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Completely iterative algebras and completely iterative monads

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Abstract

Completely iterative theories of Calvin Elgot formalize (potentially infinite) computations as solutions of recursive equations. One of the main results of Elgot and his coauthors is that infinite trees form a free completely iterative theory. Their algebraic proof of this result is extremely complicated. We present completely iterative algebras as a new approach to the description of free completely iterative theories. Examples of completely iterative algebras include algebras on complete metric spaces. It is shown that a functor admits an initial completely iterative algebra iff it has a final coalgebra. The monad given by free completely iterative algebras is proved to be the free completely iterative monad on the given endofunctor. This simplifies substantially all previous descriptions of these monads. Moreover, the new approach is much more general than the classical one of Elgot et al. A necessary and sufficient condition for the existence of a free completely iterative monad is proved.

Keywords: Completely iterative algebra; Coalgebra; Completely iterative theory; Monad

1. Introduction

The goal of the current paper is the study of completely iterative algebras (cia), i.e., algebras in which every system of recursive equations has a unique solution. This study allows a new approach to completely iterative theories, which were introduced and studied by Elgot et al. [10]. Completely iterative theories allow the treatment of the semantics of potentially infinite computations of a

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computer program in an algebraic setting abstracting away from the nature of the external memory. They are algebraic theories (in the sense of Lawvere [13] and Linton [14]) that allow for unique solutions of fixed point equations. An important example of a completely iterative theory is the theory of finite and infinite trees over a signature Σ. In [10] it is shown that this is the free completely iterative theory over Σ.

In recent years it has been realized that a more abstract categorical approach to completely iterative theories allows to generalize the classical results beyond the universal algebra setting. Moreover, the proofs become substantially simpler and conceptually much clearer, see the work of Moss [17] and the work of Aczel et al. [1]. To be a bit more precise, in lieu of a signature one starts with an endofunctor H on Set (or more generally, any category $\mathcal{A}$ with binary coproducts) having "enough final coalgebras," i.e., for any object Y there exists a final coalgebra TY of $H(\_) + Y$. The main result of [1] is that T is a free completely iterative monad on H.

In the present paper, we add completely iterative algebra to the picture, and we establish for every category $\mathcal{A}$ with binary coproducts, and every endofunctor H on $\mathcal{A}$ that given an object mapping T of $\mathcal{A}$ the following three statements are equivalent:

(a) for every object Y, TY is a final coalgebra of $H(\_) + Y$,
(b) for every object Y, TY is a free completely iterative H-algebra on Y, and
(c) T is a free completely iterative monad on H.

The implication that (a) implies (c) is the main result of [1]. The converse (c) implies (a) is a new result. It has appeared before in the extended abstract [16] but not in a journal article. The main contribution of the current paper is to add (b) to the above list. Here we shall first establish the equivalence of (a) and (b), and then we prove that (b) implies (c). This leads to a substantial simplification of the proof of [1]. For the converse (c) implies (b) we use the technical material from [16], and we take here the opportunity to streamline it a bit. More on the technical side this material will allow us to drop an annoying little side condition of our results in [1]—there coproduct injections were assumed to be monomorphic—and the freeness in (c) can be slightly extended.

In Section 1, we shall restrict ourselves to the classical case to clarify our results a bit more. So suppose we are given a *polynomial* endofunctor H_Σ on the category Set, i.e., one that is obtained from a signature $\Sigma = (\Sigma_n)_{n<\omega}$ as follows:

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots .$$

Thus, the classical Σ-algebras are precisely the algebras of the functor H_Σ. From Section 2 on we shall work more generally with an endofunctor on an arbitrary category with binary coproducts.

A Σ-algebra A is called *completely iterative*, if every system

$$x_i \approx t_i, \quad i \in I, \tag{1.1}$$

where I is some (possibly infinite) set, $X = \{ x_i \mid i \in I \}$ is a set of variables and the t_i are terms over $X + A$, none of which is just a single variable, has a unique solution in A. By a *solution* we mean a set $\{ x_i{}^\dagger \mid i \in I \}$ of elements of A such that the above formal equations (1.1) become actual identities in A when the variables are substituted by the solutions and the terms t_i are interpreted in A, i.e.,

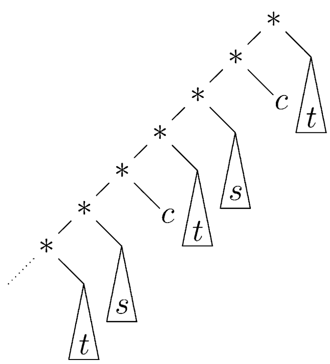
$$x_i{}^\dagger \equiv t_i \left(\{ x_j{}^\dagger / x_j \mid j \in I \} \right), \quad i \in I.$$

Example. Suppose we have a signature Σ. The algebra $A = T_\Sigma$ of all finite and infinite Σ-trees, i.e., trees whose nodes with n children are labelled by n-ary operation symbols from Σ, is completely iterative. For example, let Σ consist of a binary operation symbol $*$ and a constant symbol c. Then the following system:

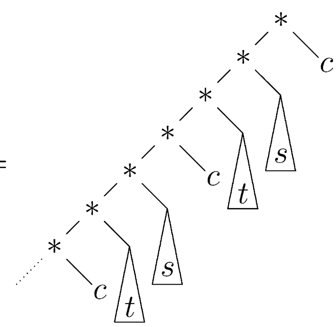
$$x_1 \approx x_2 * t \quad x_2 \approx (x_1 * s) * c, \tag{1.2}$$

where s and t are some trees in T_Σ has the following solution:

$x_1{}^\dagger =$



and $x_2{}^\dagger =$



Observe that it is sufficient to allow for the right-hand side in (1.2) only so-called *flat terms*, i.e., terms t that are either

$$t = \sigma(x_1, \ldots, x_n), \qquad \sigma \in \Sigma_n, \quad x_1, \ldots, x_n \in X,$$

or

$$t \in A.$$

In fact, for every system (1.1) one can give a system with only flat terms on the right-hand side, which has the same solution. This is done by introducing (possibly infinitely many) new variables. For example for the system (1.2) we get the following flat one:

$$\begin{array}{ll} x_1 \approx x_2 * z_1 & z_2 \approx x_1 * z_4 \\ x_2 \approx z_2 * z_3 & z_3 \approx c \\ z_1 \approx t & z_4 \approx s \end{array}$$

Obviously, the solutions $x_1{}^\dagger$ and $x_2{}^\dagger$ are the same trees as before.

Clearly, one can write every system with flat right-hand sides as a single map

$$e : X \longrightarrow H_\Sigma X + A$$

and a solution is a map $e^\dagger : X \longrightarrow A$ such that the following square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ H_\Sigma X + A & \xrightarrow[H_\Sigma e^\dagger + A]{} & H_\Sigma A + A \end{array}$$

where a denotes the algebra structure of A, commutes. We call an algebra A *completely iterative* if any flat equation morphism e has a unique solution $e^\dagger$. Among classical algebras the property of being completely iterative seems to be quite rare. However, there exist interesting examples of completely iterative algebras, e.g., the algebras

$$T_\Sigma$$

of finite and infinite Σ-trees form a completely iterative algebra, in fact, we prove below that T_Σ is the initial completely iterative Σ-algebra. It follows that for any set Y the algebra

$$T_\Sigma Y$$

of all finite and infinite Σ-trees with leaves labelled by constant symbols from Σ or variables from Y is a free cia on Y. The free cias define a monad $\mathbb{T}_\Sigma$ on Set, and this monad is the free completely iterative monad on H_Σ.

In our proof we work with an arbitrary endofunctor H on Set (or, more generally, on every category with binary coproducts), which has free cias on every set Y. In Section 2, we shall introduce completely iterative algebras in this general setting. And we will prove the equivalence of the above statements (a) and (b). In Section 3, we prove an extension of the Solution Theorem of [1] to all completely iterative algebras. In Section 4, we prove (b) implies (c) (see above): Let H be an endofunctor on a category with binary coproducts (with monomorphic injections), which has free cias on every object Y. Then these free cias define a monad $\mathbb{T}$, and this monad is a free completely iterative monad on H. In Section 5, we show how the technical assumption of having monomorphic coproduct injections in the base category used in Section 4 can be avoided at the expense of being slightly more careful with some technical notions. This also leads to an extension of the freeness result. Finally, we shall prove in Section 6 that any free completely iterative monad is given by free completely iterative algebras, i.e., (c) implies (b) above. More precisely, if $\mathbb{T} = (T, \eta, \mu)$ is a free completely iterative monad on H, then for every object Y, TY is a free cia on Y, or, equivalently, TY is a final coalgebra of $H(\_) + Y$.

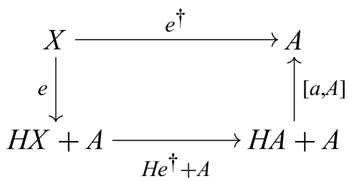
Related Work. The study of completely iterative algebras and completely iterative monads is very closely linked to the study of iterative algebras and iterative monads. In fact, historically, iterative theories were introduced by Elgot [9] before completely iterative theories. They are, roughly speaking, algebraic theories such that finitary recursive systems of equations, i.e., with a finite set of variables only, have unique solutions. Adámek et al. [2,3] have given a categorical approach to iterative theories. Similar ideas as those we use in the current paper for a simplified approach to completely iterative monads apply to the iterative case. In the latter case one starts by investigating iterative algebras, i.e., algebras that admit unique solutions of finitary systems of recursive equations. This leads to a construction of free iterative algebras using coalgebras, and these algebras yield the free iterative monad. This simplified approach to iterative theories can be found in [4]. That paper developed simultaneously with the current one.

In the classical setting of polynomial endofunctors on Set, iterative algebras were introduced by Nelson [18] to obtain a short proof of Elgot's description of free iterative theories. Also Tiuryn [20] introduced and studied a concept of iterative algebras with the aim of relating iterative theories to properties of algebras. Our notion of completely iterative algebras is an extension and generalization of the notion of iterative algebra of [18].

2. Completely iterative algebras for an endofunctor

Let $H : \mathcal{A} \longrightarrow \mathcal{A}$ be an endofunctor on a category $\mathcal{A}$ with binary coproducts. We denote by $\mathsf{inl} : X \longrightarrow X + Y$ and $\mathsf{inr} : Y \longrightarrow X + Y$ the coproduct injections and we shall write $\mathsf{can} : HX + HY \longrightarrow H(X + Y)$ for the canonical arrow $[H\mathsf{inl}, H\mathsf{inr}]$.

Definition 2.1. A morphism $e : X \longrightarrow HX + A$ of $\mathcal{A}$ is called a *flat equation morphism* in (the object of parameters) A. Suppose that A is the underlying object of an H-algebra $a : HA \longrightarrow A$. Then a *solution* of e in A is a morphism $e^\dagger : X \longrightarrow A$ such that the diagram



(2.1)

commutes.

An H-algebra is called *completely iterative* (or shortly, *cia*) if every flat equation morphism in it has a unique solution.

Notation 2.2. For any flat equation morphism $e : X \longrightarrow HX + Y$ and any morphism $f : Y \longrightarrow Z$ we get a flat equation morphism $f \bullet e$ as the "renaming of parameters by f":

$$f \bullet e \equiv X \xrightarrow{e} HX + Y \xrightarrow{HX+f} HX + Z.$$

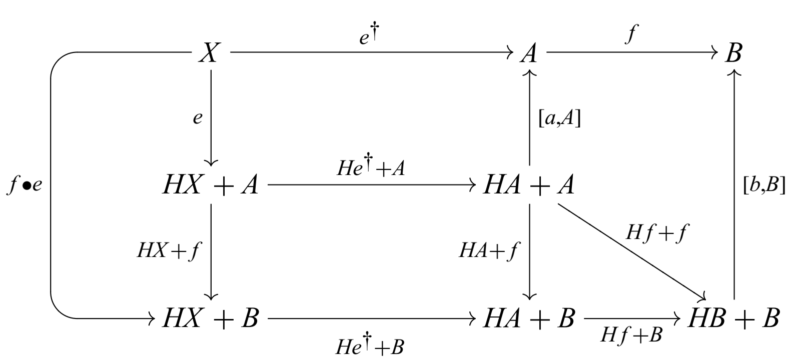
Homomorphisms of H-algebras are precisely the solution-preserving morphisms as we prove now:

Proposition 2.3. *Let (A, a) and (B, b) be completely iterative H-algebras, and let $f : A \longrightarrow B$ be a morphism. Then the following are equivalent:*

(*i*) *$f : (A, a) \longrightarrow (B, b)$ is an H-algebra homomorphism,*
(*ii*) *f is solution-preserving, i.e., for all $e : X \longrightarrow HX + A$ we have*

$$(f \bullet e)^\dagger = f \cdot e^\dagger.$$

Proof. (i) ⇒ (ii): Consider the following commutative diagram:

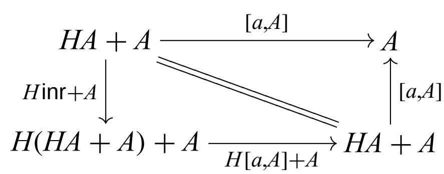


In fact, the upper middle square commutes since $e^\dagger$ is a solution of e, and the upper right-hand part since f is an H-algebra homomorphism. The other three parts are obvious. Thus, the outer square commutes proving that $f \cdot e^\dagger$ is a solution of $f \bullet e$. The result follows from the unicity of solutions in B.

(ii) $\Rightarrow$ (i): Suppose that $f : A \longrightarrow B$ is a solution-preserving morphism. We have to show that f is an H-algebra homomorphism, i.e., $f \cdot a = b \cdot Hf$. To prove it we use the uniqueness of solutions. First, consider the equation morphism

$$e \equiv HA + A \xrightarrow{H\mathsf{inr}+A} H(HA+A) + A.$$

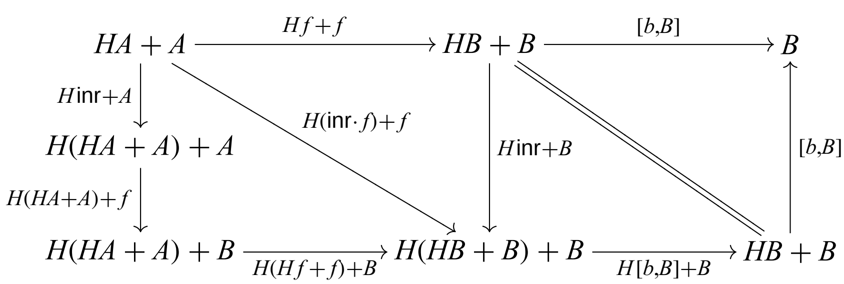
Its unique solution is $[a, A] : HA + A \longrightarrow A$. In fact, the following diagram



commutes. Since f is solution-preserving we know that $f \cdot a$ is the left-hand component of the unique solution of the following equation morphism:

$$f \bullet e \equiv HA + A \xrightarrow{H\mathsf{inr}+A} H(HA+A) + A \xrightarrow{H(HA+A)+f} H(HA+A) + B,$$

in symbols, $f \cdot a = (f \bullet e)^\dagger \cdot \mathsf{inl}$. Now consider the following commutative diagram:



It shows that $[b, B] \cdot (Hf + f) = (f \bullet e)^\dagger$; thus, we obtain

$$f \cdot a = (f \bullet e)^\dagger \cdot \mathsf{inl} = b \cdot Hf,$$

which completes the proof. $\square$

Notation 2.4. We denote by $\mathsf{CIA}\,H$ the category of all completely iterative algebras and H-algebra homomorphisms. It is a full subcategory of $\mathsf{Alg}\,H$, the category of all H-algebras and homomorphisms.

Examples 2.5.

(i) Classical algebras are seldom cias. For example, let $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ be the functor expressing one binary operation, $H_\Sigma X = X \times X$. Then a group is a cia iff its unique element is the unit 1, since the recursive equation $x \approx x \cdot 1$ has a unique solution. A lattice is a cia iff it has a unique element; consider $x \approx x \vee x$.

(ii) In [4] it was proved that the algebra of addition on

$$\widetilde{\mathbb{N}} = \{1, 2, 3, \ldots\} \cup \{\infty\}$$

is a cia w.r.t. the functor H_Σ of (i).

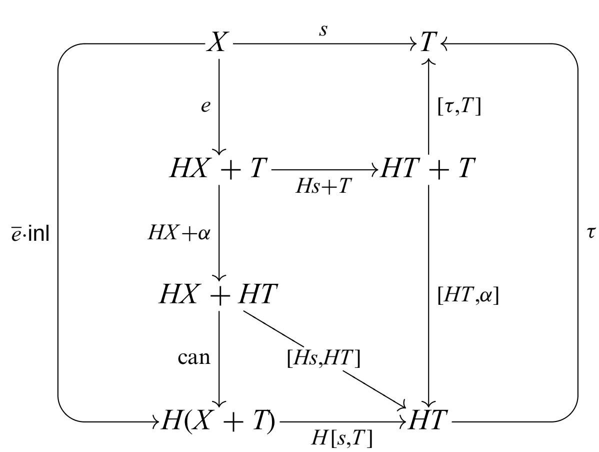
(iii) Final coalgebras are completely iterative algebras. More precisely, denote by (T, α) a final coalgebra of H, i.e., for any coalgebra (C, γ) there exists a unique coalgebra homomorphism $\gamma^\sharp : (C, \gamma) \longrightarrow (T, \alpha)$ so that $\alpha \cdot \gamma^\sharp = H(\gamma^\sharp) \cdot \gamma$. Recall that by Lambek's Lemma [12], the structure map α is an isomorphism, whose inverse we denote by $\tau : HT \longrightarrow T$. Then this H-algebra (T, τ) is completely iterative. In fact, consider an equation morphism

$$e : X \longrightarrow HX + T,$$

and form the H-coalgebra

$$\overline{e} \equiv X + T \xrightarrow{[e,\mathsf{inr}]} HX + T \xrightarrow{HX+\alpha} HX + HT \xrightarrow{\mathsf{can}} H(X + T).$$

We claim that the left-hand component of $\overline{e}^\sharp : X + T \longrightarrow T$ is the desired solution of e, and that it is unique. Indeed, any coalgebra homomorphism $(X + T, \overline{e}) \longrightarrow (T, \alpha)$ must have as its right-hand component a coalgebra homomorphism from (T, α) to itself, whence the identity on T. Then we get the following commutative diagram for the left-hand component:



If s is the left-hand component of $\overline{e}^\sharp$, then the outer shape commutes, whence so does the upper square, which shows that s solves e. Conversely, if s is a solution of e, then the upper square commutes and therefore the outer shape does, too. Thus, $[s, T] : X + T \longrightarrow T$ is a coalgebra homomorphism, and so we have $[s, T] = \bar{e}^\sharp$.

(iv) Infinite trees form completely iterative algebras. Let Σ be a signature. It is well-known that the Σ-algebra T_Σ of all (finite and infinite) Σ-trees is a final H_Σ-coalgebra. Thus, T_Σ is a cia.

(v) Finitely branching strongly extensional trees. The final coalgebra of $\mathcal{P}_{\mathrm{fin}} : \mathsf{Set} \longrightarrow \mathsf{Set}$, the finite power-set functor, has been described by Worrell [21]. It is the algebra T of all strongly extensional finitely branching trees (i.e., unordered trees such that the subtrees defined by any pair of siblings are not bisimilar). It follows from (iii) that T is a cia.

(vi) Algebras over complete metric spaces as a tool for the semantics of infinite computation have been investigated by America and Rutten [6]. Those algebras yield cias. Take $\mathcal{A} = \mathsf{CMS}$, the category whose objects are complete metric spaces (i.e., such that each Cauchy sequence has a limit), where distances are measured in the interval $[0, 1]$. The morphisms of CMS are the non-expanding maps, i.e., functions $f : (X, d_X) \longrightarrow (Y, d_Y)$ such that $d_Y(f(x), f(y)) \leqslant d_X(x, y)$ for all $x, y \in X$. Recall that for given complete metric spaces (X, d_X) and (Y, d_Y) the hom-set in CMS is a complete metric space with the metric given by

$$d_{X,Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Now suppose we have a functor $H : \mathsf{CMS} \longrightarrow \mathsf{CMS}$ which is *contracting*, i.e., there exists a constant $\varepsilon < 1$ such that for any non-expanding maps $f, g : (X, d_X) \longrightarrow (Y, d_Y)$ between complete metric spaces we have

$$d_{HX,HY}(Hf, Hg) \leqslant \varepsilon \cdot d_{X,Y}(f, g).$$

Then any non-empty H-algebra (A, a) is completely iterative. In fact, given any flat equation morphism $e : X \longrightarrow HX + A$ in CMS, choose some element $a \in A$ and define a Cauchy sequence $(e_n^\dagger)_{n \in \mathbb{N}}$ in $\mathsf{CMS}(X, A)$ inductively as follows: let $e_0^\dagger = \mathsf{const}_a$, and given $e_n^\dagger$ define $e_{n+1}^\dagger$ by the commutativity of the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;e_{n+1}^\dagger\;} & A \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle [a,A]} \\ HX + A & \xrightarrow[\;He_n^\dagger + A\;]{} & HA + A \end{array}$$

In [5] it is proved that this is indeed a Cauchy sequence in $\mathsf{CMS}(X, A)$ and that its limit yields a unique solution of e.

(vii) (Unary algebras over Set)
Here we have $\mathcal{A} = \mathsf{Set}$ and $H = Id$. A unary algebra (A, α_A) is completely iterative if and only if

(a) there exists a unique fixed point $a_0 \in A$ of all $\alpha_A^k : A \longrightarrow A$, $k \geqslant 1$,
(b) for any sequence $(b_i)_{i<\omega}$ in A with $b_i = \alpha_A(b_{i+1})$ we have $b_i = a_0$ for every $i < \omega$ (i.e., for any $a \neq a_0$ in A there is no infinite α-chain of elements of A ending in a).

To see that (a) and (b) are necessary, solve the equation $x \approx \alpha x$ to obtain the fixed point a_0. Furthermore, the system

$$x_i \approx \alpha x_{i+1}, \quad i < \omega,$$

has as solutions any sequence as in (b); in particular, the constant sequence at a_0 is a solution, and this must be the unique one.

For the sufficiency, suppose that (A, α_A) satisfies (a) and (b). Given any equation morphism $e : X \longrightarrow H_\Sigma X + A$ there is a unique solution $e^\dagger : X \longrightarrow A$: If $x \in X$ is such that there exist equations

$$\begin{aligned} x = x_0 &\approx \alpha x_1 \\ x_1 &\approx \alpha x_2 \\ &\vdots \\ x_{k-1} &\approx \alpha x_k \\ x_k &\approx a, \end{aligned}$$

where $a \in A$, then $e^\dagger(x_k) = a$ and therefore $e^\dagger(x) = \alpha^k(a)$. Otherwise we have equations

$$\begin{aligned} x = x_0 &\approx \alpha x_1 \\ x_1 &\approx \alpha x_2 \\ x_2 &\approx \alpha x_3 \\ &\vdots \end{aligned}$$

and (a) and (b) ensure that the unique solution is given by $e^\dagger(x_i) = a_0$, for all i.

We shall now show that final H-coalgebras are precisely the initial completely iterative H-algebras. This is the first step towards proving the equivalence of the statements (a) and (b) of the introduction. First, we establish two auxiliary results. For the first one observe that any endofunctor H lifts to one on the category $\mathsf{Alg}\, H$ of algebras. The lifted endofunctor acts on objects by $(A, a) \longmapsto (HA, Ha)$, and on morphisms its action is that of H. The same is true for completely iterative algebras.

Proposition 2.6. *Any endofunctor H lifts to the category of completely iterative H-algebras, i.e., for any cia (A, a) the H-algebra (HA, Ha) is completely iterative, too.*

Proof. Suppose we are given an equation morphism $e : X \longrightarrow HX + HA$, we have to produce a solution $e^\dagger : X \longrightarrow HA$, and show its uniqueness. Let us form an equation morphism

$$\overline{e} \equiv X \xrightarrow{\ e\ } HX + HA \xrightarrow{\ HX+a\ } HX + A$$

w.r.t. (A, a). Then its solution $\overline{e}^{\dagger}$ makes the diagram

$$
\begin{array}{ccccccc}
X & & & \xrightarrow{\quad\overline{e}^{\dagger}\quad} & & & A \\
 & \searrow^{\overline{e}} & & & & \nearrow^{[a,A]} & \\
\Big\downarrow e & & HX + A & \xrightarrow[H\overline{e}^{\dagger}+A]{} & HA + A & & \Big\uparrow a \\
 & \nearrow_{HX+a} & & & & & \\
HX + HA & & & \xrightarrow[{[H\overline{e}^{\dagger},HA]}]{} & & & HA
\end{array}
\tag{2.2}
$$

commutative. In fact, its upper part commutes since $\overline{e}^{\dagger}$ is a solution, and the other two parts are obvious. Now define

$$
e^{\dagger} \equiv X \xrightarrow{e} HX + HA \xrightarrow{[H\overline{e}^{\dagger},HA]} HA.
$$

We prove that $e^{\dagger}$ solves e. In fact, the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{e} & HX + HA & \xrightarrow{[H\overline{e}^{\dagger},HA]} & HA \\
\Big\downarrow e & & & \nearrow_{[H\overline{e}^{\dagger},HA]} & \Big\uparrow [Ha,HA] \\
HX + HA & & \xrightarrow[H([H\overline{e}^{\dagger},HA]\cdot e)+HA]{} & & HHA + HA
\end{array}
$$

commutes; the upper left-hand triangle is obvious, and so is the right-hand coproduct component of the lower right-hand one. The left-hand coproduct component of the latter triangle yields the outer square of Diagram (2.2) after H is removed. This proves the existence of a solution.

For the uniqueness, suppose that $s : X \longrightarrow HA$ solves e. Then $a \cdot s$ solves $\overline{e}$. In fact, notice that $a : (HA, Ha) \longrightarrow (A, a)$ is an H-algebra homomorphism and then use a similar argument as in the first part of the proof of Proposition 2.3. Thus, by uniqueness of solutions we have $a \cdot s = \overline{e}^{\dagger}$, and we obtain

$$
\begin{aligned}
s &= [Ha, HA] \cdot (Hs + HA) \cdot e \\
&= [H(a \cdot s), HA] \cdot e \\
&= [H\overline{e}^{\dagger}, HA] \cdot e \\
&= e^{\dagger}. \qquad \square
\end{aligned}
$$

Lambek's Lemma [12] states that the structure map of an initial H-algebra is an isomorphism. The same is true in the completely iterative case.

Lemma 2.7. *If (T, τ) is an initial completely iterative H-algebra, then the structure morphism τ is an isomorphism.*

Proof. By Proposition 2.6 we have a cia $(HT, H\tau)$. Then by initiality we obtain a unique H-algebra homomorphism $i : (T, \tau) \longrightarrow (HT, H\tau)$, i.e., such that the following square:

$$\begin{array}{ccc} HT & \xrightarrow{\ \tau\ } & T \\ {\scriptstyle Hi}\downarrow & & \downarrow{\scriptstyle i} \\ HHT & \xrightarrow[H\tau]{} & HT \end{array}$$

commutes. Clearly, $\tau : (HT, H\tau) \longrightarrow (T, \tau)$ is an H-algebra homomorphism. Thus, by initiality we conclude that $\tau \cdot i = 1_T$. But then also $i \cdot \tau = H\tau \cdot Hi = H1_T = 1_{HT}$. □

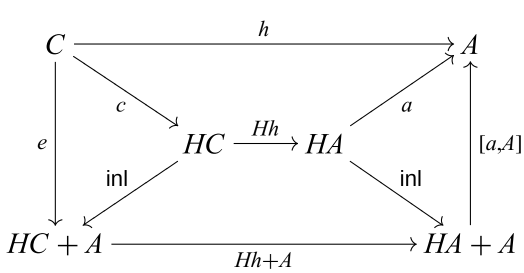
We are now ready to prove the main result of this section.

Theorem 2.8. *Let $H : \mathcal{A} \longrightarrow \mathcal{A}$ be any endofunctor.*
(i) If (T, α) is a final H-coalgebra, then (T, τ) with $\tau = \alpha^{-1}$ is an initial completely iterative H-algebra.
(ii) Conversely, if (T, τ) is an initial completely iterative H-algebra, then (T, α) with $\alpha = \tau^{-1}$ is a final H-coalgebra.

Proof. Before we prove the two statements we shall establish one useful fact about the relation between H-coalgebras and cia's. Suppose that (C, c) is any H-coalgebra and (A, a) is a cia. We can form an equation morphism

$$e \equiv C \xrightarrow{\ c\ } HC \xrightarrow{\ \mathsf{inl}\ } HC + A.$$

Then there is a one-to-one correspondence between solutions of e and morphisms $h : C \longrightarrow A$ such that $h = a \cdot Hh \cdot c$ (the so-called coalgebra to algebra homomorphisms). Indeed, this follows easily by inspection of the following diagram:



Since there exists a unique solution $e^\dagger$ for e, there exists a unique coalgebra to algebra homomorphism h. It is now quite easy to prove the theorem.

(i) We have seen in Example 2.5 that (T, τ) is completely iterative. It remains to prove the initiality. Given any cia (A, a) we have by the above considerations a unique coalgebra to algebra homomorphism $h : T \longrightarrow A$, i.e., unique H-algebra homomorphism $h : (T, \tau) \longrightarrow (A, a)$.

(ii) By Lemma 2.7 we only need to show finality of the coalgebra (T, α). Given any H-coalgebra (C, c) there exists a unique coalgebra to algebra homomorphism $h : C \longrightarrow T$, i.e., a unique H-coalgebra homomorphism $h : (C, c) \longrightarrow (T, \alpha)$. □

Remark 2.9. Observe that in the above proof of part (ii) in lieu of the full universal property of (T, τ) we have only used that the structure map τ is an isomorphism. Thus, the only cia with an isomorphic structure map is the initial one.

In the realm of H-algebras it is quite trivial to show that the initial algebra for the functor $H(\_) + Y$ is precisely the free H-algebra on the object Y. The same will now be proved for cia's, and this is the second neccessary ingredient to establish the equivalence of statements (a) and (b) from the introduction.

By a free cia on an object Y of $\mathcal{A}$ we mean, of course, a cia (TY, τ_Y) together with a morphism $\eta_Y : Y \longrightarrow TY$ in $\mathcal{A}$ such that for any cia (A, a) and any morphism $f : Y \longrightarrow A$ in $\mathcal{A}$ there exists a unique homomorphic extension $f^\sharp : (TY, \tau_Y) \longrightarrow (A, a)$, i.e., such that the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\eta_Y} & TY & \xleftarrow{\tau_Y} & HTY \\ & \searrow^{f} & \downarrow{\scriptstyle f^\sharp} & & \downarrow{\scriptstyle Hf^\sharp} \\ & & A & \xleftarrow{a} & HA \end{array}$$

commutes.

Theorem 2.10. *For any object Y of $\mathcal{A}$ the following are equivalent:*
(i) TY is an initial completely iterative $H(\_) + Y$-algebra.
(ii) TY is a free completely iterative H-algebra on Y.

Proof. First, we shall establish the following fact: To give a completely iterative H-algebra (A, a) and a morphism $f : Y \longrightarrow A$ is the same as to give a completely iterative algebra $(A, [a, f])$ of $H(\_) + Y$.

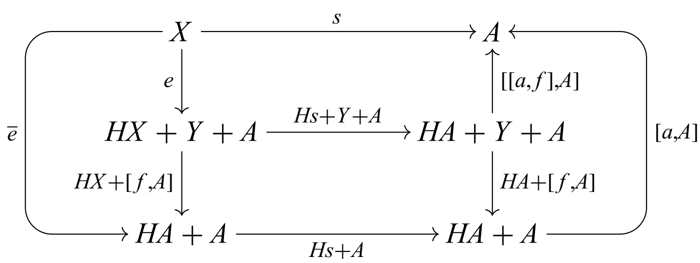
In fact, suppose we have a cia (A, a) and a morphism f. Then it is our task to find a unique solution for any equation morphism

$$e : X \longrightarrow HX + Y + A$$

for the functor $H(\_) + Y$. But e gives the following equation morphism

$$\overline{e} \equiv X \xrightarrow{e} HX + Y + A \xrightarrow{HX + [f, A]} HX + A$$

for the functor H. Now the solutions of e correspond precisely to the solutions of $\overline{e}$. Indeed, this follows by inspecting the following diagram:



The arrow s solves e if and only if the upper part commutes. Equivalently, the outer square commutes. But this says precisely that s solves $\overline{e}$. Since $\overline{e}$ has a unique solution, so has e.

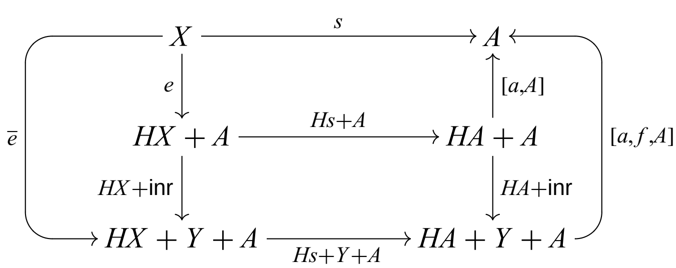
For the converse, suppose that $(A, [a, f])$ is a completely iterative algebra of $H(\_) + Y$. We must show that any equation morphism

$$e : X \longrightarrow HX + A$$

has a unique solution. We simply form an equation morphism

$$\overline{e} \equiv X \xrightarrow{e} HX + A \xrightarrow{HX+\mathsf{inr}} HX + Y + A.$$

As before, solutions of e correspond precisely to solutions of $\overline{e}$. In fact, inspect the following diagram:



The morphism s solves e precisely if the upper square commutes. This is equivalent to the commutativity of the outer shape, i.e., s solves $\overline{e}$. Hence, since $\overline{e}$ has a unique solution, so has e.

The result of the current theorem can now be proved precisely as in the case of ordinary H-algebras. This is straightforward and we leave it to the reader. □

In [1] we have called an endofunctor *iteratable*, if for any object Y of $\mathcal{A}$ there exists a final coalgebra TY of $H(\_) + Y$. Collecting the results of Theorems 2.8 and 2.10 we obtain the following characterization, i.e., the equivalence of statements (a) and (b), see Section 1.

Corollary 2.11. *For any endofuntor $H : \mathcal{A} \longrightarrow \mathcal{A}$ the following are equivalent:*
(i) *H is iteratable with final coalgebras TY of $H(\_) + Y$, for any Y in $\mathcal{A}$.*
(ii) *For any object Y there exists a free completely iterative H-algebra TY on Y.*

Example 2.12. *The free cias of* $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$.
Recall from Example 2.5(iv) the algebra T_Σ of all (finite and infinite) Σ-trees. This algebra is a cia. For every set Y the algebra $T_\Sigma Y$ of all Σ-trees over Y (i.e., trees with nodes having $n > 0$ children labelled by n-ary operation symbols and leaves labelled by constant symbols or variables from the set Y) is also a cia. It is well known that $T_\Sigma Y$ is a final coalgebra of $H_\Sigma(\_) + Y$. By Corollary 2.11, this implies that $T_\Sigma Y$ is a free completely iterative Σ-algebra on Y.

Example 2.13. *The free cias of* $\mathcal{P}_{\text{fin}} : \mathsf{Set} \longrightarrow \mathsf{Set}$.
Recall the final coalgebra T of $\mathcal{P}_{\text{fin}}$ from Example 2.5(v). Analogously, for a set Y a final coalgebra of $\mathcal{P}_{\text{fin}}(\_) + Y$ is the algebra $T(Y)$ of all finitely branching strongly extensional trees with leaves partially labelled in the set Y. By Corollary 2.11, this implies that $T(Y)$ is a free cia on Y.

Remark 2.14. A special case of a recursive equation morphism is that where no parameters appear, i.e., simply coalgebras $e : X \longrightarrow HX$. They appear in various contexts, e.g., in non-wellfounded set theory [7] or, dually, in the theory of transitive sets [19]. However, these special equation morphisms are not sufficient for our purposes. Let us (just in the present remark) call an algebra *weakly iterative* if every equation morphism $e : X \longrightarrow HX$ has a unique solution $e^\dagger : X \longrightarrow A$ (i.e., $e^\dagger = a \cdot He^\dagger \cdot e$). For example in case $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ represents a binary operation, $H_\Sigma X = X \times X$, the free cia $T_\Sigma\{\, a \,\}$ on one generator has the property that every equation $e : X \longrightarrow X \times X$ has the unique solution $e^\dagger : x \longmapsto t_0$, the constant function to the complete binary tree t_0. Consequently, every subalgebra of $T_\Sigma\{\, a \,\}$ containing t_0 is weakly iterative. However, not every such subalgebra is completely iterative; for example, the smallest subalgebra of $T_\Sigma\{\, a \,\}$ containing t_0 and all finite Σ-trees is weakly iterative but not completely iterative.

3. The solution theorem

In Section 1, we considered non-flat system (1.1) of formal recursive equations for Σ-algebras. And we argued that, due to the possibility of flattening such a system it suffices to consider only the flat equation morphisms $X \longrightarrow H_\Sigma X + A$. In this section, we shall make that statement precise by showing that in completely iterative algebras (not only in Set) much more general systems of recursive equations are uniquely solvable. This result illustrates that for polynomial endofunctors on Set cias are an extension and generalization of iterative algebras as presented by Nelson [18]. Applied to free cias our result implies the solution theorem of [1], which was also discovered independently by Moss [17] under the name Parametric Corecursion.

Let us remark first that the condition stated in (1.1) that no right-hand side of a system is a variable is important; for example, the equation $x \approx x$ has a unique solution only in the trivial terminal algebra. Systems satisfying the above condition are called *guarded.*

In this section we assume that $H : \mathcal{A} \longrightarrow \mathcal{A}$ is an iteratable endofunctor on a category $\mathcal{A}$ with binary coproducts. By Corollary 2.11, there exists a free cia TY on every object Y. In other words, we have an adjoint situation

$$\mathsf{CIA}\, H \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{A}.$$

This adjunction creates a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{A}$. More detailed, for every object Y denote by TY the (underlying object of a) free cia on Y with universal arrow

$$\eta_Y : Y \longrightarrow TY$$

and algebra structure

$$\tau_Y : HTY \longrightarrow TY.$$

Use the freeness of the cia TTY on TY to obtain $\mu_Y : TTY \longrightarrow TY$ as the unique homomorphism of H-algebras with $\mu_Y \cdot \eta_{TY} = 1_{TY}$. It is easy to check the naturality of η, τ, and μ as well as the three monad laws. Notice also that it follows from Theorems 2.8 and 2.10 that the morphism $[\tau_Y, \eta_Y] : HTY + Y \longrightarrow TY$ is an isomorphism whose inverse is the structure map of a final coalgebra of $H(\_) + Y$.

Finally, observe that the following Substitution Theorem proved in [1] using coinduction is now a trivial consequence of the freeness of the cias TY:

Theorem 3.1 (Substitution theorem)**.** *For any morphism $s : X \longrightarrow TY$ there exists a unique homomorphism $\widehat{s} : TX \longrightarrow TY$ of H-algebras extending s, i.e., with $\widehat{s} \cdot \eta_X = s$.*

Remark 3.2. In case of a polynomial endofunctor on Set induced by a signature Σ Theorem 3.1 states that substitution works for infinite Σ-trees in precisely the same way as for terms (i.e., finite trees): for a set X of variables the mapping $s : X \longrightarrow T_\Sigma Y$ assigns to each variable its substitute, which is a Σ-tree over the set Y, and the extension $\widehat{s} : T_\Sigma X \longrightarrow T_\Sigma Y$ performs on any tree t of $T_\Sigma X$ the substitution s, to obtain a tree of $T_\Sigma Y$.

Notice that the fact that each $\widehat{s}$ is an H-algebra homomorphism results in the following property of substitution of infinite trees: for each tree t which is not just a leaf labelled by a variable, i.e., for all elements of the left-hand coproduct component $H_\Sigma T_\Sigma X$ of $T_\Sigma X$, the result of any substitution will never be just a leaf labelled in Y, i.e., $\widehat{s}(t)$ lies in $H_\Sigma T_\Sigma Y$. Or, more shortly, non-variables are preserved by substitution.

Whereas the concept of variables and substitution is appropriately captured categorically by the concept of a monad, the idea of "non-variable" and its preservation by substitution is not. However, we will need such a concept when we speak of guarded systems of equations below. In fact, in the setting of algebraic theories (i.e., monads on Set) Elgot [9] introduced the concept of an ideal theory. In [1] we proved that the following concept is equivalent to this.

For a monad $\mathbb{S} = (S, \eta, \mu)$ over Set we can form the complements of the image $\eta_X[X]$ of X under η_X in SX, say,

$$\sigma_X : S'X \longrightarrow SX$$

for all objects X.

The monad is called *ideal* provided $\sigma : S' \longrightarrow S$ is a subfunctor of S, and the monad multiplication has a domain-codomain restriction $\mu' : S'S \longrightarrow S'$. For general base categories the corresponding concept is as follows:

Definition 3.3. By an *ideal monad* is understood a six-tuple

$$\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$$

consisting of a monad (S, η, μ), a subfunctor $\sigma : S' \hookrightarrow S$ and a natural transformation $\mu' : S'S \longrightarrow S'$ such that

(i) $S = S' + Id$ with coproduct injections σ and η, and
(ii) μ restricts to μ' along σ, i.e., the following square

$$\begin{array}{ccc} S'S & \xrightarrow{\mu'} & S' \\ {\scriptstyle \sigma S}\downarrow & & \downarrow{\scriptstyle \sigma} \\ SS & \xrightarrow[\mu]{} & S \end{array}$$

commutes.

Examples 3.4.

(i) Free monads are ideal. If H is a *varietor*, i.e., there exist free H-algebras FY on every object Y of $\mathcal{A}$, then this is the object assignment of a free monad F on H, and this monad is ideal. In fact, it is well-known that we have a coproduct $FY = HFY + Y$ with injections $\varphi_Y : HFY \longrightarrow FY$ and $\eta_Y : Y \longrightarrow FY$ given by the structure and the universal arrow of the free H-algebra. Thus, since coproduct injections are monomorphic, we have the subfunctor

$$\varphi : HF \hookrightarrow F.$$

The restriction of μ is

$$\mu' = H\mu : HFF \longrightarrow HF$$

and the square

$$\begin{array}{ccc} HFF & \xrightarrow{H\mu} & HF \\ {\scriptstyle \varphi F}\downarrow & & \downarrow{\scriptstyle \varphi} \\ FF & \xrightarrow[\mu]{} & F \end{array}$$

commutes since μ_Y is defined as the unique H-algebra homomorphism with $\mu_Y \cdot \eta_{FY} = 1_{FY}$.

(ii) Similarly, the free cia monad $\mathbb{T} = (T, \eta, \mu)$ together with the endofunctor HT and the natural transformation

$$\tau : HT \hookrightarrow T$$

expressing the H-algebra structure $\tau_Y : HTY \longrightarrow TY$ of each TY is ideal. The restriction of μ is $\mu' = H\mu$ again.

(iii) The monad on Set given by the free algebras with a binary commutative operation is ideal. In fact, this is the free monad on the endofunctor that assigns to every set X the set of unordered pairs from X.
(iv) The free semigroup monad $X \longmapsto X^+$ on Set is ideal. Here $S'X \hookrightarrow X^+$ is the subset of words of length at least 2, and μ' is the obvious restriction of the concatenation of words to that subset.
(v) The free monoid monad $X \longmapsto X^*$ on Set is not ideal. In fact, recall that the unit η_X maps elements of X to words of length 1. Now consider the word xx' in $\{x, x'\}^*$ and the substitution s that subsitutes x by itself and x' by the empty word. Then $\widehat{s}(xx') = x$ whence μ cannot have the necessary restriction.
(vi) Classical algebraic theories (groups, lattices, etc.) are usually not ideal.
(vii) For a polynomial endofunctor on Set the algebras $R_\Sigma Y$ of *rational trees*, i.e., those finite and infinite Σ-trees over Y that have (up to isomorphism) finitely many subtrees only, yield an ideal monad R_Σ on Set. More generally, we have shown in [4] that any finitary functor H on a locally presentable category $\mathcal{A}$ generates a rational monad R, and that this monad is ideal.
(viii) Coproducts of ideal monads exist and are ideal. Assume that $\mathcal{A}$ has colimits of ω-chains and let $S = S' + Id$ and $M = M' + Id$ be ideal monads so that S' and M' are ω-cocontinuous, i.e., they preserve colimits of ω-chains. Then a coproduct of S and M in the category of monads of $\mathcal{A}$ exists and is an ideal monad, see [11].

Remark 3.5. In [1] we defined an *equation morphism* to be a morphism

$$e : X \longrightarrow T(X + Y)$$

generalizing and extending the notion of a non-flat system, see (1.1). An equation morphism e is called *guarded* whenever there exists a factorization through $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]$:

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & T(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\tau,\eta\cdot\mathsf{inr}]} \\ & & HT(X+Y)+Y \end{array}$$

We proved that any guarded equation morphism has a unique solution, i.e., a unique morphism $e^\dagger : X \longrightarrow TY$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ T(X+Y) & \xrightarrow[T[e^\dagger,\eta_Y]]{} & TTY \end{array} \tag{3.1}$$

commutes. It is easy to extend the notion of equation morphisms and their solution to any monad, and the notion of guardedness to any ideal monad $\mathbb{S}$, see [1], Definition 4.7. In the current paper, we go one step further, and we introduce solutions in any Eilenberg–Moore algebra of $\mathbb{S}$, and we prove that any cia considered as an algebra of $\mathbb{T}$ admits unique solutions of guarded equation morphisms.

Definition 3.6. Let $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ be an ideal monad on $\mathcal{A}$.

(i) By an *equation morphism* is meant a morphism

$$e : X \longrightarrow S(X + Y)$$

in $\mathcal{A}$ where X is any object ("of variables") and Y is any object ("of parameters").

(ii) The equation morphism e is called *guarded* if it factors through the morphism $[\sigma_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]$:

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & S(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\sigma,\eta\cdot\mathsf{inr}]} \\ & & S'(X+Y)+Y \end{array}$$

(iii) Given an Eilenberg–Moore algebra $\alpha : SA \longrightarrow A$ and a morphism $f : Y \longrightarrow A$ (interpreting parameters in A), we call a morphism $e^\dagger : X \longrightarrow A$ a *solution of e induced by f* provided that the square

$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \alpha} \\ S(X+Y) & \xrightarrow[S[e^\dagger, f]]{} & SA \end{array} \tag{3.2}$$

commutes.

Notation 3.7. For any cia $a : HA \longrightarrow A$ we denote by

$$\widetilde{a} : TA \longrightarrow A$$

the unique H-algebra homomorphism with $\widetilde{a} \cdot \eta_A = 1_A$. It is easy to check that $\widetilde{a}$ is the structure of an Eilenberg–Moore algebra of the monad $\mathbb{T}$. Notice that in case of a polynomial functor this can be thought of as computations of finite and infinite Σ-trees over A in the Σ-algebra A.

Remark 3.8. For the free cia monad $\mathbb{T}$ obtained from a polynomial functor of Set and a cia (A, a) considered as an Eilenberg–Moore algebra $\widetilde{a} : TA \longrightarrow A$ the commutativity of square (3.2) means that the assignment $e^\dagger$ of variables of X to elements of A has the following property: form first the "substitution" mapping $[e^\dagger, f] : X + Y \longrightarrow A$ (which interprets variables according to the solution $e^\dagger$ and parameters according to f). Apply this substitution to the right-hand side of the given system e of formal equations, and compute the resulting infinite trees in A. This yields the same assignment of variables to elements of A as $e^\dagger$. That means that the formal equations $x \approx e(x)$ become actual identities in A after the substitution $x \longmapsto e^\dagger(x)$ is performed on both sides of the equations and the right-hand side is evaluated in A.

Formally, one extends $[e^\dagger, f]$ to the unique homomorphism

$$\widetilde{a} \cdot T[e^\dagger, f] : T(X + Y) \longrightarrow A$$

from the free cia on $X + Y$ to A. Precomposed with e it yields the morphism $e^\dagger$.

Theorem 3.9. *In a completely iterative algebra, for any guarded equation morphism and every interpretation of its parameters there exists a unique solution.*

Remark. More precisely, let $a : HA \longrightarrow A$ be a cia considered as an Eilenberg–Moore algebra $\widetilde{a} : TA \longrightarrow A$. Suppose that we have a guarded equation morphism

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T(X+Y) \\ & {\scriptstyle e_0}\searrow & \big\uparrow {\scriptstyle [\tau,\eta\cdot\mathsf{inr}]} \\ & & HT(X+Y)+Y \end{array} \tag{3.3}$$

and an interpretation $f : Y \longrightarrow A$. Then there exists a unique morphism $e^\dagger : X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle \widetilde{a}} \\ T(X+Y) & \xrightarrow[T[e^\dagger,f]]{} & TA \end{array}$$

commutes.

Proof. We form the following flat equation morphism

$$\overline{e} \equiv T(X+Y) \xrightarrow{[\tau,\eta]^{-1}} HT(X+Y)+X+Y \xrightarrow{[\mathsf{inl},e_0,\mathsf{inr}]} HT(X+Y)+Y \xrightarrow{HT(X+Y)+f} HT(X+Y)+A$$

w.r.t. the cia A. Let us denote by s the unique solution of $\overline{e}$, i.e., s is the unique morphism such that the following diagram

$$\begin{array}{ccc} T(X+Y) & \xrightarrow{\;s\;} & A \\ {\scriptstyle [\tau,\eta]^{-1}}\big\downarrow\big\uparrow{\scriptstyle [\tau,\eta]} & & \\ HT(X+Y)+X+Y & & \\ {\scriptstyle [\mathsf{inl},e_0,\mathsf{inr}]}\big\downarrow & & \big\uparrow{\scriptstyle [a,A]} \\ HT(X+Y)+Y & & \\ {\scriptstyle HT(X+Y)+f}\big\downarrow & & \\ HT(X+Y)+A & \xrightarrow[Hs+A]{} & HA+A \end{array} \tag{3.4}$$

commutes. Consider the coproduct components of $HT(X+Y)+X+Y$ separately to conclude that s is uniquely determined by the following three equations:

(i) $s \cdot \tau_{X+Y} \quad\quad = a \cdot Hs$,
(ii) $s \cdot \eta_{X+Y} \cdot \mathsf{inr} = f$,
(iii) $s \cdot \eta_{X+Y} \cdot \mathsf{inl} = [a, A] \cdot (Hs + A) \cdot (HT(X+Y) + f) \cdot e_0$.

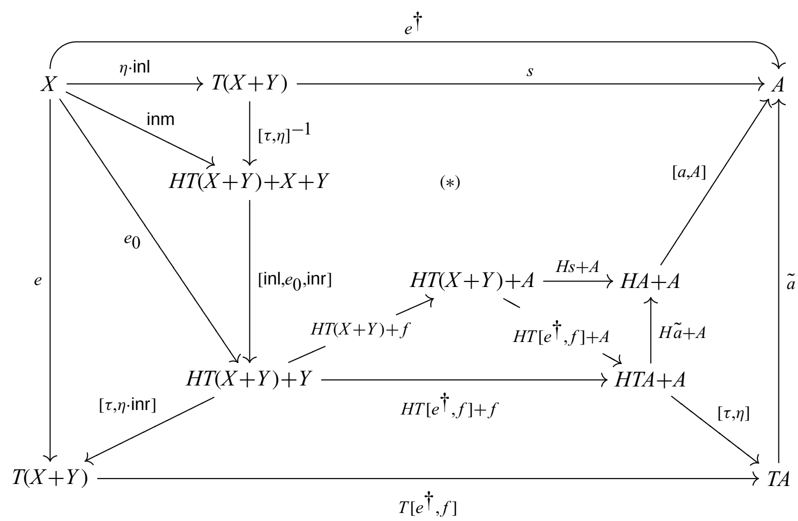
Existence of a solution of e induced by f. We define the morphism

$$e^\dagger \equiv X \xrightarrow{\mathsf{inl}} X + Y \xrightarrow{\eta_{X+Y}} T(X+Y) \xrightarrow{s} A.$$

It follows immediately that s is a homomorphism of H-algebras with $s \cdot \eta_{X+Y} = [e^\dagger, f] : X + Y \longrightarrow A$, see (i) and (ii) above. We invoke the freeness of the cia $T(X+Y)$ to conclude that the equation

$$s = \widetilde{a} \cdot T[e^\dagger, f] \tag{3.5}$$

holds. To establish that $e^\dagger$ is a solution of e induced by f consider the following commutative diagram



(3.6)

The upper most part is the definition of $e^\dagger$, part $(*)$ is diagram (3.4), the lower left-hand triangle is (3.3), and the right-hand part commutes due to the definition of $\widetilde{a}$. The inner right-hand triangle commutes because of (3.5), and the remaining parts are obvious. Thus, the outer square commutes as desired.

Uniqueness. Suppose that $e^\dagger$ is any solution of e induced by f. Let $s = \widetilde{a} \cdot T[e^\dagger, f] : T(X+Y) \longrightarrow A$. To complete the proof it suffices to show that s solves $\overline{e}$ w.r.t. the cia A (equivalently, the above equations (i), (ii), and (iii) hold). In fact, the morphism s determines $e^\dagger$ since clearly we have the equation $e^\dagger = s \cdot \eta_{X+Y} \cdot \mathsf{inl}$. Observe first that s is an H-algebra homomorphism with $s \cdot \eta_{X+Y} = [e^\dagger, f]$. Thus, the equations (i) and (ii) hold. To see that equation (iii) holds consider again diagram (3.6). Its outer square commutes since $e^\dagger$ is a solution of e induced by f. Since all other parts clearly commute, so does part $(*)$ when extended by $\eta_{X+Y} \cdot \mathsf{inl}$. But this is precisely the desired equation (iii). □

Remark 3.10. Theorem 3.9 shows that the Eilenberg–Moore algebra $\widetilde{a} : TA \longrightarrow A$ arising from a cia (A, a) admits unique solutions. For general Eilenberg–Moore algebras of $\mathbb{T}$ uniqueness of solutions fails. In fact, for the endofunctor $H_\Sigma = Id$ of Set (expressing one unary operation) the two element set $\{0,1\}$ carries an Eilenberg–Moore algebra as follows: Notice that for any set X, $T_\Sigma X = \mathrm{N} \times X + \{\infty\}$. It is easy to check that the map

$$T_\Sigma A \longrightarrow A, \quad (n,i) \longmapsto i,\ i = 0,1, \quad \infty \longmapsto 0$$

is a structure of an Eilenberg–Moore algebra, see also [5], Example 3.8 and Theorem 5.5. However, the equation $x \approx x$ expressed by the guarded equation morphism

$$\{x\} \longrightarrow \mathrm{N} \times \{x\} + \{\infty\} = T_\Sigma(\{x\} + \emptyset), \quad x \longmapsto (1,x),$$

has for the unique interpretation $\emptyset \longrightarrow A$ two solutions 0 and 1.

The following result was first proved independently by Moss [17] and by Aczel et al. [1]. We obtain it as a corollary of Theorem 3.9.

Theorem 3.11 (Solution theorem)**.** *For any guarded equation morphism $e : X \longrightarrow T(X+Y)$ there exists a unique solution in the algebra TY, i.e., a unique morphism $e^\dagger : X \longrightarrow TY$ such that Diagram (3.1) commutes.*

Proof. Apply Theorem 3.9 to e and $f = \eta_Y : Y \longrightarrow TY$ and observe that $\widetilde{\tau_Y} = \mu_Y : TTY \longrightarrow TY$. $\square$

4. Free completely iterative monad

In this section, we still assume that $H : \mathcal{A} \longrightarrow \mathcal{A}$ is an iteratable endofunctor on a category $\mathcal{A}$ with binary coproducts (equivalently, H has free cias on every object of $\mathcal{A}$, see Corollary 2.11). We also assume that coproduct injections are monomorphic; this can be avoided (see Section 5).

As the main result of [1] it was proved that the monad $\mathbb{T}$, which is given by the final coalgebras TY of $H(\_) + Y$, is a free completely iterative monad on H. The proof given there is technically quite complicated, involving an unpleasant amount of rather unintuitive diagram chasing arguments. Here we will give a much simpler proof. Recall the statements (a), (b), and (c) from the introduction. In lieu of proving (a) implies (c) directly we use the equivalence of (a) and (b) established in Section 2 and prove (b) implies (c). In fact, the universal property of the free cias TY, for every object Y, more easily yields the desired universal property of the monad $\mathbb{T}$.

We start by recalling the definition of a completely iterative monad from [1].

Definition 4.1. An ideal monad $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ is called *completely iterative* if every guarded equation morphism $e : X \longrightarrow S(X+Y)$ has a unique solution induced by $\eta_Y : Y \longrightarrow SY$ in the free Eilenberg–Moore algebra $\mu_Y : SSY \longrightarrow SY$, i.e., for each guarded e there exists a unique solution $e^\dagger : X \longrightarrow SY$ so that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & SY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ S(X+Y) & \xrightarrow[S[e^\dagger, \eta_Y]]{} & SSY \end{array} \tag{4.1}$$

commutes.

An *ideal monad morphism* from an ideal monad $(S, \eta^S, \mu^S, S', \sigma, \mu'^S)$ to an ideal monad $(U, \eta^U, \mu^U, U', \omega, \mu'^U)$ is a monad morphism $\lambda : (S, \eta^S, \mu^S) \longrightarrow (U, \eta^U, \mu^U)$ which has a domain-codomain restriction to the ideals (i.e., there exists a natural transformation $\lambda' : S' \longrightarrow U'$ with $\lambda \cdot \sigma = \omega \cdot \lambda'$).

Given a functor H, a natural transformation $\lambda : H \longrightarrow S$ is called *ideal* provided that it factors through $\sigma : S' \hookrightarrow S$.

Example 4.2. The monad $\mathbb{T}$ is ideal (w.r.t. $T \cong HT + Id$), and the Solution Theorem 3.11 states that $\mathbb{T}$ is completely iterative. Notice that $\mathbb{T}$ comes with the following canonical natural transformation

$$\kappa \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T,$$

which is ideal.

Theorem 4.3. *The monad $\mathbb{T}$ is a free completely iterative monad. That is, for any completely iterative monad $\mathbb{S}$ and every ideal natural transformation $\lambda : H \longrightarrow S$ there exists a unique monad morphism $\overline{\lambda} : \mathbb{T} \longrightarrow \mathbb{S}$ with $\overline{\lambda} \cdot \kappa = \lambda$. And the induced $\overline{\lambda}$ is an ideal monad morphism.*

Remark 4.4.

(i) Notice that the statement of the Theorem is slightly stronger than in [1]. Here we do not require that the monad morphism $\overline{\lambda}$ be ideal in order to obtain its uniqueness. And the proof is substantially simpler.

(ii) For the category $\mathsf{CIM}(\mathcal{A})$ of all completely iterative monads and ideal monad morphisms we have a forgetful functor

$$U : \mathsf{CIM}(\mathcal{A}) \longrightarrow [\mathcal{A}, \mathcal{A}], \quad \mathbb{S} \longmapsto S'.$$

The theorem states that there exists a universal arrow at each iteratable endofunctor H. However, notice that this does not imply the existence of a left adjoint to U. (A left adjoint may not even exist if one restricts the codomain of U to iteratable functors. It is not clear that for an iteratable functor H the ideal HT of the completely iterative monad $\mathbb{T}$ is iteratable again.) If $\mathcal{A}$ is a locally presentable category and we restrict the codomain of U to $\mathsf{Acc}[\mathcal{A}, \mathcal{A}]$, the category of accessible endofunctors on $\mathcal{A}$, and the domain to the category $\mathsf{CIAM}(\mathcal{A})$ of accessible completely iterative monads $\mathbb{S}$ (i.e., such that both S and S' are accessible) then this restriction has a left adjoint, viz. the functor $H \longmapsto \mathbb{T}$. In fact, T is then accessible, see [4],

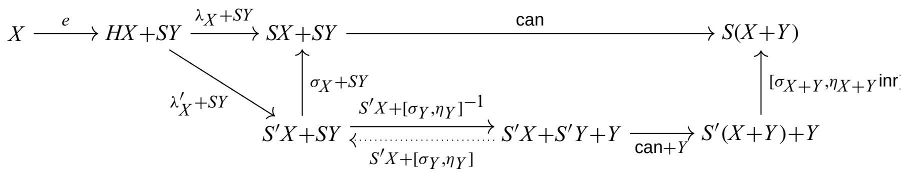
Proof. (1) For every object Y consider SY as an H-algebra as follows:

$$HSY \xrightarrow{\lambda_{SY}} SSY \xrightarrow{\mu_Y} SY.$$

It is completely iterative. In fact, every equation morphism $e : X \longrightarrow HX + SY$ yields the following equation morphism w.r.t. $\mathbb{S}$:

$$\overline{e} \equiv X \xrightarrow{e} HX + SY \xrightarrow{\lambda_X + SY} SX + SY \xrightarrow{\mathsf{can}} S(X + Y).$$

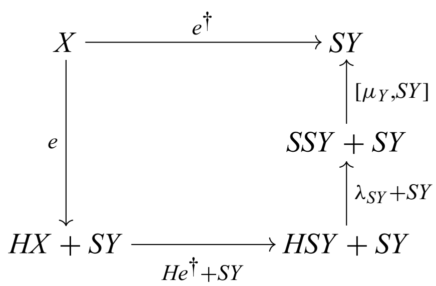
To verify that $\overline{e}$ is guarded, use the restriction $\lambda' : H \longrightarrow S'$ of λ and consider the commutative diagram



To see the commutativity of the square, consider the three components of $S'X + S'Y + Y$ separately, and use naturality of σ and η.

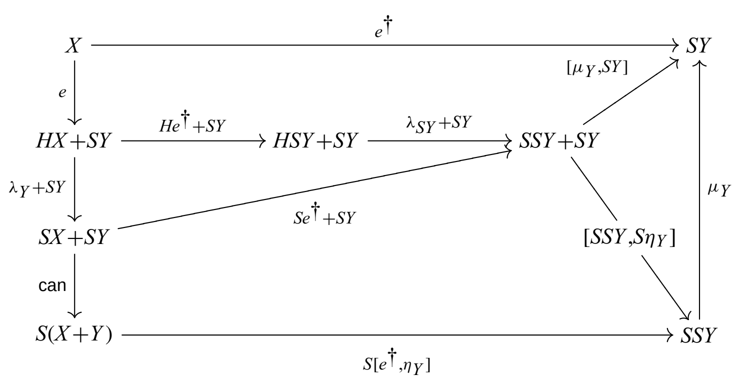
We prove that a morphism $e^\dagger : X \longrightarrow SY$ is a solution of e in the H-algebra SY if and only if it is a solution of $\overline{e}$ w.r.t. the iterative monad $\mathbb{S}$.

(1a) Let $e^\dagger$ be a solution of e in the algebra SY, i.e., let



(4.2)

commute. We are to show that the following diagram:



(4.3)

has the outward square commutative. The upper part is (4.2), the one directly below it is the naturality of λ. The lower part is obvious as is the right-hand triangle due to $\mu_Y \cdot S\eta_Y = 1_{SY}$.

(1b) Let the outward square of (4.3) commute. Then (4.2) commutes because it forms the upper part of (4.3), where the two adjacent parts and the lower part commute.

(2) *Existence of an ideal monad morphism* $\overline{\lambda}$ *with* $\overline{\lambda} \cdot \kappa = \lambda$. Denote by

$$\overline{\lambda}_Y : TY \longrightarrow SY$$

the unique homomorphism of H-algebras with

$$\overline{\lambda}_Y \cdot \eta_Y = \eta^S_Y.$$

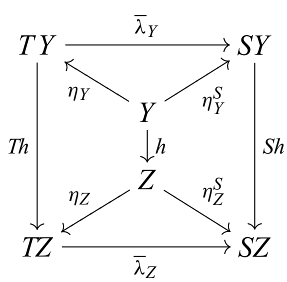
We first observe that $\overline{\lambda}$ is a natural transformation. Given a morphism $h : Y \longrightarrow Z$ then Sh is a homomorphism of H-algebras from SY to SZ:

$$\begin{array}{ccccc} HSY & \xrightarrow{\lambda_{SY}} & SSY & \xrightarrow{\mu_Y} & SY \\ {\scriptstyle HSh}\downarrow & & \downarrow{\scriptstyle SSh} & & \downarrow{\scriptstyle Sh} \\ HSZ & \xrightarrow[\lambda_{SZ}]{} & SSZ & \xrightarrow[\mu_Z]{} & SZ \end{array} \tag{4.4}$$

Thus, we have two parallel homomorphisms of H-algebras

$$Sh \cdot \overline{\lambda}_Y, \overline{\lambda}_Z \cdot Th : TY \longrightarrow SZ.$$

They agree when precomposed with η_Y; in fact, the following diagram commutes:

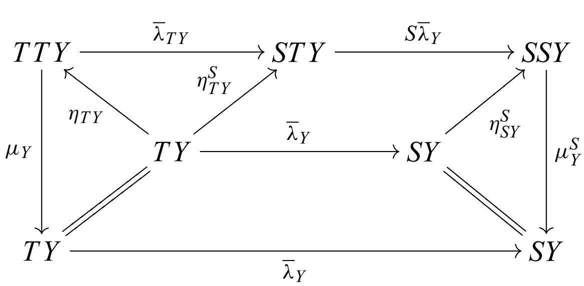


By the universal property of η_Y, and since SZ is a completely iterative H-algebra, this proves that the above naturality square commutes.

Let us prove that $\overline{\lambda}$ is a monad morphism. Since $\overline{\lambda} \cdot \eta = \eta^S$ by definition, it only remains to prove the commutativity of the following diagram:

$$\begin{array}{ccccc} TTY & \xrightarrow{\overline{\lambda}_{TY}} & STY & \xrightarrow{S\overline{\lambda}_Y} & SSY \\ {\scriptstyle \mu_Y}\downarrow & & & & \downarrow{\scriptstyle \mu^S_Y} \\ TY & & \xrightarrow[\overline{\lambda}_Y]{} & & SY \end{array} \tag{4.5}$$

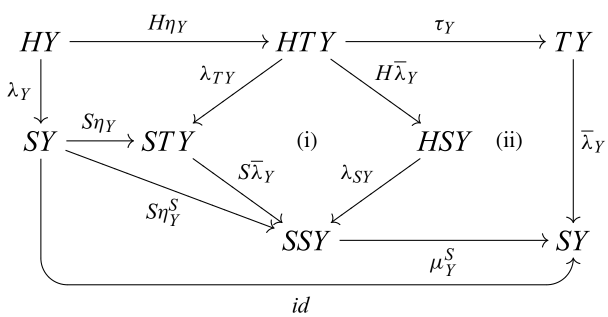
By (4.4), applied to $h = \overline{\lambda}_Y$, we see that $S\overline{\lambda}_Y$ is a homomorphism of H-algebras. By the universal property of η_{TY} it is sufficient to prove that (4.5) commutes when precomposed with η_{TY}:



Finally, the equation

$$\lambda = \overline{\lambda} \cdot \kappa = \overline{\lambda} \cdot \tau \cdot H\eta$$

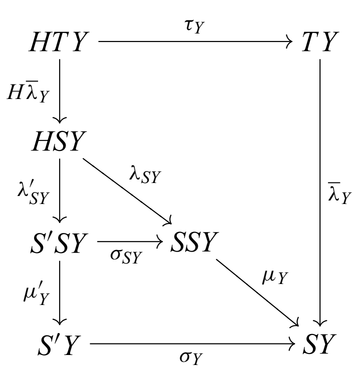
follows from the commutativity of the diagram



(4.6)

where (i) is naturality of λ, (ii) is the definition of $\overline{\lambda}$, the upper left-hand part is the naturality of λ, and the triangle below it uses the unit law for $\overline{\lambda}$. For the lowest part use the monad law $\mu^S_Y \cdot S\eta^S_Y = 1_{SY}$.

Thus, we have found a monad morphism $\overline{\lambda} : \mathbb{T} \longrightarrow \mathbb{S}$ with $\overline{\lambda} \cdot \kappa = \lambda$. It remains to verify that $\overline{\lambda}$ is ideal. To this end consider the commutative diagram



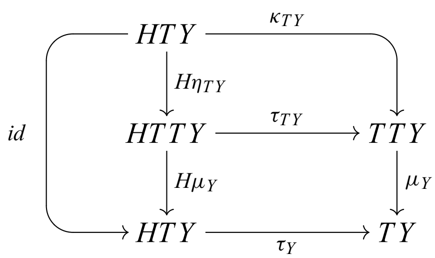
The upper right-hand part commutes by the definition of $\overline{\lambda}_Y$, the left-hand triangle commutes since λ is an ideal transformation, and for the lower part we use that μ restricts to μ'. Thus, we see that $\mu'^S \cdot \lambda' S \cdot H\overline{\lambda} : HT \longrightarrow S'$ is the desired restriction of $\overline{\lambda}$.

(3) *Uniqueness of* $\overline{\lambda}$. Suppose that $\overline{\lambda} : \mathbb{T} \longrightarrow \mathbb{S}$ is a monad morphism with $\overline{\lambda} \cdot \kappa = \lambda$. We are going to show that for any object Y, $\overline{\lambda}_Y$ is an H-algebra homomorphism extending η^S_Y, and then invoke the freeness of TY as a completely iterative H-algebra, which establishes the desired uniqueness.

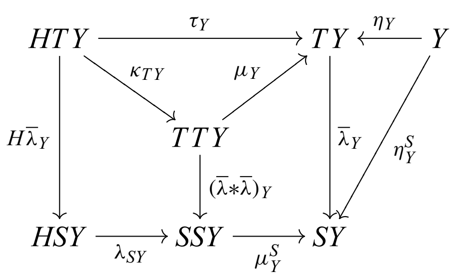
First, notice that for any object Y we have

$$\tau_Y = \mu_Y \cdot \kappa_{TY}. \qquad (4.7)$$

Indeed, the following diagram commutes:



Consequently, the following diagram

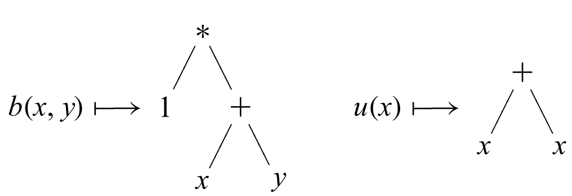


commutes: the right-hand triangle and the lower right-hand part commute since $\overline{\lambda}$ is a monad morphism, the lower left-hand part commutes since $\overline{\lambda} \cdot \kappa = \lambda$ and by naturality, and the upper triangle is (4.7).

Thus, $\overline{\lambda}_Y : TY \longrightarrow SY$ is an H-algebra homomorphism between completely iterative H-algebras such that $\overline{\lambda}_Y \cdot \eta_Y = \eta^S_Y$. This determines $\overline{\lambda}_Y$ uniquely. □

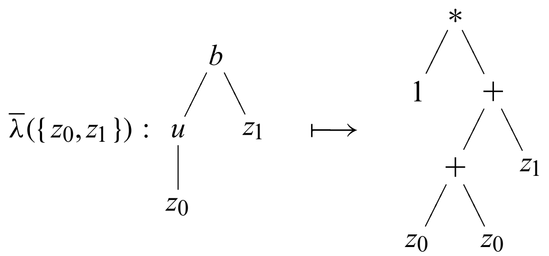
Remark 4.5. For polynomial endofunctors on Set, the freeness of $\mathbb{T}$ specializes to *second order substitution*, see [8], i.e., substitution of finite or infinite trees for operation symbols.

For example, consider a signature Σ with a binary operation symbol b, and a unary one u, and another signature Γ with two binary operation symbols $+$ and $*$ and a constant symbol 1. The following assignment:



(4.8)

of operation symbols in Σ to Γ-trees gives rise to a natural transformation $\lambda : H_\Sigma \longrightarrow T_\Gamma$. The induced ideal monad morphism $\overline{\lambda} : \mathbb{T}_\Sigma \longrightarrow \mathbb{T}_\Gamma$ replaces, for any set of variables X, the operation symbols in trees of $T_\Sigma X$ according to λ. Example:



The requirement that λ be an ideal transformation means that no operation symbol of Σ is replaced by a single variable, i.e., that λ is a so called *non-erasing* substitution.

5. Idealized monads

In this section, we show how to prove the results of the previous section in full generality, i.e., for any category $\mathcal{A}$ with binary coproducts (not necessarily having monomorphic coproduct injections) and every iteratable endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$. The proof ideas remain essentially unchanged, although the technical difficulty is somewhat increased due to the fact that ideal monads should be replaced with *idealized monads*, which we introduce below, but, on the other hand, the freeness result of Theorem 4.3 can be extended a little further.

The main technical tool of this section is an ideal coreflection of any idealized monad. The ideas for the proofs of the respective results are essentially those used in the technical material of [16]. We shall need that material in Section 6 below where we complete the proof of the equivalence of the three statements (a), (b), and (c) from the introduction. Here we will use an ideal coreflection to extend the result of Theorem 4.3 to idealized monads, more precisely, we prove in Theorem 5.14 below that the free cia monad $\mathbb{T}$ is a free w.r.t. all idealized completely iterative monads, thus we establish (b) implies (c) in full generality.

Remark 5.1.

(i) Recall that in Example 3.4(ii) we showed that the free cia monad $\mathbb{T}$ is ideal. A quick inspection of all previous proofs reveals that this was the only place where we used the assumption that coproduct injections are monomorphic. However, when we drop that assumption, it is no longer sufficient to have in an ideal monad $\mathbb{S}$ just a "restriction" $\mu' : S'S \longrightarrow S'$ of μ. It is natural to assume additionally that μ' obeys certain laws similar to the ones for the monad multiplication μ; this leads to the requirement that (S', μ') be an S-module, see Definition 5.5 below.

(ii) Another restriction in the previous section was the requirement that an ideal monad $\mathbb{S}$ should satisfy $S' + Id$ so that intuitively S' gives an abstract notion of "non-variables" to be used as the allowed right-hand sides of guarded equation morphisms. We have used that property of S in part (1) of the proof of Theorem 4.3. A different point of view is that of equipping a monad with some abstract notion of "allowed right-hand sides of equations", which leads us to the notion of idealized monad as introduced below.

Definition 5.2. Let (M, η, μ) be a monad on $\mathcal{A}$. A *(right) M-module* is a pair (F, f) consisting of an endofunctor F on $\mathcal{A}$ and a natural transformation $f : FM \longrightarrow F$ such that the following diagrams:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FM \\ & \searrow^{=} & \downarrow{\scriptstyle f} \\ & & F \end{array} \qquad \text{and} \qquad \begin{array}{ccc} FMM & \xrightarrow{fM} & FM \\ {\scriptstyle F\mu}\downarrow & & \downarrow{\scriptstyle f} \\ FM & \xrightarrow[f]{} & F \end{array}$$

commute.

If (F, f) and (G, g) are M-modules, then a natural transformation $h : F \longrightarrow G$ such that the square

$$\begin{array}{ccc} FM & \xrightarrow{hM} & GM \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xrightarrow[h]{} & G \end{array}$$

commutes is called a *module homomorphism.*

Remark 5.3. In [15], Section VII.4, (left) modules are defined under the name action for any monoidal category. The above Definition 5.2 states that definition for the special case of the monoidal category of endofunctors of $\mathcal{A}$ with composition as tensor product and the identity functor as unit. Here we chose the name module since in the monoidal category of abelian groups monoids are precisely rings and modules are the usual R-modules for a ring R.

Examples 5.4.

(i) Any monad (M, η, μ) is trivially an M-module (M, μ).

(ii) If $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ is an ideal monad in the sense of Definition 3.3, then (S', μ') is an S-module. This follows easily from the monad laws for S using the fact that the coproduct injections $\sigma_Y : S'Y \longrightarrow SY$ are monomorphic.

Definition 5.5. An *idealized monad* $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ consists of a monad (S, η, μ), an S-module (S', μ'), and a module homomorphism $\sigma : (S', \mu') \longrightarrow (S, \mu)$.

We call $\mathbb{S}$ *ideal* if $S = S' + Id$ with coproduct injections σ and η.

An idealized monad $\mathbb{S}$ is called *completely iterative* if any guarded equation morphism

$$\begin{array}{ccc} X & \xrightarrow{e} & S(X + Y) \\ & \searrow & \uparrow{\scriptstyle [\sigma, \eta \cdot \mathsf{inr}]} \\ & & S'(X + Y) + Y \end{array}$$

has a unique solution $e^{\dagger} : X \longrightarrow SY$ (i.e., such that Diagram (4.1) commutes).

A *morphism of idealized monads* between $\mathbb{S}$ and $\mathbb{M} = (M, \eta^M, \mu^M, M', m, \mu'^M)$ is a pair (h, h') consisting of a monad morphism $h : (S, \eta, \mu) \longrightarrow (M, \eta^M, \mu^M)$ and a natural transformation $h' : S' \longrightarrow M'$ such that the squares

$$\begin{array}{ccc} S'S & \xrightarrow{h' * h} & M'M \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \mu'^M} \\ S' & \xrightarrow[h']{} & M' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S' & \xrightarrow{h'} & M' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow[h]{} & M \end{array}$$

commute. (Notice that the left-hand square means that h' is a module homomorphism with change of base h.)

Remark 5.6. Notice that idealized monad morphisms $(h, h') : \mathbb{S} \longrightarrow \mathbb{M}$ between ideal monads are determined by their second components. In fact, since the equations $S = S' + Id$ and $M = M' + Id$ hold, the two equation $m \cdot h' = h \cdot \sigma$ and $h \cdot \eta = \eta^M$ imply that $h = h' + Id$.

Examples 5.7.

(i) Any ideal monad in the sense of Definition 3.3 is an ideal monad in the sense of Definition 5.5.
(ii) The monad $\mathbb{T}$ given by the free completely iterative H-algebras is an ideal monad.
(iii) The free semigroup monad $X \longmapsto X^+$ together with S' assigning to X the set of words in X of length at least n, for some $n > 2$, is an idealized monad which is not ideal. It is trivial to check that the restriction of the monad multiplication to S' satisfies the necessary laws.
(iv) Let Σ be a signature and let Σ' be a subsignature of Σ. Then T_Σ together with $S' = H_{\Sigma'} T_\Sigma$ is an idealized monad. Note that S' assigns to a set Y all finite and infinite Σ-trees over Y whose root node is labelled by a symbol from Σ'. Once again, the laws of an idealized monad are easy to check, and this is another example which is not ideal whenever Σ' is a proper subsignature of Σ.

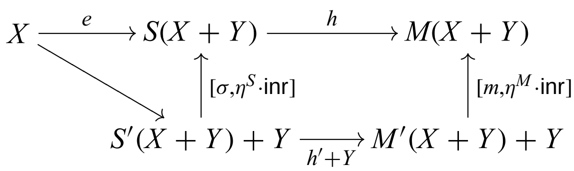
Notation 5.8. We denote by $\mathsf{CIzM}(\mathcal{A})$ the category of all idealized completely iterative monads and all idealized monad morphisms. By $\mathsf{CIM}(\mathcal{A})$ we denote its full subcategory consisting of all ideal completely iterative monads.

We also use $\mathsf{CIzAM}(\mathcal{A})$ to denote the full subcategory of $\mathsf{CIzM}(\mathcal{A})$ consisting of all *accessible* idealized monads $\mathbb{S}$, i.e., such that S and S' are accessible functors. Analogously $\mathsf{CIAM}(\mathcal{A})$.

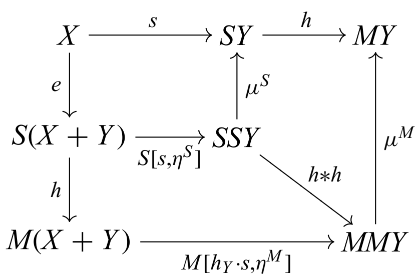
Proposition 5.9. *Idealized monad morphisms preserve solutions.*

Remark. More precisely, let $(h, h') : \mathbb{S} \longrightarrow \mathbb{M}$ be an idealized monad morphism between idealized monads. Then for every guarded equation morphism $e : X \longrightarrow S(X + Y)$ we get a guarded equation morphism $h_{X+Y} \cdot e$, and any solution s of e yields a solution $h_Y \cdot s$ of $h_{X+Y} \cdot e$. In particular, if $\mathbb{S}$ and $\mathbb{M}$ are completely iterative, we have $(h_{X+Y} \cdot e)^\dagger = h_Y \cdot e^\dagger$.

Proof. To see that $h_{X+Y} \cdot e$ is guarded, consider the commutative diagram



To see that $h_Y \cdot s$ solves $\overline{e}$, inspect the commutative diagram



where $*$ denotes parallel composition. □

Lemma 5.10. *If $\mathbb{S}$ is an idealized monad, then $S' + Id$ yields an ideal monad.*

Remark. By this we mean, of course, the sixtuple

$$\widetilde{\mathbb{S}} = (S' + Id, \widetilde{\eta}, \widetilde{\mu}, S', \mathsf{inl}, \widetilde{\mu}'),$$

where

$$\widetilde{\eta} \equiv Id \xrightarrow{\mathsf{inr}} S' + Id\,,$$

$$\widetilde{\mu}' \equiv S'(S' + Id) \xrightarrow{S'[\sigma,\eta]} S'S \xrightarrow{\mu'} S'\,,$$

$$\widetilde{\mu} \equiv (S' + Id)^2 = S'(S' + Id) + S' + Id \xrightarrow{[\widetilde{\mu}', S'] + Id} S' + Id\,.$$

The proof of this result is essentially straightforward and involves only diagram chasing arguments using the axioms of the given idealized monad $\mathbb{S}$. For the sake of brevity we leave it to the reader. A very similar result was proved as Lemma 3.4 in [16].

Proposition 5.11. *The natural transformation $[\sigma, \eta] : S' + Id \longrightarrow S$ yields a morphism*

$$([\sigma, \eta], 1_{S'}) : \widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$$

of idealized monads. And this is a coreflection of $\mathbb{S}$ in the category of ideal monads and morphisms of idealized monads.

Remark 5.12. More precisely, for any ideal monad

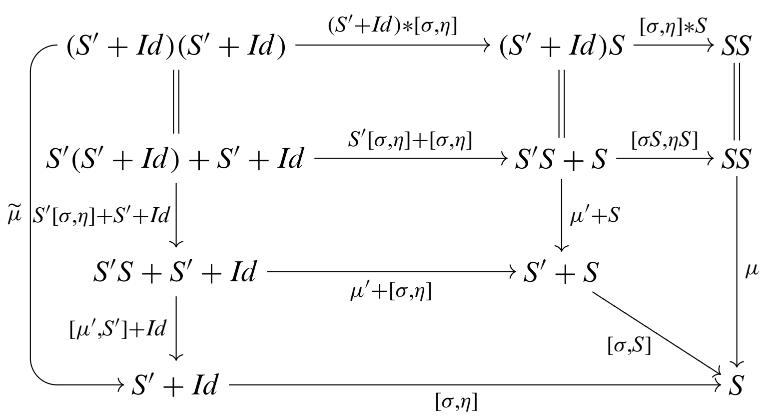
$$\mathbb{M} = (M, \eta^M, \mu^M, M', m, \mu'^M)$$

and any morphism of idealized monads $(h, h') : \mathbb{M} \longrightarrow \mathbb{S}$ there exists a unique idealized monad morphism $(\overline{h}, \overline{h}') : \mathbb{M} \longrightarrow \widetilde{\mathbb{S}}$ such that

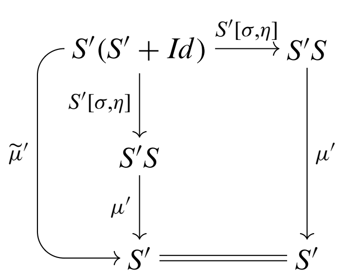
$$[\sigma, \eta] \cdot \overline{h} = h \quad \text{and} \quad \overline{h}' = h'. \tag{5.1}$$

That means that the inclusion of the full subcategory of ideal monads in the category of idealized monads has a right adjoint.

Proof. (1) We start by showing that $[\sigma, \eta] : S' + Id \longrightarrow S$ is a monad morphism. In fact, the unit law is obvious. For the associativity consider the commutative diagram

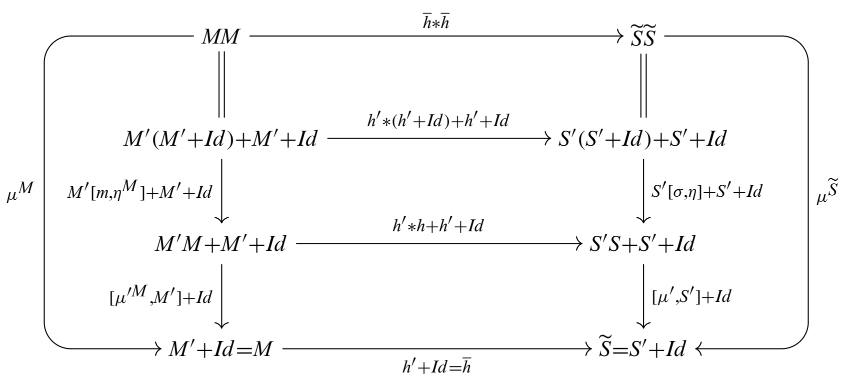


That $1_{S'}$ is a “restriction” of $[\sigma, \eta]$ is trivial, we have $[\sigma, \eta] \cdot \mathsf{inl} = \sigma = \sigma \cdot 1_{S'}$. Finally, $1_{S'}$ is a module homomorphism with change of base $[\sigma, \eta]$:



Thus, $([\sigma, \eta], 1_{S'})$ is a morphism of idealized monads.

(2) *Existence.* Put $\widetilde{S} = S' + Id$. Given $\mathbb{M}$ and (h, h'), then $(h' + Id, h') : \mathbb{M} \longrightarrow \widetilde{\mathbb{S}}$ is the desired ideal monad morphism. In fact, preservation of units is obvious. For the multiplication consider the following diagram



whose commutativity easily follows from axioms for ideal(ized) monads.

We leave the task to check that h' is a module homomorphism with change of base $h' + Id$, i.e., a restriction of h (see Definition 5.5), to the reader. This follows easily from the corresponding properties of (h, h').

Finally, we need to check the first equation of (5.1):

$$\begin{aligned}[][\sigma, \eta] \cdot \overline{h} &= [\sigma, \eta] \cdot (h' + Id) \\ &= [\sigma \cdot h', \eta] \\ &= [h \cdot m, h \cdot \eta^M] \\ &= h \cdot [m, \eta^M] \\ &= h.\end{aligned}$$

(3) *Uniqueness.* Given any morphism of idealized monads $(\overline{h}, \overline{h}') : \mathbb{M} \longrightarrow \widetilde{S}$ satisfying (5.1), we immediately have $\overline{h}' = h'$, and therefore $\overline{h} = \overline{h}' + Id = h' + Id$. □

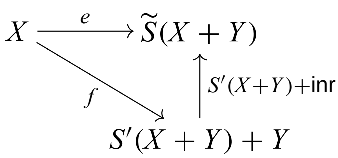
Lemma 5.13. *If* $\mathbb{S}$ *is a completely iterative monad, then so is its coreflection* $\widetilde{\mathbb{S}}$.

Remark. This result means, that the restriction $\mathsf{CIM}(\mathcal{A}) \longrightarrow \mathsf{CIzM}(\mathcal{A})$ of the embedding of Remark 5.12 also has a right adjoint. Notice also that this adjunction also clearly holds for the respective subcategories of accessible completely iterative monads:

$$\mathsf{CIAM}(\mathcal{A}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CIzAM}(\mathcal{A}).$$

The proof of this result is similar to the proof of Lemma 3.5 in [16]. In our current setting it can be simplified due to Proposition 5.9.

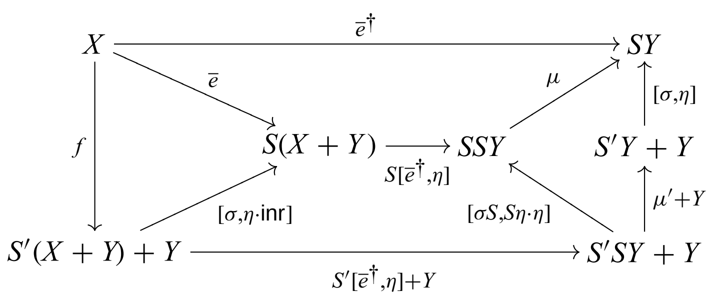
Proof. Denote for any object X by $\widetilde{S}X$ the coproduct $S'X + X$. We have to show that any guarded equation morphism



has a unique solution $e^\dagger : X \longrightarrow \widetilde{S}Y$. Define another guarded equation morphism w.r.t. $\mathbb{S}$ by composing with the coreflection arrow:

$$\overline{e} \equiv X \xrightarrow{e} \widetilde{S}(X+Y) \xrightarrow{[\sigma,\eta]} S(X+Y).$$

That $\overline{e}$ is indeed guarded follows from Proposition 5.9. Solve $\overline{e}$ to obtain a unique arrow $\overline{e}^\dagger : X \longrightarrow SY$ such that the upper part of the following diagram commutes:



(5.2)

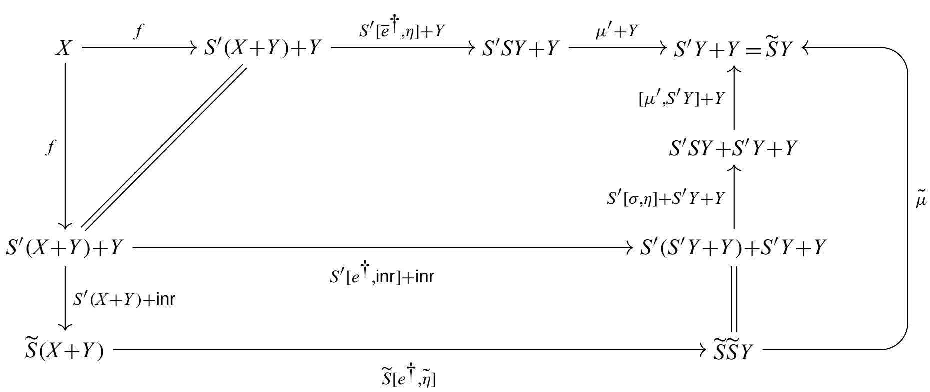
Then the outer square commutes, since the other three inner parts clearly do.

We shall prove that the following morphism

$$e^\dagger \equiv X \xrightarrow{f} S'(X+Y)+Y \xrightarrow{S'[\overline{e}^\dagger,\eta]+Y} S'SY+Y \xrightarrow{\mu'+Y} S'Y+Y = \widetilde{S}Y \tag{5.3}$$

is a unique solution of e.

That this morphism solves e follows from inspection of the diagram



It commutes except, perhaps, for the upper middle part, which we consider componentwise. The right-hand coproduct component is obvious, and for the left-hand one notice that the last arrow is μ' on both paths. We show that the rest already commutes, even when S' is removed, i.e., we plug in the definition (5.3) of $e^\dagger$ and obtain the commutative diagram:

$$\begin{array}{ccccc} X+Y & & \xrightarrow{[\overline{e}^\dagger,\eta]} & & SY \\ {\scriptstyle [f,\mathsf{inr}]}\downarrow & & & & \uparrow{\scriptstyle [\sigma,\eta]} \\ S'(X+Y)+Y & \xrightarrow[S'[\overline{e}^\dagger,\eta]+Y]{} & S'SY+Y & \xrightarrow[\mu'+Y]{} & S'Y+Y \end{array}$$

Its right-hand component is obvious, and the left-hand one is the outer square Diagram (5.2).

We have proved existence of a solution of e so far. As for the unicity suppose that $s : X \longrightarrow \widetilde{S}Y$ is any solution of e, i.e., the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{s} & S'Y+Y = \widetilde{S}Y \\ & & \uparrow{\scriptstyle [\mu',S'Y]+Y} \\ {\scriptstyle f}\downarrow & & S'SY+S'Y+Y \\ & & \uparrow{\scriptstyle S'[\sigma,\eta]+S'Y+Y} \\ S'(X+Y)+Y & \xrightarrow[S'[s,\mathsf{inr}]+\mathsf{inr}]{} & S'(S'Y+Y)+S'Y+Y \\ {\scriptstyle [\mathsf{inl},\mathsf{inr}]}\downarrow & & \Vert \\ \widetilde{S}(X+Y) & \xrightarrow[\widetilde{S}[s,\widetilde{\eta}]]{} & \widetilde{S}\widetilde{S} \end{array} \qquad \widetilde{S}\widetilde{S} \xrightarrow{\widetilde{\mu}} \widetilde{S}Y \tag{5.4}$$

Since $[\sigma, \eta] : \widetilde{S} \longrightarrow S$ is the first component of an idealized monad morphism (the coreflection arrow), the following morphism:

$$X \xrightarrow{s} S'Y + Y \xrightarrow{[\sigma,\eta]} SY \tag{5.5}$$

solves $\overline{e}$, see Proposition 5.9. Then it is not difficult to show that $s = e^\dagger$. In fact, start with the definition of the solution $e^\dagger$

$$e^\dagger = (\mu'_Y + Y) \cdot (S'[\overline{e}^\dagger, \eta_Y] + Y) \cdot f,$$

then substitute (5.5) for $\overline{e}^\dagger$ to obtain

$$(\mu'_Y + Y) \cdot (S'[[\sigma, \eta] \cdot s, \eta_Y] + Y) \cdot f, \tag{5.6}$$

and finally use the equation $\eta_Y = [\sigma_Y, \eta_Y] \cdot \mathsf{inr}$ in order to see that (5.6) is the same as

$$(\mu'_Y + Y) \cdot (S'[\sigma, \eta] + Y) \cdot (S'[s, \mathsf{inr}] + Y) \cdot f,$$

which is just s due to the upper left-hand part of Diagram (5.4). $\square$

At this point we are ready to prove the main result of this section, i.e., we extend the freeness result of Theorem 4.3 to all idealized monads. Thus, we establish that (b) implies (c) (see Section 1) in full generality.

Theorem 5.14. *For every completely iterative monad $\mathbb{S}$ and every ideal natural transformation λ: $H \longrightarrow S$ there exists a unique idealized monad morphism $(\overline{\lambda}, \overline{\lambda}') : \mathbb{T} \longrightarrow \mathbb{S}$ such that the following diagram:*

$$\begin{array}{ccccc} H & \xrightarrow{H\eta} & HT & \xrightarrow{\tau} & T \\ & \searrow^{\lambda'} & \downarrow{\scriptstyle \overline{\lambda}'} & & \\ & & S' & & \downarrow{\scriptstyle \overline{\lambda}} \\ & & & \searrow^{\sigma} & \\ & & & & S \end{array} \tag{5.7}$$

commutes.

Proof. First, suppose that $\mathbb{S}$ is an ideal monad. Theorem 4.3 states that the monad $\mathbb{T}$ of free cias is free w.r.t. all ideal completely iterative monads $\mathbb{S}$, whenever coproduct injections are monomorphic in $\mathcal{A}$. The same proof works in our current setting (i.e., with coproduct injections not necessarily monomorphic) with some minor modifications only. For the existence part we must verify that the induced pair $(\overline{\lambda}, \overline{\lambda}')$, where

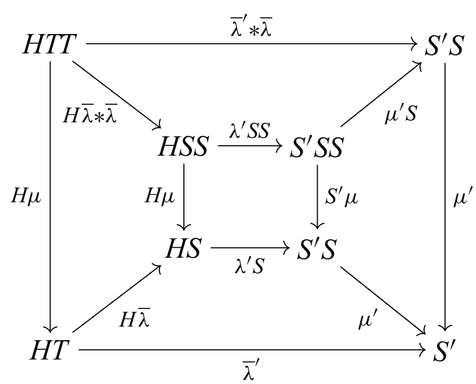
$$\overline{\lambda}' \equiv HT \xrightarrow{H\overline{\lambda}} HS \xrightarrow{\lambda' S} S'S \xrightarrow{\mu'} S', \tag{5.8}$$

is a morphism of idealized monads, and that the left-hand triangle in (5.7) commutes. For the latter, consider the following diagram

$$\begin{array}{ccc} H & \xrightarrow{H\eta} & HT \\ & \searrow^{H\eta^S} & \downarrow{\scriptstyle H\overline{\lambda}} \\ \downarrow{\scriptstyle \lambda'} & & HS \\ & & \downarrow{\scriptstyle \lambda' S} \\ S' & \xrightarrow{S'\eta^S} & S'S \\ & \searrow^{=} & \downarrow{\scriptstyle \mu'} \\ & & S' \end{array}$$

It commutes: for the upper triangle use the fact that $\overline{\lambda}$ is a monad morphism, for the middle part use the naturality of λ', and the lower triangle is the unit law of the $\mathbb{S}$-module S'.

To see that $(\overline{\lambda}, \overline{\lambda}')$ is an idealized monad morphism, it only remains to show that $\overline{\lambda}'$ is a module homomorphism with change of base $\overline{\lambda}$. Consider the following diagram

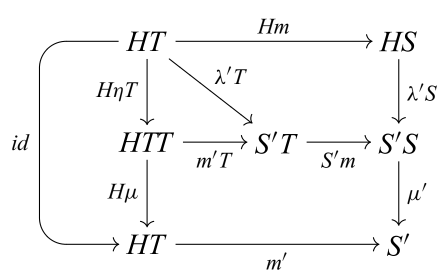


For the upper part it suffices to consider the parallel components for HT and T separately. The HT-part is (5.8), the other one is trivial. The other parts of the diagram are clear. For the middle three parts use—from left to right—that $\overline{\lambda}$ is a monad morphism, naturality of λ' and the module laws for S', and the lower part is (5.8) again.

For the uniqueness of $(\overline{\lambda}, \overline{\lambda}')$ assume that $(m, m') : \mathbb{T} \longrightarrow \mathbb{S}$ is an idealized monad morphism such that (5.7) with $(\overline{\lambda}, \overline{\lambda}')$ replaced by (m, m') commutes. Then from the proof of Theorem 4.3 we conclude that $m = \overline{\lambda}$ and from this it follows that:

$$m' = \mu' \cdot \lambda' S \cdot Hm = \mu' \cdot \lambda' S \cdot H\overline{\lambda} = \overline{\lambda}'.$$

In fact, to see the first equality consider the diagram



The lower square commutes since m' is a module homomorphism with change of base m, the left-hand part does by the unit law of the monad $\mathbb{T}$, the upper triangle by (5.7) and the upper right-hand part by naturality of λ'.

We have established the desired result for all ideal completely iterative monads $\mathbb{S}$. Now if $\mathbb{S}$ is an arbitrary (idealized) completely iterative monad, form its ideal coreflection $\widetilde{\mathbb{S}}$, which is completely iterative by Lemma 5.13. We have an ideal transformation

$$t \equiv H \xrightarrow{\lambda'} S' \xrightarrow{\mathsf{inl}} \widetilde{S}$$

inducing a unique morphism of idealized monads $(\bar{t}, \bar{t}') : \mathbb{T} \longrightarrow \widetilde{\mathbb{S}}$ which extends t. Use the adjunction of Lemma 5.13 to see that the composition of $(\bar{t}, \bar{t}')$ with the coreflection arrow $\widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$ yields the desired unique idealized monad morphism extending the given ideal transformation λ.

Remark 5.15. For accessible functors the last part of the proof just composes the two adjunctions from Remark 4.4 and Lemma 5.13 (see 5.8 for notation)

$$\mathsf{ClzAM}(\mathcal{A}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{CIAM}(\mathcal{A}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{Acc}[\mathcal{A},\mathcal{A}]$$

and it is clear that this extends to all completely iterative monads using the freeness of $\mathbb{T}$ for ideal completely iterative monads and (the full strength of) the adjunction from Lemma 5.13.

For the record we note the following result.

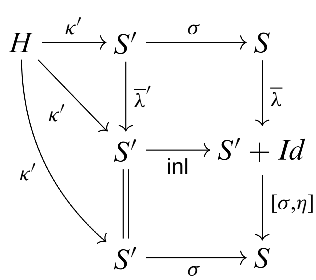
Proposition 5.16. *If $\mathbb{S}$ is a free completely iterative monad on H then it is ideal.*

Proof. We will show that the ideal coreflection $\widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$ is an isomorphism. Recall that $\widetilde{\mathbb{S}}$ is given by $\widetilde{S} = S' + Id$. Since $\mathbb{S}$ is completely iterative, so is $\widetilde{\mathbb{S}}$ by Lemma 5.13. Moreover, from the universal arrow $\kappa : H \longrightarrow S$ we get an ideal transformation

$$\lambda \equiv H \xrightarrow{\kappa'} S' \xrightarrow{\mathsf{inl}} S' + Id.$$

Thus, by the freeness of $\mathbb{S}$ we have a unique idealized monad morphism $\alpha = (\overline{\lambda}, \overline{\lambda}') : \mathbb{S} \longrightarrow \widetilde{\mathbb{S}}$ extending λ. We shall show that this is an inverse of the coreflection arrow $\beta = ([\sigma,\eta], 1_{S'}) : \widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$.

(i) $\beta \cdot \alpha = 1_{\mathbb{S}}$: Consider the following commutative diagram:



It shows that $\beta \cdot \alpha$ is an idealized monad morphism extending $\kappa = \sigma \cdot \kappa'$, whence it must be the identity on $\mathbb{S}$.
(ii) $\alpha \cdot \beta = 1_{\widetilde{\mathbb{S}}}$: From $\beta \cdot \alpha = 1_{\mathbb{S}}$ we get in the second component $\overline{\lambda}' = 1_{S'}$. Hence, we must only check the first component of $\alpha \cdot \beta$. Consider the coproduct components of $S' + Id$ separately to see that $\overline{\lambda} \cdot [\sigma, \eta] = [\mathsf{inl} \cdot \overline{\lambda}', \widetilde{\eta}] = [\mathsf{inl}, \mathsf{inr}] = 1_{S'+Id}$. □

6. Iterability is necessary

In this section, we assume that $\mathcal{A}$ is a category with binary coproducts. We have seen above that any iteratable endofunctor on $\mathcal{A}$ admits a free completely iterative monad. In this section, we prove that, conversely, every endofunctor admitting a free completely iterative monad is iteratable. This is a new result, which has only appeared in the extended abstract [16]. It is the last ingredient we need to complete the main task of the current paper, i.e., to establish that the statements (a), (b), and (c) from the introduction are equivalent. Fortunately, as compared to [16] the proof is now relatively short since all the necessary technical auxiliary results have already been established in Section 5. Also because of the equivalence of statements (a) and (b), the proof can be somewhat simplified.

Theorem 6.1. *Every endofunctor generating a free completely iterative monad is iteratable.*

Remark 6.2. More detailed, suppose that H is an endofunctor on $\mathcal{A}$ and

$$\kappa : H \longrightarrow S$$

is a free completely iterative monad on H (where κ is an ideal transformation), then for all objects Y of $\mathcal{A}$, SY is a free completely iterative H-algebra on Y with universal arrow $\eta_Y : Y \longrightarrow SY$, and it follows that H is iteratable, see Corollary 2.11.

Proof. Let a free completely iterative monad $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ on H with universal arrow κ: $H \longrightarrow S$ be given. Observe first that $(HS, H\mu)$ forms a right S-module. In fact, the module laws follow trivially from the monad laws for S. The following natural tranformation:

$$s \equiv HS \xrightarrow{\kappa S} SS \xrightarrow{\mu} S$$

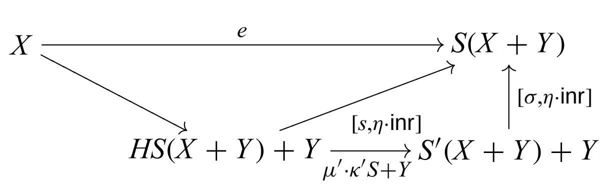
is a module homomorphism $(HS, H\mu) \longrightarrow (S, \mu)$. To see this inspect the commutative diagram

$$\begin{CD} HSS @>{\kappa SS}>> SSS @>{\mu S}>> SS \\ @V{H\mu}VV @V{S\mu}VV @VV{\mu}V \\ HS @>>{\kappa S}> SS @>>{\mu}> S \end{CD}$$

Thus, we have an idealized monad

$$\overline{\mathbb{S}} = (S, \eta, \mu, HS, s, H\mu).$$

This monad is completely iterative, since any guarded equation morphism e for $\overline{\mathbb{S}}$ is also guarded for $\mathbb{S}$. To see this consider the commutative diagram

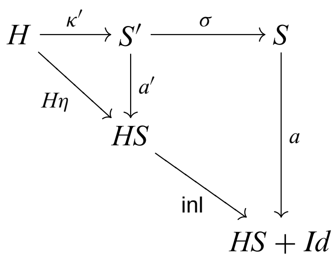


Thus, e has a unique solution.

Now denote by $\widetilde{\mathbb{S}}$ the ideal coreflection of $\overline{\mathbb{S}}$ whose underlying functor is given by $HS + Id$. We clearly have an ideal transformation

$$\lambda \equiv H \xrightarrow{H\eta} HS \xrightarrow{\mathsf{inl}} HS + Id.$$

Since $\mathbb{S}$ is free on H, we obtain a unique idealized monad morphism $\alpha = (a, a') : \mathbb{S} \longrightarrow \widetilde{\mathbb{S}}$ extending λ, i.e., such that the diagram

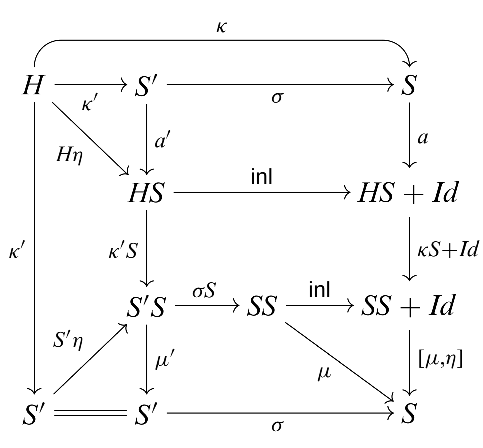


commutes. Let us prove that α is an isomorphism with an inverse given by

$$\beta = (b, b') \equiv \widetilde{\mathbb{S}} \xrightarrow{([\mu \cdot \kappa S, \eta], 1_{HS})} \overline{\mathbb{S}} \xrightarrow{(1_S, \mu' \cdot \kappa' S)} \mathbb{S}$$

It is not difficult to see that β is an idealized monad morphism. The first morphism is the coreflection arrow, and for the second one it is clear that 1_S is a monad morphism and it is easy to see that $\mu' \cdot \kappa' S$: $HS \longrightarrow S'S \longrightarrow S'$ is a module homomorphism (with change of base 1_S, i.e., no change of base).

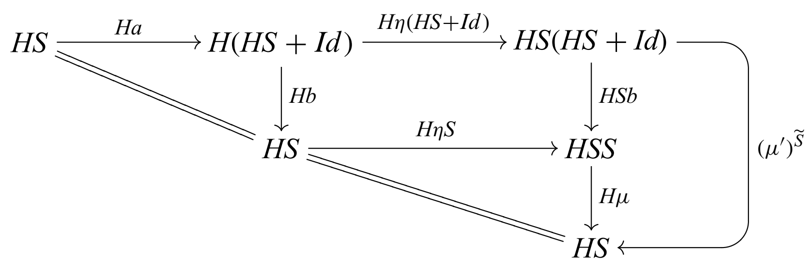
(1) $\beta \cdot \alpha = 1_{\mathbb{S}}$: Notice that $\beta \cdot \alpha$ is an idealized monad morphism extending the universal arrow κ, i.e., the following diagram



commutes. By the freeness of $\mathbb{S}$, $\beta \cdot \alpha$ must be the identity on $\mathbb{S}$.
(2) $\alpha \cdot \beta = 1_{\widetilde{\mathbb{S}}}$: Since $\widetilde{\mathbb{S}}$ is an ideal monad, it suffices to check the second component of $\alpha \cdot \beta$ (see Remark 5.6). Hence, we show that $a' \cdot b' = 1_{HS}$:

$$\begin{aligned} a' \cdot b' &= a' \cdot \mu' \cdot \kappa' S && \text{(definition of } b'\text{)} \\ &= (\mu')^{\widetilde{S}} \cdot (a' * a) \cdot \kappa' S && (a' \text{ is a module homomorphism}) \\ &= (\mu')^{\widetilde{S}} \cdot (H\eta * a) && (a' \cdot \kappa' S = H\eta), \end{aligned}$$

where $*$ denotes parallel composition. Analyzing the last arrow further, we finally get the following commutative diagram



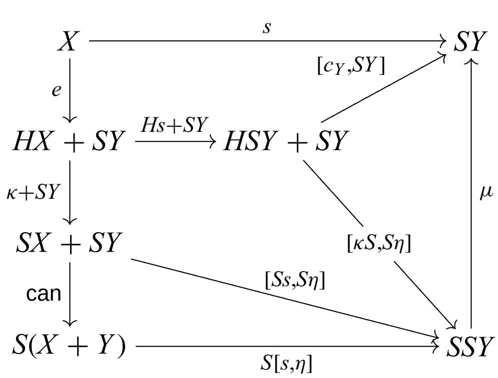
For the left-hand triangle use that $\beta \cdot \alpha = 1_{\mathbb{S}}$, the lower triangle is one of the monad laws of S, the upper square is naturality of η, and the right-hand part is the definition of μ' for the ideal coreflection $\widetilde{\mathbb{S}}$ (see Lemma 5.10). Thus $a' \cdot b' = 1_{HS}$, and therefore $\alpha \cdot \beta$ is the identity on $\widetilde{\mathbb{S}}$ as desired.

To complete the proof we show now that SY carries the structure of a free cia on Y. In fact, notice first that $b_Y \cdot \mathsf{inr} = \eta_Y$. We shall prove below that SY with the structure map $c_Y = b_Y \cdot \mathsf{inl} = \mu_Y \cdot \kappa_{SY}$ is a completely iterative H-algebra. Then, it follows from the proof of Theorem 2.10 that (SY, b_Y) is a completely iterative algebra of $H(\_) + Y$, and since b_Y is an isomorphism this cia is initial, see Remark 2.9. Thus, by Theorem 2.10, (SY, c_Y) is a free cia with universal arrow $\eta_Y : Y \longrightarrow SY$.

Now in order to see that (SY, c_Y) is a cia let $e : X \longrightarrow HX + SY$ be a flat equation morphism. Then form the following equation morphism:

$$\overline{e} \equiv X \xrightarrow{e} HX + SY \xrightarrow{\kappa + SY} SX + SY \xrightarrow{\mathsf{can}} S(X + Y)$$

for the monad $\mathbb{S}$. Since κ is an ideal transformation, $\overline{e}$ is guarded. Solutions of $\overline{e}$ w.r.t. the completely iterative monad $\mathbb{S}$ are in one-to-one correspondence with solutions of e w.r.t. the algebra (SY, c_Y). Indeed, consider the diagram



The arrow s is a solution of $\overline{e}$ if and only if the outer square of the diagram commutes. Equivalently, the upper part commutes since all other parts obviously do. But this is precisely the case if s solves e. Thus, since $\overline{e}$ has a unique solution so does e. □

To conclude the paper let us collect the results of Corollary 2.11, and Theorems 5.14 and 6.1 to state our main result compactly.

Corollary 6.3. *Let H be an endofunctor on $\mathcal{A}$, and let T be an object assignment of $\mathcal{A}$. Then the following are equivalent:*

(i) for every object Y of $\mathcal{A}$, TY is a final coalgebra of $H(\_) + Y$, i.e., H is iteratable,
(ii) for every object Y of $\mathcal{A}$, TY is a free completely iterative algebra of H on Y, and
(iii) T is a free completely iterative monad on H.

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