Coalgebraic Predicate Logic

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Abstract. We propose a generalization of first-order logic originating in a neglected work by C.C. Chang: a natural and generic correspondence language for any types of structures which can be recast as Set-coalgebras. We discuss axiomatization and completeness results for two natural classes of such logics. Moreover, we show that an entirely general completeness result is not possible. We study the expressive power of our language, contrasting it with both coalgebraic modal logic and existing first-order proposals for special classes of Set-coalgebras (apart for relational structures, also neighbourhood frames and topological spaces). The semantic characterization of expressivity is based on the fact that our language inherits a coalgebraic variant of the Van Benthem-Rosen Theorem. Basic model-theoretic constructions and results, in particular ultraproducts, obtain for the two classes which allow for completeness—and in some cases beyond that.

1 Introduction

Non-relational semantics play an important and ever-increasing role in computer science, e.g. in concurrency, reasoning about knowledge and agency, description logics and ontologies (see e.g. [1, 6, 20, 16]). Nevertheless, the expressivity of ordinary modal logic is somewhat limited. Just as reasoning about relational structures, reasoning about probabilities, agency, social interactions, or conditionals may require variable binding, interaction of local and global information, or reference to individual states. Moreover, a natural and well-tailored predicate language would allow a transfer of (or at least a comparison with) methods, tools and results of classical and finite model theory.

Thus motivated, we propose *coalgebraic predicate logic (CPL)*: a generic and natural first-order language to reason about such diverse structures as neighbourhood frames, discrete Markov chains, conditional frames, multigraphs and indeed any type of structure that can be understood in terms of $\mathbf{Set}$-coalgebras. In particular, the interpretation of CPL over Kripke frames (sets with a binary relation) recovers the standard semantics of first-order logic.

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Our proposal originates in a largely forgotten paper by C.C. Chang [4] which in contemporary terms can be described as an early contribution to the model theory of *Scott-Montague neighbourhood frames*, i.e., coalgebras for the doubly contravariant powerset functor $\mathcal{N} = \mathcal{Q}\mathcal{Q}$. Chang's original motivation was to simplify model theory for what Montague called *pragmatics* and to replace Montague's many-sorted setting by a single-sorted one. Chang's contributions were primarily of model-theoretic nature. He provided adaptations of (elementary) submodel/extension, elementary chain of models and ultraproduct and established Tarski-Vaught, downward and upward Löwenheim-Skolem theorems. One of the main notable points of [4] are its lucid motivation, natural examples and concise syntax, with only one sort of variables and no need for explicit quantification over neighbourhoods or successors. Here we are going to work with a notational variant of Chang's original syntax which we find even more readable.
The semantics uses the fact that coalgebraic structures can be naturally described in terms of *modal operators*. For example, relational semantics yield an operator $\Diamond$: *there exists a successor* ..., and probability distributions an operator L_p: *with probability* $\geq p$ .... More abstractly, (n-ary) modal operators $\heartsuit$ come equipped with a coalgebraic interpretation taking an n-tuple of predicates as arguments. Each operator induces (in the unary case) an atomic formula $t\heartsuit\lceil z : \phi\rceil$ where t is a term, ϕ is a formula of coalgebraic predicate logic and z is a (comprehension) variable. Intuitively, the above formula stipulates that (the denotation of the term) t satisfies property $\heartsuit$, which may parametrically depend on the set of all z that satisfy ϕ. For example, standard modal logic over relational semantics provides a formula $x\Diamond\lceil z : z = y\rceil$ which is semantically equivalent to stipulating that x has y as a successor, i.e., $y \in R(x)$. In the probabilistic setting, validity of $xL_p\lceil y : y \neq x\rceil$ forces that the probability of moving from x to a different state is $\geq p$.
Our aim is to convince the reader that CPL is a fruitful common generalization of both first-order logic and coalgebraic modal logic. Section 2 introduces syntax, semantics and a number of intuitive examples. Section 3 discusses axiomatization and completeness results for two natural classes of structures, including neighbourhood and Kripke frames as extremal cases. Moreover, we show that a fully general completeness result must necessarily fail even for rather natural classes of structures (e.g., Markov chains with non-standard probabilities). Section 4 gives both syntactic and semantic characterizations of coalgebraic modal logic as a fragment of CPL. The semantic characterisation naturally generalizes the van Benthem-Rosen characterization of ordinary modal logic. Section 5 takes first steps in the model theory of CPL and Section 6 concludes.

Related Work. We have already discussed Chang's paper [4] not only in terms of the inspiration of the approach presented here, but also in terms of concrete results on the first-order logic of neighbourhood frames. An alternative, two-sorted language for neighbourhood frames has been proposed in [12, Section 5]. Over neighbourhood frames, the language studied in the present work is a fragment of that of [12]. Without giving full syntactic details, our $x\heartsuit\lceil y : \phi(y)\rceil$ (we restrict the attention to the unary case to keep things simple) can be translated as $\exists u.(x\mathsf{N}u \wedge \forall y.(u\mathsf{E}y \leftrightarrow \phi(y)))$.
First-order formalisms have also been considered for topological spaces, which happen to be particular instances of neighbourhood frames when defined in terms of local neighbourhood bases. In particular, Sgro [28] studies interior operator logic in topology together with interior modalities also for all finite topological powers of the space, which do not seem meaningful in the topological context. This language is the weakest

one in the hierarchy of topological languages considered in the classical overview paper [30]. However, the closest reference in this line of work seems to be [17], which does in fact provide a completeness result for the Chang language itself, i.e., a special version of Theorem 7 below. See also [3] for a more contemporary reference.
The relationship between coalgebraic logic and first order logic is the subject of [24], albeit using involved three-sorted syntax and not giving an axiomatization. The technical results of [24] remain valid—indeed, we use them below in Section 4 to establish a van Benthem-Rosen theorem for our language. An explicit embedding of our language into that of *op.cit* is given in the proof of Theorem 15 below. However, one-sorted coalgebraic predicate logic as presented in this paper seems a more natural common generalization of first-order logic and coalgebraic modal logic. It can be shown that our language is a *proper* fragment of that of [24] using, e.g., Example 27 in *op.cit*.
Finally, a different generic first-order logic largely concerned with the Kleisli category of a monad rather than with coalgebras for a functor is introduced and studied in [14]. Of all the languages discussed above, this one seems least related to the present one; indeed, the study of connections with languages like that of [24] is mentioned in [14] as a subject for future research. We also believe the study of possible connections could be of interest.

2 Syntax, Semantics and Examples

We fix a modal similarity type Λ consisting of modal operators $\heartsuit$ and a set Σ of predicate symbols; every $\heartsuit \in \Lambda$ and $P \in \Sigma$ comes with a fixed arity, but instead of writing $\mathsf{ar}P$ or $\mathsf{ar}\heartsuit$, we will just use natural numbers for readability (typically n for $\mathsf{ar}\heartsuit$ and k for $\mathsf{ar}P$). Formulas of *coalgebraic predicate logic (CPL)* over Λ and Σ (denoted as $\mathcal{CPL}_\Lambda\Sigma$, but we will drop Σ wherever possible) are given by the grammar

$$\phi, \psi \quad ::= \quad y_1 = y_2 \mid P(\boldsymbol{x}) \mid \bot \mid \phi \to \psi \mid \forall x.\phi \mid x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil$$

where $\heartsuit \in \Lambda$ is an n-ary modal operator and $P \in \Sigma$ a k-ary predicate symbol, x, y_i are variables from a fixed set iVar we keep implicit. Booleans and the existential quantifier are defined in the standard way. We do not include function symbols which can be added at no extra cost [4]. In the $\lceil y_i : \phi_i\rceil$ component, y_i is used as a comprehension variable, i.e., $\lceil y_i : \phi_i\rceil$ denotes a subset of the carrier of the model, to which modal operators can be applied in the usual way. In $x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil$, x is free and y_i is bound in ϕ_i (not elsewhere though!), otherwise the notions of freeness and boundedness are standard. A variable is *fresh* for a formula if it does not have free occurrences in it. A *sentence*, as usual, is a formula without free variables. The notion of a (capture-avoiding) substitution is defined in the expected way: all the usual caveats for quantified variables have to apply now to comprehension variables as well.
Formally, elements of $\mathcal{CPL}_\Lambda$ are interpreted over *coalgebras*, that is, pairs $(C, \gamma : C \to TC)$ consisting of a carrier set C and a transition function γ that maps every world into a set TC of *structured successors*, where $T : \mathbf{Set} \to \mathbf{Set}$ is an endofunctor extending to a *Λ-structure*, i.e. equipped with a set-indexed family of mappings $[\![\heartsuit]\!]_C : (\mathcal{Q}C)^n \to \mathcal{Q}TC$ for every n-ary modal operator $\heartsuit \in \Lambda$ ($\mathcal{Q}$ is the contravariant powerset functor) subject to *naturality*, i.e. $(Tf)^{-1} \circ [\![\heartsuit]\!]_C = [\![\heartsuit]\!]_D \circ (f^{-1})^n$ for every set-theoretic function $f : C \to D$.

A pair $\mathfrak{M} = (C, \gamma, I)$ consisting of a coalgebra $\gamma : C \to TC$ and a predicate interpretation $I : \Sigma \to \bigcup_{n\in\omega} \mathcal{Q}(C^n)$ respecting arities of symbols will be called a *(coalgebraic) model*. In other words, a coalgebraic model consists simply of a $\mathbf{Set}$-coalgebra and an ordinary first-order model whose universe coincides with the carrier of the coalgebra. Given a model $\mathfrak{M} = (C, \gamma, I)$ and a valuation $v : \mathsf{iVar} \to C$, we define satisfaction $\mathfrak{M}, v \models \phi$ in the standard way for first-order connectives and for $\heartsuit$ by the clause

$$\mathfrak{M}, v \models x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil \iff \gamma(v(x)) \in [\![\heartsuit]\!]_C([\![\phi_1]\!]_C^{y_1}, \ldots, [\![\phi_n]\!]_C^{y_n})$$

where $[\![\phi]\!]_C^y := \{c \in C \mid \mathfrak{M}, v[c/y] \models \phi\}$ and $v[c/y]$ is v modified by mapping y to c.

We have the following examples of our setting.

Social Situations and Neighbourhood Frames. The modifications proposed in [4] probably would not be accepted in Montague's account of *pragmatics*, but as noted by Chang himself, the resulting language is particularly well-tailored for reasoning about social situations and relationships between an individual and sets of individuals. The semantics is given in terms of neighbourhood frames, which we capture coalgebraically using $\Lambda := \{\Box\}$ and by putting $TC := \mathcal{Q}\mathcal{Q}C$ (the doubly contravariant powerset functor) which extends to a Λ-structure by $[\![\Box]\!]_C(A) := \{\sigma \in TC \mid A \in \sigma\}$. In the presence of a binary relation $S(x, y)$ that we read as 'x speaks to y' and interpreting $\Box$ as 'enjoyable', the formula $\exists y_1.\exists y_2.(x\Box\lceil z : S(z, y_1)\rceil \wedge x\Box\lceil z : S(z, y_2)\rceil \wedge y_1 \neq y_2)$ reads as 'there are at least two people such that x finds it enjoyable to speak to them' where x determines the truth of this sentence by inspecting the set $\{z : S(z, y_i)\}$ of people speaking to y_i.

Relational first-order logic. As already discussed, for $TC := \mathcal{P}C$, i.e., covariant powerset endofunctor, we get a notational variant of ordinary FOL over relational structures.

Facebook Friends and Graded Modal Logic. We obtain a variant of graded modal logic [9] if we consider the similarity type $\Lambda = \{\langle k\rangle \mid k \geq 0\}$ where $\langle k\rangle$ reads as 'more than k successors satisfy …'. We interpret the ensuing logic over multigraphs: coalgebras for $\mathcal{B}C := \{f : C \to \mathbb{N} \mid f(c) \neq 0$ only finitely often$\}$, extending $\mathcal{B}$ to a Λ-structure by stipulating $[\![\langle k\rangle]\!]_X(A) = \{f \in \mathcal{B}X \mid \sum_{x\in A} f(x) > k\}$ to express that more than k successors (counted with multiplicities) have property A. Given a $\mathcal{B}$-coalgebra $C \xrightarrow{\gamma} \mathcal{B}C$, we can think of elements of C as individuals, and of $\gamma(c)(c')$ as the number of 'likes' (in the sense of Facebook) that c' has received from c. In other words, $\gamma(c)(c') = n$ models the fact that c has pressed the 'like'-button on c's page n times. In the presence of a binary relation $F(x, y)$ expressing that y is a Facebook-friend of x, the formula $x\langle k\rangle\lceil z : \exists y.F(x, y) \wedge F(y, z)\rceil$ expresses that x likes more than k activities of friends of his/her friends.

Presburger modal logic and arithmetic. A more general set of operators than graded modal logic is that of positive Presburger modal logic [7], which admits integer linear inequalities $\sum a_i \cdot \#(\phi_i) > k$ among formulas (we assume that $a_i \geq 0$). By keeping the same functor $\mathcal{B}$, we can also give the corresponding predicate lifting in a natural way. As before, let C be the supply of individuals but $\gamma(c)(c')$ will be now the number of *posts* of c to c''s wall. In addition to $F(x, y)$ as above, we introduce $T(x, y)$ expressing that y is a follower of x in Twitter and $I(x)$ expressing that x is influential. Then, the formula $\forall x.(x(3 \cdot \#\lceil y : F(x, y)\rceil + 1 \cdot \#\lceil y : T(x, y)\rceil > 10000) \to I(x))$ means that, if x's weighted number of wall posts to his/her Facebook friends and Twitter-followers is greater than ten thousands, then x is influential, provided that Facebook is three times as influential as Twitter.

Combination of Frame Classes. Frame classes can be combined: instead of using the relation symbol R in the previous example, we could consider coalgebras $(C, \gamma : C \to TC)$ where $TC := \mathcal{B}C \times \mathcal{P}C$ gives a multigraph structure and a relational structure, and interpret the operators $\langle k \rangle$ and $\Box$ by projecting out the components. We leave it to the reader to express 'x likes more than k activities of friends of his/her friends' in this setting. Alternatively, we can take $T := \mathcal{B} \times \mathcal{Q}\mathcal{Q}$ and combine operators for the Facebook sense of 'like' and Chang's modalities for social situations. A formula $\neg x \heartsuit \lceil y : y\langle 3 \rangle \lceil y : y = z \rceil \rceil$ expresses then that x does not fancy the perspective of liking strictly more than 2 of Facebook activities of z (or, to be more precise, the general company of people who do so). The reader may find it entertaining to compare our Facebook examples with these of [27].

Agents and Coalition Logic. Coalgebraically, the semantics of coalition logic [20] or, equivalently, alternating time temporal logic [1] is formulated over game frames $\mathcal{G}(X) = \{(S_i)_{i \in P}, f : \prod_{i \in P} S_i \to X \mid \emptyset \neq S_i \subseteq \mathbb{N}\}$ where P is a (fixed) set of players, S_i is the set of strategies available to player $i \in A$ and f is an *outcome function* that determines the next move of the game, depending on the strategy chosen by each player. We use the modalities $\Lambda = \{[Q] \mid Q \subseteq A\}$ where $[Q]$ reads 'the coalition Q of players can achieve ...'. The functor T extends to a Λ-structure via $[\![[Q]]\!]_X(A) = \{(f, (S_i)) \in \mathcal{G}X \mid \exists (s_i)_{i \in P} \forall (s_j)_{j \in P \setminus Q} (f(s_i)_{i \in P} \in A)\}$ which gives the standard semantics of coalition logic and alternating time temporal logic. Given a coalgebra $(C, \gamma : C \to \mathcal{G}C)$, we think of C as being the positions of a strategic game, and $\gamma(c)$ as describing the different strategies available to the agents, and their ramifications. In this context, the formula $x[\emptyset]\lceil y : y = x \rceil$ describes that the state x is a dead end: independent of the choice of strategies of the players, the next position will be x itself. The formula $\forall y (x[Q]\lceil z : z = y \rceil \to y[Q]\lceil z : z = x \rceil)$ expresses that—given position x on the board—whenever coalition Q can force a position y on the game board, they also have a (collective) strategy to revert back to x. Universal quantification over x would then ensure that coalition Q enjoys this power, irrespective of the state of the game.

Ludo and Probabilistic Modal Logic. Taking the similarity type $\Lambda = \{\langle p \rangle \mid p \in [0,1] \cap \mathbf{Q}$ and reading $\langle p \rangle$ as 'with probability at least p', we obtain a localised version of Halpern's probabilistic first-order logic [11] and $\Lambda_k = \{\langle n/k \rangle \mid n = 0, \dots, k\}$ restricts to probabilities in the set of multiples of $1/k$. Both logics are interpreted over (local) probability distributions, that is, the Λ-structure given by $\mathcal{D}X = \{\mu : X \to [0,1] \mid \mu$ has finite support and $\sum_x \mu(x) = 1\}$ where $[\![\langle p \rangle]\!]_X = \{\mu \in \mathcal{D}X \mid \sum_{x \in A} \mu(x) \geq p\}$. If all possible probabilities are contained in some finite set (such as when rolling a die) we consider the sub-structure $\mathcal{D}_k X = \{\mu : X \to \{0, 1/k, \dots, k/k\} \mid \sum_x \mu(x) = 1\}$ with the same interpretation of the modal operator. Taking the carrier of a model to consist of the positions of a ludo board, the (true) formula $x\langle 1/2 \rangle \lceil y : \langle 1/2 \rangle \lceil z : z = x \rceil \rceil$ expresses the fact that x can capture, with probability $\geq 1/2$ all pieces that could capture x (with the same probability).

Party Invitations and Non-Monotonic Conditionals. An example of a binary modality is provided by (conditional) implication $\Rightarrow$, written in infix notation. We interpret $\Rightarrow$ on selection function frames $\mathcal{S}X = \{f : \mathcal{P}(X) \to \mathcal{P}(X)\}$ using $[\![\Rightarrow]\!]_X(A, B) = \{f \in \mathcal{S}X \mid f(A) \cap f(B) \neq \emptyset\}$. The formula $\phi \Rightarrow \psi$ expresses that ψ is possible under condition ϕ. This presentation of conditional logic is equivalent to (but not identical) to the standard presentation [5] and has the technical advantage of boundedness in the second argument (that we will use in Section 3). In the spirit of Chang's original examples concerning social situations, we may read the antecedent of the conditional as 'invited' and the consequent as 'happy'. Given a binary relation ff ('facebook friend') the formula $\exists y(x(\lceil y : \mathsf{ff}(x,y)\rceil \Rightarrow \lceil z : z = y \rceil)$ describes that there is a person (y – possibly Mark Zuckerberg) who is happy if x invites *precisely* her facebook friends to her birthday party. If x also invites non-facebook friends, then the non-monotonicity of the conditional does not allow us to infer anything about y's emotional state.

3 Completeness

In order to state our axiomatization and completeness results, we need an auxiliary notion of *one-step satisfiability*.

Definition 1. Given any supply of primitive symbols D (which can be any set), define $\mathcal{M}^0(D)$ as $\boldsymbol{A}, \boldsymbol{B} ::= d \mid \boldsymbol{A} \to \boldsymbol{B} \mid \bot$ where $d \in D$, $\mathcal{M}^{\curlywedge}_{\Lambda}(D)$ as $\boldsymbol{W}, \boldsymbol{V} ::= \heartsuit d_1 \dots d_n \mid \boldsymbol{W} \to \boldsymbol{V} \mid \bot$ and $\mathcal{M}^1_\Lambda(D)$ as $\boldsymbol{X}, \boldsymbol{Y} ::= \heartsuit \boldsymbol{A}_1 \dots \boldsymbol{A}_n \mid \boldsymbol{X} \to \boldsymbol{Y} \mid \bot$; in other words, $\mathcal{M}^1_\Lambda(D) = \mathcal{M}^{\curlywedge}_{\Lambda}(\mathcal{M}^0(D))$. For any $C \in \mathbf{Set}$, given a valuation $\tau : D \to \mathcal{P}(C)$, we write $C, \tau \models \boldsymbol{A}$ if $\tau(\boldsymbol{A}) = \top$. We also set $[\![\boldsymbol{X}]\!]_{TC,\tau}$, i.e., the interpretation of $\boldsymbol{X}$ in the boolean algebra $\mathcal{P}(TC)$ under τ, to be the inductive extension of the assignment $[\![\heartsuit \boldsymbol{A}_1 \dots \boldsymbol{A}_n]\!]_{TX,\tau} = [\![\heartsuit]\!]_C(\tau(\boldsymbol{A}_1), \dots, \tau(\boldsymbol{A}_n))$. We write $TC, \tau \models \boldsymbol{X}$ if $[\![\boldsymbol{X}]\!]_{TC,\tau} = TC$, and $t \models_{TC,\tau} \boldsymbol{X}$ if $t \in [\![\boldsymbol{X}]\!]_{TC,\tau}$. A set $\Xi \subseteq \mathcal{M}^1_\Lambda$ is *one-step satisfiable* w.r.t. τ if $\bigcap_{\boldsymbol{X} \in \Xi} [\![\boldsymbol{X}]\!]_{TC,\tau} \neq \emptyset$. If $D \subseteq \mathcal{P}(C)$ and τ is just the inclusion, we will usually drop it from the notation.

Just like in case of coalgebraic modal logic (see Section 4 below), proof systems for CPL are best described in terms of rank-1 rules—or, more precisely, rule schemes.

Definition 2. Fix a collection sVar of schematic variables $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \dots$. A *one-step rule* is of the form $\boldsymbol{A}/\boldsymbol{X}$, $\boldsymbol{A} \in \mathcal{M}^0(\mathsf{sVar})$ and $\boldsymbol{X} \in \mathcal{M}^1_\Lambda(\mathsf{sVar})$. A one-step rule will be called a *one-step axiom scheme* if its premise is empty. A rule is *one-step sound* if $TC, \tau \models \boldsymbol{X}$ whenever $C, \tau \models \boldsymbol{A}$ for a valuation $\tau : \mathsf{sVar} \to \mathcal{P}(C)$. Given a set $\mathcal{R}$ of one-step rules and a valuation $\tau : \mathsf{sVar} \to \mathcal{P}(C)$, a set $\Xi \subseteq \mathcal{M}^1_\Lambda(\mathsf{sVar})$ is *one-step consistent (with respect to τ)* [26] if the set $\Xi \cup \{\boldsymbol{X}\sigma \mid \sigma : \mathsf{sVar} \to \mathcal{M}^0; \boldsymbol{A}/\boldsymbol{X} \text{ a rule in } \mathcal{R}; C, \tau \models \boldsymbol{A}\sigma\}$ is propositionally consistent.

From now on, we will only consider rule sets one-step sound relatively to a given Λ-structure, so the assumption of one-step soundness will not be mentioned explicitly.

Definition 3. A rule set $\mathcal{R}$ is *strongly 1-step complete (S1SC)* for a Λ-structure if for every $C \in \mathbf{Set}$, any $\Xi \subseteq \mathcal{M}^1_\Lambda x$ and any $\tau : \mathsf{sVar} \to \mathcal{P}(C)$, Ξ is one-step satisfiable wrt τ whenever it is one-step consistent wrt τ. We say that a set of rules is *finitary S1SC* if the above holds whenever $\tau : \mathsf{sVar} \to \mathcal{P}_{\mathit{fin}}(C)$ (but not necessarily for arbitrary τ).

Full S1SC is a somewhat restrictive condition; of all examples in Section 2, it is satisfied by neighbourhood and coalition logic modalities, but not by the remaining ones, which only enjoy finitary S1SC. However, the latter property in itself is too weak to ensure completeness results; we need an additional property of associated predicate liftings.

Tab. 1: Enderton-style [8] Axioms for CPL

Everywhere below, $\forall\overline{y}.$ denotes a sequence of universal quantifiers of arbitrary length, possibly empty.

Axiom schemes valid for arbitrary structures	
EG1	tautologies of propositional logic, axiomatized for example by: $\forall\overline{y}.\,((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi)))$ $\forall\overline{y}.\,(((\phi \to \bot) \to \phi) \to \phi)$ $\forall\overline{y}.\,(\phi \to ((\phi \to \bot) \to \psi))$
EG2	$\forall\overline{y}.\,(\forall x.\phi \to \phi)$
EG3	$\forall\overline{y}.\,(\forall x.\,(\phi \to \psi) \to (\forall x.\phi \to \forall x.\psi))$
EG4	$\forall\overline{y}.\,(\phi \to \forall x.\phi)$ (x fresh for ϕ)
EG5	$\forall\overline{y}.\,(x = x)$
EG6	$\forall\overline{y}.\,(x = z \to (P(\overline{u}, x, \overline{v}) \to P(\overline{u}, z, \overline{v})))$ $(P \in \Sigma)$ $\forall\overline{y}.\,(x = z \to (x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil \to z\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil))$
CONG	$\forall\overline{y}.(\forall x.\,((\phi_1 \leftrightarrow \psi_1) \wedge \ldots \wedge (\phi_n \leftrightarrow \psi_n)) \to$ $\to \forall x.\,(x\heartsuit\lceil x : \phi_1\rceil \ldots \lceil x : \phi_n\rceil \leftrightarrow x\heartsuit\lceil x : \psi_1\rceil \ldots \lceil x : \psi_n\rceil))$
ONESTEP	$\forall\overline{y}.\forall z.(\boldsymbol{A}\sigma) \to \forall x.[\sigma, x, z](\boldsymbol{X})$ ($\boldsymbol{A}/\boldsymbol{X}$ a rule in $\mathcal{R}$, $\sigma : \mathsf{sVar} \to \mathcal{CPL}_\Lambda$ and $[\sigma, x, z] : \mathcal{M}^1_\Lambda(\mathsf{sVar}) \to \mathcal{CPL}_\Lambda$ as in Def. 5)
An additional axiom scheme for predicate liftings k-bounded in argument i	
BDPL$_{k,i}$	$\forall\overline{y}.(x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil \leftrightarrow \exists z_1 \ldots z_k.(x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_{i-1} : \phi_{i-1}\rceil$ $\lceil y_i : y_i = z_1 \vee \cdots \vee y_i = z_k\rceil\lceil y_{i+1} : \phi_{i+1}\rceil \ldots \lceil y_n : \phi_n\rceil \wedge \bigwedge_{j \leq k} \phi_i[y_i/z_j])$ ($\overline{z}$ fresh for $y_i, \overline{\phi}$)

Definition 4. A modal operator $\heartsuit$ is *k-bounded* in i-th argument for $k \in \mathbb{N}$ and with respect to a Λ-structure T if for every $C \in \mathbf{Set}$ and every $\overline{A} \subseteq C$,

$$[\![\heartsuit]\!]_C(A_1, \ldots, A_n) = \bigcup_{B \subseteq A_i, \#B \leq k} [\![\heartsuit]\!]_C(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n).$$

(This implies in particular that $\heartsuit$ is monotonic in the i-th argument.) We say that Λ is bounded w.r.t. T if every modal operator $\heartsuit \in \Lambda$ for every i smaller than its arity is $k_{\heartsuit,i}$-bounded in i for some $k_{\heartsuit,i}$.

Examples of such operators include—apart from Kripke frames (1-bounded)—graded operators over multigraphs and **positive** Presburger logic. See [25] for details. Note that, e.g., the neighbourhood modality clearly fails to be k-bounded; boundedness is a "Kripke-like" property. The notions of strong one-step completeness and boundedness can be combined for n-ary operators. For example, the binary operator $\Rightarrow$ of conditional logic is strongly one-step complete in the first argument and finitary one-step complete in the second which is expressed by restricting valuations of the second argument to finite sets.
In our axiomatization, we will have to translate one-step rules into predicate axioms. Here is an auxiliary notion:

Definition 5. Let $\sigma : \mathsf{sVar} \to \mathcal{CPL}_\Lambda$ be a substitution. Then for any $x, y \in \mathsf{iVar}$, let $[\sigma, y, x]$ denote the mapping $\mathcal{M}^1_\Lambda(\mathsf{sVar}) \to \mathcal{CPL}_\Lambda$ defined as the inductive extension of the mapping sending each $\heartsuit(\boldsymbol{A}_1 \ldots \boldsymbol{A}_n)$ to $x\heartsuit\lceil y : \widehat{\sigma}(\boldsymbol{A}_1)\rceil \ldots \lceil y : \widehat{\sigma}(\boldsymbol{A}_n)\rceil$, where $\widehat{\sigma}$ is the inductive extension of σ to $\mathcal{M}^0$.

Let $\Gamma, \Delta \subseteq \mathcal{CPL}_\Lambda$, let $\mathcal{R}$ be a set of one-step rules and $\phi \in \mathcal{CPL}_\Lambda$. Write $\Gamma \vdash_{\Delta,\mathcal{R}} \phi$ if there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ s.t. $\gamma_1 \to \ldots \to \gamma_n \to \phi$ can be deduced from Δ, EG1–EG6, CONG and ONESTEP in Table 1 using **only Modus Ponens**. This clearly defines *a finitary deducibility relation* in the sense of Goldblatt [10, Sec. 8.1] and being *$\vdash_{\Delta,\mathcal{R}}$-consistent* is equivalent with being *finitely $\vdash_{\Delta,\mathcal{R}}$-consistent* in his sense, that is, $\Gamma \vdash_{\Delta,\mathcal{R}} \bot$ iff there is $\Gamma_0 \subseteq_{fin} \Gamma$ s.t. $\Gamma_0 \vdash_{\Delta,\mathcal{R}} \bot$. Note that the axiom CONG is in fact (a syntactic variant of) an axiom already introduced by Chang in [4].

Definition 6. For any set of additional axioms $\Delta \subseteq \mathcal{CPL}_\Lambda$ and any rule set $\mathcal{R}$, we say that a logic given by Δ and $\mathcal{R}$ is *strongly complete* wrt a given Λ-structure if for any set of sentences $\Gamma \in \mathcal{CPL}_\Lambda$, $\Gamma \nvdash_{\Delta,\mathcal{R}} \bot$ holds **if and only if** there is a coalgebraic Λ-model for Γ where axioms given by Δ hold (and, obviously, the rules in $\mathcal{R}$ are sound) under the reading of all $\heartsuit \in \Lambda$ given by the structure.

Theorem 7 (Completeness). *The set of axioms given in Table 1 is a strongly complete axiomatization of $\mathcal{CPL}_\Lambda$ whenever the Λ-structure satisfies either of the following conditions:*
- *there exists a S1SC rank-1 rule set.*
- *there exists a finitary S1SC rank-1 rule set and each $\heartsuit \in \Lambda$ is bounded.*

Example 8. For the examples discussed in Section 2 the situation is as follows. Completeness holds for neighbourhood models as they have a strongly one-step complete axiomatisation. For all others, but excluding non-monotonic conditionals, finitary one-step complete axiomatisations exist. Boundedness holds for relational models, graded modal logic and the logic of finite probabilities (interpreted over $\mathcal{D}_k$-coalgebras) which gives completeness using Theorem 7. The binary operator $\Rightarrow$ of conditional logic is strongly one-step complete in the first argument and 1-bounded in the second, and, as a consequence, the first-order logic of non-monotonic conditionals is also complete, see [25, Section 2.3] for more details.

Remark 9. The Omitting Types Theorem is a standard result of model theory. Goldblatt [10, Section 8.2] shows how to establish it wherever a Henkin-style completeness proof is available. This covers the two classes of structures in the statement of Theorem 7. Since both formulation and proof are entirely analogous to the standard relational case, we omit the details and refer the reader to [10, Section 8.2]; let us only note that the fact we used variables instead of Henkin constants (making use of advantages of an Enderton-style axiomatization) does not lead to any complications in the proof, in fact making it even simpler in some cases.

We briefly consider those cases where boundedness does not apply. In order to show both how completeness fails and what are possible alternative means to handle such a situation, we introduce a new class of functors/Λ-structures. We believe it to be of independent interest in coalgebraic logic. In the whole subsection, to keep things simple we work with unary $\heartsuit \in \Lambda$.

Definition 10 (ω-Bounded operators). A modal operator $\heartsuit$ is *ω-bounded* if for each set X and each $A \subseteq X$, $[\![\heartsuit]\!]_X(A) = \bigcup_{B \subseteq_{fin} A} [\![\heartsuit]\!]_X(B)$.

Example 11. Let D^h be the discrete distributions functor with probabilities taken from hyperreal fields. Explicitly: we intend to model Markov chains with non-standard probabilities; these consist of a set X of states, and at each state x an R_x-valued transition distribution μ_x, where R_x is a hyperreal field (we take this to mean a model of the first-order theory of the reals).These structures are coalgebras for the functor T which maps a set X to the set of pairs (R, μ) where R is a hyperreal field and μ is an R-valued probability measure. This functor is in fact class-valued, which however does not affect the applicability of our coalgebraic analysis (which never requires iterated application of the coalgebraic type functor). We take the modal signature Λ to consist of the operators M_p ('with probability more than p') for $p \in [0, 1] \cap \mathbb{Q}$.

Theorem 12. *Whenever a Λ-structure makes some $\heartsuit \in \Lambda$ ω-bounded without being k-bounded for any $k \in \omega$, strong completeness fails for any non-empty supply of predicate symbols Σ.*

Completeness for the specific case of ω-bounded operators (possibly with some additional assumptions, like a variant of S1SC property) could be restored by means of a deduction system equipped with an explicit ω-rule. A natural candidate is

$$\{\forall y_1, ..., y_k.(\phi[y/y_1] \wedge \cdots \wedge \phi[y/y_k] \rightarrow \neg x\heartsuit\lceil y : y = y_1 \vee ... \vee y = y_k\rceil) \mid k \in \omega\}/\neg x\heartsuit\lceil y : \phi\rceil.$$

In fact, Henkin-style completeness proofs for logics with infinitary rules work quite naturally in the framework of [10]. We are not pursuing this option here. As we will see below, there are other positive results which can be proved about ω-bounded operators.

4 Correspondence with Coalgebraic Modal Logic

The formulas $\mathcal{CML}_\Lambda(\Sigma)$ of pure (coalgebraic) modal logic in the modal signature Λ over Σ (now all elements of Σ are assumed to be of arity 1) are given by the grammar: $\phi, \psi ::= P \mid \bot \mid \phi \rightarrow \psi \mid \heartsuit\phi_1 \dots \phi_n$. Satisfaction is defined wrt $\mathfrak{M} = (\gamma, I)$ and a specific point $c \in C$ in a standard way, see e.g. [24, 25].

Definition and Proposition 13. Define the *coalgebraic standard translation* as $ST_x(P) := P(x)$, $ST_x(\heartsuit\phi_1 \dots \phi_n) := x\heartsuit\lceil x : ST_x(\phi_1)\rceil \dots \lceil x : ST_x(\phi_n)\rceil$, $ST_x(\bot) = \bot$, $ST_x(\phi \rightarrow \psi) = ST_x(\phi) \rightarrow ST_x(\psi)$. Then for any $\phi \in \mathcal{CML}_\Lambda(\Sigma)$, and any $\mathfrak{M} = (\gamma, I), v, c$, we have $\mathfrak{M}, c \vDash \phi \quad$ iff $\quad \mathfrak{M}, v[c/x] \vDash ST_x(\phi)$.

For example, $ST_x(\heartsuit\heartsuit P) = x\heartsuit\lceil x : x\heartsuit\lceil x : P(x)\rceil\rceil$. This definition is more straightforward than the standard translation into FOL of modal logic over ordinary Kripke frames. Moreover, ST_x uses only one variable from iVar, namely x itself. In the context of standard Kripke models, expressiveness of modal logic is characterized by van Benthem's theorem: modal logic is the bisimulation invariant fragment of first-order logic in the corresponding signature. The finitary analogue of this theorem [21] states that every formula that is bisimulation invariant *over finite models* is equivalent *over finite models* to a modal formula. In the coalgebraic context, replace bisimilarity with behavioural equivalence [29]. Moreover, we need to assume that the language has 'enough' expressive power; e.g., we cannot expect that bisimulation invariant formulas are equivalent to CML formulas over the empty similarity type. This is made precise as follows:

Definition 14. The Λ-structure T is *separating* if, for every set X, every element $t \in TX$ is uniquely determined by the set $\{(\heartsuit, A) \mid \heartsuit \in \Lambda\ n\text{-ary}, A \in \mathcal{P}(X)^n, t \in [\![\heartsuit]\!]_X(A)\}$.

Separation is in general a less restrictive condition than those we needed for completeness proofs. Of all examples introduced in Section 2, the only one which fails it is coalition logic. It was first used to establish the Hennessy-Milner property for coalgebraic logics [18, 23] and it is easy to see that all our examples are indeed separating. In particular, separation automatically obtains for Kripke semantics.

Theorem 15. *Suppose that T is separating and $\phi(x)$ is a CPL formula with one free variable. Then ϕ is invariant under behavioural equivalence (over finite models) iff it is equivalent to an infinitary CML formula with finite modal rank (over finite models).*

If we deal with finite similarity types only, the conclusion can be strengthened:

Theorem 16. *Suppose that T is separating, Λ is finite and $\phi(x)$ is a CPL formula with one free variable. Then ϕ is invariant under behavioural equivalence (over finite models) iff ϕ is equivalent to a **finite** CML formula (over finite models).*

The proof uses [24, Theorem 24], which in turn relies on a somewhat less natural three-sorted language. Instantiated to the case of Kripke models, we recover the classical results of [2, 21].

5 Beginning Model Theory

We proceed to develop some basic notions of coalgebraic model theory: we introduce an ultraproduct construction on coalgebras, and we prove a downward Lowenheim-Skolem theorem. As is often the case in coalgebraizations of classical model constructions, the structure on the ultraproduct is not uniquely determined, so we refer to the candidate structures as quasi-ultraproducts. Since ultraproducts imply compactness, they will exist only under restrictive conditions, specifically a semantic version of the alternative conditions needed for the completeness theorem. The assumptions needed for the Lowenheim-Skolem theorem are slightly more relaxed.

Observe that if $X = \prod_{\mathfrak{U}} X_i$ is an ultraproduct of sets and (A_i) is a family of subsets $A_i \subseteq X_i$, then $A = \prod_{\mathfrak{U}} A_i := \{x \mid \{i \mid x_i \in A_i\} \in \mathfrak{U}\}$ is a well-defined subset of X.

Definition 17 (Quasi-Ultraproducts of Coalgebras). Let $(C_i) = (X_i, \xi_i)_{i \in I}$ be a family of T-coalgebras, and let $\mathfrak{U}$ be an ultrafilter on I. A coalgebra ξ on the set-ultraproduct $X = \prod_{\mathfrak{U}} X_i$ is called a *quasi-ultraproduct* of the C_i if for every family (A_i) of subsets $A_i \subseteq X_i$, every $x \in \prod_{\mathfrak{U}} X_i$, and every $\heartsuit \in \Lambda$,

$$\xi(x) \in [\![\heartsuit]\!]_X \prod\nolimits_{\mathfrak{U}} A_i \iff \{i \in I \mid \xi_i(x_i) \in [\![\heartsuit]\!]_{C_i}(A_i)\} \in \mathfrak{U}. \qquad (1)$$

The notion of quasi-ultraproduct extends naturally to coalgebraic models.

Theorem 18 (Coalgebraic Łoś's Theorem). *If $\mathfrak{M} = (C, \gamma, V)$ is a quasi-ultraproduct of $\mathfrak{M}_i = (C_i, \gamma_i, V_i)$ for the ultrafilter $\mathfrak{U}$, then for every tuple $(a^1, \dots, a^n)$ of states in C, where $a^k = (a_i^k)_{i \in I}$, and for every CPL formula $\phi(x_1, \dots, x_n)$, $C \models \phi(a^1, \dots, a^n) \iff \{i \mid C_i \models \phi(a_i^1, \dots, a_i^k)\} \in \mathfrak{U}$.*

From this theorem, we obtain the usual applications, in particular compactness. The question is, of course, when quasi-ultraproducts exist. A core observation is

Lemma 19. *In the notation of Definition 17, the demands placed on $\xi(x)$ by (1) constitute a finitely satisfiable set of one-step formulas.*

The lemma immediately implies that the quasi-ultraproducts exist if the logic is one-step compact, e.g. for neighbourhood logic and coalition logic. This is a mild generalization of the corresponding construction in [4]. Alternatively, we can use bounded operators, along with a semantic version of finitary S1SC axiomatizability:

Definition 20. A Λ-structure is *finitary one-step compact* if for every set X, every finitely satisfiable set $\Phi \subseteq \mathcal{M}_\Lambda^\curlywedge(\mathcal{P}_{fin}(X))$ of one-step formulas is satisfiable.

Remark 21. Finitary one-step compactness is clearly a consequence of finitary S1SC, hence all our "Kripke-like" cases enjoy this property. Interestingly enough, Example 11 also happens to be finitary one-step compact although its operators are only ω-bounded but not k-bounded. While the ultraproduct construction cannot be available in such cases (cf. Theorem 12), counterparts of some other standard results fare better, notably Lowenheim-Skolem.

Theorem 22. *If a Λ-structure is finitary one-step compact and all its operators are bounded, then it has quasi-ultraproducts.*

Theorem 23 (The Downward Löwenheim-Skolem Theorem). *If a Λ-structure is ω-bounded and finitary one-step compact, then $\mathcal{CPL}_\Lambda$ satisfies the downward Löwenheim-Skolem theorem.*

Theorem 24. *If a Λ-structure is one-step compact, then $\mathcal{CPL}_\Lambda$ satisfies the downward Löwenheim-Skolem.*

A special case of Theorem 24 for neighbourhood logic has been proved in [4].

6 Conclusions and Further Work

We believe this work opens up several new research avenues. The route towards coalgebraic finite model theory has already been paved in [24], and our Van Benthem-Rosen result is based on the spadework done therein. It is worth observing that Van Benthem-Rosen is a rare instance of a model-theoretic characterization of a fragment of FOL which remains valid over finite models. The only other major one we are aware of is the characterization of existential-positive formulas as exactly those preserved under homomorphisms [22]. The result is relevant to constraint satisfaction problems and to database theory, as existential-positive formulas correspond to unions of conjunctive queries. Interestingly, the proof of Rossman's result relies on Gaifman graphs, which also play a central role in the proof of the Rosen theorem used in [24]. A general CPL variant of Rossman result and development of non-relational database theory seem thus natural research directions.

Generalizations of standard results of *classical* model theory like Beth definability or interpolation and the Keisler-Shelah characterization theorem also seem an interesting research problem. A Herbrand theorem could lead towards an investigation of logic programming in a general coalgebraic setting.

While we are rather satisfied with the shape of our Hilbert-style axiomatization, it would certainly be of interest to study Gentzen-style proof systems. A natural route to explore would be to marry ordinary proof systems for first-order logic with one-step Gentzen systems for CML [19]. This will be in fact the subject of our forthcoming paper.

It remains to be seen which results of *modal model theory* building upon the interplay between modal and predicate languages can be generalized. Specific potential examples include Sahlqvist-type results for suitably well-behaved structures and analogues of results by Fine (does elementary generation imply canonicity, at least wherever the coalgebraic Jónsson-Tarski theorem [15] obtains?) or Hodkinson [13] (is there an algorithm generating a CML axiomatization for CPL-definable classes of coalgebras?). Finally, a very natural future work from the point of view of the coalgebraic community would be to study models based on coalgebras for endofunctors on other categories than $\mathbf{Set}$ and variants of CPL with non-boolean propositional bases.

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