NP Reasoning in the Monotone μ-Calculus

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Satisfiability Checking

Complexities of satisfiability checking for some modal logics:

$\mathbf{K}$ / $\mathcal{ALC}$	PSPACE
$\mathbf{K}$ + global axioms (universal modality)	EXPTIME
modal μ-calculus	EXPTIME
monotone modal logic	NP
monotone modal logic + global axioms	?
monotone μ-calculus	?

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▶ $\mathbf{K}$ + global axioms (universal modality)	EXPTIME
▶ modal μ-calculus	EXPTIME
▶ monotone modal logic	NP
▶ monotone modal logic + global axioms	NP
▶ alternation-free monotone μ-calculus + global axioms	NP

Monotone Modal Logic

Syntax

$$\phi, \psi ::= \bot \mid \top \mid p \mid \phi \wedge \psi \mid \phi \vee \psi \mid [a]\phi \mid \langle a \rangle \phi \quad (p \in \mathsf{At},\ a \in \mathsf{Act})$$

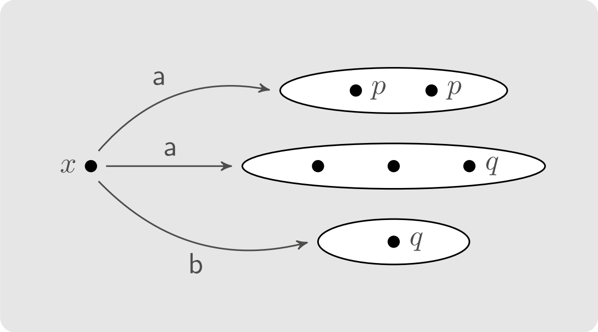
Semantics

Interpret formulae over neighbourhood structures $M = (W, N, I)$,
$N : \mathsf{Act} \times W \to \mathcal{P}(\mathcal{P}(W))$, $I : \mathsf{At} \to \mathcal{P}(W)$

$$[\![[a]\phi]\!] = \{w \in W \mid \forall S \in N(a, w).\, S \cap [\![\phi]\!] \neq \emptyset\}$$
$$[\![\langle a \rangle \phi]\!] = \{w \in W \mid \exists S \in N(a, w).\, S \subseteq [\![\phi]\!]\}$$

Vardi, 1989: Satisfiability checking is in NP (by non-normality)

Monotone Modal Logic, example



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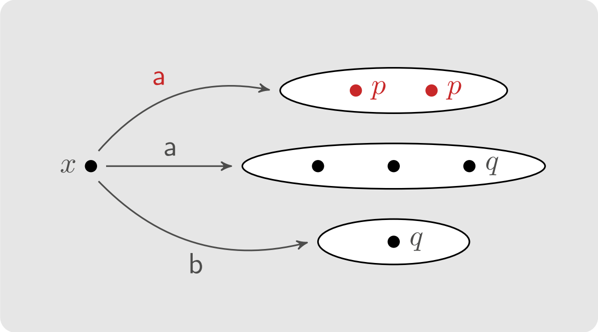
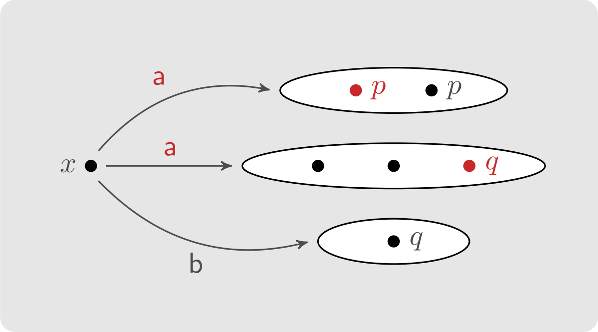


$x \in [\![\langle a \rangle p]\!]$

Monotone Modal Logic, example



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$x \in [\![[a](p \vee q)]\!]$

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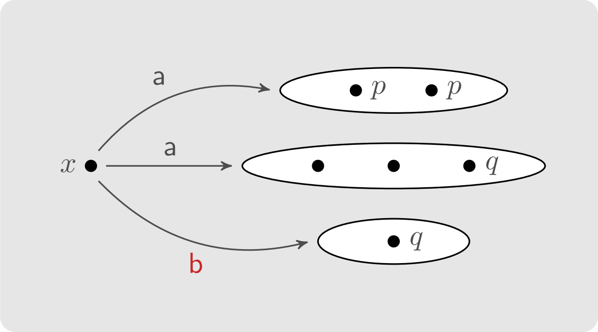
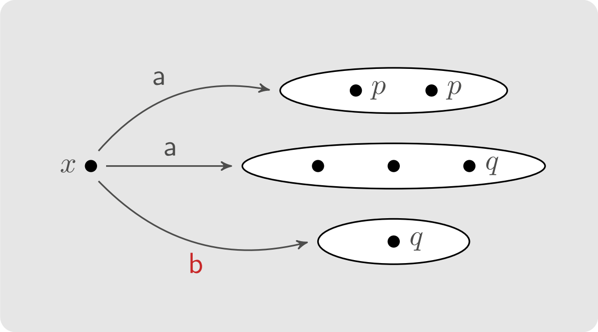


$x \in [\![\langle a \rangle p]\!]$

$x \in [\![[a](p \vee q)]\!]$

$x \notin [\![\langle b \rangle p]\!]$

Monotone Modal Logic, example



$x \in [\![\langle a \rangle p]\!]$

$x \in [\![[a](p \vee q)]\!]$

$x \notin [\![\langle b \rangle p]\!]$

Cannot express e.g. "p holds in every successor state"
"p holds in at least one successor state"

The Monotone μ-Calculus

Syntax

$$\phi, \psi ::= \bot \mid \top \mid p \mid \phi \wedge \psi \mid \phi \vee \psi \mid [a]\phi \mid \langle a \rangle \phi \mid X \mid \nu X.\phi \mid \mu X.\phi$$

$(p \in \mathsf{At},\ a \in \mathsf{Act},\ X \in \mathsf{Var})$

Semantics

Valuation $\sigma : \mathsf{Var} \to \mathcal{P}(W)$

$$[\![X]\!]_\sigma = \sigma(X) \quad [\![\mu X.\phi]\!]_\sigma = \mathsf{LFP}[\![\phi]\!]_\sigma^X \quad [\![\nu X.\phi]\!]_\sigma = \mathsf{GFP}[\![\phi]\!]_\sigma^X$$

where $[\![\phi]\!]_\sigma^X(A) = [\![\phi]\!]_{\sigma[X \mapsto A]}$ for $A \subseteq W$

The Monotone μ-Calculus, ctd.

Readings:

- Epistemic Logic

  $\langle a \rangle \phi$ – "Agent a knows ϕ"

- Concurrent PDL (CPDL), Peleg (1987)

  $\langle \alpha \rangle \phi$ – "There is execution of program α in parallel, nondeterministic system s.t. all end states satisfy ϕ"

- Game Logic, Parikh (1983)

  $\langle \alpha \rangle \phi$ – "Player Angel has strategy to achieve ϕ in game α"

Embedding CPDL into Monotone μ-Calculus

Programs: $\alpha ::= a \mid ?\psi \mid \alpha_1; \alpha_2 \mid \alpha_1 \cup \alpha_2 \mid \alpha_1 \cap \alpha_2 \mid \alpha^*$

Embedding functions $(-)^\sharp$ and τ_γ (mutually recursive):

$$\begin{aligned}
(\langle\gamma\rangle\phi)^\sharp &= \tau_\gamma(\phi^\sharp) \text{ (plus propositional cases)} \\
\tau_{?\psi}(\phi) &= \psi^\sharp \\
\tau_a(\phi) &= \langle a\rangle\phi \\
\tau_{\gamma\cup\delta}(\phi) &= \tau_\gamma(\phi) \vee \tau_\delta(\phi) \\
\tau_{\gamma\cap\delta}(\phi) &= \tau_\gamma(\phi) \wedge \tau_\delta(\phi) \\
\tau_{\gamma;\delta}(\phi) &= \tau_\gamma(\tau_\delta(\phi)) \\
\tau_{\gamma^*}(\phi) &= \mu X.\phi \vee \tau_\gamma(X)
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← X may be unguarded!

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X may be unguarded!

Take size of ϕ to be $|\mathsf{cl}(\phi)|$ (cardinality of *closure*)

$\rightsquigarrow$ *Guarded transformation* has only polynomial blowup

(Bruse et al. 2015)

Embedding Game Logic into Monotone μ-Calculus

Games: $\alpha ::= a \mid ?\psi \mid \alpha_1 ; \alpha_2 \mid \alpha_1 \cup \alpha_2 \mid \alpha_1 \cap \alpha_2 \mid \alpha^* \mid \alpha^\times$

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Embedding Game Logic into Monotone μ-Calculus

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CPDL $\subseteq$ Game Logic (alternation free) $\subseteq$ af monotone μ-calculus

Main Result

Peleg (1987): Satisfiability checking for CPDL is $\mathrm{EXPTIME}$-complete.

But: requires restricting to relational models $\rightsquigarrow$ PDL $\subseteq$ CPDL

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Main Theorem

The satisfiability problem for the alternation-free monotone μ-calculus with global assumptions is NP-complete.

Proof sketch:

ϕ is satisfiable	$\Leftrightarrow$	there is tableau for ϕ	$\Leftrightarrow$	Eloise wins satisfiability game for ϕ

Fix alternation-free monotone μ-calculus formula ϕ, $n ::= |\mathsf{cl}(\phi)|$

Tableau Rules

$$(\bot)\quad \frac{\Gamma, \bot}{} \qquad (\lightning)\quad \frac{\Gamma, p, \neg p}{}$$

$$(\wedge)\quad \frac{\Gamma, \phi_0 \wedge \phi_1}{\Gamma, \phi_0, \phi_1} \qquad (\vee)\quad \frac{\Gamma, \phi_0 \vee \phi_1}{\Gamma, \phi_0 \qquad \Gamma, \phi_1}$$

$$(\langle a\rangle)\quad \frac{\Gamma, \langle a\rangle\phi_0, [a]\phi_1}{\phi_0, \phi_1} \qquad (\eta)\quad \frac{\Gamma, \eta X.\, \phi_0}{\Gamma, \phi_0[\eta X.\, \phi_0/X]}$$

where $a \in \mathsf{Act}$, $p \in \mathsf{At}$, $X \in \mathsf{Var}$, $\eta \in \{\mu, \nu\}$

Alphabet identifying rule applications (and choice of conclusion):

$$\Sigma = \{(\phi_0 \wedge \phi_1), (\phi_0 \vee \phi_1, b), (\eta X.\, \phi_0), (\langle a\rangle\phi_0, [a]\phi_1) \mid b \in \{0,1\}, a \in \mathsf{A}\}$$

Formula Tracking

Nondeterministic tracking function $\gamma : \mathsf{cl}(\phi) \times \Sigma \to \mathcal{P}(\mathsf{cl}(\phi))$:

$$\gamma(\psi, a) = \begin{cases} \{\phi_0, \phi_1\} & \psi = \phi_0 \wedge \phi_1 = a \\ \{\phi_b\} & \psi = \phi_0 \vee \phi_1, a = (\psi, b) \\ \{\phi_0[\eta X.\, \phi_0]\} & \psi = \eta X.\, \phi_0 = a \\ \{\phi_0\} & a = (\langle a \rangle \phi_0, [a]\phi_1), \psi = \langle a \rangle \phi_0 \\ \{\phi_1\} & a = (\langle a \rangle \phi_0, [a]\phi_1), \psi = [a]\phi_1 \end{cases}$$

Deferrals $\mathsf{dfr} \subseteq \mathsf{cl}(\phi)$: Formulae originating from least fixpoints

Define $\delta : \mathsf{dfr} \times \Sigma \to \mathcal{P}(\mathsf{dfr})$ by $\delta(\psi, a) = \gamma(\psi, a) \cap \mathsf{dfr}$

- δ can be seen as transition function of nondeterministic Co-Büchi automaton (accepting *bad branches*)

Definition

Tableau: Finite graph constructed by tableau rules in which all δ-traces of formulae along paths are finite.

Theorem

ϕ is satisfiable $\Leftrightarrow$ There is tableau for ϕ.

Proof: Standard for μ-calculi, but with monotone modality

Büchi Satisfiability Games

$$U = \{\Psi \subseteq \mathsf{cl}(\phi) \mid 2 \geq |\Psi|\} \quad V_\exists = U^2 \quad V_\forall = \mathsf{states}^2 \quad F = \{(\Psi, \emptyset) \in V_\exists\}$$

Node	Moves
$(\Psi, \Phi) \in V_\exists$	$\{(\gamma(\Psi, w), \delta(\Phi, w)) \in V_\forall \mid w \in (\Sigma_p)^*, \lvert w\rvert \leq 3n\}$
$(\Gamma, \Phi) \in V_\forall$	$\{(\{\phi_0, \phi_1\}, \Phi') \in V_\exists \mid \{\langle a\rangle\phi_0, [a]\phi_1\} \subseteq \Gamma\}$,
	if $\Phi \neq \emptyset$, then $\Phi' = \delta(\Phi, (\langle a\rangle\phi_0, [a]\phi_1))$,
	if $\Phi = \emptyset$, then $\Phi' = \{\phi_0, \phi_1\}$ }

- Propositional reasoning condensed into single Eloise-moves
- Modal steps track at most two formulae
- Implicit: economic variant of Miyano/Hayashi Co-Büchi automata determinization

Theorem

Eloise wins satisfiability game $\Leftrightarrow$ There is tableau for ϕ.

Büchi Satisfiability Games

saturated sets of formulae ↘

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$\mathcal{O}(n^2)$ states

Theorem

Eloise wins satisfiability game $\Leftrightarrow$ There is tableau for ϕ.

Victory!

Conclusion

Results:

- Satisfiability checking for
  - CPDL
  - alternation-free Game Logic
  - alternation-free monotone μ-calculus with global axioms

  is only NP-complete!
- Polynomial bound on model size ($\mathcal{O}(n^2)$)

Future work:

- How about full monotone μ-calculus / Game Logic?