{Pr}(P, p, e_p) = (P, p)$ is final.

Finally, to see that $(FX, Fe) = \text{colim} D$ one uses that the forgetful functor $U: \text{Coalg} F((-,) + A) \rightarrow \mathcal{A}$ creates colimit and obtains $FX = \text{colim} UD$ using the finality of $\text{Pr}$ and the following equation

$$UD = \left( \mathcal{D} \xrightarrow{\text{Pr}} \mathcal{A}_{fp}/FX \xrightarrow{DFX} \mathcal{A} \right),$$

where $DFX$ is the canonical diagram (see Remark 10.1.12(3)).

### 10.4 The Rational Fixed Point of a Set Functor

In this section we mention, in addition to the description in Proposition 10.2.16, two descriptions of the rational fixed point in the case where $\mathcal{A} = \text{Set}$. The first one is as the coalgebra of all finite well-pointed coalgebras (see Section 9.3), and the latter is related to a presentation of the functor $F$ by operations and basic equations (see Section 4.3), which means that $F$ is the quotient of a polynomial functor $H_\Sigma$.

**Notation 10.4.1.** Let $R$ be a set representing all finite well-pointed coalgebra (see Definition 9.3.2). Thus, $R$ can be chosen as a subset of $T$ in Notation 9.3.4. In that notation $F$ was assumed to preserve intersections. Since here we work with a finitary $F$, its Trnková hull $\bar{F}$ preserves them and the choices of $T$ and $R$ are the same for $F$ and $\bar{F}$ (see Remark 8.1.16 and Corollary 8.1.17).

Recall from Theorem 9.3.8 that $T$ carries the final coalgebra $\nu F$.

**Proposition 10.4.2.** For every finitary set functor $F$ the set $R$ of all finite well-pointed coalgebras is the rational fixed point $\rho F$.

In particular, $R$ is a subcoalgebra of the terminal coalgebra $\nu F$ by Proposition 10.2.16.

**Proof.** Recall the coalgebra structure $\tau: T \rightarrow FT$ from (9.6). Given an element of $R$ viz. a finite well-pointed coalgebra $(A, \alpha, x)$, then the well-founded modification $\alpha^+(x)$ is also finite, and so can be chosen to lie in $R$, in symbols: $\tau[R] \subseteq FR$. Let $\omega: R \rightarrow FR$ be the restriction of $\tau$. For every finite coalgebra $(A, \alpha)$ the unique homomorphism $\alpha^+: (A, \alpha) \rightarrow (T, \tau)$ yields a finite subcoalgebra $\alpha^+[A]$. Hence its image lies in $R$. It is clear that the corresponding restriction of $\alpha^+$ is the unique homomorphism from $(A, \alpha)$ to $(R, \omega)$. $\square$
Example 10.4.3. For $FX = \{0, 1\} \times X^\Sigma$ we get that $\varrho F$ consists of all finite minimal deterministic automata. This coalgebra is isomorphic to that formed by all regular languages on $\Sigma$.

Our second description of $\varrho F$ is based on the notion of a presentation (cf. Definition 4.3.7).

Recall from Proposition 4.3.19 that every finitary functor $F$ can be presented by a finitary signature $\Sigma$ and a set $E$ of basic equations, i.e. $F \cong H\Sigma/E$. As before we denote by $\varepsilon: H\Sigma \rightarrow F$ the corresponding natural epi-transformation.

We have seen in Proposition 4.3.22 and Theorem 4.3.26 that the initial algebra and the terminal coalgebra for $F$ arise as the quotients of the initial algebra and terminal coalgebra for $H\Sigma$ (carried by the sets of all finite $\Sigma$-trees and all $\Sigma$-trees, respectively) modulo the equivalence given by application of the basic equations in $E$:

$$\mu F = \mu H\Sigma/\sim \quad \text{and} \quad \nu F = \nu H\Sigma/\approx.$$  

Here $\sim$ is the equivalence of finitely many applications of the basic equations in $E$ (see (4.3)) and $\approx$ that of finite and infinite applications (see (4.4)). It turns out that the equivalence $\approx$ restricts to the rational fixed point $\varrho H\Sigma$, which is carried by the set of all rational $\Sigma$-trees (see Example 10.2.17(2)). In fact, we consider $\varrho H\Sigma$ as an $F$-coalgebra and obtain the following result:

**Theorem 10.4.4** [25, Thm. 5.7]. Given a presentation of a finitary set functor $F$, the rational fixed point of $F$ is a quotient of the $F$-coalgebra $\varrho H\Sigma$ of all rational $\Sigma$-trees modulo the equivalence $\approx$; in symbols:

$$\varrho F = \varrho H\Sigma/\approx.$$  

**Corollary 10.4.5.** Given a presentation of a finitary set functor $F$, we have the following commutative square of $F$-coalgebra homomorphisms:

$$\begin{array}{ccc}
\varrho H\Sigma & \xrightarrow{\varepsilon} & H\Sigma(\varrho H\Sigma) \\
\downarrow & & \downarrow \\
\nu H\Sigma & \xrightarrow{\varrho} & \nu F
\end{array}$$  

Each of the above homomorphisms is a uniquely determined and represents a canonical subcoalgebra or quotient coalgebra, respectively.

**Examples 10.4.6.** (1) For the finite power-set functor $\mathcal{P}_f$ we have the presentation of Example 4.3.27 with $H\Sigma X = X^*$. Hence we obtain $\varrho \mathcal{P}_f = R\Sigma/\approx$, where $R\Sigma$ is the coalgebra of all rational finitely branching ordered trees and $\approx$ the equivalence of Example 4.2.10.
10.5 Full Abstractness and Finitely Generated Objects

We have seen in Proposition 10.2.16 that the rational fixed point of a finitary set functor is fully abstract, i.e. it is a subcoalgebra of the terminal coalgebra. We shall see that for a finitary functor $F$ on a general locally finitely presentable category $\mathcal{A}$ this is not always the case. In this section we will study sufficient conditions on $\mathcal{A}$ and $F$ that ensure full abstractness of the rational fixed point. Those conditions are closely tied to an alternative notion of ‘finite’ object one may consider in a locally finitely presentable category. We have so far considered finitely presentable objects, which are a very useful formalization of ‘finiteness’. In this section we use also finitely generated objects, which are a broader concept. We will see that $\varrho F$ is fully abstract whenever the classes of finitely presentable and finitely generated objects coincide. Besides being our key for the full abstractness of $\varrho F$, we shall see that finitely generated objects lead to another interesting fixed point of $F$. In the special case where $\mathcal{A}$ is a finitary variety we will also consider the more restrictive concept of free finitely generated algebras (aka. ffg objects) in lieu of finitely presentable objects modelling ‘finiteness’. This yields a third fixed point for finite behaviour.

**Definition 10.5.1.** An object $X$ is called finitely generated if $\mathcal{A}(X, -)$ preserves filtered colimits of monomorphisms, i.e. colimits of filtered diagrams $D : \mathcal{D} \to \mathcal{A}$ such that $Dh$ is a monomorphism for all morphisms $h$ in $\mathcal{D}$.

Like finite presentability, this concept stems from general algebra, where it designates those algebras which are generated by a finite subset.

**Remark 10.5.2.** (1) In general, every finitely presentable object is finitely generated. But the converse may fail (see e.g. Example 10.5.4(6) and (7)).

(2) An object is finitely generated iff it is the strong quotient of a finitely presentable object [43, Prop. 1.69], i.e. there exists a finitely presentable object $X_0$ and a strong epimorphism $X_0 \to X$. It follows that finitely generated objects are closed under strong quotients and that $\mathcal{A}_{fg} = \mathcal{A}_{fp}$ iff finitely presentable objects are closed under strong quotients.

(3) Since $\mathcal{A}$ is locally finitely presentable, it is co-well-powered (see [43, Thm. 1.58]). Thus it has only a set of finitely generated objects (up to isomorphism).
**Notation 10.5.3.** We denote by \( \mathcal{A}_{fg} \) the (essentially small) full subcategory given by all finitely generated objects of \( \mathcal{A} \).

**Examples 10.5.4.** 
1. In the categories \( \text{Set}, \text{Pos}, \text{Gra} \), and \( K\text{-Vec} \) every finitely generated object is finitely presentable, i.e. the finitely generated objects are the finite sets, posets, graphs, and finite-dimensional vector spaces, respectively. Indeed, we used only filtered diagrams of monomorphisms in Example 10.1.7(1)–(3).

2. In the category \( \text{Nom} \) of nominal sets we have that \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \) are the orbit-finite sets (cf. Example 10.1.7(6)). This follows from Remark 10.5.2(2).

3. In the category \( \text{Set}^\mathcal{F} \) of sets in context we have that \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \) [30, Cor. 3.34], and this is the class of super-finitary presheaves (see Example 10.1.7(7)).

4. In a finitary variety of algebras (see Example 10.1.7(4)), the finitely generated objects are precisely those algebras which are generated by a finite subset, i.e. they can be presented by finitely many generators and (possibly) infinitely many relations [43, Theorem 3.12]. In other words, finitely generated algebras are precisely the quotients of free algebras on finitely many generators.

5. Every locally finite variety (see Example 10.1.7(5)) clearly satisfies \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \) by Remark 10.5.2(2).

6. For a simple example where \( \mathcal{A}_{fp} \neq \mathcal{A}_{fg} \), consider the signature \( \Sigma \) consisting of constant symbols \( c_n \ (n \in \mathbb{N}) \). Then a \( \Sigma \)-algebra is a set \( A \) with chosen elements \( a_n \ (n \in \mathbb{N}) \). It is finitely generated if the set \( A \setminus \{a_n\} \) is finite, and it is finitely presentable, if, moreover, there exists \( k \in \mathbb{N} \) such that all \( a_n \) with \( k \geq n \) are pairwise distinct. For example, the terminal (one-element) algebra is finitely generated but not finitely presentable.

7. In the categories of groups, monoids, or lattices we have \( \mathcal{A}_{fp} \neq \mathcal{A}_{fg} \). The examples demonstrating this are non-trivial. For example, there are only countably many finitely presented groups, while there are already \( 2^{\aleph_0} \) groups on 2 generators. A simple example of a finitely generated group which is not finitely presented is the standard wreath product \( \mathbb{Z} \wr \mathbb{Z} \) [204, §14.1]. For monoids see Campbell et al. [78, Example 4.5].

We now proceed to investigate the case where \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \) holds. It turns out that the fact proved in Section 10.2 for finitary set functors smoothly generalize to finitary functors on \( \mathcal{A} \). In the following we call a subobject \( s: S \to A \) finitely presentable if \( S \) is a finitely presentable object.

**Proposition 10.5.5.** Suppose that \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \) and that \( F \) preserves monomorphisms. Then a coalgebra \( (A, \alpha) \) is lfp iff every finitely presentable subobject of \( A \) is contained in a subcoalgebra from \( \text{Coalg}_F \).

More precisely, \((A, \alpha)\) is lfp iff for every subobject \( s: S \to A \) with \( S \) finitely presentable there exists a subcoalgebra \( m: (B, \beta) \to (A, \alpha) \) with \((B, \beta)\) in \( \text{Coalg}_F \) with \( s \leq m \), i.e. there is a monomorphism \( s': S' \to B \) such that \( m \cdot s' = s \).

**Proof.** By Remark 10.1.12(4), a coalgebra \( (A, \alpha) \) is lfp iff the following two conditions
10.5 Full Abstractness and Finitely Generated Objects

are met:

(1) For every morphism $p: P \to A$ where $P$ is a finitely presentable object, there exist a coalgebra $(B, \beta)$ in $\text{Coalg}_f F$, a morphism $p': P \to B$ and a coalgebra homomorphism $h: (A, \alpha) \to (B, \beta)$ such that $p = h \cdot p'$:

$$
\begin{align*}
B & \xrightarrow{\beta} FB \\
P & \xrightarrow{p} A & \xrightarrow{\alpha} FA \\
 & \xrightarrow{h} Fh & \xrightarrow{Fh} FB \\
 & \xrightarrow{p'} B
\end{align*}
$$

(2) Given two such factorizations, i.e. $p_t: P \to B$ ($t = 1, 2$) with $h \cdot p_1 = h \cdot p_2$, there exist a coalgebra homomorphisms $h': (B', \beta') \to (A, \alpha)$ and $g: (B, \beta) \to (B', \beta')$ such that $h' \cdot g = h$ and $g \cdot p_1 = g \cdot p_2$.

In order to prove the proposition we proceed in two steps:

(i) we prove that under our current assumptions (1) implies (2), and
(ii) we prove that (1) is equivalent to the condition in the statement of the proposition.

Ad (i). Suppose that $(A, \alpha)$ satisfies conditions (1). Let $p_t: P \to A$ ($t = 1, 2$) be a pair of morphisms with $P$ finitely presentable, and let $h: (B, \beta) \to (A, \alpha)$ be a homomorphism with $(B, \beta)$ in $\text{Coalg}_f F$ such that $h \cdot p_1 = h \cdot p_2$. Take the coequalizer $c: B \to C$ of $p_1, p_2$ in $\mathcal{A}$. Since $P$ and $B$ are finitely presentable so is $C$ (cf. Remark 10.1.12(2)). By the universal property of $c$ we obtain a morphism $g: C \to A$ such that $g \cdot c = h$. We now apply condition (1) to $g$, and we obtain a coalgebra $(B', \beta')$ in $\text{Coalg}_f F$, a morphism $g': C \to B'$ and a homomorphism $h': (B', \beta') \to (A, \alpha)$ such that $h' \cdot g' = g$. Next we factorize the homomorphism $h'$ into a homomorphism $e: (B', \beta') \to (B'', \beta'')$ carried by a strong epimorphism followed by $m: (B'', \beta'') \to (A, \alpha)$ carried by a monomorphism (see Remark 10.1.12(5)). In summary we have the following commutative diagram:

$$
\begin{align*}
P & \xrightarrow{p_1} B & \xrightarrow{e} C & \xrightarrow{g'} B' & \xrightarrow{m} B'' \\
P & \xrightarrow{p_2} B & \xrightarrow{h} A & \xrightarrow{h'} B' & \xrightarrow{m} B'' \\
 & \xrightarrow{h-p_2} A & \xrightarrow{m} B''
\end{align*}
$$

Clearly, $e \cdot g' \cdot c$ merges $p_1$ and $p_2$ since so does $c$. To see that $e \cdot g' \cdot c$ is a homomorphism (in $\text{Coalg}_f F$) consider the diagram below:

$$
\begin{align*}
B & \xrightarrow{\beta} FB \\
 & \xrightarrow{e \cdot g' \cdot c} H(e \cdot g' \cdot c) F B \\
 & \xrightarrow{Hh} Hh F B \\
 & \xrightarrow{m} FM \\
A & \xrightarrow{\alpha} FA
\end{align*}
$$

Its outside commutes since $h$ is a homomorphism, the left- and right-hand parts commute by (10.12), and the lower inner square commutes since $m$ is a homomorphism.
Thus the desired upper inner square commutes when extended by \( Fm \). Since \( Fm \) is a monomorphism by assumption, we see that the upper inner square commutes.

Ad (ii). Suppose that (1) holds and let \( s : S \rightarrow A \) be a finitely presentable subobject. We obtain a homomorphism \( h : (B_0, \beta_0) \rightarrow (A, \alpha) \) and a morphism \( s'_0 : S \rightarrow B_0 \) with \( h \cdot s'_0 = s \). Then the image \( m : (B, \beta) \rightarrow (A, \alpha) \) of \( h \) (cf. Remark 8.2.4) yields the desired subcoalgebra.

Conversely, assume that the condition in the statement of the lemma holds and suppose towards a proof of (1) that \( p : P \rightarrow A \) is a morphism with \( P \) finitely presentable. Take the (strong epi, mono)-factorization \( p = s \cdot e \) and apply the condition to \( s \) to obtain \( s' \) and a subcoalgebra \( m : (B, \beta) \rightarrow (A, \alpha) \) with \( m \cdot s' = s \). With \( p' = s' \cdot e \) this yields the desired factorization of \( p \).

The next result presents sufficient conditions on \( \mathcal{A} \) and \( F \) for the rational fixed point to be fully abstract w.r.t. final semantics, i.e. a canonical subcoalgebra of the terminal coalgebra. It also entails the concrete descriptions of \( \rho F \) in the subsequent examples.

**Proposition 10.5.6.** Suppose that \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \) and that \( F \) preserves monomorphisms. Then \( \rho F \) is the subcoalgebra of \( \nu F \) given by the union of images of all \( F \)-coalgebra morphisms \( (A, \alpha) \rightarrow (\nu F, \tau) \) where \( (A, \alpha) \) ranges over \( \text{Coalg}_F \).

**Proof.** Let \( \mathcal{D} \) be the full subcategory formed by all subcoalgebras of \( \nu F \) in \( \text{Coalg}_F \), i.e. all coalgebras \( (A, \alpha) \) with \( A \) finitely presentable and such that the unique coalgebra homomorphism \( (A, \alpha) \rightarrow (\nu F, \tau) \) is a monomorphism. By Corollary 10.2.15, it is sufficient to show that the inclusion \( \mathcal{D} \hookrightarrow \text{Coalg}_F \) is final. Since \( \mathcal{D} \) is a full subcategory and \( \text{Coalg}_F \) is filtered (Remark 10.2.4) it suffices that for every \( (A, \alpha) \) in \( \text{Coalg}_F \) we have a homomorphism \( (A, \alpha) \rightarrow (A', \alpha') \) with \( (A', \alpha') \) in \( \mathcal{D} \). Indeed, given \( (A, \alpha) \) let \( (A', \alpha') \) be its image under the unique homomorphism \( (A, \alpha) \rightarrow (\nu F, \tau) \) (see Remark 8.2.4). Then we have a homomorphism \( e : (A, \alpha) \rightarrow (A', \alpha') \) where \( e \) is a strong epimorphism in \( \mathcal{A} \). Since \( \mathcal{A}_{fp} = \mathcal{A}_{fg} \), we conclude that \( A' \) is finitely presentable (see Remark 10.5.2(2)), whence \( (A', \alpha') \) lies in \( \mathcal{D} \).

**Corollary 10.5.7.** Under the assumptions in Proposition 10.5.6, the rational fixed point \( (\rho F, \omega) \) is a cartesian subcoalgebra of the terminal coalgebra:

\[
\begin{array}{ccc}
\rho F & \xrightarrow{\omega} & F(\rho F) \\
\downarrow & & \downarrow \\
\nu F & \xrightarrow{\tau} & F(\nu F)
\end{array}
\]

Indeed, the above square is a pullback since it has the two isomorphisms \( \omega \) and \( \tau \) on opposite sides.

Before presenting further examples of rational fixed points, we are going to prove that in the case where \( \mathcal{A} \) is a variety of algebras (see Example 10.1.7(4)) the union in Proposition 10.5.6 may be restricted to range only over \( F \)-coalgebra whose carrier is a free finitely generated algebra in \( \mathcal{A} \), i.e. an algebra which is free on a finite set of generators.
10.5 Full Abstractness and Finitely Generated Objects

Remark 10.5.8. Let $\mathcal{A}$ be a variety of algebras. Then every free algebra $A$ is projective, i.e. for every surjective morphism $e: B \to C$ and every morphism $f: A \to C$ there exists a morphism $f': A \to B$ with $e \cdot f' = f$:

\[
\begin{array}{ccc}
A & \xrightarrow{f'} & B \\
\downarrow f & & \downarrow e \\
C & & C
\end{array}
\]

Indeed, let $A$ be free on the set $X$ of generators with the universal map $\eta: X \to A$. Since $e$ is surjective, we have a map $m: C \to A$ such that $e \cdot m = \text{id}_C$ (in $\text{Set}$). Now extend the map $m \cdot f \cdot \eta: X \to B$ to a morphism $f': A \to B$ in $\mathcal{A}$ using the freeness of $A$. Then $e \cdot f' = f$ because this holds when we restrict to the generators of $A$ (i.e. we precompose with $\eta$):

\[e \cdot f' \cdot \eta = e \cdot m \cdot f \cdot \eta = f \cdot \eta.\]

Corollary 10.5.9. Let $\mathcal{A}$ be a variety such that $\mathcal{A}_{fp} = \mathcal{A}_{fg}$, and let $F: \mathcal{A} \to \mathcal{A}$ be finitary and preserve injective as well as surjective morphisms. Then $\varrho F$ is the subcoalgebra of $\nu F$ given by the union of all $F$-coalgebra morphisms $(A, \alpha) \to (\nu F, \tau)$ where $A$ is free finitely generated.

Proof. By Proposition 10.5.6 we know that $\varrho F$ is the union of images of homomorphisms $h: (B, \beta) \to (\nu F, \tau)$ where $B$ ranges over finitely generated algebras in $\mathcal{A}$. Given any such coalgebra $(B, \beta)$ we know that $B$ is the quotient of a free finitely generated algebra $A$ (cf. Example 10.1.7(4)), via a surjective morphism $e: A \to B$ say. Using projectivity (see Remark 10.5.8) and that $F$ preserves surjective morphisms we obtain some coalgebra structure on $A$ such that $e$ is a coalgebra homomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow e & & \downarrow Fe \\
B & \xrightarrow{\beta} & FB
\end{array}
\]

Thus the image of $B$ is the same as the image of $A$ in $\nu F$, which proves the claim. \qed

We proceed to presenting further examples of rational fixed points, in addition to the rational fixed point for set functors we have seen in Example 10.2.17. In all of the following examples, $\varrho F$ is fully abstract by Proposition 10.5.6.

Examples 10.5.10. (1) We know that for the set functor $FX = \mathbb{R} \times X$ the rational fixed point consists of eventually periodic streams of real numbers (cf. Example 10.2.17(6)). Let us consider $F$ as an endofunctor on the category of real vector spaces. The following interesting description of $\varrho F$ then follows from the work of Rutten [211]. The convolution product of two streams is given by

\[(\sigma \times \tau)(n) = \sum_{i=0}^{n} \sigma(i) \cdot \tau(n-i).\]
For streams $\sigma$ with $\sigma(0) \neq 0$ there exists an inverse $\sigma^{-1}$, i.e. $\sigma \times \sigma^{-1} = (1, 0, 0, 0, \ldots)$. A stream is called \textit{rational} if it has the form $\sigma \times \tau^{-1}$, where $\sigma$ and $\tau$ have finitely many non-zero entries and $\tau(0) \neq 0$. The rational fixed point for $F$ consists of all rational streams. This follows from Proposition 10.5.6 using Rutten’s characterization [211] of the rational streams as the images in the terminal coalgebra of coalgebras with finite-dimensional carrier.

(2) The previous example can be generalized from real streams to formal power-series (also called \textit{weighted languages}) over a given semiring $S$. Recall from Example 2.4.5 that one considers the functor $FX = S \times X^S$ on the category of $S$-semimodules and that the $F$-coalgebras carried by the free finitely generated $S$-semimodules $S^n$ can be identified with weighted automata. We also saw in Example 2.5.12 that the terminal coalgebra for $F$ is carried by the set $S^{X^S}$ of formal power series. A formal power-series is called \textit{recognizable} if it is recognized by a finite weighted automaton (see e.g. [91]). By the Kleene-Schützenberger theorem [216] (see also [64, Prop. 6.1 & Thm. 7.1]) it follows that recognizable power-series are, equivalently, the \textit{rational} formal power-series. For those semirings for which finitely generated semimodules coincide with finitely presentable ones, $S$-$\text{Mod}_{fg}$, Corollary 10.5.9 yields that the rational fixed point $\varphi F$ is the subcoalgebra of $\nu F = S^{X^S}$ given by the rational formal power series. This coincidence holds, for example, for

(a) every field,
(b) every finite semiring,
(c) every principal ideal domain (e.g. the ring $\mathbb{Z}$ of integers), and therefore
(d) every finitely generated commutative ring (by Hilbert’s basis theorem).

More generally, whenever $S$ is a \textit{Noetherian} semiring in the sense of Ésik and Maletti [98] (i.e. every subsemimodule of a finitely generated semimodule is itself finitely generated) then the classes of finitely generated semimodules and finitely presented ones coincide [73, Prop. 2.6].

The tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is not Noetherian [97]. The condition of being Noetherian is not necessary for the coincidence of finitely generated and finitely presentable semimodules. For example, the usual semiring of natural numbers is also not Noetherian: the $\mathbb{N}$-semimodule $\mathbb{N} \times \mathbb{N}$ is finitely generated but its subsemimodule generated by the infinite set $\{(n, n + 1) \mid n \geq 1\}$ is not. However, $\mathbb{N}$-semimodules are precisely the commutative monoids, and finitely generated monoids are finitely presentable (this is known as Redei’s theorem [203]; see Freyd [101] for a very short proof).

(3) Let $F = V + X \times X + \delta X$ on the category $\text{Set}^\mathcal{F}$ of sets in context (see Example 10.1.7(7)). As we have seen in Example 2.2.18(2), the initial algebra $\mu F$ is the presheaf of $\lambda$-terms (or finite $\lambda$-trees) up to $\alpha$-equivalence. Similarly, $\nu F$ is the presheaf of finite and infinite $\lambda$-trees up to $\alpha$-equivalence (see Example 2.5.15). The finitely generated and finitely presentable objects coincide in $\text{Set}^\mathcal{F}$ [30, Cor. 3.34]. Thus, the rational fixed point is fully abstract. Moreover, $g F$ is the subcoalgebra of $\nu F$ given by \textit{rational} $\lambda$-trees, i.e. those $\alpha$-equivalence classes of $\lambda$-trees which contain a tree having only finitely many subtrees (up to isomorphism) [39, Thm. 2.17].
10.5 Full Abstractness and Finitely Generated Objects

(4) Similarly, for the functor $FX = \mathbb{A} + X \times X + [\mathbb{A}]X$ on the category $\text{Nom}$, we have seen that $\mu F$ is formed by $\alpha$-equivalence classes of $\lambda$-terms (see Example 2.2.18(1)), and $\nu F$ is formed by $\alpha$-equivalence classes of all those $\lambda$-trees with finitely many free variables (see Example 2.5.14(3)). The class of orbit-finite nominal sets is clearly closed under equi-variant quotients. Thus, the classes of finitely presentable and finitely generated objects coincide in $\text{Nom}$, and so that the rational fixed point of $F$ is fully abstract. In fact, $\varrho F$ is the nominal set of all rational $\lambda$-trees modulo $\alpha$-equivalence [184]. Similarly, for any functor on nominal sets arising from a binding signature [183].

(5) For the functor $FX = \{0, 1\} \times X^\mathbb{A}$ on $\text{Nom}$ (see Example 2.5.14(1)) the category $\text{Coalg}_F$ consists precisely of the orbit-finite deterministic nominal automata. It follows from Proposition 10.5.6 that $\varrho F$ is the subcoalgebra of all finitely supported languages over $\mathbb{A}$ accepted by these automata.

Similarly, for $FX = \{0, 1\} \times X^\mathbb{A} \times [\mathbb{A}]X$ (see Example 2.5.14(2)) we have that $\varrho F$ consists precisely of all bar languages accepted by the deterministic orbit-finite automata with variable binding transitions considered by Kozen et al. [157].

All categories considered in Example 10.5.10 satisfy $\mathcal{A}_p = \mathcal{A}_g$. We now turn to categories $\mathcal{A}$ in which $\mathcal{A}_p$ and $\mathcal{A}_g$ do not coincide. This is known to be the case for a number of categories relevant for the coalgebraic modelling of systems. For example, certain varieties of algebras (e.g. for the categories of groups, monoids, or semimodules for the semiring $S = (\mathbb{Z}_2)^N$), the target categories of generalized determinization [221]. Another example is the category of finitary monads on sets [29, Cor. 4.13] used in the categorical study of Courcelle’s algebraic trees [38].

There are also categories for which it is not known whether $\mathcal{A}_p = \mathcal{A}_g$, e.g. the category of idempotent semirings (used in the treatment of context-free grammars [241]) or the category of algebras for the stack monad (used for modelling configurations of stack machines [116, 117]).

In such categories the rational fixed point may fail to be fully abstract, i.e. $\varrho F$ is not a canonical subcoalgebra of $\nu F$ (cf. [73, Ex. 3.15] for a related example):

Example 10.5.11 [176, Ex. 2.18(1)]. Let $\mathcal{A}$ be the category of algebras for the signature $\Sigma$ with two unary operation symbols $u$ and $v$. The set of natural numbers $\mathbb{N}$ with the successor function as both operations and with the coalgebra structure given by the usual head and tail functions.

Note that the free $\Sigma$-algebra on a set $X$ of generators is $TX = \{u, v\}^* \times X$; we denote its elements by $w(x)$ for $w \in \{u, v\}^*$ and $x \in X$. The operations are given by prefixing words by the letters $u$ and $v$, respectively: $s^{TX} \colon w(x) \mapsto sw(x)$ for $s = u$ or $v$.

Now one considers the $F$-coalgebra $\alpha \colon A \to FA$, where $A = T\{x\}$ is free $\Sigma$-algebra on one generator $x$ and $\alpha$ is determined by $\alpha(x) = (0, u(x))$. We write $\mathcal{A}_\alpha \colon A \to \nu F$ for the unique coalgebra morphism. Clearly, $\mathcal{A}_\alpha(x)$ is the stream $(0, 1, 2, 3, \cdot \cdot \cdot)$ of all natural
numbers, and since \( \dagger \alpha \) is a \( \Sigma \)-algebra morphism we have
\[
\dagger \alpha(u(x)) = \dagger \alpha(v(x)) = (1, 2, 3, 4, \cdots).
\]
Since \( A \) is (free) finitely generated, it is of course, finitely presentable as well. Thus, 
\((A, \alpha)\) is a coalgebra in \( \text{Coalg}_F \).

However, one can prove that the (unique) \( F \)-coalgebra morphism \( \alpha^\#: A \to \varrho F \) satisfies
\[
\alpha^\#(u(x)) \neq \alpha^\#(v(x)).
\]
For details see op. cit.

The Locally Finite Fixed Point. One problem we encountered with the rational fixed point is that, in general, it need not be fully abstract, which means that it is not a subcoalgebra of the terminal coalgebra (see Example 10.5.11). However, in all applications, the regular behaviour of systems is defined by taking the image of all the behaviours of finite(ly presented) systems in the semantic domain, i.e. the terminal coalgebra. Hence, in applications one wants the rational fixed point to be subcoalgebra of \( \nu F \). Unfortunately, the condition in Proposition 10.5.6 that finitely generated objects be finitely presentable is, if true at all, often difficult to verify in cases where it is unknown.

Technically, the a priori choice of taking finitely presentable objects as the right abstraction of ‘finite set’ is not completely canonical. One may ask what happens if one chooses finitely generated objects instead. This has been investigated by Milius, Pattion, and Wißmann [182]. In fact, one can rework much of the theory we have seen by systematically replacing finitely presentable by finitely generated objects. We now briefly mention the key definitions and results; for details see op. cit.

**Assumption 10.5.12.** In addition to Assumption 10.2.1 we now assume that \( F: \mathcal{A} \to \mathcal{A} \) preserves nonempty monomorphisms (see Remark 8.2.6(2)).

First one replaces lfp coalgebras by the corresponding concept based on finitely generated objects. The definition given in op. cit. is equivalent to the following

**Definition 10.5.13** [182, Cor. 3.6]. An \( F \)-coalgebra \((A, \alpha)\) is called *locally finitely generated* (or \( \text{lf} \), for short) if it is the colimit of of a diagram of coalgebras with a finitely generated carrier.

**Theorem 10.5.14** [182, Thm. 3.8]. Suppose that \( F: \mathcal{A} \to \mathcal{A} \) preserves nonempty monomorphisms. The full subcategory of \( \text{Coalg} F \) given by \( \text{lf}_g \) coalgebras has a terminal object which is given by the colimit of all coalgebras with finitely generated carrier.

**Definition 10.5.15.** The terminal \( \text{lf}_g \) coalgebra is called the *locally finite fixed point* of \( F \) and denoted by \( \vartheta F \).

That the name is justified follows from the first part of the following result. Furthermore, an advantage of \( \vartheta F \) is that, unlike the rational fixed point, it is always fully abstract w.r.t. behavioural equivalence.

**Theorem 10.5.16** [182, Thm. 3.12 & 3.14]. (1) The coalgebra structure of \( \vartheta F \) is an isomorphism. (2) The locally finite fixed point \( \vartheta F \) is a (canonical) subcoalgebra of \( \nu F \).
10.5 Full Abstractness and Finitely Generated Objects

Corollary 10.5.17 [182, Cor. 3.10]. If $\mathcal{A}_{fp} = \mathcal{A}_{fg}$, then the locally finite fixed point coincides with the rational fixed point: $\vartheta F \cong \rho F$.

Indeed, the colimits in Corollary 10.2.15 and Theorem 10.5.14 are the same.

Under additional assumptions on $\mathcal{A}$ (which hold e.g. for all varieties $\mathcal{A}$) there is a strong connection of $\vartheta F$ to the rational fixed point: $\vartheta F$ is the image of $\rho F$ in the terminal coalgebra [182, Thm. 5.4]; in symbols we have

$$\rho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F.$$

Examples 10.5.18. (1) By Corollary 10.5.17, all instances of rational fixed points in Examples 10.2.17 and 10.5.10 are examples of locally finite fixed points, too. Hence, all examples of rational fixed points which are of widespread interest may be studied as locally finite fixed points.

(2) In addition, we also obtain several examples of finite behaviour domains that could not so far be obtained as instances of the rational fixed point, e.g. context-free languages (and their (real-time deterministic and non-deterministic variants, resp.), constructively $S$-algebraic formal power-series (and any other instance of the generalized power-set construction by Silva et al. [221]) and the monad of Courcelle’s algebraic trees (see [84]). That description of algebraic trees as a locally finite fixed point yields the first characterization of those trees by a universal property, solving an open problem from [38]. For details see [182, Sec. 6].

Finally, the locally finite fixed point $\vartheta F$ is also characterized by a universal property as an algebra in a similar way as $\rho F$: define fg-iterative algebras for $F$ by replacing the finitely presentable object $X$ by a finitely generated one in Definition 10.3.1. Then one obtains the following result.

Theorem 10.5.19 [182, Cor. 4.9]. The locally finite fixed point is the initial fg-iterative algebra.

The Locally fgf Fixed Point. In the case where the base category $\mathcal{A}$ is a finitary (many-sorted) variety of algebras one has yet another choice of the notion of a ‘finite object’, namely the class of all free finitely generated algebras, i.e. free algebras on a finite set of generators; we call them fgf objects, for short. The fgf objects form a proper subclass of the class of all finitely generated objects. This class is interesting because it consists precisely of the carriers of coalgebras arising as the targets of the generalized power-set construction (see Silva et al. [221]). Again one could attempt to rework the theory of the rational fixed point, this time replacing finitely presentable by fgf objects.

For that a slightly stronger condition needs to be required of $F$, namely that it is strongly finitary which means finitary and preserving reflexive coequalizers. These are coequalizers of parallel pairs $f, g: X \rightarrow Y$ having a joint spitting $s: Y \rightarrow X$ (with $f \cdot s = g \cdot s = \text{id}_Y$). This is a fairly mild restriction; for example, every finitary set functor is strongly finitary (this follows from [44, Cor. 6.30]). Furthermore, every functor $F$ which is a lifting of a finitary set functor to the variety $\mathcal{A}$ is strongly finitary. This
follows from the fact [44, Cor. 11.9] that the forgetful functor $\mathcal{A} \to \text{Set}$ preserves and reflects filtered colimit and reflexive coequalizers. (However, in general, not every finitary functor is strongly finitary [44, Ex. 7.11].)

**Definition 10.5.20** [176]. Let $F$ be a strongly finitary endofunctor on a variety. We define the coalgebra $\varphi F$ as the colimit of all coalgebras carried by ffg objects. It is called the *locally ffg fixed point* of $F$.

**Theorem 10.5.21** [235, Lem. 4.5]. *The coalgebra structure of $\varphi F$ is an isomorphism.* Somewhat surprisingly, the coalgebra $\varphi F$ fails to have the finality property w.r.t. to coalgebras with ffg carriers [177, Section 2.5]. This also shows that $\varphi F$ cannot have a universal property as some kind of iterative algebra (i.e. where solutions are unique). However, one can adjust the notion of an Elgot algebra (see Section 10.6 below), and prove that $\varphi F$ is the initial ffg-Elgot algebra [177]. In addition, in op. cit. it is shown that free ffg-Elgot algebras exist, and for an ffg object $X$ the free ffg-Elgot algebra is given by $\varphi(F(\cdot) + X)$.

**Relations Between Four Fixed Points.** Still in the setting of the previous paragraph, where $\mathcal{A}$ is a finitary variety, let $F: \mathcal{A} \to \mathcal{A}$ be finitary and preserve surjective as well as nonempty injective morphisms of $\mathcal{A}$. Then we already know from (10.13) that the subcoalgebra $\vartheta F$ of $\nu F$ is a quotient of $\varrho F$. In addition, $\varrho F$ is a quotient of $\varphi F$ [176, Prop. 3.9]. Hence, (10.13) extends to the following picture:

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F.$$  

Whenever $\mathcal{A}_{fp} = \mathcal{A}_{fg}$, we have additionally that $\varrho F \cong \vartheta F$ by Corollary 10.5.17, i.e. $\varrho F$ is fully abstract w.r.t. behavioral equivalence. Moreover, if $\mathcal{A}_{fp} = \mathcal{A}_{fg}$ coincide with the class of all ffg objects, $\varrho F$ and $\vartheta F$ coincide with $\varphi F$ as well. However, this only happens in very rare cases, e.g. in $\text{Set}$ and $\text{K-Vec}$. Milius [176] introduced the notion of a *proper* functor (generalizing the notion of a proper semiring of ´Esik and Maletti [97]) and proved that a functor $F$ is proper if and only if the first three fixed points coincide, i.e. the picture above collapses to $\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$. Finally, there are instances where $\varrho F \cong \vartheta F \cong \nu F \cong 1$ are trivial but $\varphi F$ is non trivial [177, Section 2.5].

**Unified Fixed Points.** The technical development for the four fixed points above shows a number of parallels. Urbat [235] gives a uniform account of several results on fixed points that are based on coalgebras. His theory is parametric in a class $\mathcal{I}$ of diagram schemes and a class $\mathcal{M}$ of morphisms coming from a factorization system $(\mathcal{E}, \mathcal{M})$ on the base category $\mathcal{A}$, where $\mathcal{E}$ is some class of epimorphisms. For an endofunctor $F: \mathcal{A} \to \mathcal{A}$ he considers those coalgebras that have an $(\mathcal{I}, \mathcal{M})$-presentable carrier, i.e. a carrier $X$ such that $\mathcal{A}(X, \cdot)$ preserves colimits in $\mathcal{A}$ that have a diagram scheme in $\mathcal{I}$ and whose colimit injections lie in $\mathcal{M}$.

---

Footnote 1: These are mild assumptions; e.g. if $\mathcal{A}$ is single-sorted and $F$ a lifting of a finitary set functor, then these conditions are fulfilled.
Under suitable conditions on \( I, M \) and a given endofunctor \( F : \mathcal{A} \to \mathcal{A} \), Urbat proves that the colimit \( T_F \) of all coalgebras with an \((I, M)\)-presentable carrier is a fixed point of \( F \) and that this is characterized by a universal property both as a coalgebra and as an algebra (the proofs of these characterizations use that all diagrams in \( I \) are filtered). As instances one obtains \( \nu F \), \( \rho F \), \( \vartheta F \) and \( \phi F \) as shown in the following table:

<table>
<thead>
<tr>
<th>( I )</th>
<th>( (\mathcal{E}, M) )</th>
<th>( T_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>categories with a terminal object</td>
<td>(iso, all)</td>
<td>( \nu F )</td>
</tr>
<tr>
<td>small filtered categories</td>
<td>(iso, all)</td>
<td>( \rho F )</td>
</tr>
<tr>
<td>small filtered categories (strong epis, monos)</td>
<td>(iso, all)</td>
<td>( \vartheta F )</td>
</tr>
<tr>
<td>small sifted categories</td>
<td>(iso, all)</td>
<td>( \phi F )</td>
</tr>
</tbody>
</table>

Note that in the last case \( I \) does not consist of filtered diagrams, and hence, the characterization of \( \varphi F \) by a universal property does not follow from the uniform proofs.

### 10.6 Beyond the Rational Fixed Point

In this section we briefly discuss further work which is based on the ideas and results on the rational fixed point we have seen in the two previous sections.

**Iterative Monads.** We already saw in Corollary 10.3.7 that for a finitary functor \( F \) on a locally finitely presentable category \( \mathcal{A} \) free iterative algebras \( RX \) exist on every every object \( X \) of \( \mathcal{A} \). They are, equivalently, the rational fixed points for the functors \( F(\cdot) + X \) (see [34, Prop. 2.24]). More precisely, given the rational fixed point \( \omega_X : RX \to FRX + X \), then precomposing \( \omega_X^{-1} \) by the two coproduct injections yields the structure \( FRX \to RX \) and the universal morphism \( X \to RX \) of a free iterative algebra on \( X \). It follows that \( R \) is the object assignment of monad on \( \mathcal{A} \). This monad is called the rational monad of \( F \), and it is characterized by a universal property: it is the free iterative monad on \( F \) [34, Thm. 5.12]. This generalizes and extends classical work on iterative theories by Elgot [93] and on iterative algebras for a signature by Nelson [191] and Tiuryn [230].

**Elgot Algebras.** In a similar way as we discussed in Remark 7.2.20(3) one is lead to the notion of an Elgot algebra \((A, \alpha, (\cdot)^{\dagger})\), which is simply the variation of the notion of a complete Elgot algebras where the solution operation \((\cdot)^{\dagger}\) is restricted to finitary flat equation morphisms \( e : X \to FX + A \). Every iterative algebra is an Elgot algebra, and continuous algebras in \( \mathcal{A} = \text{CPO} \) are examples of Elgot algebras not admitting unique solutions \( e^{1} \) in general.

Theorem 10.3.9 can be augmented to state that the rational fixed point is the initial Elgot algebra (in addition, the rational fixed point of \( F(\cdot) + X \) is the free Elgot algebra on \( X \)) [33, Prop. 4.7]. The main result [33, Thm. 4.8] states that the Elgot algebras form precisely the category of Eilenberg-Moore algebras for the rational monad \( R \) above.

**Elgot Monads and Iteration Theories.** Going one level up from the locally finitely presentable category \( \mathcal{A} \) to the category \([\mathcal{A}, \mathcal{A}]_{\text{fin}}\) of finitary endofunctors on \( \mathcal{A} \) one sees...
that the assignment of a free iterative monad \( R_F \) to a given finitary functor \( F : \mathcal{A} \to \mathcal{A} \) is the object assignment of a monad \( R \) on \([\mathcal{A}, \mathcal{A}]_{\text{fin}}\). It is natural to ask what the Eilenberg-Moore algebras for this monad are: these turn out to be closely related to Bloom and Ésik’s iteration theories [69]. In fact, this relationship can be clearly stated for \( \mathcal{A} = \text{Set} \): assigning to a signature \( \Sigma \) the rational monad \( R(H_{\Sigma} + \{\bot\}) \) (of rational trees over \( \Sigma \) augmented with a new constant symbol \( \bot \)) yields a the free iteration theory on \( \Sigma \). Morever this assignment yields a monad on the category of signatures whose Eilenberg-Moore algebras are precisely Bloom and Ésik’s iteration theories [35].

The move from Elgot’s iterative theories (featuring unique solutions of recursive equations) to Bloom and Ésik’s iteration theories (featuring solution operators \((\_)^\dagger\) obeying natural equational laws) can also be performed in a more general category theoretic setting. For a locally finitely presentable category \( \mathcal{A} \) having well-behaved coproducts (see [15] for a precise definition) one considers the assignment that maps a finitary functor \( F : \mathcal{A} \to \mathcal{A} \) to the free iterative monad on \( F + C_1 \), where \( C_1 \) is the constant functor on \( 1 \). This yields a monad on \([\mathcal{A}, \mathcal{A}]_{\text{fin}}\) whose Eilenberg-Moore algebras are Elgot monads, the appropriate categorical generalization of iteration theories [36, 37].

10.7 Summary of this chapter

We have seen that besides the initial algebra and the terminal coalgebra a number of interesting fixed points of a finitary functor arise from ‘finite’ coalgebras. We have looked in detail at the rational fixed point for a finitary functor \( F \) on a locally finitely presentable category \( \mathcal{A} \). In this setting the role of ‘finite’ objects is played by the finitely presentable objects (a notion suitably generalizing that of finite sets or of algebras presented by finitely many generators and relations). We considered all those coalgebras whose carrier is finitely presentable and obtain lfp coalgebras as their (filtered) colimits. The rational fixed point \( \rho_F \) is then defined to be the terminal lfp coalgebra for \( F \). We proved that it is indeed a fixed point of \( F \). Furthermore, it can be constructed as the colimit of all coalgebras with finitely presentable carrier. In this sense \( \rho F \) is determined by ‘finite behaviour’ as the title of the chapter suggests. In addition, we showed that whenever \( \mathcal{A}_{\text{fp}} = \mathcal{A}_{\text{fg}} \) and \( F \) preserves monomorphisms, the rational fixed point is fully abstract w.r.t. behavioural equivalence, i.e. it is a subcoalgebra of the terminal coalgebra \( \nu F \). This result entails concrete descriptions of rational fixed points for endofunctors on sets and many other categories of interest in the coalgebraic semantics of systems.

We then turned to iterative algebras and proved that free iterative algebras exist and that the rational fixed point is an initial iterative algebra for \( F \). Thus, the initial iterative algebra is precisely the terminal lfp coalgebra.

Finally, we saw two explicit description of the rational fixed point for a finitary endofunctor \( F \) on sets. First, \( \rho F \) consists of all finite well-pointed coalgebras for \( F \). Second, using a presentation of \( F \) by a finitary signature \( \Sigma \) and a set \( E \) of basic equations, we saw that \( \rho F \) is the quotient of the rational fixed point \( \rho H_S \) of all rational \( \Sigma \)-trees modulo finite and infinite applications of the basic equations in \( E \).
We concluded the chapter with a discussion of further work inspired by results and ideas arising from the theory of the rational fixed point.
Index

$HC$, 153
$HF$, 153
$\Sigma$-algebra
  on Set, 21
$\text{Sub}_M(A)$, 155
$\lambda$-accessible, 162
smooth class, 153

algebra, 11
  completely iterative, 185–195
  corecursive, 179–185
  free, 33
  initial, 11, 12, 27, 98
  quotient, 98
algebraically complete, 76, 149
analytic, 102
analytic functor, 98
automaton, 44, 208
  deterministic, 40
  Mealy, 41
  Moore, 41
  non-deterministic, 41, 113, 128
  weighted, 44

bag, 74
bisimulation, 58, 90, 128, 243, 255
  tree $\sim$, 110, 111
Brzozowski derivative, 49

cardinal, 146
category
  BiP, 76
  Clat, 119
  CMS, 70, 72, 131–140, 188
  CMS-enriched, 132
  CPO, 72, 118

  CPO-enriched, 118–173
  CPO$_*$, 129
  CPO$_\perp$, 79
  Class, 257
  DCPO, 119
  DCPO$_\perp$, 125
  Gra, 156, 169, 231
  KMS, 140
  $K$-Vec, 154, 158, 159, 230
  MS, 70–72
  Nom, 25, 33, 45, 46, 265, 266
  Pfn, 119, 154, 158, 159
  Pos, 69, 72, 79
  Pred, 231
  Rel, 119, 128, 158
  Set, 12, 17, 27, 72, 133, 140, 158
  Set$^S$, 158, 159, 164
  pointed, 117
  presheaf, 265
chain, 147, 160
  convergence ordinal, 149, 161
  initial-algebra, 149
  op, 160
  op-, 147
  terminal-coalgebra, 161
coalgebra, 13
  $I$-pointed, 254
  canonical graph of, 210
  homomorphism, 13
  pointed, 245
  reachable, 245
  recursive, 196
  simple, 241–244
  terminal, 13
  well-founded, 205–236
Index

well-founded part, 222
well-pointed, 254–260
coalgebra-to-algebra morphism, 196
cocone, 66
cofinal, 146
coinduction, 14
colimit, 66
congruence, 90
contracting
  chain, 134
  function, 132
corecursion, 14
cpo
  ideal, 44
depth, 29
diagram
  \lambda\text{-directed}, 162
duality, 46
embedding-projection pair, 119
epi-transformations, 94
epimorphism
  strong, 213, 242
extensional quotient, 74
finitary
  functor, 93
fixed point
  canonical, 133
fixed point, 11, 27, 165
  canonical, 117, 131, 140, 173–175
  unique, 132, 137, 139
formal language, 128
free algebra, 33
function
  non-expanding, 70
functor
  \mathcal{R}, 212
  \lambda\text{-accessible}, 162
  \lambda\text{-bounded}, 162
  \mathcal{P}_f, 16
  accessible, 103
  Aczel-Mendler, 96
  analytic, 75, 101

bag \mathcal{D}, 74, 96, 101, 188
constant, 117, 124
contracting, 136, 138, 188
discrete probability measure \mathcal{D}, 18
filter, 245
filter \mathcal{F}, 164
finite, 85–107, 266
finite power-set \mathcal{P}_f, 18, 88–188
Hausdorff \mathcal{H}, 133
locally continuous, 124, 175
polynomial, 17, 124, 133, 152, 180
presentation of, 92–101
quotient, 94
standard, 193
weakly contracting, 137

graph, 217

ideal, 44, 124
initial object
  strict, 215
initial-algebra chain, 66

kernel equivalence, 89

labelled transition system, 41
Lambek’s Lemma, 27
language, formal, 49
lifting, 128, 173
  contracting, 133, 138, 189
limit-colimit coincidence, 118, 136
linear weighted automaton, 208

measure
  discrete probability, 17
metric space
  discrete, 72
metric space, 70
monad, 35
monomorphism
  nonempty, 215
monomorphisms
  smooth class, 153
  universally smooth, 226
multiset, 74

428
op-chain, 147
ordinal, 145
pointed, 117, 245
polynomial functor, 21
presheaf category, 164, 231
quotient
  algebra, 98
  strong, 242
quotient coalgebra, 90
quotient functor, 167–173
reachable, 245
reachable part, 245
recursion, 11
recursive coalgebra, 196
relation
  well-founded, 196
root, 29
Scott induction, 200
semiring, 44, 208
set
  hereditarily countable, 153
  hereditarily finite, 153
signature, 21, 80, 84, 93–98
  constant symbols, 21
  finitary, 73, 75, 95, 212, 266, 286, 298
  infinitary, 81, 152
  many-sorted, 82, 84
  nullary symbols, 21
simple, 241–244
solution, 180
state transition graph, 207
stream, 79
subcoalgebra, 47, 112
  cartesian, 206
  generated, 256
subfunctor, 28, 167–173
  M, 169
subnatural transformation, 210
subobject, 26, 155
subobject classifier, 234
terminal chain, 77
theorem
  Banach Fixed Point, 132
  Cantor’s, 27
  Kleene’s, 65, 67
tree, 29
  Σ, 152
  cutting ∂ₙ, 92
  extensional, 74, 92, 111
  leaf of, 29
  ordered, 29
  saturated, 114
  strongly extensional, 110, 111, 188
  strongly extensional quotient, 111
  unordered, 29
universally smooth class, 226
varietor, 159
weakly terminal, 88
well-founded part, 222
well-pointed, 254–260
well-powered, 26, 155
zero object, 118
Bibliography

[1] Samson Abramsky. A Cook’s Tour of the finitary non-well-founded sets. In We
Will Show Them: Essays in honour of Dov Gabbay, volume 1, pages 1–18. College

[2] Samson Abramsky and Achim Jung. Handbook of Logic in Computer Science,


[4] Peter Aczel, Jiří Adámek, Stefan Milius, and Jiří Velebil. Infinite trees and
2003.

and Computer Science (CTCS), volume 389 of Lecture Notes Comput. Sci., pages


[12] Jiří Adámek. Final coalgebras are ideal completions of initial algebras. J. Logic

Bibliography


Bibliography


Bibliography


435


Bibliography


Bibliography


Bibliography


