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Subverting the Classical Picture: (Co)algebra, Constructivity and Modalities*

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1. Subverting the Classical Picture

> *[L]ogic has permeated through computer science during the past thirty years much more than it has through mathematics during the past one hundred years. Indeed, at present concepts and methods of logic occupy a central place in computer science, insomuch that logic has been called "the calculus of computer science"* [190] *… [L]ogic provides computer science with both a unifying foundational framework and a powerful tool for modeling and reasoning about aspects of computation.*
>
> Joseph Y. Halpern, Robert Harper, Neil Immerman, Phokion G. Kolaitis, Moshe Y. Vardi and Victor Vianu [124, pp. 215, 233]

> *In academia, in industry, and in the commercial world, there is a widespread belief that computing science as such has been all but completed and that, consequently, computing has "matured" from a theoretical topic for the scientists to a practical issue for the engineers, the managers and the entrepreneurs, i.e. mostly people—and there are many of those!—who can accept the application of science for the obvious benefits, but feel rather uncomfortable with its creation because they don't understand what the doing of research, with its intangible goals and its uncertain rewards, entails. This widespread belief, however, is only correct if we identify the goals of computing science with what has been accomplished and forget those goals that we have failed to reach, even if they are too important to be ignored. I would therefore like to posit that computing's central challenge, viz. "How not to make a mess of it", has* not *been met.*
>
> Edsger W. Dijkstra [78]

Logic as traditionally presented in typical introductory courses—in mathematics, philosophy or computer science—has four distinctive features:

BOOL It relies on classical, boolean laws: *law of excluded middle*, *the law of double negation* and *Peirce's syllogism*. In particular, any formula can be transformed to a suitable negation normal form, either a conjunctive one

(CNF) or a disjunctive one (DNF). It also entails a high degree of dependence between basic connectives: $\to$, $\vee$, $\wedge$, $\neg$, $\top$, $\bot$, $\forall$ and $\exists$.

SETAC Furthermore, in modelling and formalising any phenomenon, one can use not only these boolean assumptions, but also standard set-theoretic principles such as the Axiom of Choice or at least the Boolean Prime Ideal Theorem.

RELST It is supposed to deal with various types of *relational structures*, either in the form of *transition systems* in operational semantics of programming languages or in the form of the *relational model/algebra* in database theory. Furthermore, these relations are supposed to be modelled in a standard set-theoretic way.

SYFOL Finally, its syntactic paradigm is supposed to be provided by standard (mostly *first-order*) *predicate logic*, which reflects and enforces its relational orientation mentioned in the preceding point. The syntax of predicate logic underlies not only languages used in logic programming, but also the concept of dependent types.

Such an understanding of logic goes back to authors brought up in the nineteenth century and reacting to philosophical and mathematical challenges of their day. Its respective components were provided, in varying contexts and for different reasons, by the great pioneers of the field: Gottlob Frege, Bertrand Russell, Charles S. Peirce, George Boole and Georg Cantor. In a later generation of logicians, Willard Van Orman Quine can be seen as a militant proponent of an entire set of views combining these ingredients.[1] In what follows, I am going to call it the *Classical Picture*. It is not nearly adequate for the evolving needs of contemporary computer science; we are going to recall the reasons for subverting *SYFOL* from § 2 on, for subverting *RELST* from § 3 on, for subverting *SETAC* from § 4 on and for subverting *BOOL* from § 6 on. This overview summarizes my recent (as of mid-2018) contributions to an ever-growing body of work illustrating the fruitfulness of lifting each of the above assumptions.

Remark 1.1. As a community, we still seem far away from having an adequate synthesis of this emerging new picture, something akin to a new *Organon* of logic in computer science. This is related to another question: how should the logical education of CS students evolve to close the growing gap between teaching and research? Apart from presenting my own work, this overview provides some insight into the choice of subjects which in my opinion such a synthesis should incorporate. These include elements of category theory, type theory, proof theory, finite model theory, (co)algebraic methods, the study of effectiveness, computability and complexity, and last but not the least, automated reasoning and interactive theorem proving.

I focus on my papers accepted or published between 2014 and the first half of 2018, with occasional references to earlier ones. In more detail:

- Even in the case of relational structures with classical metatheory, *modalities* have been well-understood to provide an attractive alternative to predicate

[1] To avoid getting into finer historical points too early in this introduction, I provide a somewhat more detailed discussion of Quine's (and Tarski's) views in Appendix D.

first- and higher-order syntax. This perspective lies at the heart of what one may call the *Amsterdam approach* to modal logic [17, 38] and is broadly accepted in contemporary CS [292, 124, 192]. Some discussion of my work on related subjects prior to the time interval covered by this overview [O12, O7, O6, O2, O3, O1] can be found at the end of § 2.

- Furthermore, the research of the past two decades has shown that modal formalisms arise naturally in the context of *coalgebraic logic* (§ 3), which provides a categorical umbrella for various *state-based* but *non-relational* systems. Incorporating modalities directly into predicate syntax—*not* in the sense of creating a variant of quantified modal logic!—is at the center of our generalization of FOL to the coalgebraic setting: *coalgebraic predicate logic* (CPL) [I7, S3, S2, I11] (§§ 3.2 and 4.1).
- In recent years, another generalization of Kripke semantics is gaining popularity, namely so-called *possibility semantics*. The research on the subject is led mostly by the team of Wesley H. Holliday at UC Berkeley. While Kripke coalgebras are dual to *Boolean Algebras with operators* (BAOs) which may fail to be *completely additive*, possibility frames are dual to BAOs which may fail to be *atomic* (§ 5, especially § 5.3). My 2005 PhD Thesis [O8] and a series of related publications [O10, O9, O4] investigate a whole hierarchy of *"sub-Kripkean" notions of completeness* (§ 5.2). The insights provided by research on possibility semantics have helped Holliday and myself [I1] to solve an open question posed in my earlier work: the existence of logics incomplete with respect to completely additive BAOs (§ 5.3).
- Moreover, our investigation of modal definability/axiomatizability of complete additivity [I1, §§ 8–9] has lead us to the invention of the *global quantificational modality* $[\forall p]$. Unlike the propositional quantifier $\forall p$ by itself, the global quantificational modality $[\forall p]$ can be straightforwardly interpreted in any *Boolean Algebra Expansion* (BAE); for example, it does not require any form of *lattice-completeness*. Our paper [I2] presents a Hilbert-style calculus GQM, shows its completeness, studies its relationship with the theory of discriminator BAEs, and shows that GQM formulas valid over lattice-complete BAEs cannot be recursively axiomatized. The proof of the last statement provides a generalized prenex-form result for the full second-order propositional modal logic (SOPML). Our formalism enables a conceptual shift: what have traditionally been called different "modal logics" now become $[\forall p]$-universal theories over the base logic GQM (§ 5.4). There is a formalization of this work in the Coq proof assistant developed by my student (cf. Footnote 8).
- Computer science provides numerous natural instances of structures departing even more drastically from the Classical Picture, i.e., giving up the boolean base altogether. An important class of applications is provided by the *Curry-Howard correspondence* between proofs in extensions of intuitionistic logic (or even substructural logic) and computer programs (§ 6.1).
  - My paper [I5] (briefly discussed in § 6.3) provides a general overview of *constructive modalities for guarded recursion*, focusing in particular on the overlooked rôle of the intuitionistic axioms mHC and KM.

  - In a joint work with Stefan Milius [I10, S4], we provide a general framework for *categorical models of guarded recursion*, which can be equivalently seen as models for constructive strong Löb logic (§ 6.2).
  - Finally, we have studied [I8] how *negative translations* relate intuitionistic modal logics and their classical counterparts (§ 6.4). My student coauthors provided a Coq formalization (cf. Footnote 11).

- Furthermore, moving to the intuitionistic propositional calculus (IPC) allows discovering distinctions invisible in the classical setting. A striking example is provided by the study of *constructive strict implication* $\prec\!\!3$ in our joint paper with Albert Visser [I9] (§ 7.1 of this overview). In the intuitionistic setting, this original modal connective of C. I. Lewis [183, 184] has greater expressive power than unary $\Box$. Apart from discussing completeness, axiomatization and correspondence results, we illustrate its various appearances, ranging from schematic logics of arithmetical theories—more specifically, in terms of *preservativity* in extensions of Heyting Arithmetic (HA)—to functional programming. Indeed, a strong variant of $\prec\!\!3$ has been also discovered by the functional programming community in their study of "arrows" as contrasted with "idioms" [141, 187, 186, 193] (see § 7.1, especially Remark 7.5)
- Another way of adding a separate implication-like connective $-\!\!*$ in the intuitionistic setting is suggested by the *logic of bunched implications* (BI). Our overview with Peter Jipsen [I3] proposes *generalized BI algebras* (GBI-*algebras*) as a common framework for algebras arising via "declarative resource reading", intuitionistic generalizations of relation algebras (i.e., *weakening relations*), formal languages, or a fine-grained treatment of labelled trees and semistructured data. We discuss the lattice of subvarieties of GBI, some dualities for GBI-algebras and an algebraic route towards generic results on decidability, both positive and negative ones. The paper provides an account of the theory behind state-of-art tools, culminating with an algebraic and proof-theoretic presentation of (*bi*-)*abduction* [51] (§ 7.2).
- An even more drastic revision of the Classical Picture dispenses altogether not only with the boolean axioms, but even with the distributivity of the underlying lattice structure. Surprisingly enough, it leads to a new perspective on the relational model in the form of *relational lattices*. Our work with Szabolcs Mikulás and Jan Hidders [I6, S1] proposes an axiomatization of the abstract class, shows pseudoelementarity of the concrete class and undecidability of the quasiequational theory, and proves representation results (§ 8.1). There is also a modal perspective on this work (cf. Remark 8.5).
- Finally, questions regarding acceptable axioms, rules of inference and means of proof arise on the level of *metatheory*, even when the underlying logic is assumed to be boolean. Our work on CPL (with Lutz Schröder, Katsuhiko Sano, and Dirk Pattinson) offers a striking illustration of this fact. Our results [I7] presented in § 3.2 mimic classical model theory and hence apply only in the restricted setting of coalgebraic structures which are either "sufficiently Kripke-like" or "sufficiently neighbourhood-like". On the other hand, our results [I11] presented in § 4.1 rely on *effective* methods of *finite model theory* and apply across the coalgebraic spectrum. As discussed in my paper

[I4], similar problems can be seen in another important area of applications of logics: that of formal economics (§ 4.2). It is surprisingly common in *social choice*, *welfare analysis* and *aggregation theory* to fall back on non-effective axioms, in particular the Axiom of Choice (AC), and prove results whose real meaning is far from obvious. By contrast, potential links with finite model theory and database theory do not seem fully explored.

Publications submitted along with this overview are listed in § 9.1.1, i.e., major papers accepted or published between 2014 and the first half of 2018. § 9.1.2 lists some workshop and conference papers completed during the first half of my appointment at FAU (since 2012) and made obsolete by later journal papers included in § 9.1.1. Specifically,

- the ICALP 2012 [S3] and the TbiLLC 2013 [S2] papers have been superseded by the journal version in *Logical Methods in Computer Science* [I7] (with Lutz Schröder, Katsuhiko Sano, and Dirk Pattinson),
- the RAMiCS 2014 [S1] paper was among the five contributions invited to the postconference issue of *Journal of Logical and Algebraic Methods in Programming* and hence superseded by the resulting paper [I6] (with Szabolcs Mikulás and Jan Hidders),
- and the FiCS 2013 paper [S4] was superseded by its journal version in *Fundamenta Informaticae* [I10] (with Stefan Milius).

Several journal papers or book chapters listed in § 9.1.1 and covered by this overview have had no conference/workshop forerunners. This includes:

- a paper forthcoming in *Review of Symbolic Logic* [I1] (with Wesley H. Holliday),
- a paper forthcoming in *Studia Logica* [I4],
- a paper in the special issue of *Indagationes Mathematicae* on "L.E.J. Brouwer, fifty years later" [I9] (with Albert Visser),
- two chapters in books published in the series *Outstanding Contributions in Logic*: *Leo Esakia on Duality in Modal and Intuitionistic Logics* [I5] and *Hiroakira Ono on Residuated Lattices and Substructural Logics* [I3] (the latter jointly with Peter Jipsen).

The paper [I11] in *Journal of Logic and Computation* with Schröder and Pattinson continues a conference paper of Schröder and Pattinson [262] which preceded (in fact, to a certain extent triggered; cf. § 3.2 below) my involvement in the development of Coalgebraic Predicate Logic. Finally, this overview includes two conference papers:

- *Formal Structures in Computation and Deduction* (FSCD) 2017 [I8] (with Miriam Polzer and Ulrich Rabenstein) and
- *Advances in Modal Logic* (AiML) 2018 [I2] (with W. H. Holliday).

2. Beyond Frege and Russell: Modalities Instead of Predicates

There are logicians, myself among them, to whom the ideas of modal logic (e.g. Lewis's) are not intuitively clear until explained in non-modal terms ... [235]
Everything is what it is, ask not what it may or must be. [239, p. 174]

Willard Van Orman Quine 1947, 1986

[T]here is no one fundamental logical notion of necessity, nor consequently of possibility. If this conclusion is valid, the subject of modality ought to be banished from logic, since propositions are simply true or false.

Bertrand Russell 1905, quoted following Goldblatt [115]

This section presents introductory historical material, although in its final part some of my papers prior to 2012 are briefly discussed. In the words of Wilfrid Hodges, in this overview "'I' means I, 'we' means we" [136, p. xiii] and care is taken to use *pluralis modestiae* instead of *pluralis maiestatis*. Finally, a warning is due that I chose some quotes and epigraphs with a somewhat provocative intention.

Various structures of certain importance in computer science—(labelled) transition systems, (labelled) trees, relational databases—can be seen as examples of *first-order models* or *structures* in the sense of classical model theory. Hence, standard first-order logic (FOL) provides a natural *correspondence language* for them. Nevertheless, despite its ubiquity and well-developed metatheory, first-order logic is far from being an ideal tool in this respect. Let us list potential issues, enumerating them for the sake of later comparison with modal logic:

- fol1 Its satisfiability problem over arbitrary structures is not decidable for almost any meaningful signature, the only exception being monadic ones.
- fol2 Complexity of its model checking is not always feasible for practical purposes.
- fol3 Despite being far from computationally straightforward, FOL cannot express many concepts of fundamental importance, such as well-foundedness or finiteness.
- fol4 Indeed, when it comes to *finite* structures, the validity problem becomes even worse: it is not even recursively enumerable, meaning that it does not allow any effective axiomatization. Furthermore, there is a dramatic discrepancy between classical and finite model theory for FOL, each focusing on a largely different set of tools and results. In particular, few standard *preservation theorems* of classical model theory are valid in the finite. This is a theme which is going to play a major rôle in § 4.1 below.
- fol5 Games used to characterize first-order notions of preservation and invariance do not capture most interesting notions of equivalence from a CS point of view, such as *bisimilarity*.
- fol6 It is not obvious how to generalize ordinary FOL syntax to accommodate important non-relational transition structures (see § 3 and Appendix A for examples).
- fol7 Finally, the presence of quantifiers and binding in first-order formulas adds extra layers of complexity to any fully or partially automated treatment via theorem provers or proof assistants. While such syntactic constructs are hard to avoid in any language expressive enough to formalize workable metatheory (see, however, Tarski and Givant [284]), it is often desirable to have a sufficiently expressive alternative for everyday CS purposes.

In the second half of the XXth century, modal logic turned out to provide an attractive alternative to FOL in the light of the above difficulties. The groundwork, interestingly enough, was laid in philosophy, even though most philosophy departments until at least 1960's were hostile environments for modal logic, with most philosophers of the period holding a firmly rooted belief in the supremacy of first-order syntax and in the general correctness of the Classical Picture.

Remark 2.1. Some quotes illustrating this belief were used as epigraphs of this section. As an illustration of its former ubiquity, recall that teachers of Saul Kripke for years advised against publication of his completeness results, claiming that publications on modal logic would harm his career (see, e.g., [115, § 4.8]). One can also attribute to such philosophical influences the fact that Alfred Tarski, coauthor of the work [154] which in hindsight laid the foundations for the theory of modal completeness and duality, refused to see a connection between his work and modal logic, reportedly even in a direct discussion with Kripke [66, p. 13] (quoted in [115, § 3.4], [38, p. 47]). See Appendix D for more.

Kripke's completeness results, in some ways preceded by or parallel with the work of Jónsson, Tarski, Beth, Carnap, Hintikka, Bayart, Kanger, Meredith, Prior, Geach, Montague or Grzegorczyk (detailed history and references are provided by Goldblatt [115]), opened up the possibility of seeing modal logic as a natural language for relational structures, rather than an unorthodox syntax for intensional metaphysics (although see § 7.1 for a modern perspective on modal logic as originally proposed by Lewis). One should see in a similar vein the early work of Prior [229, 230, 231] on temporal logic. Not only did it sow the seeds of numerous later developments such as *hybrid logic* (cf., e.g., [38, § 7.3],[7]), it also led directly to Kamp's result [158] that temporal formalism has the same expressive power as first-order logic over the real line (although there is a price to pay in terms of succinctness). Finally, in the next decade, it inspired the monograph of Rescher and Urquhart [243] (sometimes rather unfairly overlooked in contemporary overviews) and subsequently the seminal model-checking work by Pnueli [226, 227].

Remark 2.2. In the interest of historical accuracy, it is worth pointing out that Pnueli's first conference paper [226], most commonly quoted as the original reference for the use of temporal logic in model checking, used basic modal syntax rather than that of full LTL! "Rescher and Urquhart in their book *Temporal Logic* [243] give a survey of different logical systems which increasingly capture more and more of the properties of time. Out of this selection we adopted a fragment of the tense logic ..." [226, p. 52], which as explicitly claimed several pages later [226, p. 54] (and is easy to check) is just equivalent to ordinary S4. Only the later journal paper [227] uses the full LTL syntax and explicitly quotes Prior's book [230], likely influenced by Pnueli's collaboration with Gabbay, Shelah and Stavi [106], which just like the earlier work of Kamp [158] highlighted the importance of Until and related connectives.

Apart from introducing temporal logic to model checking, the 1970's saw several other important developments. One needs to include here Burstall [47], suggesting the use of both modal logic and substructural logic in reasoning about shared mutable data structures (anticipating later successs of the logic of bunched implications and separation logic, cf. Reynolds [244, 245] or our forthcoming overview [I3]), or the creation of Propositional Dynamic Logic (PDL) [228]. Less obviously so, the 1970's also brought the realization of *limitations* of Kripke semantics, in particular in the form of *incompleteness results*. Tellingly, the monograph of Blackburn et al. [38] proposes the date of publication of the first paper on modal incompleteness as the beginning of the *modern era* in modal logic; such incompleteness results will play an important rôle in § 5 below.

In the of the 1970's, van Benthem [15, 17] systematized the study of modal logic as a well-behaved language for relational structures. While several founding ideas had been provided by seminal papers such as Sahlqvist [252] or Fine [99], van Benthem initiated an entire research program of *modal correspondence theory*, heavily relying on tools and techniques of classical model theory.

- In the *frame*-oriented perspective, the standard translation renders modal logic a well-behaved fragment of *monadic second-order logic*; we will say more about this in § 5 below.
- In the *model*-oriented perspective, modal logic becomes a well-behaved fragment of first-order logic.

Regarding the second of these items, van Benthem [15, 17] showed that modal logic corresponds to the fragment of first-order logic *invariant under bisimulations*. These days, the concept of bisimulation is ubiquitous in CS, in areas as diverse as concurrency theory, reactive programming or coalgebra, but van Benthem's theorem seems to have been the first result of major importance where bisimulations played a central rôle, followed soon by Park [220]. The result displays remarkable stability and robustness under various generalizations:

- it extends to a characterization of *program safety* [18, 137, 103] [38, § 2.7];
- unlike earlier characterization results for subclasses for first-order formulas in classical model theory, it survives both "in the finite" and in the coalgebraic context. These two survival skills seem intertwined, and we are going to elaborate on this in § 4.1;
- moreover, the methods which yield the in-the-finite variant are also amenable to another generalization: they allow analogues over restricted classes of frames. See Dawar and Otto [73, 74] and § 4.1 below. Note that variants of the van Benthem result for broader classes of frames do not transfer automatically to classes satisfying additional conditions;
- finally, it can be reproduced for modal μ-calculus, which analogously can be shown to be the bisimulation-invariant fragment of monadic second-order logic; this is the content of the celebrated Janin-Walukiewicz theorem [149].

Let us pause here and briefly summarize the most salient results characterizing modal logic as an alternative correspondence language for relational structures, contrasting it with our previous list of challenges facing first-order logic:

ml1 Basic modal logic over arbitrary frames is known to be PSPACE complete, ditto for numerous other standard classes of frames. In some cases, it is even possible to improve on this, e.g., all finitely axiomatizable tense logics of linear and transitive relations are coNP-complete [O12]. Similar results survive for various extensions of the basic modal language, such as tense logics or hybrid logic with satisfaction operators. Even very powerful extensions such as the modal μ-calculus remain decidable.

ml2 Complexity of model checking remains polynomial both in the input formula and the input model even for powerful formalisms such as CTL.

ml3 Over *frames*, modal logic can express important properties inexpressible in first-order language, such as the combination of Noetherianity (converse well-foundedness) and transitivity expressed by the *Löb axiom* $\Box(\Box p \to p) \to \Box p$, which is going to play a central rôle in this overview.

ml4 There is no difference between modal validity on arbitrary frames and in the finite; the basic modal logic K and numerous extensions corresponding to standard frame conditions have *the finite model property*. The same applies to very powerful extensions of K, such as modal μ-calculus.

ml5 Modal equivalence is intimately connected to bisimilarity. This connection is displayed not only by the van Benthem theorem and its generalizations [18, 137, 103, 149, 247, 73, 74], but also results such as the Hennessy-Milner theorem [128] (see [38, Th. 2.24] for the basic modal variant).

ml6 As we are going to see in § 3 and § 5, modal syntax allows one to travel far from the standard setup of possible worlds and relational transitions, preserving a surprisingly large body of results and providing unexpected insights on the interplay between Kripkean and non-Kripkean worlds.

ml7 Not only the basic modal syntax, but its numerous extensions remain binder- and quantifier-free. Furthermore, even the addition of certain powerful binders preserves decidability, with μ-calculus being again the flagship example. On the other hand, as we are going to see below, modal modifications of predicate syntax (§ 3.2) and quantifiers (§ 5.4) open up unexpected avenues.

In my own research before 2012 (selected papers listed in § 9.1.3), I contributed on several fronts to research capitalizing on the advantages of suspending point *SYFOL* of the Classical Picture, including point ml1 above (a joint paper with Frank Wolter [O12] on co-NP complexity of finitely axiomatizable tense logics of linear logics of time flows), and a line of papers investigating hybrid extensions of modal syntax [O7, O6, O2]). Perhaps the most interesting CS example is provided by publications [O3, O1] supported by my NWO Rubicon grant *Algebraic Optimization of XPath Queries*, which contain axiomatization, completeness and complexity results for well-behaved fragments of the modal logic of finite trees, understood as the navigational core of query languages for semi-structured (XML) databases. This is to be contrasted with another approach to (standard relational!) databases breaking with the Classical Picture and described in § 8.1 below. Three aspects of these results and their proofs deserve mention here.

- First, a central rôle in relevant axiomatizations is played by the Löb axiom (ml3 above), which we are going to see repeatedly in §§ 6.2, 7.1 and 8.2.
- Second, these papers provide an analogue of van Benthem (and Rosen [247]) results for logics of finite trees (ml5 above) by combining the Janin-Walukiewicz characterization of μ-calculus [149] with the Sambin-de Jongh [253, 42] fixpoint theorem for the Löb logic (cf. § 6.3 and [I5]).

- ▸ Third, the completeness proofs used therein are more constructive (or at least *quasiconstructive* or *agnostic* [112, 257], cf. § 4.2) than those in most standard references. We are going to return to this issue in § 4.3.

Remark 2.3. Some references in the above paragraph depart from the first order syntax in a somewhat different way than standard modal logic does. As discussed in our papers [O3, O1], the syntax of Core XPath path expressions can be understood in terms of reducts of Tarski's relation algebras [283]: there exists a linear time translation [191, 285] into *dynamic relation algebras with unions* (UDRAs) [19, 138] or *antidomain semirings* [77, 76]. From a two-sorted perspective, incorporating also Core XPath node expressions, the syntax in question is contained between *boolean modules* [134] and *semiring modules* [92]; the most appropriate name here would seem to be *antidomain semiring modules* or *boolean modules over idempotent semirings*. *Modal Kleene algebras* [200] provide an example of a closely related class of relational structures with recent CS applications. Yet another one, with a somewhat different motivation, is that of *concurrent Kleene algebras* (cf., e.g., [135] and § 7.2 below).

Nevertheless, the developments discussed above were mostly abolishing *SYFOL* from the Classical Picture, leaving its remaining pillars largely intact—although we already touched on the issue of the validity of *RELST*. Now it is time to address this problem more systematically.

3. Beyond Peirce, Tarski and Kripke: Coalgebras and CPL

Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial.
Peter Freyd [105, p. 122, Footnote 1]

The first pillar of the Classical Picture to be attacked is *RELST*. We are going to learn how standard transition systems are generalized to coalgebras. On the other hand, we are also going to learn that attacking *RELST* does not entail dispensing with *SYFOL* altogether; rather, we can use modal insights to provide a natural generalization of the predicate syntax. More specifically, I present *Coalgebraic Predicate Logic* (CPL), the focus of several papers I coauthored.

3.1. Introduction to the Coalgebraic Framework

Before focusing on the papers on CPL covered by this overview [I7, S3, S2, I11], we need a general introduction on coalgebraic semantics of modal logic.

Coalgebras as generalized transition systems A Kripke structure with carrier set W interprets an unary $\Box$ as a binary relation $R_\Box \subseteq W \times W$; $R_\Box$ can be equivalently seen as a function $W \to \mathcal{P}W$, where $\mathcal{P}$ is the powerset of W. The powerset operation can be lifted to an *endofunctor* on the category Set. In fact, there are two ways of doing so:

- ▸ One approach is to respect the direction of arrows, i.e., define the *covariant* functor lifting $f : W \to V$ to $\mathcal{P}f : \mathcal{P}W \to \mathcal{P}V$. This is achieved by the *direct image*, i.e., for any $X \subseteq W$, $\mathcal{P}f(X) = \{f(x) \mid x \in X\}$.

- Nevertheless, it is equally natural to *reverse* arrows via *inverse image* (or *preimage*) by setting for any $Y \subseteq V$, $\mathcal{Q}f(Y) = \{x \mid f(x) \in Y\}$. Strictly speaking, it is a *contravariant* functor from Set to its *opposite category* Set^{op}, while its composition with itself $\mathcal{QQ}$ is a covariant endofunctor on Set.

Whenever C is a category, T an endofunctor on C, and W an object of C, entities of the form $W \to TW$ are called T-coalgebras. Given any two coalgebras $\gamma : W \to TW$ and $\delta : V \to TV$, we say that $f : W \to V$ is a *coalgebraic morphism* (or *T-morphism*) from γ to δ if $Tf \cdot \gamma = \delta \cdot f$. For $T = \mathcal{P}$, this yields that f is a bounded morphism of Kripke frames (i.e., one whose graph is a bisimulation). Thus, a standard categorical construction leads us directly to modal morphisms. § A.1 contains examples of other Set-endofunctors other than $\mathcal{P}$ and their resulting coalgebras: multigraphs, selection function frames or neighbourhood frames. Numerous other ones, including variants of weighted automata or Markov chains, can be found in references scattered through this section. Furthermore, the categorical definition of coalgebra immediately inspires futher generalizations and modular constructions. An example is provided by products of functors, which allow a natural way of combining frame classes (see Schröder and Pattinson [265] and Remark 3.2 below).

Coalgebras as modal semantics Now we need to learn how to generalize the Kripkean notion of satisfaction for standard modal languages.

Remark 3.1. The adjective "standard" is used here to exclude the "nabla" syntax proposed originally by Moss [205], essentially introducing the functor itself into logic. There are detailed studies [173, 172] of its relationship with standard modal syntax and the predicate lifting approach presented below.

An adequate solution has been proposed by Pattinson [222]: fix a *modal signature* Λ, i.e., a supply of modal connectives and their arities. An n-ary modal connective $\heartsuit \in \Lambda$ is interpreted by its corresponding *predicate lifting* associating with every n-ary modal operator $\heartsuit \in \Lambda$ a set-indexed family of mappings $[\![\heartsuit]\!]_W : (\mathcal{Q}W)^n \to \mathcal{Q}TW$ respecting the defining condition of a natural transformation, i.e., *naturality*:

$$(Tf)^{-1} \cdot [\![\heartsuit]\!]_W = [\![\heartsuit]\!]_V \cdot (f^{-1})^n$$

for every set-theoretic function $f : W \to V$. Thus, given any coalgebra $\gamma : W \to TW$ and formulas φ_1, ..., φ_n, we get $[\![\heartsuit\varphi_1 \dots \varphi_n]\!]_W$ as

$$\mathcal{Q}\gamma \cdot [\![\heartsuit]\!]_W([\![\varphi_1]\!]_W, \dots, [\![\varphi_n]\!]_W).$$

In other words, while the coalgebraic structure γ generalizes the Kripke accessibility relation, the predicate lifting $[\![\heartsuit]\!]$ generalizes the notion of Kripke-style (or forcing-style) satisfaction. A pair $(T, ([\![\heartsuit]\!])_{\heartsuit\in\Lambda})$ is called a Λ-*structure based* on T (or a Λ-structure *over* T).

Appendix A.2 illustrates how the predicate liftings approach works for the examples of coalgebras and functors in Appendix A.1. The reader is also encouraged to contrast the presentation in this section with a more algebraic approach presented in § 5.1 below; as discussed therein, an alternative way of presenting predicate liftings is to cast them as a generic way to *naturally* transform coalgebraic semantics into neighbourhood semantics.

Remark 3.2. We have not mentioned the notions of *model* and its associated *valuation*, which provides the interpretation of propositional atoms. Indeed, one route would be to distinguish between *coalgebraic frames* and *coalgebraic models*. But the coalgebraic setting allows introducing propositional atoms directly into modal signature: a nullary modality $p \in \Lambda$ is a *propositional atom* if T decomposes as $T = T' \times 2$ and under this decomposition, $[\![p]\!]_X = T'X \times \{\top\}$.

Thus, *coalgebraic models* can be seen as coalgebras for the product of the base "frame" functor T and the "model" functor, i.e., the constant functor $\mathcal{P}(\mathsf{At})$. This is an example of *combination of frame classes* [265]. Note that the language corresponding to "frame" covariant powerset functor under this perspective is the variable-free *Hennessy-Milner logic*. There are, however, occasions where the more traditional approach is the more natural one, e.g., in the study of (uniform) interpolation in coalgebraic logic [268]. Cf. also the distinction between definability by *variable-free modal theory* and *definability by modal theory* in papers on the coalgebraic Goldblatt-Thomason theorem [174, 175]. In § 3.2, we will return to this discussion.

Rank-1 axiomatizations Coalgebraic logic is focused mostly on the logic of *all* coalgebras for a given functor. In the Kripke case, for example, this means K. In general, the logics which fit naturally into the coalgebraic paradigm are those which can be axiomatized over the boolean base by *rank-1 formulas* (every atom occurs within the scope of exactly one modality) or *rank-1 rules* (premises are modality-free and the conclusion is a rank-1 formula); in the presence of the congruence rule, such rules can be converted into rank-1 axioms. The general apparatus for axiomatizing all coalgebras for a given functor produces by default rank-1 formulas [175, 264]. Appendix A.3 illustrates coalgebraic rank-1 axiomatizations of the structures and predicate liftings of Appendices A.1 and A.2.

There is a way of broadening the paradigm to *shallow* formulas (also known as *non-iterative* or *rank 0-1*, i.e., those where every atom is within the scope of *at most* one modality) [261]. There is also a coalgebraic perspective on logics determined by classes of coalgebras, i.e., on the issue of modal definability (cf. the discussion of the Goldblatt-Thomason Theorem in Remark 3.2). There are even extended conference abstracts discussing coalgebraic completeness beyond rank-1 [223], but little systematic study, partly because of the difficulty of proving general results of this kind.

Coalgebraic success story It is not the goal of this section to provide a broad overview of the coalgebraic literature, or to advertise the advantages of the coalgebraic framework. But even in such a short sketch, especially given the remarks in the preceding paragraph, we should note that—at least inasmuch as base logics of generalized transition systems are concerned—coalgebraic logic has been remarkably successful in establishing generic results, including for example (mostly) PSPACE upper bounds for complexity [259, 264] or construction of cut-free sequent systems [224], Hennessy-Milner results on expressivity [260, 264], Goldblatt-Thomason results on definability [174, 175], uniform interpolation [268], coalgebraic generalizations of Janin-Walukiewicz expressivity [87] and Kozen-Walukiewicz completeness results for modal μ-calculus [86] etc. etc. The reader is referred to programmatic and overview papers [61, 172, 264, 175] for more details, references and examples.

3.2. Coalgebraic Predicate Logic

This beginning list of results in modal model theory probably gives the misleading impression that one can now keep on churning out new model-theoretic results one after the other, and, at worst, one can always …reduce things to first order model theory. This is not so. The sample questions below are not answered. All of them have answers in first order logic … The point is to find the largest classes of models …which are "natural" and in which some answers may be provided.

Chen Chung Chang [57, pp. 616–617]

So far, we have been looking at purely modal coalgebraic languages. We have not seen yet anything resembling a coalgebraic generalization of FOL. Prior to my involvement, a proposal for such a language had been made by Pattinson and Schröder [262]. The language was tailored to obtained an early variant of the van Benthem/Rosen result (see § 4.1 below). Its syntax is rather unwieldy.

Remark 3.3. This first language proposed by Pattinson and Schröder involved

- three sorts s, t, n ranging respectively over W (states), a subset of TW (admissible successors) and a subset of $\mathcal{P}W$ (admissible neighbourhoods);
- the relation symbol $\mathsf{tr} : s \times t$ for the coalgebraic transition structure;
- the relation symbol $\mathsf{supp} : t \times n$ interpreted by a *support* relation picking for every $t \in TW$ a set $A \subseteq W$ s.t. $t \in TA$;
- the relation symbol for each modality between t and suitably many copies of n representing predicate liftings *relative to the support*;
- and the relation symbol $\in: s \times n$ for membership in admissible neighbourhoods.

Moreover, the semantics in question is a complicated Henkin-style one, requiring not only the choice of admissible successors and admissible neighbourhoods, but also of the support relation. Investigating which part of this language is genuinely necessary to define a coalgebraic analogue of the Standard Translation, I realized that a suitable, much simpler sublanguage can be traced back to a syntax proposed by C.C. Chang [57] in the neighbourhood context as a simplification of Montague's account of *pragmatics* [203]; later on, it reappeared as a weak logic of topological spaces [269, 306, 189, 53] (which are obviously a special case of neighbourhood semantics). This language, which we baptised the *Coalgebraic Predicate Logic* (CPL), adds a single construct to the usual first-order syntax: for each n-ary $\heartsuit \in \Lambda$, each term t, formulas $\varphi_1, \ldots, \varphi_n$ and individual variables x_1, …, x_n, write

$$t\heartsuit\lceil x_1 : \varphi_1\rceil \ldots \lceil x_n : \varphi_n\rceil$$

to denote that given a coalgebra $\gamma : W \to TW$, a valuation v of indvidual variables and formulas $\varphi_1(x_1)$, ..., $\varphi_n(x_n)$ denoting subsets of W (with x_i's being used as *comprehension variables*, hence bound in $\lceil x_i : \varphi_i \rceil$), the denotation of t in W belongs to

$$(\mathcal{Q}\gamma \cdot [\![\heartsuit]\!]_W)[\![\varphi_1]\!]_{W,v/\{x_1\}} \cdots [\![\varphi_n]\!]_{W,v/\{x_n\}},$$

where $[\![\varphi_i]\!]_{W,v/\{x_i\}}$ denotes the set of those $w \in W$ s.t. $v[w/x_i] \Vdash \varphi$.[2] As discussed in Remark 3.2, predicate symbols can either be incorporated into the modal signature and corresponding predicate liftings or treated in the standard FOL way.

Chang proposed his formalism as a tool for reasoning about social situations. Our suggested examples are merging these ideas with computer science applications: we propose, on the one hand, a series of examples utilizing Facebook, Twitter and social networks [S3, S2, I11], and on the other hand—on *delay-* or *disruption-tolerant* networking (DTN) [I7]: routing and forwarding protocols backed by social insights as suggested by Hui et al. [142].

Example 3.4. Let us use the selection-function frames and conditional logic to illustrate the "social-networking" vs. "packet-networking" readings. As an example of the first one [I11, S3, S2], given a binary relation ff ("facebook friend") the formula

$$\exists y_1, y_2. (y_1 \neq y_2 \wedge x > \lceil z : \mathsf{ff}(x,z) \rceil \lceil z : z = y_1 \rceil \wedge x > \lceil z : \mathsf{ff}(x,z) \rceil \lceil z : z = y_2 \rceil)$$

states that there are at least two persons y_1, y_2 who are made happy by x if x invites *precisely* his/her facebook friends to his/her birthday party. As an example of the second one [I7],

$$\forall x, u.(\varphi(u) \rightarrow x > \lceil y : \varphi(y) \rceil \lceil z : z = u \rceil)$$

says that if x is currently active in a subcommunity delineated by the formula $\varphi(y)$ and u belongs to that subcommunity, then u is normally a possible target for packets forwarded by x. For more, see our papers [I11, S3, S2, I7].

Our paper [I7] and its earlier incarnations [S3, S2] stay within the paradigm of *SETAC*, i.e., continue Chang's original program of investigating *classical* model theory for this language. We have discovered that results developed in this paradigm are limited to structures where all modalities are either "neighbourhood-like" (*strongly one-step complete* or *S1SC*; cf. [I7, Def. 3.4] and Remark 5.2 below for an explanation of this property) or "Kripke-like" (*finitary S1SC* and *bounded*) in each coördinate [I7, Def. 3.15].

Example 3.5. As these alternative names indicate, neighbourhood frames are S1SC and Kripke frames are finitary S1SC and bounded. As clarified by Example 3.8 in our paper [I7], the last class covers also graded modal logic and Pressburger logics. Conditional logic provides an interesting mixed case, with the first coördinate being "neighbourhood-like" and the other "Kripke-like" [I7, Def. 3.15 & Ex. 3.16].

[2] Note that the notation used in our papers [I7, I11, S3, S2] is slightly different.

Definition and Remark 3.6. For a given operator $\heartsuit$, being *bounded* by k means that for any W and any $A \subseteq W$, $\heartsuit A$ is obtained as the sum of $\heartsuit A'$ for all $A' \subseteq A$ of cardinality at most k. This is indeed a paradigm property of Kripke or graded diamonds. Being k-bounded on i-th coördinate is expressible in CPL as follows:

$$x\heartsuit \ldots \lceil y_i : \varphi_i \rceil \ldots \leftrightarrow \exists z_1 \ldots z_k.(\bigwedge_{j\leq k} \varphi_i[z_j/y_i] \wedge x\heartsuit \ldots \lceil y_i : \bigvee_{j\leq k} y_i = z_j \rceil).$$

For Kripkean diamond, i.e., unary and 1-bounded $\Diamond$, this scheme instantiates to the following statement of the fact that their dual algebras are $\mathcal{AV}$-BAEs, i.e., combine *atomicity* with *complete additivity* (cf. § 5.2 below):

$$x\Diamond\lceil y : \varphi \rceil \leftrightarrow \exists z.(\varphi[z/y] \wedge x\Diamond\lceil y : y = z \rceil).$$

With $x\Diamond\lceil y : y = z \rceil$ rewritten as $xR_\Diamond z$, this illustrates how CPL over relational structures reduces to first-order logic [I7, § 2].

Here is a brief overview of results for neigbourhood-like and Kripke-like modalities we obtained by following the classical model-theoretic paradigm of Chang:

- the Strong Completeness Theorem [I7, Th. 3.19] for a suitable Hilbert-style axiomatization [I7, Tab. 1];
- (a suitable variant of) the Omitting Types Theorem [I7, Th. 3.30];
- the existence of ultraproducts [I7, Th. 5.6 & 5.8];
- (a suitable variant of) the Downward Löwenheim-Skolem Theorem [I7, Th. 5.10 & 5.13]. In this case, the Kripke-like restrictions can be somewhat broadened, allowing examples such as non-standard or zero-dimensional probability subdistributions [I7, Ex. 3.35 & 3.36], which are ω-bounded but fail to be k-bounded for any finite k.

Finally, we propose a syntactic cut elimination for a Gentzen system corresponding to the neighbourhood-like case [I7, § 6]. The effective nature of this result illustrates that there is a potential for lifting *SETAC* in the coalgebraic setting. This is the direction in which we turn our attention now.

4. Beyond Cantor: Effectivity, Computability and Determinacy

We always "know" that the axiom of choice is true. In addition we had thought that there is no interesting general combinatorial set theory without AC *(though equivalence of version of choice, inner model theory and some other exist). Concerning the second point, since* [270] *our opinion changed ...*

Saharon Shelah [271]

[T]he full power of the Axiom of Choice is almost never used in formal economics or in classical analysis for that matter. What is used is a much weaker axiom, the Axiom of Dependent Choice.

William Zame [304, p. 195]

In this section, we are attacking the *SETAC* pillar of the Classical Picture. There is a genuine danger in over-reliance on nonconstructive, infinitary set-theoretic and model-theoretic methods, tools and axioms. To speak only somewhat metaphorically: just like with every other addiction, the short-term shock of withdrawal can be more than offset by long-term benefits. Quick nonconstructive fixes can provide a fleeting delusion of power. After going clean, however, one can not only see things clearly, but also achieve more.

This point is illustrated using two different threads developing on the themes of § 3.2. In § 4.1 based on our paper [I11], I discuss the right choice of model-theoretic toolbox and metatheory for CPL. It appears that the apparatus of finite model theory is more amenable to coalgebraic generalizations than its unrestricted counterpart. § 4.2, based on my paper [I4], builds on the theme of "logic for social reasoning". This is another application domain where acceptance of nonconstructive principles has sometimes tempted mathematicians and logicians into proving rather confusing results. My work proposes the Hildenbrand(–Aumann) criterion ruling out such abuses of ZFC. The *quasiconstructive* [257, Ch. 14] or *agnostic* [112] methodology allowed by this criterion turns out to be relevant for modal logic; some examples are given in § 4.3.

4.1. Finite vs. Classical Model Theory: The Case of CPL

> *Moshe Vardi has suggested to me a classification of finite-model theory into three lines of research. ... The first line of research he calls* negative*; here we consider theorems of model theory that fail for finite-model theory. ... The second line of research could be called* preservative*; here we consider theorems of model theory that continue to hold for finite-model theory. ... The final line of research Vardi calls* positive*; here we consider results that are unique to finite-model theory.*
>
> Ronald Fagin [93, pp. 26–27]

As we have seen in § 3.2, adapting classical model theory to the full generality of CPL has its limits. The strategy works for those structures which are either "Kripke-like" or "neighbourhood-like". While such structures form the opposite extremes of the coalgebraic spectrum, we have seen that most familiar features of their model theory, in particular compactness, are far from being representative.

Remark 4.1. § 5.1 and Remark 5.3 below clarify the "extreme" character of, respectively, the neighbourhood semantics and the Kripke semantics.

On the other hand, we would like some model-theoretic results to hold for as broad class of signatures as possible. In particular, we noted how van Benthem's characterization of the relationship between first-order and modal logic survives under various generalizations and in different settings. It seemed reasonable to expect a suitably general result relating CPL and coalgebraic modal logic under minimal assumptions on the functor and the collection of predicate liftings.

Remark 4.2. In order to find such a result, we first need to know which of possible coalgebraic notions of bisimulation should be used in its statement [250, 280, 119]. As suggested by these references, the following notion proves most versatile: given T-coalgebras (C, γ) and (D, δ), two states $(c, d) \in C \times D$ are called *behaviourally equivalent* if they can be identified by morphisms of T-coalgebras, i.e., there are morphisms $f : (C, \gamma) \to (E, \epsilon)$ and $g : (D, \delta) \to (E, \epsilon)$ into a T-coalgebra (E, ϵ) such that $f(c) = g(d)$. A property P of states is *invariant* under behavioural equivalence, in short *behavioural-equivalence invariant*, if whenever a state x has property P and y is behaviourally equivalent to x then y has property P.

As stated in § 2, a set of tools similar to (but still much broader than) the one which allows Rosen [247] to prove his finite-model variant also allows Dawar and Otto [73, 74] to produce variants of the van Benthem theorem for various other (non-elementary, well-founded, transitive, rooted...) classes of frames.

Moreover, we have mentioned in § 3.2 that an early version of a coalgebraic correspondence language proposed by Schröder and Pattinson [262] (cf. Remark 3.3) has been specifically tailored for such a result, allowing techniques used by Dawar and Otto [73, 218, 74] at the cost of being three-sorted, with Henkin-type restrictions on its semantics and an additional level of parametricity induced by the notion of a support. The question is thus if there is a CPL-native version of this result and its proof. This is precisely what our paper [I11] achieves.

Remark 4.3. It appears one cannot entirely do away with supports, but in the CPL case the user does not have to bother defining/choosing the support and the models do not have to come explicitly equipped with it as a part of the signature. Instead, the proofs are using "under the hood" the notion of a *supporting Kripke frame* for a given coalgebra and an extension of CPL to "support-CPL" as an intermediate language, invisible from the user's point of view and not involved in a suitable version of the standard translation [I11, Def. 4 & § 4].

Our techniques are based on generalizations of standard notions of finite model theory: Gaifman distance, Gaifman graphs and Gaifman locality [I11, § 4]. In fact, we [I11, Th. 27] provide a variant of the Gaifman theorem for the "support-CPL" mentioned in Remark 4.3.

Unlike compactness-oriented restrictions necessary in § 3.2,[3] our restriction to *separating* signatures is most natural in the coalgebraic context.

Definition 4.4. We say that Λ is *separating* if for all sets X, every $t \in TX$ is uniquely determined by the set $\{\heartsuit A \in \Lambda(\mathcal{P}(X)) \mid t \Vdash \heartsuit A\}$.

A more nontrivial restriction is that our variant of van Benthem/Rosen provides a characterization for finitary modal coalgebraic formulas only when the signature itself is *finitary* [I11, Cor. 36]; otherwise, one needs infinitary formulas of finitary rank in its statement [I11, Cor. 35 & Rem. 37]. A version of our result avoiding

[3] A variant of the Van Benthem theorem for neighbourhood models in Hansen et al. [125] also relies on machinery from classical model theory, in particular compactness and saturation [I11, Rem. 39].

this restriction would in particular cover graded modal logic over multigraphs and probabilistic modal logic. Note that a van Benthem-type result for graded modal logic over Kripke frames is provided by de Rijke [246].

Enqvist et al. [87] prove a coalgebraic variant of the Janin-Walukiewicz theorem, which is a second-order counterpart of van Benthem-type results. Yamamoto [302] pursues further the approach based on classical model theory by addressing our challenge [S3, § 6], [I7, § 7] to investigate the status of Fine's theorem. On the finite model theory front, a major challenge is to examine the status of the other preservation theorem surviving in the finite, i.e., that of Rossman [248].

4.2. Effectivity, Choice and Determinacy in Formal Economics

> *[T]he measurability assumption … is of technical significance only and constitutes no real economic restriction. Nonmeasurable sets are extremely "pathological"; it is unlikely that they would occur in the context of an economic model.*
>
> Robert J. Aumann [10, p. 44]

> *The Axiom of Choice and its negation cannot coexist in one proof, but they can certainly coexist in one mind. It may be convenient to accept* AC *on some days—e.g., for compactness arguments—and to accept some alternative reality, such as* ZF + DC + BP *on other days—e.g., for thinking about complete metric spaces.*
>
> Eric Schechter [257, § 14.74, p. 402]

As we have seen in § 3.2, even in its computer science applications, CPL lends itself particularly well to examples based on social intuitions; in fact, we can trace back its origins to such inspirations. It is thus perhaps to be expected that the main theme of this section—subverting *SETAC* and highlighting the problematic rôle played by nonconstructive assumptions in metatheory—deserves attention also in the context of social welfare analysis, social choice and preference aggregation. This is the subject matter of my paper [I4] presented in this subsection.

It might be surprising that such principles enter into discussion at all; a natural expectation would be that social choice and related areas deal with finite populations. Yet, in fact, there are numerous references and several lines of research where this assumption of finiteness have been often lifted. The paper [I4] begins with a critical analysis of motivations for considering inifinite populations. Subsequently, inspired by the following quote from Hildenbrand [133]:

> Instead of considering a sequence of economies and looking for an asymptotic identity one may reason "in the limit," i.e., one considers economic systems with more than finitely many participants and proves that the identity holds in this case. … But, as an economist, our interest in these "ideal economies" is proportional to *how much new information can be derived for large but finite economies*. In other words, *the relevance of the ideal case to the finite case has to be established* [133, p. 162, emphasis mine–T.L.].

it provides the following *Hildenbrand criterion* (or the *Hildenbrand–Aumann criterion*) distinguishing fruitful and problematic uses of infinite populations:

> Even if a theoretical economist decides to consider infinite populations of agents as "limit" or "ideal" generalizations of very large finite ones, the results ob-

> tained via such an idealization process should remain *effective*. *Moving to the limit* should mean precisely this and nothing more: making the rôle of any particular individual, or perhaps even any particular generation, infinitesimally negligible. It cannot be an excuse to introduce by a sleight of hand—or, if one prefers, a magical or metaphysical trick—a pseudo-solution, which by nature cannot correspond to any meaningful algorithm or definable strategy. [I4, § 2]

The literature on infinitary "solutions" to Arrow's impossibility theorem in voting theory [102, 164, 201, 181, 60, 194, 195, 45, 132, 130, 131, 11] and, to a certain extent, also on intergenerational equity [282, 304, 179, 180] provides numerous examples that better understanding of this criterion is urgently needed in the field. From a TCS perspective, it would seem natural to move to some constructive/type-theoretic setting (see § 6), but this step is not likely to be understood, much less accepted by most experts in the area, whose training tends to be ZFC-based. On the other hand, this training is not always advanced enough to appreciate independence proofs, although some references quoted above are indeed using or at least mentioning forcing and related techniques. Hence, my paper [I4], apart from isolating the Hildenbrand(–Aumann) criterion itself, proposed its specific interpretation—or rather, a simple set-theoretic solution for establishing the failure of comformity of a given result with that criterion. More specifically, the paper suggests revisiting the *Axiom of Determinacy* (AD) for that purpose.

Remark 4.5. Consider any set X and $A \subseteq X^\omega$, i.e., any given subset of the space of infinite sequences of elements of A; if X is taken to be ω, X^ω can be identified with $\mathbb{R}$ via a standard argument. With every such X and A, we can associate naturally an infinite two-person game $G_X(A)$ with perfect information, where two players $\forall$ and $\exists$ take turns to choose elements of X. The $\exists$-player, who makes the first move, wins iff the sequence created this way belongs to A. In other words, A is what is usually called the *payoff* for $G_X(A)$. If either of the two players has a winning strategy, $G_X(A)$ is *determined*. AD says simply that *all games of the form $G_\omega(A)$, i.e., whose payoffs are (identifiable with) subsets of $\mathbb{R}$, are determined*. Early references used the name *Axiom of Determinateness* instead. It was proposed by Mycielski and Steinhaus in 1962 [210] and has been studied ever since [211, 208, 209, 151, 163, 159].

AD itself is, of course, not more constructive than AC is: the rôle it should play is purely negative. The point is that if a given result fails in every model ZF satisfying AD this entails its non-derivability from safe, economically meaningful weakenings of AC such as DC.

Remark 4.6. This is assuming, of course, the consistency ZF+ AD, but these days it is rather commonly accepted among set-theorists thanks to seminal results of Woodin. See Kanamori [159, Th. 32.14–16] or my paper [I4, Th. 5.1] for a summary.

Kechris [163], for example, shows that if the model of ZF based on $L(\mathbb{R})$ is assumed to satisfy AD, then DC holds therein. The primary usefulness of AD for a formal economists is that it provides a *game-theoretic* route for disproving conformity with the Hildenbrand(–Aumann) criterion. Unlike techniques based on forcing or analysis of fine-structure of non-standard models of ZF, it can be directly and intuitively understood by researchers in the area.

In other words, my paper [I4] argues that the appropriate metatheory for social choice should be at most what Schechter [257, § 14.76, p. 404] calls "*quasiconstructive* mathematics", and Garnir [112]—"*agnostic* mathematics". Game-oriented techniques showing inconsistency with AD are simply a promising way to exclude results and principles which cannot be accommodated in a quasiconstructive/agnostic setting. Some examples provided therein are:

- Theorem 6.2 [I4] illustrates how adapting the standard "strategy-stealing" argument (cf. [159, Prop. 28.1]) extends the famous Arrow's impossibility theorem to an arbitrary countable collection of voters.

Remark 4.7. The rôle of the assumption of countability is an interesting separate question that we do not address here.

- Theorem 7.1 [I4] shows that AD can be used for a similar purpose in the discussion of intergenerational equity [282, 304, 179, 180].

Remark 4.8. I was not aware of Garnir [112] and Schechter [257] at the time, hence their terminology is not used in my paper [I4]. I found the second (and subsequently the first) of these references via a recent draft of N. Bezhanishvili and W. H. Holliday on choice-free Stone duality, a subject highly relevant to our concerns. Another recommended reference, illustrating vividly the discussions surrounding the use of AC, is the monograph of Horst Herrlich [129].

A broader research program connecting metamathematics of social choice and of computer science appears long overdue. For that matter, it is hard to escape the impression that methods and tools of finite model theory or database theory have not been fully utilized in formal economy either.[4]

4.3. Quasiconstructivity and Modal Logic

For the sake of justice, it should be noted that the first results on Kripke completeness were obtained without using such nonconstructive reasoning. The technique used in the works of S. Kripke himself clearly aimed towards finiteness of the means employed, being in general close to Gentzen methods. However, quite quickly there came a transition to the technique of canonical models, which became the most common one.

Alexander Chagrov [54, p. 228, translated from original Russian]

Chagrov's paper [54] addresses the issue of quasiconstructivity/agnosticism in modal logic without mentioning these terms or the above-quoted references [112, 257]. It postulates the development of *reverse mathematics of modal logic*, focusing in particular on the question *which of the results of modal logic can be proved*

[4]In a related context, promising work in progress of two young researchers seems worth mentioning. The first is the study of assumptions of constructivity and the rôle of AC in foundations of probability by Kenny Easwaran (Texas A&M University). The second is recent research on models satisfying AD in social welfare by Giorgio Laguzzi (ALU Freiburg).

by finitary means, and also in which cases what variants of infinitary means are used [54, p. 234]. Our research reported in § 4.1 fits under this umbrella. Let us have a quick look at more case studies, involving several papers I (co)authored.

- One of Chagrov's examples involves the *Blok Dichotomy* [39] (cf. §§ 5.2 and 5.3 below). Chagrov points out that the proof of this result [55, Ch. 10] avoiding Blok's original non-quasiconstructive techniques (e.g., Jónsson's Lemma) allows Chagrova's generalization to (duals of) neigbourhood frames [56]. In fact, the question whether Blok's original proof can be adjusted to the neighbourhood setting had been asked already in 1978 by Dziobiak [83] and answered in the negative by me three decades later [O8, § 6.1] [O4]. By contrast, our papers [O8, O4, I1] (see §§ 5.2–5.3 below) show that Chagrov & Zakharyaschev's strategy establishes the Blok Dichotomy for very large classes of algebras, such as ω-complete or completely additive ones.
- Another example, perhaps closest in focus to the above epigraph, is provided by Fine-style modal completeness proofs via *normal forms* [98, 113, 204, 29, 28, 67] as contrasted with standard proofs via the non-quasiconstructive method of canonical models. This includes our papers [O3, O1] (cf. § 2) on modal aspects of fragments of XPath, a language for navigating and querying semi-structured databases. The use of the normal-form technique allows us, e.g., to lift completeness results for *node* expressions to completeness for *path* expressions (in some cases, modulo a non-orthodox rule).

Remark 4.9. Moss [204] illustrates educational advantages of this technique and recalls that it is used in the Kozen & Parikh [167] completeness proof for the logic of programs. Several references address dualities inherent in normal-form approaches [113, 29, 67, 28], and Bezhanishvili and Ghilardi [28] also discuss proof-theoretic corollaries such as the *bounded proof property*.

Remark 4.10. There are more areas where the issue of (quasi)constructive proof methods arises in modal logic. Apart from further examples provided by Chagrov [54], which are of a rather technical nature (logics without immediate predecessors, axiomatizations of tabular logics), let us point out surprising set-theoretic aspects of modal correspondence theory. E.g., a fairly recent paper [140], written without any mention of modal logic, partially addresses the question of the frame counterpart of the transitive McKinsey axiom ($\mathsf{K4.1}$) in the absence of AC.[a] In the area of topological semantics of extensions of $\mathsf{S4}$, classification of logics of various classes of spaces (e.g., *extremally disconnected* ones [25, 26]) requires strong set-theoretic assumptions, at times going beyond ZFC; to the best of my knowledge, there is no research how such results would fare under AD.

[a] I would like to thank Ilya Shapirovsky for bringing this paper to my attention. More specifically, it investigates the deductive strength of principles akin to CS: "Every partially ordered set without a maximal element has two disjoint cofinal subsets".

5. Beyond Scott, Montague and Moss: Algebras and Possibilities

> *Such uses of modal algebra are a joy to some; to others they show that the algebraic approach is merely 'syntax in disguise'. … Nevertheless, the algebraic perspective has further uses, which are being discovered only gradually. First, notice that it offers a more general framework than Kripke semantics.*
>
> Johan van Benthem [20, p. 358]

Since § 2, I have been arguing that modalities are central to the enterprise of subverting the Classical Picture, beginning with *SYFOL*. Furthermore in § 3, we have seen coalgebras as modal semantics departing from *RELST*. Now it is time to see how algebra offers a still broader umbrella for our subversive efforts. Let us first revisit (§ 5.1) coalgebraic semantics to understand better how it relates to the algebraic one and recall briefly (§ 5.2) my earlier work on algebraically motivated hierarchy of incompleteness notions. Then in § 5.3 and § 5.4, I am going to present my two recent joint papers [I1, I2] building on this earlier work.

5.1. Dual Algebras, Neighbourhood Frames and $\mathcal{CA}$-BAEs

Recall from § 3.1 that a n-ary modal connective $\heartsuit \in \Lambda$ is interpreted by its corresponding *predicate lifting*: a set-indexed family of mappings

$$[\![\heartsuit]\!]_W : (\mathcal{Q}W)^n \to \mathcal{Q}TW$$

(subject to naturality) and that for any coalgebra $\gamma : W \to TW$ and formulas φ_1, ..., φ_n, we set $[\![\heartsuit\varphi_1 \ldots \varphi_n]\!]_W = \mathcal{Q}\gamma \cdot [\![\heartsuit]\!]_W([\![\varphi_1]\!]_W, \ldots, [\![\varphi_n]\!]_W)$. In other words, for any $\gamma : W \to TW$ and any $\heartsuit \in \Lambda$, we get that $\mathcal{Q}\gamma \cdot [\![\heartsuit]\!]_W$ is an n-ary operation on the powerset algebra of W, which we can denote as $\heartsuit_\gamma$. Thus, for any Set-functor T and any supply of modal operators Λ, every Λ-structure over T, i.e., any pair of the form $(T, ([\![\heartsuit]\!])_{\heartsuit\in\Lambda})$ yields a recipe for assigning to every T-coalgebra its *dual algebra* (relative to the structure) in the signature of *Boolean Algebra Expansions* (BAEs) determined by Λ. As this algebra has a powerset algebra as its carrier set, it exhibits the following defining properties of algebras obtained thus way:

($\mathcal{C}$) *lattice-completeness*: given any set X of elements of the algebra, its *join* $\bigvee X$ exists as well. This also implies the existence of arbitrary meets.

($\mathcal{A}$) *atomicity*: any non-bottom element is above an *atom*, i.e., a minimal non-bottom element.

Remark 5.1. In the case of dual algebras of coalgebras, where every subset is admissible, $\bigvee$ is simply $\bigcup$ and the atoms are obviously singleton sets $\{x\}$. Importantly, in proper subalgebras of such set algebras, lattice completeness does *not* imply that infinite suprema coincide with set-theoretic sums.

There is a way of bringing one-step axioms into this picture as well, raising slightly the level of abstraction, following the approach exemplified by Kurz and Rosický [175]. The discussion can be found in existing references [264, Rem. 55]

[I7, Rem. 3.10], but let us quote it here to make this introduction more self-contained. Every signature Λ together with a given set of one-step axiom schemes (equivalently, one-step rules) can be encoded disregarding concrete syntax by its *functorial presentation* [171] (cf. also [264, Def. 28]) as an endofunctor L_Λ on the category BA of Boolean algebras. BA is dually adjoint to Set, with the adjunction given by the contravariant powerset functor[5] $\overline{\mathcal{Q}}$ and the functor $\mathcal{S}$ taking a Boolean algebra to the set of its ultrafilters:

$$L_\Lambda \circlearrowright \mathsf{BA} \underset{\mathcal{S}}{\overset{\overline{\mathcal{Q}}}{\leftrightarrows}} \mathsf{Set} \circlearrowleft T \tag{5.1}$$

The information contained in each Λ-structure can be more abstractly encoded by $\delta : L_\Lambda\overline{\mathcal{Q}} \to \overline{\mathcal{Q}}T$ [171] and the *canonical* structure for Λ is given by $M_\Lambda = \mathcal{S}L_\Lambda\overline{\mathcal{Q}}$. Let us use the term Λ-*neighbourhood frames* for structures which assign n-ary neighbourhoods, i.e. subsets of the n-th power of the state set, to every n-ary modality in Λ. Coalgebras for the canonical structure can be thus described as Λ-neighbourhood frames subject to satisfaction of the frame conditions embodied in the given one-step rules [264, Remark 34].

Remark 5.2. For every Λ-structure, we can define a canonical *structure morphism* [264, p. 1121] to M_Λ by composing the counit of the above adjunction with $\mathcal{S}\delta$, and S1SC effectively requires that this structure morphism is surjective. In other words, a Λ-structure is S1SC iff its functor surjects onto the canonical neighbourhood semantics; it is for this reason that we refer to the S1SC case as "neighbourhood-like".

Thus, the predicate lifting approach not only provides a generic method for constructing dual algebras, but also highlights that neighbourhood (Scott-Montague [202, 267, 58]) semantics of modal logic is most general among state- or set-based ones. Is there nothing in between this semantics and the full generality of arbitrary algebras? In fact, the picture turns out to be more complicated and interesting even in the restricted case of normal modal logics.

5.2. A Hierachy of (In)completeness Notions for Normal Modal Logics

The approach described in the preceding subsection yields, in particular, a somewhat more abstract way of describing the usual construction of the dual algebra of a Kripke frame via the predicate lifting given in § A.2. As observed already by Jónsson and Tarski [154], such a dual algebra, in addition to being $\mathcal{CA}$, has the following property:

($\mathcal{V}$) *complete additivity*: for any set X of elements, if $\bigvee X$ exists, then $\bigvee\{\Diamond x \mid x \in X\}$ exists and

$$\Diamond \bigvee X = \bigvee\{\Diamond x \mid x \in X\}. \tag{5.2}$$

For $\mathcal{C}$-BAEs, $\mathcal{V}$ is just the validity of (5.2). Of course, $\mathcal{V}$ can be equivalently stated with $\bigwedge$ replacing $\bigvee$ and $\Box$ replacing $\Diamond$.

[5] The use of notation $\overline{\mathcal{Q}}$ highlights that we change the target category from Set^{op} to BA.

Actually, the combination of the three properties above is a defining feature of duals of Kripke frames. One can say even more: the category of Kripke frames with bounded morphisms is dually equivalent to that of $\mathcal{CAV}$-BAOs with *complete* morphisms [286]. In particular, taking any Kripke frame/$\mathcal{CAV}$-BAO, converting it into its dual $\mathcal{CAV}$-BAO/Kripke frame, and then going back produces an output isomorphic to the original input.

Remark 5.3. An exhaustive discussion of complete additivity (in its dual form for $\Box$, called "continuity" therein) from the predicate lifting perspective in coalgebraic logic is contained between Theorem 28 and Example 37 of Schröder [258]. Recall that the notion of "boundedness" we needed in § 3.2 above is a generalization of $\mathcal{V}$.

While complete additivity of dual $\mathcal{CA}$-BAEs of coalgebras can be expressed in the language of CPL (cf. Def. & Rem 3.6), it is inexpressible over arbitrary BAEs/BAOs in the language of (finitary) propositional modal logic (although see Example 5.8). It can only capture its finitary trace: preservation of finite joins by $\Diamond$ or, equivalently, preservation of finite meets by $\Box$.[6] As we see in Appendix A.3, this property is expressible by one-step rules, or—equivalently—by one-step axioms modulo the rule of congruence. Modalities satisfying this property are called *normal* and the algebraic operations they induce are called *operators*; hence *Boolean Algebras with Operators* (BAOs) in the title of the original reference on the subject [154] and in the subsequent literature.[7]

It follows from § 5.1 that we have not only a functorial presentation of normality via an endofuctor on boolean algebras, but also a corresponding canonical neighbourhood structure. These are *normal* (or *filter*) *neighbourhood frames*: those where the family of neighbourhoods of any point/world/state forms a non-empty filter of the powerset algebra. The duality between $\mathcal{CA}$-BAOs and such structures is discussed in detail by Došen [79], who provides neighbourhood analogues of results of Thomason [286] mentioned above.

Additional frame conditions and additional axioms arising from principles of computation, provability, knowledge, belief, obligation etc. are bread-and-butter of our field. This makes the issue of *completeness*—that is, the question whether derivable formulas coincide with those valid in the intended semantics—challenging and important. While the modal community has identified large classes of complete formulas [252, 100], several limitative results [39, 287] show that in a deep sense, the incompleteness phenomenon cannot be overcome. In particular, Thomason [287] shows that the Kripke consequence relation not only cannot be recursively axiomatized, but lies beyond the entire arithmetical hierarchy. Coalgebraic rank-1 axioms with their generic completeness properties are dramatically non-representative of the complexity of the full modal realm.

From an algebraic point of view, normal modal logics correspond to *varieties* (equationally definable classes) of BAOs, and Kripke incompleteness is the phe-

[6] Recall that the join and meet of $\emptyset$ are, respectively, $\bot$ and $\top$.

[7] Strictly speaking, *operators* of Jónsson and Tarski [154] were modal *diamonds*, i.e., they distributed over finite joins. In the original terminology, boxes are *dual operators*. In the boolean setting, it is common to be cavalier about this distinction; cf. § 6 below.

nomenon that not every variety of BAOs can be generated as the smallest variety containing some class of $\mathcal{CAV}$-BAOs. A series of natural questions arises: what happens if we drop or weaken one or more of the properties $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{V}$? Do we thereby obtain distinct notions of completeness for normal modal logics? Can we represent the resulting BAOs with some kind of frames with richer structure than Kripke frames? Does incompleteness persist even if we retain only one of the properties $\mathcal{C}$, $\mathcal{A}$, or $\mathcal{V}$?

In my PhD [O8] and in a series of publications prior to my employment at FAU [O10, O9, O4], I have launched a systematic investigation into these questions, building on earlier results by Thomason, Fine, Gerson, van Benthem, Chagrova, Chagrov, Wolter, Zakharyaschev, Venema and other researchers. My main discoveries, as summarized in our paper [I1, § 3], have been as follows:

> - almost any conceivable combination of the above properties of BAOs and several closely related ones (such as $\mathcal{T}$, i.e., *admissibility of residuals/conjugates*) leads to a distinct notion of completeness. In other words, for almost any pair of such combinations, there is a logic complete in one sense, but not in the other;
> - the Blok Dichotomy [39] i.e., the result of Wim Blok showing that Kripke incompleteness is in a certain mathematical sense the norm rather than an exception among normal modal logics, generalizes to most of these weaker notions of completeness;
> - many of these notions admit syntactic characterizations in terms of conservativity of certain types of extensions, e.g., $\mathcal{AV}$-completeness in terms of conservativity of minimal nominal extensions or $\mathcal{T}$-completeness in terms of conservativity of minimal tense extensions (…)
>
> The possibility of $\mathcal{V}$-incompleteness, however, was left completely open (…) [O10], [O8, Ch. 9], [O4, § 7]. … Venema [293, § 6.1] singled out a slightly stronger version of the same question: whether there are $\mathcal{V}$-inconsistent logics, i.e., normal modal logics that are not sound over any $\mathcal{V}$-BAO. [I1, pp. 14–15]

Why was this question interesting and nontrivial to attack? And, actually, does complete additivity deserve our interest in its own right?

5.3. Possibility Semantics and Complete Additivity

$\mathcal{V}$-incompleteness is a curious property. While a free algebra on infinitely many generators in any variety of BAOs can never be lattice-complete or atomic, it can be completely additive (see Footnote 19 in our paper [I1] and references therein). As we point out [I1, § 3], one can imagine that as $\mathcal{V}$ is not inconsistent with freeness, there could be a general way of turning any BAO into a completely additive one without changing the set of valid equations. Furthermore prior to our paper [I1], there were other ways in which $\mathcal{V}$ seemed somewhat intangible: unlike its closest relatives $\mathcal{AV}$ and $\mathcal{T}$, it did not seem definable in a language with a usable model theory. Nevertheless, we have discovered that it is, in fact, first-order definable.

Example 5.4. To wit, with the following abbreviations:

$a \in (b]$	stands for	$a \neq \bot$ & $a \leq b$
$\exists a \in (b].\alpha$	stands for	$\exists a.a \in (b]$ & α
$\forall a \in (b].\alpha$	stands for	$\forall a.a \in (b] \Rightarrow \alpha$,

we can define [I1, Cor. 4.5] the class of $\mathcal{V}$-BAOs using the following first-order sentence

$$\forall a, b.\ a \wedge \Diamond b \neq \bot \Rightarrow \exists c \in (b].\forall d \in (c].\ a \wedge \Diamond d \neq \bot.$$

An important incentive for revisiting completeness w.r.t. $\mathcal{V}$-BAOs, as well as the insight that led to overcoming its puzzling aspects came from Holliday's [139] study of *possibility frames*. More specifically, $\mathcal{CV}$-completeness is equivalent to completeness with respect to *full possibility frames* and $\mathcal{V}$-completeness is equivalent to completeness with respect to *principal possibility frames*; these results follow from dualities mirroring those of Thomason [286] mentioned in § 5.2.

Remark 5.5. The reader is referred to Holliday [139] for a detailed presentation of possibility frames: as stated therein, they

> can be seen as extending to modal logic the semantics for classical logic used in weak forcing in set theory, or as semanticizing a negative translation of classical modal logic into intuitionistic modal logic. [139, p. 3]

The latter subject, i.e., that of negative translations of classical modal logics, will occupy a central position in § 6.4 below.

Our investigation showed that:

- Several logics previously known to be incomplete w.r.t. more restricted semantics, including not only [I1, Th. 5.2] but even more strikingly, the bimodal version of provability logic [150, 41] [I1, Th. 5.4] are $\mathcal{V}$-incomplete. In both cases, extending the language with an additional modality yields examples of $\mathcal{V}$-*inconsistent* logics [I1, Th. 5.3 & 5.5].

Definition 5.6. Van Benthem's logic vB [16, 68] is axiomatized by:

$$\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p).$$

The system GLB, i.e., the bimodal version of provability logic [150, 41] (cf. § 7.1 below) capturing the combined logic of provability and ω-provability in PA is given by the following axioms:

(i) $[n]([n]p \to p) \to [n]p$ for $n = 0, 1$;

(ii) $[0]p \to [1]p$;

(iii) $\langle 0\rangle p \to [1]\langle 0\rangle p$.

- The van Benthem example can also be transformed to show that $\mathcal{V}$-inconsistent logics can have no worse complexity than classical propositional logic [I1, § 6.1]. To be more specific: it is possible to construct a coNP-complete bimodal logic based on a general frame used to show incompleteness of vB, whose associated variety does not contain any $\mathcal{V}$-BAOs (cf. Venema's [293, § 6.1] challenge mentioned above).

- Just like other modal completeness properties, $\mathcal{V}$-completeness is undecidable [I1, § 6.2].
- The Blok Dichotomy generalizes to degrees of $\mathcal{V}$-incompleteness [I1, § 7].
- Examples of $\mathcal{V}$-incompleteness discussed above, in particular vB and GLB, can be traced to the failure of admissibility of suitably chosen instances of

$$\frac{p \to \chi(p) \qquad \boxdot_i \chi(p) \to p \qquad \alpha \to \Box_j \chi(p) \qquad p \text{ fresh for } \alpha}{\alpha \to \Box_j \bot}. \qquad (\mathcal{V}\text{-mod})$$

  There is a further generalization of this rule scheme in terms of *necessity forms*, whose admissibility may yet turn out to provide a necessary and sufficient characterization of $\mathcal{V}$-completeness. For details, see ($\mathcal{V}_\ell$-mod) in our paper [I1, § 9.2]. Similar connections between incompleteness and the failure of admissibility of suitable rule schemes are investigated in my earliest papers [O11, O10].

The *syntactic internalization* of $\mathcal{V}$-completeness mentioned in the last point involved an intermediate step [I1, § 9.1] in an extended language we invented (cf. Example 5.8 below). This has led to the work presented in the next subsection.

5.4. GQM: One Modal Logic to Rule Them All?

> *Another suggestion is that the great proliferation of modal logics is an epidemy from which modal logic ought to be cured.*
>
> Robert Bull and Krister Segerberg [46, p. 25]

> *[T]hese systems are not "different modal logics", but different special theories of particular kinds of accessibility relation. We do not speak of "different first-order logics" when we vary the underlying model class. There is no good reason for that here, either.*
>
> Johan van Benthem [21, p. 93]

Our next paper [I2] addresses two related problems: *proliferation of modal logics* mentioned by the first epigraph of this subsection and *conservative handling of propositional quantifiers*. Clearly, when one sees modal "logics" as de facto *theories*, as suggested by the second epigraph above, one thinks of propositional variables occurring in their axioms as being implicitly universally quantified. However, extending modal syntax with propositional quantification leads to several problems. In particular, the most obvious way of interpreting such quantifiers is via infima and suprema in (dual) algebras. As we have learned in § 5.2, however, not all modal logics are complete w.r.t. $\mathcal{C}$-BAEs.

We propose a form of propositional quantification sound over arbitrary BAEs. Namely, we extend the modal language with what we call the *global quantificational modality* $[\forall p]$. In essence, this modality combines the propositional quantifier $\forall p$ with the global modality A: $[\forall p]$ plays the same role as the compound modality $\forall p\mathsf{A}$. Clearly, we can switch between $\forall$ and $\exists$ and between A and E, thus obtaining other global quantificational modalities—$\langle\exists p\rangle$, $[\exists p]$ and $\langle\forall p\rangle$—but in classical logic they are all definable using $[\forall p]$. Note that $\mathsf{A}\varphi$ itself is definable in this new language as $[\forall p]\varphi$ for a fresh p (thus, "vacuous" quantification has nontrivial effects!). Hence, we have:

$$\langle\exists p\rangle\varphi := \neg[\forall p]\neg\varphi \quad (= \exists p\mathsf{E}\varphi),$$
$$[\exists p]\varphi := \langle\exists p\rangle\mathsf{A}\varphi \quad (= \exists p\mathsf{A}\varphi),$$
$$\langle\forall p\rangle\varphi := \neg[\exists p]\neg\varphi \quad (= \forall p\mathsf{E}\varphi).$$

Unlike the propositional quantifier $\forall p$ by itself, $[\forall p]$ can be straightforwardly interpreted in any BAE: whatever the valuation of propositional variables in φ, $[\forall p]\varphi$ can only be either $\top$ or $\bot$.

- Our paper [I2, § 5] presents a Hilbert-style calculus GQM for this language and shows that it is complete with respect to the intended algebraic semantics by exhibiting an equivalence between a suitable global consequence relation of GQM and the first-order theory of discriminator BAEs [I2, § 8]. Our formalism, however, is more succinct.

Definition 5.7. The logic GQM is the smallest set of formulas containing the following axioms and closed under the following rules. Below, '$\vdash_{\mathsf{GQM}} \varphi$' means $\varphi \in \mathsf{GQM}$.

- all classical propositional tautologies;
- distribution: $[\forall p](\varphi \to \psi) \to ([\forall p]\varphi \to [\forall p]\psi)$;
- instantiation: $[\forall p]\varphi \to \varphi^p_\psi$ where ψ is substitutable for p in φ;
- global instantiation: $[\forall p]\varphi \to [\forall r]\varphi^p_\psi$ where ψ is substitutable for p in φ and r is not free in φ^p_ψ;
- quantificational 5 axiom: $\neg[\forall p]\varphi \to [\forall r]\neg[\forall p]\varphi$ where r is not free in $[\forall p]\varphi$;
- $\Box$-congruence: $[\forall p](\varphi \leftrightarrow \psi) \to (\Box\varphi \leftrightarrow \Box\psi)$;
- modus ponens: if $\vdash_{\mathsf{GQM}} \varphi$ and $\vdash_{\mathsf{GQM}} \varphi \to \psi$, then $\vdash \psi$;
- $[\forall p]$-necessitation: if $\vdash_{\mathsf{GQM}} \varphi$, then $\vdash_{\mathsf{GQM}} [\forall p]\varphi$;
- universal generalization: if $\vdash_{\mathsf{GQM}} \alpha \to [\forall p]\varphi$ and q is not free in α, then $\vdash_{\mathsf{GQM}} \alpha \to [\forall q][\forall p]\varphi$.

- On the other hand, GQM formulas valid over $\mathcal{C}$-BAEs cannot be recursively axiomatized. In fact, in the presence of lattice-completeness our formalism becomes as bad as the full second-order propositional modal logic (SOPML) with A [I2, § 3]. This result is proved by generalizing prenex form results for SOPML obtained by ten Cate [52] and is of interest in its own right.
- Furthermore, we discuss [I2, § 4] a novel approach to paradoxes of propositional quantification such as the one discovered by Kaplan [161].
- What have traditionally been called different "modal logics" now become $[\forall p]$-universal theories over the base logic GQM [I2, § 6]: instead of defining

a new logic with an axiom schema such as $\Box\varphi \to \Box\Box\varphi$, one reasons in GQM about what follows from the globally quantified formula $[\forall p](\Box p \to \Box\Box p)$.

Example 5.8. To connect with the theme of § 5.3, in particular Example 5.4, complete additivity is definable by a GQM formula

$$[\forall r][\forall s]\big([\forall p][\exists q]\big((\mathsf{A}(s \to p) \wedge \mathsf{E}\neg p) \to \big(\mathsf{A}(p \to q) \wedge \mathsf{E}\neg q \wedge \mathsf{A}(r \to \Box q)\big)\big) \to \mathsf{A}(r \to \Box s)\big).$$

This was in fact the first non-trivial GQM formula we wrote [I1, § 9], which triggered our interest in this language.

Our present efforts focus on further "internalization" of modal theory in GQM, providing a good Gentzen-style system, and analyzing variants of GQM's based on bisimulation quantifiers rather than standard propositional ones.[8]

Remark 5.9. Given that Bull and Segerberg [46] mentioned the criticism of "the great proliferation of modal logics" (cf. the first epigraph of this subsection) in the context of difficulties of developing modal Gentzen systems, it would be very interesting to investigate how such systems for (fragments of) GQM would perform.

But perhaps the most promising route, directly related to theme of the next part of this overview, is weakening the classical propositional basis. There is no reason to insist that future versions of GQM should incorporate the excluded middle. As long as the underlying modal logic is built on a propositional base with well-behaved implication, such as the intuitionistic one playing the central rôle in the next section, we can transform the question of theoremhood in the boolean extension in the same way as in the preceding paragraph: to reasoning about what follows from the globally quantified formula $[\forall p](p \vee \neg p)$.

6. Beyond Boole and Lewis: Intuitionistic Modal Logic

The use of this term [constructivism] in mathematics is not wholly pinned down, but it may be roughly defined as the practice, project, or policy of mathematizing with one's hands tied. … Sanguine souls there have been and are, however, who envisioned and envision a constructive mathematics adequate to all scientific applications after all. L. E. J. Brouwer was at it two generations ago, but his approach involved unattractive deviations from standard logic. (pp. 33–36)

The deviant logic aggravates matters, complicating and muddying up the crystalline simplicity of classical predicate logic. … We who are unpersuaded of intuitionism accordingly remain unmoved. (pp. 55)

Willard Van Orman Quine [240]

[8] There is also a Coq formalization developed by my student Michael Sammler: `https://gitlab.cs.fau.de/lo22tobe/GQM-Coq`

> *[T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but let us suggest a point of view at least something like his. ... The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts. The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number.*
>
> Clarence Irving Lewis [182, p. 505]

So far, *BOOL* was the only pillar of the Classical Picture that remained unsubverted: we focused on formalisms based on the classical propositional calculus. In the present section, focusing on contributions of three papers I (co)athored ([I10] in § 6.2), [I5] in § 6.3 and [I8] in § 6.4), this last dogma is going to fall as well, and it will continue to crumble in sections to follow. Let us first overview the Curry-Howard correspondence, and the potential scope it leaves for modalities.

6.1. Curry-Howard Correspondence, Constructive Logic and $\Box$

> *The* Brouwer - Heyting - Kolmogorov - Schönfinkel - Curry - Meredith - Kleene - Feys - Gödel - Läuchli - Kreisel - Tait - Lawvere - Howard - de Bruijn - Scott - Martin-Löf - Girard - Reynolds - Stenlund - Constable - Coquand - Huet - ...- isomorphism *might be a more appropriate name, still not including all the contributors.*
>
> Morten Heine Sørensen and Paweł Urzyczyn [277, p. viii]

Given the rôle played in the computer science by the Curry-Howard correspondence, the reader may actually wonder why it took us so long to attack *BOOL*. Let us briefly overview that correspondence (quoting from our paper [I9, § 7.1], discussed further in § 7.1).

> The connection between intuitionistic logics and functional programming is provided by the *Curry-Howard correspondence*, also known as the *Curry-Howard isomorphism* or *proposition-as-types paradigm* (cf. [277]). ... The shortest outline is that
>
> - (intuitionistic) formulas correspond to *types*,
> - logical connectives correspond to type operators/constructors,
> - logical axioms correspond to *inhabited* types and hence deciding theoremhood corresponds to the type inhabitation problem,
> - logical proofs—e.g., in a variant of a natural deduction system or in a Hilbert-style system—are encoded by *proof terms*—in a variant of lambda calculus or of combinatory logic understood as a (functional) programming language and hence
> - proof *normalization* corresponds to reduction of these terms, understood as representing *computation*.
>
> In particular, ordinary intuitionistic implication $\varphi \to \psi$ corresponds to forming the *function space* of programs (proofs) which take data from (proofs for) φ as their input and produce members of (proofs for) ψ as their output. The introduction rule for $\to$ corresponds to λ-abstraction and its elimination rule (i.e., ordinary Modus Ponens) corresponds to function application.

This correspondence, commonly taken as a specification of the Brouwer-Heyting-Kolmogorov interpretation of propositional connectives, is well-known not to validate all axioms of classical logic; in order to see this, note that that a proof of $\varphi \vee \neg\varphi$ under that interpretation, would need to be either a proof of φ or a proof of $\neg\varphi$. Instead, the logic in question is *intuitionistic* (or *Heyting*) propositional calculus (IPC). Details can be easily found in standard references [277].

One can easily guess that there are numerous interpretations of $\Box$ in the IPC setting, depending also on additional axioms one wants to impose. First and foremost, the variety of interpretations of classical modal $\Box$ mentioned in §§ 2 and 3 can be transferred to the intuitionistic setting, often revealing a much subtler picture (see § 6.4 below). For example, the intuitionistic provability interpretation leads to reading of $\Box$ as provability in *Heyting Arithmetic* (HA) rather than Peano Arithmetic (PA), and the resulting logic is still not fully understood; we will have more to say about this in § 7.1.

Moreover, allowing a more liberal propositional base opens up classically unavailable or unnatural interpretations. As a matter of fact, these are the ones which lead to interesting extensions of the Brouwer-Heyting-Kolmogorov interpretation and the Curry-Howard correspondence. Note that such an extension needs to come equipped with a natural computational interpretation not only of the modal connective itself, but also of its associated proof rules as program constructors and proof normalization as program execution. This is not always trivial, given the complexities and idiosyncrasies of modal Gentzen systems. But it has been done fruitfully on a number of occasions.[9]

Remark 6.1. For a striking example based on an intuitionistic counterpart of a standard modal logic, let us just note the work of Davies and Pfenning [72] on the modal logic of *staged computation*, which grew out of, or in parallel with their *judgemental reconstruction* of modal logic [225], the modal logic in question being intuitionistic S4 (with both $\Box$ and $\Diamond$). That work has the potential (to the best of my knowledge, never fully explored) for generalizations towards intuitionistic K and beyond. See remarks at the end of § 2.5 and § 6.4 of that paper [225]. Moreover, it provided a decomposition of another important constructive modality: that of the *lax logic* PLL [94].[a] This system is in turn a paradigm example of a logic whose axioms classically would yield a near-degenerate logic, yet whose constructive version is a perfectly natural formalism, repeatedly rediscovered as the logic of *nuclei*, *Grothendieck topology* in toposes, intuitionistic hardware verification, access control and most importantly, *monads* [4, 24, 70, 94, 110, 111, 116, 117]. In particular, several references in the 1990's [24, 94, 165] present PLL as the Curry-Howard correspondent of Moggi's monad-based *computational metalanguage* [199].

[a] Arguably, this decomposition has been somewhat overlooked by modal logicians. Even Goldblatt [117], an otherwise exhaustive reference on discussions regarding the complicated status of the PLL modality, does not mention it.

PLL is just one example of a computationally relevant constructive modal logic

[9] Furthermore, an important recent line of references investigates computational interpretations of calculi more suited to modal proof theory than the standard, single-context sequent/ND setup. Kavvos [162] extends the Curry-Howard-Lambek correspondence to *dual-context calculi*, whereas Clouston [62] provides an analogous investigation of *Fitch-style calculi*.

which in the presence of excluded middle would collapse to near triviality. We turn our attention now to constructive logic(s) of guarded (co)recursion.

6.2. Guarded Recursion and Its Neighbours

In the last two decades, constructive modalities have found numerous applications is the study of guarded (co)recursion: as an important tool to ensure *productivity* in (co)programming with *coinductive types* [168, 169, 170, 8, 198, 37, 63] and, on the metalevel, in semantic reasoning about programs involving *higher-order store* or a combination of *impredicative quantification* with *recursive types* [80, 34, 36, 281, 272, 156, 35]. The original reference used the name *modality for recursion* [212, 213]. Other ones include *guardedness type constructor* [8], *approximation modality* [6], *next-step modality/next clock tick* [168, 169], *later operator* [23, 33, 34, 148]. Such a modality is typically assumed to satisfy the *strong Löb axiom*: $(\Box p \to p) \to \Box p$ or, equivalently, $(\Box p \to p) \to p$.

From a CS point of view, most of the time one is more interested in laws governing morphisms (program execution, proof reduction) than in pure theoremhood (inhabitation laws). In other words, the focus moves to categorical models: type theories provide *classifying theories* for categories with enough structure, and in the reverse direction categories model type theories [69, 177].

Remark 6.2. Typically, the minimal requirement is being *cartesian*, i.e., having finite products. In order to model intuitionistic implication, the category must be moreover *cartesian closed*. The presentation in Appendix B should clarify these points.

When it comes to modelling recursion and iteration, we have moreover the body of work on *iteration theories* [40, 274]. Finally, on the modal side, several references [30, 13, 219] provide a detailed account of how categorical models of proof calculi for modal logic are obtained in terms of a cartesian (closed) category with an endofunctor corresponding to the modal operator.

Thus, our paper [I10] (and its workshop forerunner [S4]), whose main goal is to study essential properties of categorical models of guarded (co)recursion, can be also seen as a study of categorical models of the strong Löb logic. This perspective is not made very explicit therein, but it has been a major motivation for the present author. Appendix B to this overview provides some previously unpublished material, which clarifies the syntactic aspects of this correspondence.

A rather striking feature of our setting, both modally and categorically, was that we only needed the assumption of monotonicity (functoriality) rather than normality (product preservation) for our main results:

- We define the notions of
  - a *guarded fixpoint operator* on a cartesian category and a *guarded fixpoint category* $(\mathcal{C}, \bullet, \dagger)$, a *unique guarded fixpoint category* $(\mathcal{C}, \bullet)$ [I10, Def. 2.3];
  - a *guarded Conway category* and *uniform guarded Conway category* [I10, Def. 3.3] satisfying additional axioms whose syntactic counterparts are presented in Appendix B.

- We show that unique guarded fixpoint categories satisfy the axioms of uniform guarded Conway categories [I10, Th. 3.4]. This covers categories used in most references as models of guarded (co)recursion, in particular complete metric spaces with r-contractive morphisms (for a fixed $r \in (0,1)$), the "topos of trees" [34, 64] and its generalizations such as categories of functors from well-founded posets to complete categories [I10, Ex. 2.4(2)–(5)].
- We show that non-parametrized (hence, not explicitly assumed to support arbitrary context, cf. Remark 6.2) notions of guarded fixpoints such as *weak model of guarded fixpoint terms* [34] allow defining our parametrized notion of fixpoint in cartesian *closed* categories [I10, Prop. 2.7], i.e., supporting ordinary implication (Remark 6.2 above). On the other hand, cartesian closure is a necessary assumption [I10, Ex. 2.8].
- We show that *completely iterative monads* [196], the categorical counterpart of the standard setting in which guarded recursive definitions are studied—Elgot's (completely) iterative theories [84, 85]—can be seen as an instance of our setting when replaced by the (duals of) their Kleisli categories [I10, § 2.2]. Ditto for the category CPO of complete partial orders with the modal guard/delay being the lifting functor $\bullet = (-)_\perp$ [I10, Th. 3.7].
- We show that some of the standard properties of unguarded Conway theories/categories, such as *dinaturality*, split into the guarded setting into several versions depending on the way the guarding functor is used [I10, § 3.2], leading to new challenges regarding their relationships [I10, Problems 3.16 & 3.17] we could only partially solve [I10, Prop. 3.15 & 3.15].
- We propose the notion of (functionally) guarded trace operator and showing that they obey an analogue of the unguarded result by Hasegawa [126]: in the cartesian setting, they are equivalent to guarded Conway operators [I10, § 4]. We will return briefly to this issue in § 8.2.

6.3. KM, mHC, LC and the Topos of Trees

My paper [I5] began its life as a study relating contributions of Leo Esakia and the Tbilisi school of modal logic and topos theory [88, 91, 89, 90] to computer science developments mentioned in § 6.2, but it also occasioned a discussion of the (co)recursion-guarding strong Löb modality in another constructive language associated with categories with enough structure: the Mitchell-Bènabou language, i.e., the internal language of toposes. Contributions of this reference include:

- the derivation of isomorphism between lattices of extensions of the logic mHC and the logic wGrz using the transfer techniques of Wolter and Zakharyaschev [298, 299, 301] (Corollary 31). This is a generalization of results obtained by Kuznetsov and Muravitsky in mid-1980's [176, 206] announced by Esakia [90] without details of the proof;
- using the same results of Wolter and Zakharyaschev to show not only the canonicity of mHC [90, 116], but also its finite model property [I5, Cor. 29], a result not claimed or proved in earlier references;

- identifying the "linear" variant of the logic KM, i.e., KM + LC, as the propositional fragment of the logic of the topos of trees [34].[10] This has been significantly extended by Clouston and Goré [64], who studied a Gentzen system for this logic;
- a self-contained, type-theoretic introduction to the Mitchell-Bènabou language, similar in spirit to Appendix B here [I5, § 5.1] and a discussion contrasting "proposition as predicates" and "proposition as types" approaches to modality, inspired by Birkedal et al. [34].

6.4. Between Boole and Heyting: Negative Translations

We have seen numerous variants of classical and constructive modal formalisms. It is time to ask if there is a systematic way of relating the classical variant and the intuitionistic variant of a given logic. In the absence of modalities, both in the propositional case and in the predicate case, the classical calculus is well-known to be embeddable into its intuitionistic counterpart via several variants of the *negative translations*, i.e., prefixing double negation in front of suitably chosen subformulas; in the extreme case of the Kolmogorov translation, in front of *every* subformula. The computational interpretation and importance of *negative translations* is well-known (cf., e.g., [120, 207, 4] and [277, Ch. 6]) and their behaviour is rather well-understood not only for IPC, but also for IQC. Modal logic also has a very well-developed metatheory, not only over the classical propositional calculus (CPC), but also over IPC [116, 43, 278, 298, 299, 303]; some uses of this metatheory have been evident in the preceding sections.

Furthermore, negative translations (mostly Glivenko) have been studied for other non-classical logics, especially substructural ones [108, 217, 95]. The Gödel-Gentzen translation also plays an important rôle in the theory of *possibility semantics* for modal logic [22, 139], which we briefly discussed in § 5.3. And another type of translations in the reverse direction, from an intuitionsitic modal logic to suitable *bi*modal logics—Gödel-McKinsey-Tarski-type—has, in fact, been exhaustively studied [43, 298, 299].

These observations provide the starting point for our paper [I8].[11] An interesting feature of our methodology was that we consciously restricted our toolbox to a very specific subset of constructive methods: we have been working almost exclusively with purely syntactic ones. In other words, we have been avoiding not only *BOOL* and *SYFOL*, but also *SETAC*. This contrasts with less constructive methods typically employed in, to borrow Wolter and Zakharyaschev's phrases, "die Modallogik als die Klassentheorie" or "the Big Research Programme" [300]. To be more specific, we were using the minimal natural deduction system of Bellin, de Paiva and Ritter [12, 13, 157, 219] (see also Appendix B) extended with additional axioms/combinators and hence also allowing a direct formalization in a proof assistant (in our case Coq). This is based on a simple observation: one does not need to postulate unrestricted substitution as an explicit rule. As first observed by Sobociński (in a Hilbert-style setting) [275, 178], propositional deductions can

[10] This result has been explicitly stated only in the online version.

[11] It was written with my students Miriam Polzer and Ulrich Rabenstein, who provided the Coq formalization: `git://git8.cs.fau.de/dnegmod` and `https://cal8.cs.fau.de/redmine/projects/dnegmod`

be put in a form where the substitution rule is *only applied to axioms*. This idea can be replayed in a Gentzen-style setup.

The specific results we have established for the class of negative translations we call *regular*[12] [I8, § 3.1] are as follows:

- No negative translation is adequate for axioms as simple as $\mathsf{C4} = \Box\Box p \to \Box p$ or $\flat\mathsf{4} = \Box\Box p \to \Box\Box\Box p$ [I8, § 4], i.e., rather basic examples of *modal reduction principles* [300, §4.5]. An even more dramatic failure [I8, § 4.1] can be observed for the *logic of bunched implications* BI [216, 232] (§ 7.2).
- In order to establish whether a negative translation is adequate for a given modal logic, it is enough to clarify the adequacy w.r.t. its axioms instead of all derivable theorems [I8, Th. 15].
- Regular translations are adequate for all logics axiomatized by a class of formulas we called *enveloped implications* [I8, Th. 18].
- For all extensions of $\mathsf{i\text{-}S}_\Box = \mathsf{i\text{-}K}_\Box + p \to \Box p$, which can be seen as the logic of *applicative functors*, also known as *idioms* [193] (see § 7.1, esp. Remark 7.5 below), not only the regular translations, but even the Glivenko translation is adequate [I8, § 6].

Example 6.3. Here are some examples of enveloped implications [I8, Cor. 19]:

R	$p \to \Box p$	CB	$\Box p \to (q \to p) \lor q$	$\flat\mathsf{em}$	$\Box p \lor \Box\neg p$
$\mathsf{4}$	$\Box p \to \Box\Box p$	NV	$\neg\Box\bot$	$\mathsf{em}\flat$	$\Box p \lor \neg\Box p$
T	$\Box p \to p$	NNV	$\neg\neg\Box\bot$	$\mathsf{T}^{\neg\neg}$	$\Box p \to \neg\neg p$
coK	$(\Box p \to \Box q) \to \Box(p \to q)$	$\flat\mathsf{R}$	$p \to \Box\Box p$	$\mathsf{T}^{\neg}$	$\Box\neg p \to \neg p$
$\flat\mathsf{Lin}$	$\Box(p \to q) \lor \Box(q \to p)$	$\mathsf{Lin}\flat$	$(\Box p \to q) \lor (\Box q \to p)$	$\mathsf{wem}\flat^{\neg}$	$\neg\Box p \lor \neg\Box\neg p$

or any superintuitionistic axiom, i.e.,
a formula in the modality-free propositional signature.

Remark 6.4. From a foundational point of view, a systematic investigation of such questions seems warranted by Simpson's [273, Ch 3.2] influential *requirements* that an intuitionistic modal logic is supposed to satisfy, in particular, Requirement 3: *The addition of $\alpha \lor \neg\alpha$ to an intuitionistic modal logic should yield a standard classical modal logic.* We claim [I8, § 1] that this requirement is disputable from at least two points of view. On the one hand, it is too restrictive: we have already seen entire classes of important intuitionistic modal logics which classically collapse to rather trivial systems, like extensions of like PLL and SL (despite the fact that even the Glivenko translation works for them); other "classically near-degenerate" examples can be found in provability logics of extensions of HA, cf., e.g., [146, 296, I9] and § 7.1 below. On the other hand, we have seen that negative translations can go wrong even on "standard" axioms. Something seems wrong with the relationship between a constructive propositional logic and its classical variant if no negative translation works in the right way. In this sense, Simpson's Requirement 3 is arguably too liberal.

[12] Roughly speaking, these are the ones using at least as many $\neg\neg$ as either the Kuroda translation or the suitably adjusted variant of the Gödel-Gentzen translation.

7. Unifying Lewis and Brouwer

> *There are several closely interrelated operators, called* modal *operators, which are characteristic of modal logic ... These various operators are easily definable in terms of one another. Thus impossibility is necessity of the negation; possibility and non-necessity are the negations of impossibility and necessity; and strict implication and strict equivalence are necessity of the material conditional and biconditional. In a philosophical examination of modal logic we may therefore conveniently limit ourselves for the most part to a single modal operator, that of* necessity. *Whatever may be said about necessity may be said also, with easy and obvious adjustments, about the other modes.*
>
> Willard Van Orman Quine [237]

The epigraph of this section expresses the received wisdom in the presence of *RELST* and especially *BOOL*.[13] It is well-known that many classical dualities, such as the one between $\forall$ and $\exists$, break down intuitionistically and the two quantifiers are not mutually definable anymore; in fact, this is what makes negative translations discussed in § 6.4 interesting. But Quine mentions here also the original connective proposed by C. I. Lewis [183, 184], the creator of modern modal logic, i.e., the *strict implication* $\dashv$. Classically, $\varphi \dashv \psi$ is indeed definable as $\Box(\varphi \to \psi)$. Intuitionistically, however—and this is the starting point of the paper [I9] presented in § 7.1—there is no reason to accept this equivalence. In fact, as discussed below, functional programming is one of the two areas where the distinction between constructive $\varphi \dashv \psi$ and $\Box(\varphi \to \psi)$ was independently discovered, the other area being metatheory of arithmetic. These days, however, computer scientists may be more familiar with another implication-like connective complementing ordinary $\to$: the *magic wand* $-\!\!*$ of the *logic of bunched implications* (BI). Our work [I3] discussed further in § 7.2 presents an algebraic framework for a more general, noncommutative system GBI and its applications.

> **Convention 7.1.** Everywhere in this work, implication-like connectives ($\to$, $\dashv$, $-\!\!*$) associate to the right. Moreover, $\dashv$ and $-\!\!*$ bind stronger than ordinary $\to$.

7.1. Constructive Strict Implication

> *Many programs and libraries involve components that are "function-like", in that they take inputs and produce outputs, but are not simple functions from inputs to outputs ... [S]uch "notions of computation" defin[e] a common interface, called "arrows". ... Monads ... serve a similar purpose, but arrows are more general. In particular, they include notions of computation with static components, independent of the input, as well as computations that consume multiple inputs.*
>
> Ross Paterson [221, p. 201]

A mere investigation for natural intuitionistic generalizations of Kripke satisfaction clauses for $\varphi \dashv \psi$ and $\Box(\varphi \to \psi)$ shows they must be semantically distinct formulas. See the discussion of $\dashv$-frames (Def. C.1) in Appendix C.[14]

[13] Note here the ironic contrast between Quine's authoritative pronouncements on "easy" mutual definability of modes and his claim [235] used as one of epigraphs of § 2 professing not to have any intuitive grasp of modes "until explained in non-modal terms". I discuss further Quine's contradictions in Appendix D.

[14] As stated therein, it is entirely based on our paper [I9, § 3] (see also [I5]); I provide it here in order to

Definition and Remark 7.2. The following schemes axiomatize over IPC the logic iA of all $\prec$-frames [146, Prop. 4.1.1], [144, Prop. 7], [305, Th. 2.1.10] [I9, Th. 6.4a]:

$\mathsf{N_a}$ $\vdash \phi \to \psi \;\Rightarrow \vdash \phi \prec \psi$,

Tr $\phi \prec \psi \to \psi \prec \chi \to \phi \prec \chi$,

$\mathsf{K_a}$ $\phi \prec \psi \to \phi \prec \chi \to \phi \prec (\psi \wedge \chi)$,

Di $\phi \prec \chi \to \psi \prec \chi \to (\phi \vee \psi) \prec \chi$.

Our paper presents several alternative axiomatizations of iA [I9, Tab 4.2 & § 4.2.1]. Importantly, the status of Di is somewhat special [I9, § 4.2.2], at least from the point of view of arithmetical interpretations [I9, § 5.4.1], and also from a functional programming point of view, as we are going to see in Def. & Rem. 7.4.

Actually, the failure of definability of constructive $\prec$ in terms of $\Box$ has been discovered in two different contexts, beginning with the area of *preservativity logic* [294, 295, 144, 145]—that is, in the study of the metatheory of Heyting Arithmetic (HA) and related systems.

Definition and Remark 7.3. Recall that Σ^0_1-*preservativity* for an arithmetical theory T is defined as follows:

- $\varphi \prec_T \psi$ if, for all Σ^0_1-sentences χ, if $T \vdash \chi \to \varphi$, then $T \vdash \chi \to \psi$.

T-provability, i.e., $\Box_T \psi$ can be thus defined as $\top \prec_T \psi$. But while provability can be reduced to this degenerate form of preservativity, the converse does not hold. The $\Box$-logic of provability of a constructive arithmetical theory provides only limited information about the behaviour of its $\prec$. Moreover, in contrast to the classical result of Solovay [276] identifying the classical variant of the Löb logic as *the* provability logic of Peano Artihmetic (PA), we are still far from knowing how an axiomatization of the $\Box$-provability logic of HA could look like. Paradoxically, in several ways we seem to be closer to axiomatizing its preservativity logic.

What is more important from our perspective, however, is that the failure of definability of $\varphi \prec \psi$ as $\Box(\varphi \to \psi)$ has also been discovered in the context of functional programming. To understand this, let us revisit the Curry-Howard correspondence in § 6.1 and recall that ordinary intuitionistic implication $\varphi \to \psi$ corresponds to forming the *function space* of programs (proofs) which take data from (proofs for) φ as their input and produce members of (proofs for) ψ as their output. The introduction rule for $\to$ corresponds to λ-abstraction and its elimination rule (i.e., ordinary Modus Ponens) corresponds to function application.

But as pointed out in the second epigraph to this section, "computations" need not be exactly co-extensional with "members of function space"; and we can use the term "arrows" for this broader notion of computation.

make this overview self-contained. An interesting separate question we discuss is to what extent combining intutionistic logic with Lewis' analysis of strict implication is philosophically viable and whether there any parallels between the fates of the philosophical projects of Lewis and Brouwer. The only known quote in which Lewis mentions intuitionism, used as the anti-Quine epigraph of § 6, is clearly positive in spirit, but it does not seem he was familiar with the work of Kolmogorov, Glivenko and Heyting, turning Brouwer's philosophical insights into a propositional calculus. See our paper [I9, § 2].

Definition and Remark 7.4. Here are the inhabitation axioms of the calculus of "classic arrows" [187, Fig. 4] over the disjunction-free fragment of IPC:[a]

$\mathsf{S_a}$ $(\varphi \to \psi) \to \varphi \multimap \psi$,

Tr $\varphi \multimap \psi \to \psi \multimap \chi \to \varphi \multimap \chi$,

$\mathsf{K'_a}$ $\varphi \multimap \psi \to (\varphi \land \chi) \multimap (\psi \land \chi)$.

Note that $\mathsf{N_a}$ of Def. & Rem. 7.2 becomes derivable in the presence of $\mathsf{S_a}$, and $\mathsf{K'_a}$ is an alternative form of $\mathsf{K_a}$ [I9, Lem. 4.1a]. This leaves Di, the only axiom of iA using $\lor$ in an essential way, as the only one whose status is unclear in the light of classic arrows/idioms interpretation. Otherwise, the inhabitation laws of the calculus for idioms [187, Fig. 3] are exactly those of i-$\mathsf{S}_\Box$ (defined in § 6.4 above).

[a] We use the notation and naming schemes of our paper [I9]. Lindley et al. [187] call these axioms arr, >>> and first, respectively. They also use $\rightsquigarrow$ in place of $\multimap$.

Remark 7.5. Lindley et al. [187] call arrows determined by a unary $\Box$ *static arrows* and show that such arrows correspond to the "idioms" or "applicative functors" of McBride and Paterson [193]. This is, however, only a special subclass of computations encoded by arrows: namely those computations "in which commands are *oblivious* to input" [187]. Lindley [186] rephrases this claim to the effect that idioms are distinguished by their static approach to *data flow*.

A special subclass of applicative functors is by far the most important one from a programming point of view: that of (*strong*) *monads*, providing the most popular framework for *effectful computations*. We have already encountered them and their Curry-Howard logic PLL in § 6.1, while the axiom $\mathsf{C4}_\Box$ made an unexpected cameo in § 6.4. Another subclass of applicative functors are recursion-guarding delay functors discussed in § 6.2 and Appendix B.

Monads can be shown [141, 187] to be in 1-1 correspondence with *higher-order arrows* or *classical arrows with apply* satisfying the law:

$\mathsf{App_a}$ $(\varphi \land (\varphi \multimap \psi)) \multimap \psi$.

This is discussed further in our paper [I9, Remarks 7.2 and 7.3].

Even in those settings where $\multimap$ happens to be definable in terms of $\Box$, there are still good *proof-theoretic* reasons to treat it as a primitive connective. This is a point made by Clouston and Goré [64], which as we remember from § 6.2 studied a Gentzen system for the modal fragment of the logic of the topos of trees; as noted therein, such reasons are already provided by the original work of Nakano [212, 213]. See § 7.2 of our paper [I9] for a detailed discussion.

Apart from overviewing these issues and noticing surprising parallels between Lewis's and Brouwer's contributions to metamathematics and logic (and their later fates), we offer the following technical contributions:

- We provide axiomatizations for numerous natural extensions of the minimal logic of Lewis's arrows [I9, § 4]; "naturality" of an axiom can be measured as its importance from, for example, a computational, arithmetical, philosophical, proof-theoretic or semantic point of view.
- As an intriguing example, we show that in the presence of

Di $\varphi \prec \chi \to \psi \prec \chi \to (\varphi \vee \psi) \prec \chi$
(which is valid on all $\prec$-frames) $\psi \prec \chi$ is provably equivalent to

$$(\psi \vee \neg\psi) \prec (\psi \to \chi)$$

[I9, Lem 4.6]. This explains why $\varphi \prec \psi$ is classically equivalent to $\Box(\varphi \to \psi)$. Cf. the discussion of validity of Di in Def. & Rem. 7.2 and 7.4 above.

Remark 7.6. Note that:

- as $(\psi \vee \neg\psi) \prec (\psi \to \chi)$ is parametric in the antecedent of strict implication, one cannot call it a satisfying reduction of $\prec$ to $\to$;
- if one adds $\prec$ to Johansson's minimal logic instead of IPC, even this transformation does not work anymore;
- there is no one-variable formula $\varphi(p)$ in the disjunction-free language s.t. $p \prec q$ is equivalent to $\varphi(p) \prec (p \to q)$ and $\mathsf{CPC} \vdash \varphi(p)$ [I9, Ex. 6.10].

- We provide alternative axiomatizations for the type inhabitation logic of classic arrows and higher-order arrows [I9, Lem. 4.1, 4.10, 4.17 & § 7.1].
- We translate to the $\prec$-setting my earlier results [O8], [O5, § 3] on failure of strong completeness in the presence of Löb-like axioms [I9, Th. 6.5].
- We significantly extend the list of known Kripke correspondents for $\prec$-axioms [I9, § 6.2, Fig. 6.2 & Th. 6.6] by noticing that the technology of Wolter and Zakharyaschev [298, 299] can be adjusted to the $\prec$-setting. Details of these proofs will be published separately [P1].
- We use Kripke semantics to show that important $\prec$-logics do differ, often even when the attention is restricted to their closed fragments [I9, § 6.3].

Remark 7.7. While I omit in this overview most of the arithmetical aspects suggested in Def. & Rem. 7.3, here are some highlights:

- an analogue of the *Orey-Hájek characterization* for constructive $\prec$: deconstruction of the validity of formulas of the form $\varphi \prec \psi$ in terms of validity of an infinite family of formulas $\Box_n\varphi \to \psi$, with $\Box_n$ denoting provability with axioms whose Gödel number is bounded by n [I9, Th. 5.5];
- a sufficient condition for the validity of Di in the preservativity logic of an arithmetical theory: its closure under *q-realizability* [I9, Th. 5.6];
- a study of conditions under which the preservativity logic of a given theory contains the "strict negation of the excluded middle": $(\varphi \vee \neg\varphi) \prec \bot$ for some suitable φ. This phenomenon turns out to be surprisingly common [I9, § 8.2];
- finding new interpretability principles valid in (extensions of) HA [I9, App. C].

7.2. Relevance, Resources and (G)BI

[L]ogic as Brouwer sees it is a relevance logic.

Mark van Atten [9]

> BI *should be considered a relevant logic, where we follow Stephen Read in considering "relevant logic" broadly, as referring to a variety of logics which control the use of structural rules, and not necessarily to ones that possess Contraction.*
>
> Peter O'Hearn and David Pym [216, p. 238]

C. I. Lewis' analysis of strict implication was a direct stimulus for the creation of early systems of *relevance* logics [I9, Footnote 15]: those where the antecedent is supposed to be genuinely *relevant* for the consequent. This fact is reasonably well-known, if too easily forgotten. Nevertheless, as pointed out by our paper [I9], one can argue that

> even the later enterprise of relevance logic would not satisfy Lewis' expectations: he wanted to *supplement* material implication with a strict one, not *replace* it altogether. In this sense, still more recent *resource-aware* formalisms with computer-science motivation where *both* a substructural *and* an intuitionistic/classical implication are present (either as an abbreviation or directly in the signature) such as linear logic [114, 288, 2, 31] or the logic of bunched implications BI [216, 232, 234] seem closer to Lewis' original idea. [I9, § 2.1]

The *logic of bunched implications* BI mentioned in this passage is obtained by enriching IPC with another implication connective $-\!\!*$, which is *residual* (or *adjoint to*) a commutative and associative connective of *separating conjunction* $*$:

$$\varphi * \psi \vdash \chi \quad \text{iff} \quad \varphi \vdash \psi -\!\!* \chi.$$

Example 7.8. The very fact that $-\!\!*$ is obtained as a residual forces it to satisfy a somewhat different set of laws and rules than those of $\multimap$. For example,

$$(\top -\!\!* \varphi) \to (\top -\!\!* (\top -\!\!* \varphi))$$

is a theorem of BI whereas its $\multimap$-counterpart is just $\mathsf{4}$: $\Box\varphi \to \Box\Box\varphi$, which is not universally valid in $\multimap$-frames (cf. Def. C.1) The axiom $\mathsf{4}$ is, of course, valid in those of them which satisfy the laws of *strong* arrows discussed in § 7.1, in particular the axiom $\mathsf{S_a}$. But then again the $-\!\!*$-counterpart of $\mathsf{S_a}$, i.e., $(\varphi \to \psi) \to \varphi -\!\!* \psi$ is not a theorem of BI; in fact, its addition would force that *weakening* for substructural connectives is valid in BI. Closing under the $-\!\!*$-variant of $\mathsf{N_a}$ in Def. & Rem. 7.2 would have a similar effect. On the other hand, there are theorems of BI such as $\varphi -\!\!* (\psi -\!\!* \chi) \to \psi -\!\!* (\varphi -\!\!* \chi)$ whose $\multimap$-counterpart is not valid even in i-$\mathsf{S}_\Box$.

BI has been explicitly introduced by O'Hearn and Pym [216, 214, 233, 232]. Its most important motivation can be summarized in one phrase: modular reasoning about *shared mutable data structures* [244, 245], i.e., "structures where an updatable field can be referenced from more than one point" [245]. In a narrow sense, this applies to heap mutation, pointer aliasing and (de)allocation: in short, dynamic memory management. However, as O'Hearn et al. [215] recall, BI originates *in the context of a broader investigation of "resource*[15] *modelling"* [234]. Recent important applications of BI and separation logic arise in the area of concurrency

[15] See O'Hearn et al. [216, pp. 240–241] for a comparison of the "declarative" approach to resource in BI with that of "proofs-as-actions" in linear logic.

[44, 215, 135]. For more about resource-oriented models, including those arising from *weakening relations*, formal languages or fine-grained treatment of labelled trees and semistructured data, see our paper [I3, § 3–4].

On the other hand, BI has obvious and explicitly admitted roots in relevance logic and substructural logic [233, §9], [216], [232, Ch. 1]; in fact, BI can be seen as an instance of Belnap's *scheme of display logic* [14]. Moreover, the *bunches* in its name come from Dunn's work on a sequent calculus for the relevance logic R [82, 14, 242]; cf. also Mints [197].

In our paper [I3], we bring together research on resource modelling and on algebraic semantics of substructural logics:

- We propose *generalized* BI *algebras* (GBI-*algebras*) as a common framework for algebras arising via "declarative resource reading", intuitionistic generalizations of *relation algebras* and *arrow logics* and the distributive Lambek calculus with intuitionistic implication [I3, § 2–4]. Cf. Def. & Rem. 7.9 below for axiomatizations of GBI and BI.
- We survey the lattice of subvarieties of GBI [I3, § 5], identifying its atoms (and their covers) and classifying non-isomorphic algebras of small cardinalities.
- We present a suitable duality for GBI along the lines of Esakia, Priestley, and Urquhart [I3, § 6] and an algebraic proof of cut elimination in the setting of residuated frames of Galatos and Jipsen [I3, § 8].
- We show how the algebraic approach allows generic results on decidability, both positive and negative ones [I3, § 7].
- We introduce the substructural audience to separation logic [I3, § 9–10] and theory underlying state-of-art tools, culminating with an algebraic and proof-theoretic presentation of (*bi-*) *abduction* [51]. It appears that (the two flavours of) the algebraic approach to (bi-)abduction presented in our paper have no precedents in the literature [I3, § 11].

Definition and Remark 7.9. A Hilbert-style system for GBI is obtained from the one for IPC by adding symbols $\cdot, \backslash, /$, all substitution instances of

$$(\varphi \cdot \psi) \cdot \chi \leftrightarrow \varphi \cdot (\psi \cdot \chi) \qquad 1 \cdot \varphi \leftrightarrow \varphi \qquad \varphi \cdot 1 \leftrightarrow \varphi$$

and the bidirectional residuation rules $\dfrac{\varphi \cdot \psi \to \chi}{\psi \to \varphi \backslash \chi}$ $\dfrac{\varphi \cdot \psi \to \chi}{\varphi \to \chi / \psi}$. A Hilbert system for BI is obtained by adding an axiom $\varphi * \psi \to \psi * \varphi$, in which case the rules for $/$ can be omitted, and the rules for $\cdot$ and $\backslash$ are rewritten with $*$ and $-\!\!*$, respectively. The resulting system is similar to the one provided by Pym [232]. However, we propose [I3, § 2.3] an equivalent Hilbert-style axiomatization of BI over IPC:

$$(\varphi * \psi) * \chi \leftrightarrow \varphi * (\psi * \chi) \qquad \varphi * \psi \to \psi * \varphi \qquad \varphi * 1 \leftrightarrow \varphi \qquad \varphi * (\varphi -\!\!* \psi) \to \psi$$

$$\varphi -\!\!* (\psi -\!\!* \chi) \leftrightarrow \varphi * \psi -\!\!* \chi \qquad \frac{\varphi \to \psi}{\varphi * \chi \to \psi * \chi} \qquad \frac{\varphi \to \psi}{1 \to \varphi -\!\!* \psi}.$$

8. Beyond Brouwer, Heyting and Kolmogorov: Nondistributivity

One day, in my first year as an assistant professor at MIT, while walking down one of the long corridors, I met Professor Z, a respected senior mathematician with a solid international reputation. He stared at me and shouted, "Admit it! All lattice theory is trivial!" I did not have the presence to answer that von Neumann's work in lattice theory is deeper than anything Professor Z has done in mathematics. Those who have reached a certain age remember the visceral and widespread hatred of lattice theory from around 1940 to 1979; this has not completely disappeared.

Gian Carlo Rota [249, p. 52]

We have already seen in § 6 that the CS status of *BOOL* can be subverted in several different ways and § 7 has accustomed us to the thought that computationally meaningful implication-like connectives ($\multimapinv$, $-\!\!*$) may differ even from the standard intuitionistic $\to$, not to mention its classical cousin. It is time to go even further and subvert such seemingly fundamental propositional laws as distributivity.

A well-informed reader may protest that such subvertion has already happened in computer science in the setting of, e.g., linear logic [2, 32, 114] (see Footnote 15). Indeed, § 8.2 revisits the work [S4, I10] discussed in § 6.2 in this context. But another line of attack presents itself in the context of standard relational model of database theory, i.e., within the *RELST* pillar of the Classical Picture. In fact, our research [I6] opens up a new perspective on basic lattice laws.

8.1. Relational Lattices

One perhaps would not expect (a reevaluation of) the relational model and Codd's relational algebra to provide interesting new foundational or algebraic insights these days; the body of work on the subject is impressive and one can be easily tricked into believing it has been well-understood since at least 1980's.

Remark 8.1. Such an illusion of having a complete or near-complete understanding of a given field is a common theme in the history of mathematics, philosophy and science. Déscartes famously claimed (in 1619!) that there was *almost nothing left to discover in geometry*, whereas Kant expressed similar beliefs concerning logic, which in his time was still restricted to Aristotelian syllogistic (in fact, one can make a good case that even Aristotelian syllogistic itself was not properly understood until the XXth century). Another striking example, though less related to logic, was the belief of numerous XIXth century physicists that there was little left to explain in their field apart from a few marginal phenomena. And the Dijkstra quote used as an epigraph of this overview points out that some computer scientists these days are also prone to similar illusions regarding their field.

It seems still less likely that not only *BOOL* but even the assumption of distributivity of lattice connectives may be subverted in the process. Yet this is precisely what happens in our paper [I6] and its conference forerunner [S1].

Why one would like to revisit the foundations of the relational model? Recall that Codd's (*named*) *relational algebra* [65] [1, Ch. 5] is only a *partial algebra*. This means that while some operations like *natural join* $\bowtie$ are total, i.e., well-defined for any pair of relations (tables), some others are defined only between relations with suitable headers, e.g., the (*set*) *union* or *the difference operator*. Apart from the issues of mathematical elegance and generality, this partial nature of operations has also unpleasant practical consequences, forcing database systems to perform at least rudimentary type-checking to avoid "crashing" queries [49].

Vadim Tropashko [289, 279, 290] from Oracle realized that generalizing set union to a total operation of *inner union* $\oplus$ not only removes partiality from relational algebra, but also yields a lattice dual to the natural join. We provide an alternative proof that $\bowtie$ and $\oplus$ form a lattice using closure operators; this representation of *relational lattices* [I6, Lem. 2.1]—or, as we suggest, *Tropashko lattices*—proved useful in subsequent references [254, 255, 256].

Our paper [I6] clarifies that, on the one hand, even the simplest database instances produce *nondistributive* lattices, but on the other hand, there are nontrivial lattice equalities which hold in all such structures. Examples include (with $\bowtie$ being the lattice meet $\wedge$, and $\oplus$ the lattice join $\vee$):

$$x\bowtie y \oplus x\bowtie z = x\bowtie(y\bowtie(x\oplus z)\oplus z\bowtie(x\oplus y)) \qquad \mathsf{RL1}$$

$$t\bowtie((x\oplus y)\bowtie(x\oplus z)\oplus(u\oplus w)\bowtie(u\oplus v)) =$$
$$= t\bowtie((x\oplus y)\bowtie(x\oplus z)\oplus u\oplus w\bowtie v) \quad \oplus \quad t\bowtie((u\oplus w)\bowtie(u\oplus v)\oplus x\oplus y\bowtie z) \qquad \mathsf{RL2}$$

In fact, the variety induced by relational lattices does not lie in the area of the lattice of varieties of lattices investigated in earlier rather exhaustive studies and monographs on the subject [152, 153]. It does not satisfy typical laws such as

- join- and meet-semidistributivity,
- almost distributivity and neardistributivity,
- semimodularity (and hence also modularity),
- the Jordan–Dedekind chain condition,
- supersolvability.

Our paper provides relevant definitions and proofs [I6, Th. 3.2]. In the opening section, we highlight the relationship with earlier database work focusing on *logical data independence* and *algebraic equivalences* as opposed to *strong equivalences* [5, 50]. Furthermore, we stress that in contrast with the embedding of Codd's relational algebras in cylindric algebras proposed by Imieliński and Lipski [147], we avoid an artificial uniformization of headers; hence queries formed with the use of proposed connectives enjoy the *domain independence property* [291], [1, Ch. 5]. In fact, our paper [I6, § 2] refines this notion to *strict domain independence*, also preserved by terms and equivalences in the signature of relational lattices. We are going to say more about the importance of domain independence in Appendix D.

Besides, we motivate [I6, § 1] the study of *quasiequations*, i.e., definite Horn clauses over equalities, in terms of reasoning over key constraints and inclusion dependencies [1, Ch. 9] (also known as *foreign keys*) and *semantic query optimization* [59, Sec. 3.1]. As made clear by our concrete examples, such applications require extending the language with, at the very least, the *header constant* H, denoting the table with empty header (empty set of attributes) and empty body. Importantly, in the resulting lattice structure H is neither $\top$ nor $\bot$.

Remark 8.2. In fact, writing concrete database queries in the presence of an explicit schema/header information would most of the time involve entire families of constants denoting relations with empty bodies, one for each specific attribute; H would be definable in such a setup. See § 1 and § 6 of the discussed paper.

Further results we establish include:

- a presentation of relational lattices in terms of the *Grothendieck construction* [I6, § 2.3], which reveals the connection to the categorical approach to relational databases proposed by Abramsky [3];

Remark 8.3. Abramsky's work suggests further surprising connections, including Bell's Theorem from theoretical physics, probabilistic databases and provenance semirings.

- a study of mutual dependences between nontrivial (quasi)equations valid in relational lattices, with and without H [I6, Th. 3.4 & 3.5];

Example 8.4. Our proposed axioms involving H still seem an optimal equational choice:

$$\mathsf{H}\bowtie x\bowtie(y\oplus z)\oplus y\bowtie z = (\mathsf{H}\bowtie x\bowtie y\oplus z)\bowtie(\mathsf{H}\bowtie x\bowtie z\oplus y) \qquad \mathsf{AxRHL1}$$
$$x\bowtie(y\oplus z) = x\bowtie(z\oplus \mathsf{H}\bowtie y)\oplus x\bowtie(y\oplus \mathsf{H}\bowtie z) \qquad \mathsf{AxRHL2}$$

In the absence of H, our proposed axiomatization of the abstract class has been since improved by Santocanale [254, 255, 256]; seeing how complicated $\mathsf{RL2}$ above is, a more perspicuous axiomatization was to be expected and desired.

- a proof that relational lattices form a *pseudoelementary* class, i.e., they are reducts of a first-order class definable in an extended language. Moreover, the axiomatization in the extended language involves finitely many axioms [I6, Th. 4.1]. This yields as corollaries: closure under ultraproducts, the fact that SP closures are equational classes, the fact that quasiequational, universal and elementary classes are recursively enumerable;
- a proof that the quasiequational theory in the signature with H is undecidable [I6, Th. 4.7]. From a logician's point of view, a particularly interesting feature of this proof is that it works by reduction to Maddux's [188] undecidability result for cylindric algebras of dimension ≥ 3, again illustrating that the lattice-theoretic setting in question is surprisingly expressive. Later on, Santocanale [255] managed to show that undecidability persists for the quasiequational theory in pure lattice signature using a different and arguably more involved technique;
- a discussion of the (dual) structure of relational lattices in terms of Formal Concept Analysis [109], i.e., their *standard contexts* [I6, § 5]. The analysis showed that relational lattices obtained from concrete database instances are

subdirectly irreducible but not *simple*. It also played a rôle in Santocanale's subsequent work [254, 255, 256];

- finally, a proposal for several extensions of signature [I6, § 6]. A modest addition of unary singleton constants (cf. Remark 8.2) yields equivalence with the monotonic relational expressions of Sagiv and Yannakakis [251, Sec. 2.2] [I6, § 6.2], and further extensions recover the full expressive power of Codd's relational algebra [I6, § 6.3].

We also state [I6, Rem. 3.8] that the *equational* theory of relational lattices is decidable, postponing a detailed, database-oriented proof to a subsequent publication. In the meantime, another proof in the pure lattice signature has been published by Santocanale [256]. It is based on more abstract mathematical techniques and does not provide an optimal complexity bound.

Remark 8.5. An interesting feature of the line of papers produced by Santocanale [254, 255, 256] in the wake of ours [I6] is that they explicitly involve encoding of relational lattices into multimodal logics (already implicit in our reduction of quasiequational undecidability to results of Maddux [188], in cylindric-algebraic disguise), thus again highlighting both the ubiquity of modal logic in CS and its revolutionary and subversive uses wherever the Classical Picture needs to be dispensed with.

8.2. Functorially Guarded Traces

Finally, the issue of distributivity necessitates a brief revisit of our paper [I10] discussed in § 6.2. At the beginning of the present section, we have already mentioned linear logic and its computational significance [2, 32, 114]. While a generic categorical semantics of linear Natural Deduction systems can be obtained by generalizing the approach of § 6 and Appendix B (with *monoidal* categories replacing cartesian ones), a more fine-grained categorical analysis of the dynamics of linear cut elimination is provided by Geometry of Interaction [122, 123], where a central rôle is played by the so-called *Int-construction*, which in turn involves the notion of a *trace* on a (suitably well-behaved) monoidal category [155].

There is a more direct way of motivating the notion of trace on a monoidal category, especially in the context of our overview. Namely, in the study of recursion and iteration, traces provide a more flexible framework than the fixpoint operators and Conway theories we have encountered § 6 and Appendix B, precisely because of their applicability to non-cartesian tensor products [I10, Rem. 4.1]. As long as the underlying category is cartesian, traces and Conway theories are two sides of the same coin by results of Hasegawa: a fixpoint operator satisfying the Conway axioms is equivalent to a trace operator w.r.t. the product on $\mathcal{C}$ [127, 126].

A *trace operator* on a (symmetric) monoidal category $(\mathcal{C}, \otimes, I, c)$ is a natural family of operations

$$\mathsf{Tr}^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \to \mathcal{C}(A, B)$$

satisfying natural conditions (Vanishing, Superimposing and Yanking). We propose a generalized notion: a *guarded trace operator* [S4, I10] of the type

$$\mathsf{Tr}^X_{A,B} : \mathcal{C}(A \otimes \bullet X, X \otimes B) \to \mathcal{C}(A, B)$$

defined on a *delmon*,[16] i.e. a (*symmetric*) *delayed monoidal category* $(\mathcal{C}, \otimes, I, c, \bullet)$ and satisfying suitably adjusted variants of the defining conditions. The "delay modality" $\bullet : \mathcal{C} \to \mathcal{C}$ in the definition of a delmon is a comonoidal pointed endofunctor.[17] In this setting, we can replay Hasegawa's result: guarded Conway categories of § 6.2 and Appendix B are just another way of presenting traced *cartesian* delmons. In fact, this equivalence can be stated in terms of isomorphim between suitable 2-categories [I10, Cor. 4.10].

We remain in the framework of *guarding by modality*, which from a categorical point of view can be also called *guardedness by functor*. In a subsequent development, Goncharov and Schröder [118] investigated a more general notion of *abstract guardedness* based on the notion of a partial trace. Some of their work covers and generalizes our unpublished results, such as suitable generalizations of results concerning tracing on compact closed categories. On the other hand, the demarcation line between cases where the added generality of abstract guardedness is really necessary and those where it is not is still far from clear, and other questions are left entirely open. Just one example: how general can one make the Int-construction in the guarded setting? More broadly, is there anything like a guarded version of the Geometry of Interations?[18]

9. References

The bibliography is divided into several subsections, each with a different numbering scheme. The most important distinction is between the papers I have (co)authored (§ 9.1), and those in which I have not been involved (§ 9.2).

9.1. Own Papers

The most important distinction here is between those either directly included in my cumulative habilitation (§ 9.1.1) or whose contents have been superseded by such papers (§ 9.1.2) and those that have been written prior to my appointment at FAU (§ 9.1.3).

9.1.1. Papers Included in This Overview

Major papers accepted or published between 2014 and the first half of 2018, to be considered as constituent parts of my cumulative habilitation. For the convenience of those reading this overview as a self-standing paper, URLs are provided (elsewhere in the bibliography, they are omitted wherever possible for compactness).

[I1] Holliday, W. H. and **Litak, T.** 2019. "Complete additivity and modal incompleteness". *Review of Symbolic Logic* 12.3. Available on arXiv: `https://arxiv.org/abs/1809.07542` and eScholarship: `https://escholarship.org/uc/item/01p9x1hv`, pp. 487–535. URL: `https://doi.org/10.1017/S1755020317000259`.

[16] This name comes from our unpublished work.

[17] The requirement of being comonoidal is trivially satisfied by any endofunctor in a cartesian setting.

[18] It is worth pointing out that both Goncharov and Schröder [118] and the work of Santocanale [256] discussed in § 8.1 have been nominated for the Best Paper award at Foundations of Software Science and Computation Structures (FoSSaCS) 2018, with the former paper even winning it. This illustrates the importance of lines of research discussed in this section.

[I2] Holliday, W. H. and **Litak, T.** 2018d. “One modal logic to rule them all?” In: *Advances in Modal Logic*. Vol. 12. Extended technical report available at `https://escholarship.org/uc/item/07v9360j`. College Publications, pp. 367–386.

[I3] Jipsen, P. and **Litak, T.** 2018e. “An algebraic glimpse at bunched implications and separation logic”. In: *Hiroakira Ono on Residuated Lattices and Substructural Logics*. Outstanding Contributions to Logic. Accepted, to appear. Springer. URL: `https://arxiv.org/abs/1709.07063`.

[I4] **Litak, T.** 2018f. “Infinite populations, choice and determinacy”. *Studia Logica* 106 (5), pp. 969–999. URL: `https://doi.org/10.1007/s11225-017-9730-3`.

[I5] **Litak, T.** 2014e. “Constructive modalities with provability smack”. In: *Leo Esakia on Duality in Modal and Intuitionistic Logics*. Outstanding Contributions to Logic. *Author's Cut* (full electronic version): `https://arxiv.org/abs/1708.05607`. Springer, pp. 179–208.

[I6] **Litak, T.**, Mikulás, S., and Hidders, J. 2016f. “Relational lattices: From databases to universal algebra”. *Journal of Logical and Algebraic Methods in Programming* 85 (4). Special issue with selected papers of RAMiCS 2014, pp. 540–573. URL: `https://doi.org/10.1016/j.jlamp.2015.11.008`.

[I7] **Litak, T.**, Pattinson, D., Sano, K., and Schröder, L. 2018g. “Model theory and proof theory of coalgebraic predicate logic”. *Logical Methods in Computer Science* 14. URL: `http://arxiv.org/abs/1701.03773`.

[I8] **Litak, T.**, Polzer, M., and Rabenstein, U. 2017d. “Negative translations and normal modality”. In: *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017)*. Vol. 84. Leibniz International Proc. in Informatics (LIPIcs), 27:1–27:18. URL: `http://drops.dagstuhl.de/opus/volltexte/2017/7741`.

[I9] **Litak, T.** and Visser, A. 2018h. “Lewis meets Brouwer: Constructive strict implication”. *Indagationes Mathematicae* 29. A special issue “L.E.J. Brouwer, fifty years later”, pp. 36–90. URL: `https://arxiv.org/abs/1708.02143`.

[I10] Milius, S. and **Litak, T.** 2017e. “Guard your daggers and traces: Properties of guarded (co-)recursion”. *Fundamenta Informaticae* 150, pp. 407–449. URL: `http://arxiv.org/abs/1603.05214`.

[I11] Schröder, L., Pattinson, D., and **Litak, T.** 2017f. “A van Benthem/Rosen theorem for coalgebraic predicate logic”. *Journal of Logic and Computation* 27 (3), pp. 749–773. URL: `https://doi.org/10.1093/logcom/exv043`.

9.1.2. Papers Superseded by Full Papers

Conference and workshop versions of papers published during my FAU appointment, but superseded by later journal versions included in the preceding list.

[S1] **Litak, T.**, Mikulás, S., and Hidders, J. 2014f. “Relational lattices”. In: *Relational and Algebraic Methods in Computer Science 2014 (RAMiCS)*. Vol. 8428. LNCS. Superseded by [I6].

[S2] **Litak, T.**, Pattinson, D., and Sano, K. 2013g. “Coalgebraic predicate logic: Equipollence results and proof theory”. In: *Logic, Language, and Computation. Revised Selected Papers of TbiLLC 2011*. Vol. 7758. LNCS. Superseded by [I7].

[S3] **Litak, T.**, Pattinson, D., Sano, K., and Schröder, L. 2012j. “Coalgebraic predicate logic”. In: *Proc. 39th International Colloquium on Automata, Languages and Programming (ICALP) 2012, Part II*. Vol. 7392. LNCS. Superseded by [I7].

[S4] Milius, S. and **Litak, T.** 2013h. “Guard your daggers and traces: On the equational properties of guarded (co-)recursion”. In: *Proc. 9th Workshop on Fixed Points in Computer Science (FiCS)*. Vol. 126. EPTCS. Superseded by [I10].

9.1.3. Selected Older Papers

Only those relevant for this overview are mentioned.

[O1] ten Cate, B., Fontaine, G., and **Litak, T.** 2010b. “Some modal aspects of XPath”. *Journal of Applied Non-Classical Logics* 20 (3). Accessible copy: `http://www8.cs.fau.de/~litak/papers/jancl20birthday.pdf`, pp. 139–171.

[O2] ten Cate, B. and **Litak, T.** 2007e. “Topological perspective on the hybrid proof rules”. In: *Proc. International Workshop on Hybrid Logic HyLo 2006*. Vol. 174. ENTCS, pp. 79–94.

[O3] ten Cate, B., **Litak, T.**, and Marx, M. 2010c. “Complete axiomatizations for XPath fragments”. *Journal of Applied Logic* 8 (2), pp. 153–172.

[O4] **Litak, T.** 2008e. “Stability of the Blok theorem”. *Algebra Universalis* 58 (4), pp. 385–411.

[O5] **Litak, T.** 2007i. “The non-reflexive counterpart of Grz”. *Bulletin of the Section of Logic* 36 (3–4), pp. 195–208.

[O6] **Litak, T.** 2006k. “Algebraization of hybrid logic with binders”. In: *Relations and Kleene Algebra in Computer Science*. Vol. 4136. LNCS, pp. 281–295.

[O7] **Litak, T.** 2006l. “Isomorphism via translation”. In: *Advances in Modal Logic 6*, pp. 333–351.

[O8] **Litak, T.** 2005c. “An Algebraic Approach to Incompleteness in Modal Logic”. PhD thesis. Japan Advanced Institute of Science and Technology.

[O9] **Litak, T.** 2005d. “On notions of completeness weaker than Kripke completeness”. In: *Advances in Modal Logic 5 (2004)*, pp. 149–169.

[O10] **Litak, T.** 2004d. “Modal incompleteness revisited”. *Studia Logica* 76 (3), pp. 329–342.

[O11] **Litak, T.** 2002c. “A continuum of incomplete intermediate logics”. *Reports on Mathematical Logic* 36. Corrected in 2018, `https://arxiv.org/abs/1808.06284`, pp. 131–141.

[O12] **Litak, T.** and Wolter, F. 2005e. “All finitely axiomatizable tense logics of linear time flows are coNP-complete”. *Studia Logica* 81 (2), pp. 153–165.

9.1.4. In Preparation

Only those relevant for this overview are mentioned.

[P1] **Litak, T.** and Visser, A. “Lewis arrow fell off the wall: Decompositions of constructive strict implication”. In preparation.

9.2. Other References

Papers and books I did not (co)author.

[1] Abiteboul, S., Hull, R., and Vianu, V. 1995a. *Foundations of Databases*. Addison-Wesley.

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A. Examples of Coalgebraic Structures

In contrast to Appendix B, this appendix uses material extracted from our published papers [I7, I11, S3, S2]. More examples of endofunctors, predicate liftings and rank-1 axiomatizations can be found in references discussed in § 3.1.

A.1. Some Set-endofunctors

- § 3.1 introduced the doubly contravariant set endofunctor $\mathcal{QQ}$. Its coalgebras are known as *neighbourhood frames* or *Scott-Montague frames* [202, 267, 58].
- The *multiset functor* $\mathcal{B}$ is given on objects by

  $$\mathcal{B}X = \{\mu : X \to \mathbb{N} \cup \{\infty\} \mid f \text{ a map}\}.$$

  Such a map $\mu : X \to \mathbb{N} \cup \{\infty\}$ can be seen as an integer-valued discrete measure on X, i.e., we write $\mu(A) = \sum_{x \in A} \mu(x)$ for $A \subseteq X$. The functor $\mathcal{B}$ acts on maps $f : X \to Y$ by taking image measures; i.e., $\mathcal{B}f(\mu)(B) = \mu(f^{-1}[B])$ for $B \subseteq Y$. Coalgebras for the multiset functor are *multigraphs* [71].
- The *selection function functor* $\mathcal{S}$ acts on objects by $\mathcal{S}X = \{f : \mathcal{Q}X \to \mathcal{P}X\}$ and on maps $f : X \to Y$ by $\mathcal{S}f = \mathcal{Q}f \to \mathcal{P}f : \mathcal{S}X \to \mathcal{S}Y$, i.e. $\mathcal{S}f(g)(A) = f[g(f^{-1}[A])]$ for $A \subseteq X$. We think of $f_x(A)$ as the set of worlds which x sees as "most typical" given a condition $A \subseteq X$. Coalgebras for $\mathcal{S}$ are *selection function frames* [185, 58].

A.2. Predicate Liftings

- Relational modal logic K *over frames* is captured by the basic modal signature $\Lambda = \{\Diamond\}$. The interpretation over the covariant powerset functor $\mathcal{P}$ is defined by means of the predicate lifting $[\![\Diamond]\!]_X(A) = \{Y \in \mathcal{P}X \mid A \cap Y \neq \emptyset\}$. This agrees with the standard Kripke semantics of $\Diamond$.
- Just to instantiate the coalgebraic approach to models suggested in Remark 3.2, relational modal logic K *over models* requires the similarity type $\Lambda = \{\Diamond\} \cup \mathsf{At}$ where At is a set of propositional atoms. Define $TX = \mathcal{P}X \times \mathcal{P}\mathsf{At}$. That is, a coalgebra $\gamma : C \to TC$ assigns to each state $c \in C$ a set of successors as well as a set of propositional atoms valid in c. Now consider the following predicate liftings

  $$[\![\Diamond]\!]_X(A) = \{(Y, U) \in \mathcal{P}X \times \mathcal{P}\mathsf{At} \mid A \cap Y \neq \emptyset\}$$
  $$[\![p]\!]_X = \{(Y, U) \in \mathcal{P}X \times \mathcal{P}\mathsf{At} \mid p \in U\}.$$

  As the mechanism is the same for other predicate liftings, let us restrict attention to the "frame-like" setting in presentation of remaining examples.
- For $\Lambda = \{\Box\}$, $\mathcal{QQ}$ extends to a Λ-structure by $[\![\Box]\!]_C(A) = \{\sigma \in \mathcal{QQ}C \mid A \in \sigma\}$.

- For multigraphs and the functor $\mathcal{B}$, take the signature of *graded modal logic* [101] $\Lambda = \{\langle k \rangle \mid k \geq 0\}$, where $\langle k \rangle$ reads as "more than k successors satisfy …". $\mathcal{B}$ extends to a -structure by stipulating $[\![\langle k \rangle]\!]_X(A) = \{\mu \in \mathcal{B}X \mid \mu(A) > k\}$.
- A set of operators more general than graded modal logic is that of positive Presburger modal logic [75], which admits integer linear inequalities $\sum_{i=1}^{n} a_i \cdot \#(\phi_i) > k$ where $a_i \geq 0$ for all i. Consider an n-ary modality $L_k(a_1, \ldots, a_n)$ interpreted over $\mathcal{B}$ by the n-ary predicate lifting

$$[\![L_k(a_1, \ldots, a_n)]\!]_X(A_1, \ldots, A_n) = \{\mu \in \mathcal{B}X \mid \textstyle\sum_{i=1}^{n} a_i \cdot \mu(A_i) > k\}.$$

- For selection function frames, interpret the conditional $>$ by

$$[\![>]\!]_X(A, B) = \{f \in \mathcal{S}X \mid f(A) \cap B \neq \emptyset\}.$$

  The formula $\phi > \psi$ expresses that ψ is *typically possible* under condition ϕ. This presentation of conditional logic is dual to the standard presentation [58] in terms of a binary operator $\Rightarrow$ "if … then normally …", related to $>$ by $a > b \equiv \neg(a \Rightarrow \neg b)$.

A.3. Rank-1 Axiomatizations

- The rule set for the normal modal logic K consists of the rules

$$\frac{p \to q_1 \vee \ldots \vee q_n}{\Diamond p \to \Diamond q_1 \vee \ldots \vee \Diamond q_n}\ \mathsf{K_n}, \qquad n \geq 0.$$

- Among normal logics with Kripke semantics, another naturally coalgebraic example is D, i.e., the logic of *nonterminating* transitions, which correspond to coalgebras for *nonempty* powerset functor $\mathcal{P}_{\geq 1}$.
- Another Kripkean example is provided by the axiomatization of coalgebras for the endofunctor $\mathcal{P}_{\leq n}$, i.e., for the restriction of the powerset functor to sets of cardinality at most n. The details can be easily worked out.[19]
- Modal neighbourhood semantics is axiomatized by the congruence rule

$$\frac{p \leftrightarrow q}{\Box p \to \Box q}\ \mathsf{C}.$$

  It is important to note here that K is sound and in fact complete w.r.t. the filter variant of the neighbourhood semantics. § 5 provides more insight into this issue.
- For graded modal logic, we have the following rule set [266, Lemma 3.10]:

$$\frac{p \to q}{\Diamond_{n+1} p \to \Diamond_n q}\ \mathsf{RG1} \qquad \frac{r \to p \vee q}{\Diamond_{n_1+n_2} r \to \Diamond_{n_1} p \vee \Diamond_{n_2} q}\ \mathsf{A1}$$

$$\frac{p \to r \quad q \to r \quad p \wedge q \to s}{\Diamond_{n_1} p \vee \Diamond_{n_2} q \to \Diamond_{n_1+n_2+1} r \vee \Diamond_0 s}\ \mathsf{A2} \qquad \frac{\neg p}{\neg \Diamond_0 p}\ \mathsf{RN}.$$

- For non-monotonic conditionals, we have [58, 264, 224, 263]:

$$\frac{p \to q_1 \vee \ldots \vee q_n}{r > p \to r > q_1 \vee \ldots \vee r > q_n}\ \mathsf{RCK} \qquad \frac{p \leftrightarrow q}{p > r \to q > r}\ \mathsf{RE}.$$

[19] Note that for $n \geq 2$, there is no modal axiomatization, rank-1 or otherwise, for frames with branching *at least* n. Correspondingly, $\mathcal{P}_{\geq n}$ is not a functor, due to the existence of functions which are not injective.

Table 1: Typing Rules and Proof-term Assignment

$\Gamma, x : A \vdash_\Sigma x : A$	$\dfrac{f \in \Sigma(A_1,\ldots,A_n;B) \quad \Gamma \vdash_\Sigma M_1 : A_1 \ldots \Gamma \vdash_\Sigma M_n : A_n}{\Gamma \vdash_\Sigma f(M_1,\ldots,M_n) : B}$
$\dfrac{\Gamma, y : \odot A \vdash_\Sigma N : A}{\Gamma \vdash_\Sigma y{\uparrow} N : A}$	$\dfrac{\Gamma \vdash_\Sigma M : \odot A \qquad x : A \vdash_\Sigma N : B}{\Gamma \vdash_\Sigma \mathfrak{b}\, N[\bullet x \rightsquigarrow M] : \odot B}$
$\dfrac{\Gamma \vdash_\Sigma M : A}{\Gamma \vdash_\Sigma \mathfrak{p} M : \odot A}$	$\dfrac{\Gamma \vdash_\Sigma M_1 : A_1 \ldots \Gamma \vdash_\Sigma M_n : A_n}{\Gamma \vdash_\Sigma \langle M_1 \ldots M_n \rangle : A_1 \times \cdots \times A_n}$
$\Gamma \vdash_\Sigma \langle\rangle : \mathsf{unit}$	$\dfrac{\Gamma \vdash_\Sigma \langle M_1 \ldots M_n \rangle : A_1 \times \cdots \times A_n}{\Gamma \vdash_\Sigma \mathsf{Pr}_i(M_i) : A_i}$

B. Sketch of a Proof System for Guarded Recursion

In this appendix based on my unpublished notes, I sketch a presentation of a proof system for guarded Conway categories as defined in my paper with Milius [I10]. Inasmuch as possible, I try to follow the style and conventions of Crole [69], Lambek and Scott [177] or Simpson and Plotkin [274], but also references dealing with modal aspects of proof-term calculi [157, 30, 13, 219]. Of course, all these sources do not agree entirely and hence one has to make choices. Crole [69] is an excellent introduction to categorical aspects of type theory and syntactic aspects of category theory. The framework of Simpson and Plotkin [274] is overall closest to our present interests and needs, the main difference being of course modal/functorial guardedness of the present setting.

Just like in Crole [69, Sec. 4, p. 156], a *signature* Σ consists of:

- a (possibly empty) collection of *base types* $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1 \ldots$ The collection of *types* Type_Σ generated by this signature is written as

$$A, B ::= \alpha \mid \mathsf{unit} \mid A_1 \times \cdots \times A_n \mid \odot A$$

- a (possibly empty) collection of *function symbols* $f, g, h, f_1, g_1, h_1 \ldots$, each with a fixed *arity* and *sorting information* over Type_Σ. We write $f \in \Sigma(A_1, \ldots, A_n; B)$ to denote that in Σ, f is a n-ary function symbol whose respective arguments are of type $A_1, \ldots, A_n$ and the output is of type B.

Note that unlike some references [69, § 3] [274] I do not restrict the sorting information for function symbols to base types, despite that fact that the absence of, e.g., exponentials. The reason is the presence of $\odot$. A *context* Γ is a partial mapping from a supply of variables $x, y, z, x_1, y_1, z_1 \ldots$ (which we keep implicit) to Type_Σ. The typing rules and proof-terms assignment are presented in Table 1. The only binding occurrences of variable x are $x{\uparrow} N$ and $\mathfrak{b}\, N[\bullet x \rightsquigarrow M]$. Define:

$$\begin{aligned}
\mathsf{cmo}^1{}_x &= \mathfrak{b}\, \mathsf{Pr}_1(y)[\bullet y \rightsquigarrow x] \\
\mathsf{cmo}^2{}_x &= \mathfrak{b}\, \mathsf{Pr}_2(y)[\bullet y \rightsquigarrow x] \\
\mathfrak{p}'\, M &= \mathsf{Pr}_2(x{\uparrow}\langle M, \mathsf{cmo}^1{}_x\rangle)
\end{aligned}$$

for x, y fresh in M. Note we can derive the rule $\dfrac{\Gamma \vdash_\Sigma M : A}{\Gamma \vdash_\Sigma \mathfrak{p}'\ M : \odot\, A}$ as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma, x : \odot\,(A \times \odot\, A) \vdash_\Sigma M : A \qquad \dfrac{\Gamma, x : \odot\,(A \times \odot\, A) \vdash_\Sigma x : \odot\,(A \times \odot\, A) \qquad y : A \times \odot\, A \vdash_\Sigma \mathsf{Pr}_1(y) : A}{\Gamma, x : \odot\,(A \times \odot\, A) \vdash_\Sigma {\mathsf{cmo}^1}_x : \odot\, A}}{\Gamma, x : \odot\,(A \times \odot\, A) \vdash_\Sigma \langle M, {\mathsf{cmo}^1}_x \rangle : A \times \odot\, A}}{\Gamma \vdash_\Sigma x {\uparrow} \langle M, {\mathsf{cmo}^1}_x \rangle : A \times \odot\, A}}{\Gamma \vdash_\Sigma \mathsf{Pr}_2(x {\uparrow} \langle M, {\mathsf{cmo}^1}_x \rangle) : \odot\, A}$$

The question of whether $\mathfrak{p}'\ M$ and $\mathfrak{p}\, M$ are or should be considered equal is a syntactical counterpart of Lemma 3.2 in our paper [I10]. In order to discuss such questions, one needs the notion of *equation-in-context*; the definition is standard, self-evident and can be found, e.g., in Crole [69, Sec 3.3, p. 127]. Furthermore, we also need the standard notion of *theory* as a pair $\mathcal{T} = (\Sigma, \mathsf{Ax})$, where Σ is a signature as above and Ax is a set of equations-in-context relative to Σ called *axioms* of $\mathcal{T}$. Just like in Crole [69], axioms are written as $\Gamma \vdash_{\mathsf{Ax}} M = M' : A$. *Theorems* are generated by rules in Table 2; the set of equations-in-context obtained in this way is called *the Conway closure* of $\mathcal{T}$.

Definition B.1. A *delayed cartesian category* $(\mathcal{C}, \bullet)$ is a pair s.t.

- $\mathcal{C}$ is a *cartesian* category, i.e., one with finite products. We denote:
  - the product of two objects as $A \xleftarrow{\pi_\ell} A \times B \xrightarrow{\pi_r} B$;
  - the diagonal as $\Delta_A : A \to A \times A$;
  - the terminal object in a cartesian category as 1 and
  - the unique morphism for each X as $! : X \to 1$.
- $\bullet$ is a pointed *delay* (or *later*) endofunctor on $\mathcal{C}$. Recall that begin pointed means the existence of natural transformation $\mathsf{p} : \mathsf{Id} \to \bullet$.

Definition B.2. [I10, Def. 2.3] A *guarded fixpoint operator* on $(\mathcal{C}, \bullet)$ is a family of operations $\dagger_{X,Y} : \mathcal{C}(Y \times \bullet X, X) \to \mathcal{C}(Y, X)$ s. t. for every $f : Y \times \bullet X \to X$,

$$\begin{array}{ccc} Y & \xrightarrow{\quad f^\dagger \quad} & X \\ {\scriptstyle \langle f^\dagger, Y \rangle} \downarrow & \circlearrowleft & \uparrow {\scriptstyle f} \\ Y \times X & \xrightarrow[\ Y \times p_X\ ]{} & Y \times \bullet X \end{array} \tag{B.1}$$

where (as usual) we drop the subscripts and write $f^\dagger : Y \to X$ in lieu of $\dagger_{X,Y}(f)$. The triple $(\mathcal{C}, \bullet, \dagger)$ is a *guarded fixpoint category*. Moreover, $(\mathcal{C}, \bullet, \dagger)$ is a *unique guarded fixpoint category* if for every $f : \bullet X \times Y \to X$, $f^\dagger$ is the unique morphism satisfying (B.1). In this case, we can just write a pair $(\mathcal{C}, \bullet)$ instead of a triple $(\mathcal{C}, \bullet, \dagger)$.

An interpretation of Σ is a triple $(\mathcal{C}, \bullet, \dagger, [\![\cdot]\!])$, where

- $(\mathcal{C}, \bullet, \dagger)$ is a guarded fixpoint category as given by Definition B.2;
- for any base type α, $[\![\alpha]\!]$ is an object of $\mathcal{C}$. This extends to all types by

$$\begin{aligned} [\![\mathsf{unit}]\!] &= 1, \\ [\![A_1 \times \cdots \times A_n]\!] &= [\![A_1]\!] \times \cdots \times [\![A_n]\!], \\ [\![\odot\, A]\!] &= \bullet [\![A]\!]; \end{aligned}$$

Table 2: Inference Rules of Guarded Conway Theories

Equational reasoning

$$\frac{\Gamma \vdash_{\mathsf{Ax}} M = M' : A}{\Gamma \vdash_{\mathcal{T}} M = M' : A} \qquad \text{(axiom)}$$

$$\frac{\Gamma \vdash_{\Sigma} M : A}{\Gamma \vdash_{\mathcal{T}} M = M : A} \quad \frac{\Gamma \vdash_{\mathcal{T}} M = N : A}{\Gamma \vdash_{\mathcal{T}} N = M : A} \qquad \text{(equation) a, b}$$

$$\frac{\Gamma \vdash_{\mathcal{T}} M = N : A \qquad \Gamma \vdash_{\mathcal{T}} N = P : A}{\Gamma \vdash_{\mathcal{T}} M = P : A} \qquad \text{(equation) c}$$

$$\frac{\Gamma \vdash_{\mathcal{T}} M = M' : A \qquad \pi \text{ a permutation of } \Gamma}{\pi\Gamma \vdash_{\mathcal{T}} M = M' : A} \qquad \text{(exchange)}$$

$$\frac{\Gamma \vdash_{\mathcal{T}} M = M' : A \qquad \Gamma \subseteq \Gamma'}{\Gamma' \vdash_{\mathcal{T}} M = M' : A} \qquad \text{(weakening)}$$

$$\frac{\Gamma, x : A \vdash_{\mathcal{T}} N = N' : B \qquad \Gamma \vdash_{\mathcal{T}} M = M' : A}{\Gamma \vdash_{\mathcal{T}} N[M/x] = N'[M'/x] : B} \qquad \text{(substitution)}$$

Cartesian reasoning

$$\frac{\Gamma \vdash_{\Sigma} M : \mathsf{unit}}{\Gamma \vdash_{\mathcal{T}} \langle\rangle = M : \mathsf{unit}} \qquad \text{(unit)}$$

$$\frac{\Gamma \vdash_{\Sigma} M_1 : A_1 \ldots \Gamma \vdash_{\Sigma} M_n : A_n}{\Gamma \vdash_{\mathcal{T}} \mathsf{Pr}_i(\langle M_1 \ldots M_n\rangle) = M_i : A_i} \qquad \text{(projection)}$$

$$\frac{\Gamma \vdash_{\Sigma} M : A_1 \times \cdots \times A_n}{\Gamma \vdash_{\mathcal{T}} \langle \mathsf{Pr}_1(M) \ldots \mathsf{Pr}_n(M)\rangle = M : A_1 \times \cdots \times A_n} \qquad \text{(product)}$$

Pointed endofunctors

$$\Gamma, x : \odot A \vdash_{\mathcal{T}} \mathfrak{b}\, y[\bullet y \rightsquigarrow x] = x : \odot A \qquad \text{(functoriality) a}$$

$$\frac{\Gamma \vdash_{\Sigma} F : \odot A \qquad x : A \vdash_{\Sigma} G : B \qquad y : B \vdash_{\Sigma} H : C}{\Gamma \vdash_{\mathcal{T}} \mathfrak{b}\,(H[G/y])[\bullet x \rightsquigarrow F] = \mathfrak{b}\, H[\bullet y \rightsquigarrow \mathfrak{b}\, G[\bullet x \rightsquigarrow F]] : \odot C} \qquad \text{(functoriality) b}$$

$$\frac{x : A \vdash_{\Sigma} M : B}{x : A \vdash_{\mathcal{T}} \mathfrak{p}\, M = \mathfrak{b}\, M[\bullet x \rightsquigarrow \mathfrak{p}\, x] : \odot B} \qquad \text{(pointedness)}$$

Dagger equalities

$$\frac{\Gamma, x : \odot A \vdash_{\Sigma} F : A}{\Gamma \vdash_{\mathcal{T}} x{\uparrow} F = F[\mathfrak{p}\, x{\uparrow} F/x] : A} \qquad \text{(fixpoint)}$$

$$\frac{\Gamma, x : \odot A \vdash_{\Sigma} F : B \qquad z : B \vdash_{\Sigma} G : A}{\Gamma \vdash_{\mathcal{T}} x{\uparrow} G[F/z] = G[z'{\uparrow} F[\mathfrak{b}\, G[\bullet z \rightsquigarrow z']/x]/z] : A} \qquad \text{(naturality)}$$

$$\frac{\Gamma, x_1 : \odot A, x_2 : \odot A \vdash_{\Sigma} F : A}{\Gamma \vdash_{\mathcal{T}} x_1{\uparrow} x_2{\uparrow} F = x{\uparrow} F[x_2/x_1] : A} \qquad \text{(diagonality)}$$

- for any $f \in \Sigma(A_1, \ldots, A_n; B)$, $[\![f]\!] \in \mathcal{C}([\![A_1]\!] \times \ldots [\![A_n]\!], [\![B]\!])$.

Given a context $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, we define $[\![\Gamma]\!] = [\![A_1]\!] \times \cdots \times [\![A_n]\!]$. This interpretation is extended to all proof-terms as shown in Table 3.

Given a theory $\mathcal{T} = (\Sigma, \mathsf{Ax})$, we say that an interpretation $(\mathcal{C}, \bullet, \dagger, [\![\cdot]\!])$ is a *model* for Ax if for every axiom $\Gamma \vdash_{\mathsf{Ax}} M = M' : A$, it holds that $[\![\Gamma \vdash_\Sigma M : A]\!] = [\![\Gamma \vdash_\Sigma M' : A]\!]$.

We thus see that the validity of B.1 above is equivalent to the validity of the following equation-in-context:

$$\frac{\Gamma, x : \bullet X \vdash F : X}{\Gamma \vdash x{\uparrow} F = F[\mathfrak{p}\, x{\uparrow} F/x]}.$$

More generally, we have a natural soundness result similar to numerous variants stated in existing references [30, 13, 69, 177, 219, 274]:

Theorem B.3 (Soundness)**.** *For any $\mathcal{T} = (\Sigma, \mathsf{Ax})$, any interpretation $(\mathcal{C}, \bullet, \dagger, [\![\cdot]\!])$, any context Γ, any terms M, M', and type expression A, we have that*

- *$[\![\Gamma \vdash_\Sigma M : A]\!]$ is an element of $\mathcal{C}([\![\Gamma]\!], [\![A]\!])$;*
- *futhermore, whenever $(\mathcal{C}, \bullet, \dagger, [\![\cdot]\!])$ is a model for Ax, $\Gamma \vdash_{\mathcal{T}} M = M' : A$ implies*

$$[\![\Gamma \vdash_\Sigma M : A]\!] = [\![\Gamma \vdash_\Sigma M' : A]\!].$$

Proof. Left as exercise in induction, category theory and application of rules from Tables 2 and 3. ❑

In fact, one could follow our references [30, 13, 69, 177, 219, 274] in proving a syntactic *completeness result*. An algebraist can see the idea of such a proof as a categorical variant of the Lindenbaum-Tarski construction: no identification is made between syntactically different type expressions (formulas), but proof-terms are identified modulo provable equality. Finally, similar results can be proved for extensions of this system with additional rules from Table 4. We have seen in § 6.2 that the guarded setting allows nontrivial distinctions between possible variants of corresponding properties of unguarded Conway theories/categories, and leads to new questions regarding their relationships.

Remark B.4. Note that it is not required that the guarded dagger endofunctor preserves products or is lax monoidal w.r.t. the cartesian structure—i.e., that the modality it interprets is normal. To enforce it, one would need to modify the $\mathfrak{b}$ typing rule to the minimal normal one [13, 219, 157]:

$$\frac{\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash_\Sigma N : B \qquad \Gamma \vdash_\Sigma M_1 : \odot A_1 \qquad \ldots \qquad \Gamma \vdash_\Sigma M_n : \odot A_n}{\Gamma \vdash_\Sigma \mathfrak{a}\, N[\bullet(x_1, \ldots, x_n) \rightsquigarrow (M_1, \ldots, M_n)] : \odot B}$$

In such a setting one can define $\mathfrak{p}\, M$ as $\mathfrak{a}\, M[\bullet() \rightsquigarrow ()]$ (the list $M_i : A_i$ can be empty). Furthermore, both modal typing rules (i.e., $\mathfrak{b}$ and μ) could be then combined in a single equivalent one:

$$\frac{\Gamma, x_1 : A_1, \ldots, x_n : A_n, y : \odot B \vdash_\Sigma N : B \qquad \Gamma \vdash_\Sigma M_1 : \odot A_1 \qquad \ldots \qquad \Gamma \vdash_\Sigma M_n : \odot A_n}{\Gamma \vdash_\Sigma \mathfrak{w} y.\, N[\bullet(x_1, \ldots, x_n) \rightsquigarrow (M_1, \ldots, M_n)] : \odot B}$$

C. Kripke Semantics for $\Box$ and $\dashv$

This section is entirely based on § 3 of our paper [I9] (see also [I5, § 2]); this material is provided to make the overview self-contained. Kripke frames for intuitionistic modal logic come equipped with two accessibility relations. One of them, denoted here by $\preceq$, is a partial ordering (actually, it would be enough to insist on a *preorder*, i.e., a reflexive and transitive relation) interpreting intuitionistic implication:

$$k \Vdash \varphi \to \psi \text{ if, for all } \ell \succeq k, \text{ if } \ell \Vdash \varphi, \text{ then } \ell \Vdash \psi. \tag{C.1}$$

Table 3: Semantics of Proof-terms

$$[\![\Gamma, x : A \vdash_\Sigma x : A]\!] = [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{\pi_r} [\![A]\!]$$

$$\frac{\begin{array}{c} f \in \Sigma(A_1, \ldots, A_n; B) \\ [\![\Gamma \vdash_\Sigma M_1 : A_1]\!] = [\![\Gamma]\!] \xrightarrow{m_1} [\![A_1]\!] \\ \vdots \\ [\![\Gamma \vdash_\Sigma M_n : A_n]\!] = [\![\Gamma]\!] \xrightarrow{m_n} [\![A_n]\!] \end{array}}{[\![\Gamma \vdash_\Sigma f(M_1, \ldots, M_n) : B]\!] = [\![\Gamma]\!] \xrightarrow{[\![f]\!] \cdot \langle m_1, \ldots, m_n \rangle} [\![B]\!]}$$

$$\frac{[\![\Gamma, y : \odot A \vdash_\Sigma N : A]\!] = [\![\Gamma]\!] \times \bullet[\![A]\!] \xrightarrow{m} [\![A]\!]}{[\![\Gamma \vdash_\Sigma y{\uparrow} N : A]\!] = [\![\Gamma]\!] \xrightarrow{m^\dagger} [\![A]\!]}$$

$$\frac{[\![\Gamma \vdash_\Sigma M : \odot A]\!] = [\![\Gamma]\!] \xrightarrow{m} \bullet[\![A]\!] \qquad [\![x : A \vdash_\Sigma N : B]\!] = [\![A]\!] \xrightarrow{n} [\![B]\!]}{[\![\Gamma \vdash_\Sigma \mathfrak{d}\, N[\bullet x \leadsto M] : \odot B]\!] = [\![\Gamma]\!] \xrightarrow{\bullet n \cdot m} \bullet[\![B]\!]}$$

$$\frac{[\![\Gamma \vdash_\Sigma M : A]\!] = [\![\Gamma]\!] \xrightarrow{m} [\![A]\!]}{[\![\Gamma \vdash_\Sigma \mathfrak{p}\, M : \odot A]\!] = [\![\Gamma]\!] \xrightarrow{\mathsf{p}_{[\![A]\!]} \cdot m} \bullet[\![A]\!]}$$

$$\frac{\begin{array}{c} f \in \Sigma(A_1, \ldots, A_n; B) \\ [\![\Gamma \vdash_\Sigma M_1 : A_1]\!] = [\![\Gamma]\!] \xrightarrow{m_1} [\![A_1]\!] \\ \vdots \\ [\![\Gamma \vdash_\Sigma M_n : A_n]\!] = [\![\Gamma]\!] \xrightarrow{m_n} [\![A_n]\!] \end{array}}{[\![\Gamma \vdash_\Sigma \langle M_1 \ldots M_n \rangle : A_1 \times \cdots \times A_n]\!] = [\![\Gamma]\!] \xrightarrow{\langle m_1, \ldots, m_n \rangle} [\![A_1 \times \cdots \times A_n]\!]}$$

$$[\![\Gamma \vdash_\Sigma \langle\rangle : \mathsf{unit}]\!] = [\![\Gamma]\!] \xrightarrow{!} 1$$

$$\frac{[\![\Gamma \vdash_\Sigma \langle M_1 \ldots M_n \rangle : A_1 \times \cdots \times A_n]\!] = [\![\Gamma]\!] \xrightarrow{m} [\![A_1]\!] \times \cdots \times [\![A_n]\!]}{[\![\Gamma \vdash_\Sigma \mathsf{Pr}_i(M_i) : A_i]\!] = [\![\Gamma]\!] \xrightarrow{\pi_i \cdot m} [\![A_i]\!]}$$

Table 4: Additional Rules of Guarded Conway Theories (not assumed in general)

$$\frac{\Gamma, x : \odot A \vdash_\Sigma F : B \qquad \Gamma, z : \odot B \vdash_\Sigma G : A}{\Gamma \vdash_\mathcal{T} x\!\uparrow G[\mathfrak{p}\, F/z] = G[\mathfrak{p}\, z\!\uparrow F[\mathfrak{p}\, G/x]/z] : A} \quad \text{(dinaturality)}$$

$$\frac{\begin{array}{c}\Gamma, x : \odot A \vdash_\Sigma F : A \quad \Gamma, z : \odot B \vdash_\Sigma G : B \quad x' : A \vdash_\Sigma H : B \\ \Gamma, x : \odot A \vdash_\mathcal{T} G[\mathfrak{b}\, H[\bullet x' \rightsquigarrow x]/z] = H[F/x'] : B\end{array}}{\Gamma \vdash_\mathcal{T} z\!\uparrow G = H[x\!\uparrow F/x'] : B} \quad \text{(uniformity)}$$

$$\frac{\Gamma \vdash_\Sigma M : A}{\Gamma \vdash_\mathcal{T} \mathfrak{p}\, M = \mathsf{Pr}_2(x\!\uparrow\langle M, \mathfrak{b}\, \mathsf{Pr}_1(y)[\bullet y \rightsquigarrow x]\rangle) : \odot A} \quad \text{(canonical point)}$$

$$\frac{\Gamma \vdash_\Sigma M : A \quad \Gamma, x : \odot A \vdash_\Sigma F : A \qquad \Gamma \vdash_\mathcal{T} M = F[\mathfrak{p}\, M/x] : A}{\Gamma \vdash_\mathcal{T} x\!\uparrow F = M : A} \quad \text{(fixpoint uniqueness)}$$

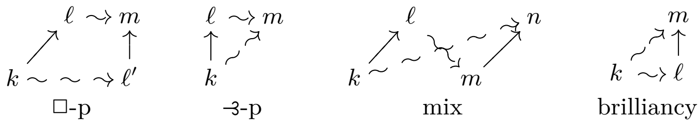

Figure 1: Minimal conditions one can impose on □-frames and ⥽-frames.

It is enough to impose (C.1) and require $\preceq$-*persistence* (*monotonicity*, *upward-closure* ...) of the forcing relation for atoms to ensure persistence for all IPC-formulas. The other accessibility relation $\sqsubset$ is the modal one. There are two choices one can make to ensure $\preceq$-persistence for $\Box$:

> One is to modify the satisfaction clauses. ... The other ... is to impose conditions on models that ensure that the monotonicity lemma does hold. [273, § 3.3]

In fact, in a unimodal language the difference between these two strategies is not essential; it becomes more consequential when a single accessibility relation is used to interpret, for example, both $\Box$ and $\Diamond$ (see Simpson [273, § 3.3] for a discussion and more references). Still, most references choose the latter one, i.e., keeping the same reading of $\Box$ as in the classical case and imposing conditions on the interaction of $\preceq$ and $\sqsubset$ to ensure persistence.

Božić and Došen [43] show that in the presence of unary $\Box$ with semantics defined by

$$k \Vdash \Box\varphi \text{ if, for all } \ell \sqsupset k,\ \ell \Vdash \psi$$

persistence is equivalent to the condition

□-p if $k \preceq \ell \sqsubset m$, then, for some ℓ', we have $k \sqsubset \ell' \preceq m$

(i.e., $\preceq \cdot \sqsubset \subseteq \sqsubset \cdot \preceq$, where " $\cdot$ " denotes relational composition). However, most references require tighter interaction, e.g., Godblatt [116] requires

⥽-p if $k \preceq \ell \sqsubset m$, then $k \sqsubset m$ (i.e., $\preceq \cdot \sqsubset \subseteq \sqsubset$).

But the most common one [278, 298, 299] is the still stronger

mix if $k \preceq \ell \sqsubset m \preceq n$, then $k \sqsubset n$ (i.e., $\preceq \cdot \sqsubset \cdot \preceq \subseteq \sqsubset$).

This condition naturally obtains in a canonical model construction à la Stone and Jónsson-Tarski for prime filters of (reducts of) Heyting algebras with normal $\Box$ [43, 278, 166, 27]. Moreover, mix is "mostly harmless" for $\Box$: it can be obtained from □-p by adding the requirement that for any ℓ, the set of its $\sqsubset$-successors is $\preceq$-upward closed, that is,

brilliancy if $k \sqsubset \ell \preceq m$, then $k \sqsubset m$ (i.e., $\sqsubset \cdot \preceq \subseteq \sqsubset$).

The name seems to have been proposed by Iemhoff [146, 143, 144, 145], another one being *strongly condensed* [43]. As noted in standard references [43, 116], not only brilliancy can*not* be defined using $\Box$, but any model satisfying $\Box$-p can be made brilliant *without changing the satisfaction relation for* $\Box$*-formulas* in a straightforward way: by replacing $\sqsubset$ by its composition with $\preceq$.

Consider now the Lewisian strict implication $\varphi \dashv\!\! \dashv \psi$. Here is the natural satisfaction clause in this semantics, directly transferring the classical one:

$$k \Vdash \varphi \dashv\!\! \dashv \psi \text{ if, for all } \ell \sqsupset k, \text{ if } \ell \Vdash \varphi, \text{ then } \ell \Vdash \psi. \tag{C.2}$$

Such an enrichment of the language makes $\Box$-p too weak to ensure persistence.

Definition C.1.
- A *preframe* is a triple $\mathcal{F} = \langle W, \preceq, \sqsubset \rangle$, where $\preceq$ is a partial order, and $\sqsubset$ is a binary relation.
- Given any *valuation* V mapping propositional variables to $\preceq$-upward closed sets, the *forcing relation* $\mathcal{F}, V, k \Vdash \varphi$ is defined in the standard way for the intuitionistic connectives and using equation (C.2) for $\dashv\!\! \dashv$.
- A ($\dashv\!\! \dashv$-)*frame* is a preframe satisfying $\dashv\!\! \dashv$-p .

Fact C.2. [305, 145] *A preframe* $\mathcal{K} = \langle W, \preceq, \sqsubset \rangle$ *is a* $\dashv\!\! \dashv$*-frame iff for any two sets* U, V *upward closed w.r.t.* $\preceq$*, the set*

$$U \dashv\!\! \dashv V := \{k \in W \mid \forall \ell \sqsupset k, \text{ if } \ell \in U, \text{ then } \ell \in V\}$$

is upward closed w.r.t. $\preceq$*, i.e., the denotation of* $\dashv\!\! \dashv$ *preserves* $\preceq$*-persistence.*

We can define in a standard way what it means for a formula to be *valid* or *refuted* in a class of models. As mentioned, for $\dashv\!\! \dashv$ the brilliancy condition does not remain “mostly harmless” in the sense described above for $\Box$ (recall Convention 7.1 for associativity):

Fact C.3. [305] *The following conditions are equivalent for a* $\dashv\!\! \dashv$*-frame:*

- *validity of* $(\varphi \wedge \psi) \dashv\!\! \dashv \chi \to \varphi \dashv\!\! \dashv (\psi \to \chi)$*;*
- *validity of* $\psi \dashv\!\! \dashv \chi \to \top \dashv\!\! \dashv (\psi \to \chi)$*;*
- *validity of brilliancy.*

One easily sees the converse implication $\varphi \dashv\!\! \dashv (\psi \to \chi) \to (\varphi \wedge \psi) \dashv\!\! \dashv \chi$ and, consequently, its special instance $\Box(\psi \to \chi) \to \psi \dashv\!\! \dashv \chi$ (take φ equal to $\top$) to be valid on any $\dashv\!\! \dashv$-frame; indeed, it is derivable from the axioms presented in Remark 7.2 [I9, § 4].

D. Quine, Tarski and the Classical Picture

I have been (ab)using the name of W. V. O. Quine as an example of an important logician being a self-appointed advocate of the Classical Picture. Quotes from several papers and books [235, 237, 239, 240] were used, with a polemical intention, as epigraphs of §§ 2, 6 and 7.1. These quotes, selected from a rich source material, hopefully illustrate that he is not a randomly chosen author. There is also another sense in which the choice of Quine as the *advocatus diaboli* was deliberate: it is hard to name any single person who influenced the perception and scope of logic accepted in post-war American philosophy departments more than he did. Still, I do not want to turn such an eminent opponent into a straw man. Let us address the seemingly disputable aspects of the characterization of Quine as a fervent advocate of what was termed here the Classical Picture:

- on the issue of *SETAC*, the Axiom of Choice, or at least its instances involving "big sets" fail in his "New Foundations" (NF) for set theory;
- on the issue of *BOOL*, despite his infamous labelling of non-classical logics as "deviant", Quine rather confusingly acknowledged in 1951 "revision even of the logical law of the excluded middle" as an empirically acceptable route towards "simplifying quantum mechanics" [236].

Remark D.1. Let us reflect on this mention of quantum mechanics as supposedly subverting "the law of excluded middle". In what sense is $\varphi \vee \neg\varphi$ refuted in orthocomplemented lattices? There are promising topos-based approaches to quantum theory, but it is definitely *not* what Quine had in mind in 1951; if anything, recent successes of these approaches can only be yet another headache for authors sharing Quine's attitude to intuitionistic logic. The truth is much simpler: Quine was systematically conflating the law of excluded middle with the principle of bivalence ("every statement is either true or false"). This can be seen explicitly in the chapter on "deviant logics" in *The Philosophy of Logic* [241, Ch. 6]; let us note that this pathological term was later used in the title of Susan Haack's book [121] and hence made undeservedly popular in some philosophy departments. Interestingly, supposed quantum considerations are *not* included among the four major objections to the law of excluded middle that Quine attempts to refute in *Quiddities* [240, pp. 55–57] (published immediately after the second edition of *The Philosophy of Logic*, the latter still mentioning quantum theory in the context of excluded middle) and yet also in that reference he proves unable or unwilling to distinguish between this law and the principle of bivalence.

We will continue on the thread of Quine's attitude to boolean laws in § D.1. Let us now focus on the former point. The status of AC in NF is not fully clarified, and so seem Quine's actual beliefs on the subject. In a detailed analysis of Thomas Forster:

> So NF refutes AC. However—so far—it seems that (apart from some weird weak formulations that make no sense outside an NF context and which all arise from refinements of Specker's proof) it is only full AC that fails, so that some strong consequences of AC remain on the cards for the moment. In particular, no-one has refuted the prime ideal theorem or countable choice or DC. Further, it is striking that all known failures of AC involve big sets. An axiom saying that all wellfounded sets are wellordered might—for all we know—be consistent with NF. In particular there is no reason to suppose that NF refutes the weak forms of AC needed in the study of the reals and complexes ("analysis"). Indeed as far as we know we can safely add to NF some axioms to say that the wellfounded sets form a model of ZFC. According to this view, NF does not so much contradict ZFC as cover extra topics, such as the universal set—phenomena about which ZF has nothing to say. This underpins the view held by both Church and Quine, albeit in different ways, that a synthesis of ZFC and NF could be obtained along these lines. [104]

Indeed, Quine certainly was not an opponent of AC, quite to the contrary:

> These and other examples … bespeak the urgency of the axiom of choice. We may therefore regret that we cannot prove it from prior assumptions, and hope that it can be added to them without contradiction. [238, § 33, p. 234]

At any rate, it would be far from fair to blame Quine, of all authors, for being most responsible for perpetuating the prejudice that we somehow platonically "know" the "truth" of AC (cf. the first epigraph of § 4). Moreover, the fact that one of the most ardent defenders of the Classical Picture developed a system of set theory where the status of AC is problematic is actually an interesting point, slyly supporting the analysis in § 4 above.

Hence, the overall evaluation of Quine's legacy looks different depending on whether one focuses on his technical contributions, or on his philosophical contributions. On the former front, his achievements including the creation of NF and related systems are, if anything, under-appreciated. On the latter front, his heritage is more problematic. And it is precisely the fact that he was no ordinary author which made his misconceptions particularly pernicious at times. Not to search very far, we have mentioned in Remark 2.1 the curious inability of Alfred Tarski to recognize the modal relevance of his own technical work, which provided a solid foundation for the study of completeness and canonicity. It is hard to ignore Quine's influence on Tarski's beliefs and interests here. Or perhaps, shall we say, their influence on each other. Cf., e.g., the following quote:

> Tarski discusses the circumstances under which he was ready to reject the logical and physical premises of a science. Here his views are similar to views later advanced by W. V. Quine in his "Two Dogmas of Empiricism". In the first footnote of that paper, Quine acknowledges a "large and indeterminate debt" to Tarski and others, and Tarski writes in this letter: "I think that I am ready to reject certain logical premisses (axioms) of our science in exactly the same circumstances in which I am ready to reject empirical premisses (e.g., physical hypotheses); and I do not think that I am an exception in this respect." By 1940, Quine reports in his autobiography, he had developed doubts about the notion of an analytic sentence. He also reports there (p. 150) that Tarski had joined him in these doubts when both of them were members of a discussion group to which Carnap tried, unsuccessfully, to read the whole of the manuscript of his *Introduction to Semantics*. [297]

It is telling that it was Carnap, perhaps the only major figure among (former) Viennese neopositivists disposed favourably towards modal logic (unless one considers Kurt Gödel to have been a "Viennese neopositivist") who was disturbed so successfully by Quine and Tarski. In fact, Tarski just like Quine can be seen as an example of a researcher prevented by his own philosophical superstitions and idiosyncratic preferences from fully utilizing the potential of his technical contributions. And failing to recognize the importance of modal logic or, say, the interesting consequences of rejecting AC (although it is telling that Tarski missed a chance to invent forcing, just like several decades earlier it was left to Gödel to destroy an earlier incarnation of the Classical Picture, despite almost all the tools being at Tarski's disposal!) are far from being the only illustrations of this problem.

A particularly striking irony is that both Quine and Tarski claimed to be interested in applications of formal logic. Yet neither of them seemed to have grasped at all the importance of computer science in this respect. Tarski's decades-long indifference to developments in modal logic happening sometimes literally right next door [38, p. 41] is only mirrored by his equally studious indifference to contributions to computer science made by his own students, including, e.g., early implementations of his own quantifier elimination algorithms [96].

His inability or unwillingness to recognize the relationship between his work on *relation* and *cylindric* algebras and Codd's *relational* algebra/calculus in database theory (see § 8.1) provides an equally dramatic example. Codd did not seem aware of most developments in algebraic logic, but we also have to keep in mind that his "algebra" was designed according to different principles. Recall again from § 8.1 above that a crucial concern was to maintain the *domain independence property* of relational queries. This property is far from being trivial; Vardi [291] shows that it is actually *undecidable* for a query written in standard first-order syntax! Thus departing from the way of thinking shaped by *SYFOL* proved crucial in the database context, even when staying squarely within *RELST*.

One would expect Tarski to have some interest in such developments. Yet, again, there is no evidence he ever showed any. The papers of Imieliński and Lipski [147] and Kanellakis [160] which made the connection explicit were published after Tarski's death (which itself happened several years after E.A. Codd had received the Turing Award!) and apparently came somewhat too late to significantly shape either the further development of "Tarskian-Hungarian" algebraic logic or the further development of database theory; by then, both fields had been set too firmly in their ways. And, while we have no space to enter into details here, this separation does not seem to have served either field particularly well. We should keep in mind, however, that there have been more papers fruitfully exploring this neglected connection (see § 8.1 in this overview, van den Bussche [48], Feferman [96] and references therein), and the logical/algebraic perspective was later more systematically explored in the study of query/navigation languages for semistructured databases (cf. papers listed at the end of § 2 and references therein).

What is the moral of this story? Quine and Tarski were to an extent right on one point during the debates in Carnap's discussion group in the 1940's and in their subsequent publications. Aims and methods of logic can and should be subject to revision (otherwise we would still be working with Aristotelian syllogisms!). The problem is that such revisions have little to do with drastic conflicts with naïvely understood empirical data, quantum physics and suchlike. In acknowledging such spurious factors and overlooking real ones, they both seem to have remained, after all, prisoners of dogmas of empiricism, neopositivism and the Vienna Circle.

Computer science or, as some prefer, *computing science*, which in a famous dictum attributed to Dijkstra[20] is "no more about computers than astronomy is about telescopes", in its more theoretical and less engineering-oriented aspects should be seen as the contemporary continuation of formal philosophy.[21] The questions of the limits of the axiomatic method, of acceptably effective means of proof, of the choice of an adequate formal modelling paradigm, or of formal languages used as contemporary counterparts of Leibniz's *calculus ratiocinator* and *characteristica universalis* can now be tested and studied like never before. It is rather an obvious point that the necessity to rethink such issues (rather than conflicts with physical experience in any meaningful sense of the word!) led to the previous great reform of logic which began in the XIXth century and which gave rise to the Classical Picture. The computer revolution should simply usher in the next stage of this process, never brought to satisfactory completion. In turn, such a change of foundational perspective may prove necessary for computer science itself to achieve "those goals that we have failed to reach", in particular finally meeting "computing's central challenge, viz. 'How not to make a mess of it' ".[22]

D.1. Afterword: Is It Possible at All to Justify The Boolean Laws?

The preceding passage can be seen as the conclusion of the entire paper. Nevertheless, Remark D.1 suggests a curious topic for an afterword: Quine's real attitude to classical logic, often obscured by excesses of his rhetoric—and, in general, the opposing ways in which contemporary authors tried to analyze and justify the boolean dogma.

The "excluded middle" entry in *Quiddities* seems to concede much more ground to the opponents of classical logic than *The Philosophy of Logic* [241] was willing to (recall again that *Quiddities* [240] were published in 1987, whereas the second edition of *The Philosophy of Logic* appeared in 1986!). It is enough to contrast the treatment of the vagueness of terms as one of the supposed challenges to this law in both references:

> Let us grant, then, that the deviant [sic! – T.L.; cf. Remark D.1] can coherently challenge our classical true-false dichotomy. But why should he want to?

[20]Dijkstra appears to have made this claim not in his own writings but, e.g., in a broadcast recorded for Dutch television in 2000. There are, however, much earlier versions of this statement made by other authors, e.g., "CS is no more about computers than astronomy is about telescopes, biology is about microscopes or chemistry is about beakers and test tubes" [97]. According to `https://larc.unt.edu/ian/research/cseducation/`, the original reference appears to have been Michael R. Fellows in 1991, visited by Dijkstra at the University of Victoria in the early 1990's.

[21]Similar views seem to have shaped the second edition of the *Handbook of Philosophical Logic* [107].

[22]Cf. Dijkstra's quote [78] used as the epigraph of this entire overview.

> Reasons over the years have ranged from bad to better. *The worst one* is that things are not just black and white; there are gradations. *It is hard to believe that this would be seen as counting against classical negation; but irresponsible literature to this effect can be cited.* [241, p. 85, emphasis mine]

versus

> So, if a sentence hinges for its truth or falsity on whether to regard some sketchily thatched old man as bald, or whether to regard some cabin a little way up the trail as on the mountain, *it is reasonable* to waive the law of excluded middle and reckon the sentence as neither true nor false. [240, p. 56, emphasis mine]

In a rather short time, Quine's accusations and epithets shifted from unbelievably "irresponsible" sophistry of a "deviant" and the "worst" possible criticism—to a "reasonable" challenge which can only be "appeased … with a double standard" [240, p. 57] … Still more importantly, the discussion in *Quiddities* ends with a surprising admission that

> the law of excluded middle is not a fact of life, but a norm governing efficient logical regimentation. [240, p. 57]

This is actually quite close to the sympathetic position expressed by C. I. Lewis in the quote [182, p. 505] used as the second epigraph of § 6. It is worthy of attention, for example, that Quine appears to suggest dividing propositional laws into those which are "facts of life" and those which are not, throwing a rather curious light on his earlier claims that it is "inane" to try to

> discern equivalence in some sense within the domain of logical truth. Logical equivalence … holds indiscriminately between all logical truths. [241, p. 83]

Let us sum it up: both C. I. Lewis and W. V. O. Quine (in his more honest moments) appear to believe that the law of excluded middle, unlike more basic logical laws, "is not writ in the heavens" [182, p. 505] or "not a fact of life" [240, p. 57]. Instead, it is just "a norm governing efficient logical regimentation" [240, p. 57]; "it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts." [182, p. 505]

Michael Dummett, on the other hand, expresses an exactly opposite view on classical logic in his passionate defence of intuitionism. It seems fair to say that according to him, the law of excluded middle might as well be "writ in heavens" but is definitely *not* a valid reasoning reasoning principle from a human point of view:

> If there are no gaps in reality, that is, no questions that have no answers, then God's logic will be classical. Those many people who favour classical over intuitionistic logic are therefore guilty of the presumption of reasoning as if they were God. [81]

These two attempts to find the foundations of boolean logic cancel each other. The law of excluded middle and its cousins appear left in a void. We have no way of knowing whether such a law is "writ in heavens" and it is questionable whether it is a "fact of life". Perhaps it can be used as a deductive principle by a being with gap-free, total knowledge of platonic reality—but then this is certainly not how our knowledge looks like. Or perhaps it is just a lazy, "simplest of all possible" patterns of reasoning we slip into by the grace of inertia, spurious familiarity with the simplest objects of everyday experience and crudest modelling templates. But then somewhat more abstract concepts, not to mention objects of mathematical study, have a curious tendency to turn naïve intuitions upside down.

By contrast, the laws of intuitionistic logic are supported by the Curry-Howard correspondence and the basic rules of computation. Moreover, we have seen that the study of resource-aware reasoning, geometry of interactions, quantum theory, social choice and infinite populations or even the relational model in database theory invites at times a still more subversive attitude towards received logical ideas, axioms, syntax and metatheory.