

FMSoft

Lecture 4 — Recursive equations and fixpoints

(pre-lecture version)

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- Recall the concept of *preorder* (sometimes also called **quasiorder**) ...
- ...a **reflexive** and **transitive** relation
- A **partial order** is a preorder which is **antisymmetric**
- A **poset**: partially ordered set. Notation (W, \sqsubseteq)

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- **Naturals/integers/rationals/reals** with natural ordering.

Note both \leq and \geq would do!

Turning a poset upside down produces a poset again

- **Divisibility ordering**: write $m \mid n$ for $\exists k.m \cdot k = n$

Variant a: $k \neq 0$

Variant b: k allowed to be 0

- A paradigm example: given any set S , its **powerset** 2^S (a.k.a. $\wp(S)$) equipped with **inclusion relation** \subseteq
- ...partial functions with inclusion order ...?

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- Note being a preorder/poset is **stable under relativization/substructure**
- What it means: assume that (W, Ξ) is a poset (or a preordered set)
- Consider now $V \subseteq W$
- Clearly, $\Xi \cap V^2$ (i.e., Ξ restricted to V) is also a partial order (or a preorder)
- This means, in particular, that when we consider any family of sets $A \subseteq 2^S$, (A, \subseteq) will still be a poset
- Return now to partial functions example ...

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This is where October 30 has ended ...

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- Recall the notion of **l.u.b. (least upper bound)** a.k.a. **supremum** a.k.a. **join** \sqcup ...
most common notation being \vee
- ...and that of **g.l.b. (greatest lower bound)** a.k.a. **infimum** a.k.a. **meet** \sqcap ...
most common notation being \wedge
- A poset where every $\{w, v\}$ has its l.u.b: **join-semilattice**
Write $w \sqcup v$ for $\sqcup\{w, v\}$
- A poset where every $\{w, v\}$ has its g.l.b: **meet-semilattice**
Write $w \sqcap v$ for $\sqcap\{w, v\}$
- If a poset is both a join-semilattice and a meet-semilattice, it is a **lattice**
- Easy blackboard (counter-)examples ...

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Our flagship examples

- **Naturals/integers/rationals/reals ...?**
- **Divisibility ordering ...?**
- ...l.u.b. $m \sqcup n$ is the **least common multiple**
a.k.a. *lowest common multiple, or smallest common multiple*
- $(2^S, \subseteq)$...?
- Partial functions ...?
- The answers will be discussed during the lecture

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Complete lattices

- **every** subset has a supremum ...
- ...every subset has an infimum ...?
- Other corollaries: bounds ...?
- **Naturals/integers/rationals/reals ...?**
- **Divisibility ordering ...?**
Variant a: $k \neq 0$
Variant b: k allowed to be 0
- $(2^S, \subseteq)$...?
- $\sqcup X = \bigcup X$ and $\sqcap X = \bigcap X$

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- Is this property stable under relativization/substructure though ...?
- Esp. in semantics of programming languages, one often considers structures where only *some* suprema exists ...
- ...e.g., those of **directed sets (dcpo)** or **ω -chains (ω -cpo)**

- Consider now $f : (W, \sqsubseteq) \rightarrow (W, \sqsubseteq)$
- w is a **fixpoint** of f if $f(w) = w$
- Assume (W, \sqsubseteq) is a lattice. Does every f have a fixpoint?
- Okay, how about **monotone** ones?
- What if both \top and \perp exist?

- We will need to find such fixpoints in our model-checking algorithms ...
- ...and later, e.g., to understand the semantics of recursive commands such as while loops
- But clearly, we need assumptions about both
 1. at least monotonicity of f or perhaps even better properties
 2. existence of certain suprema/infima in (W, \sqsubseteq) ...
...more than just those of \emptyset

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Knaster-Tarski

- This is the first method of computing, available in **complete** lattices
- When dealing with 2^S , not a problematic assumption
- In other words, Knaster-Tarski requires
 1. just monotonicity of f , but
 2. all the suprema and infima are supposed to exist in (W, \sqsubseteq)

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- Can we hope that such solutions are **unique** ...?
- What are most obviously distinguished solutions then?
- ...minimal and maximal, i.e.,
- ...the **least** and the **greatest** one!

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- Let (W, \sqsubseteq) be a complete lattice
- $f : (W, \sqsubseteq) \rightarrow (W, \sqsubseteq)$ monotone
- Let $Pre_f := \{w \in W \mid f(w) \sqsubseteq w\}$
- Let $Post_f := \{w \in W \mid w \sqsubseteq f(w)\}$
- ...how about least and greatest elements of these sets?

Theorem (Knaster-Tarski)

- *The least fixpoint of f exists and is obtained as $\sqcap Pre_f$*
- *The greatest fixpoint of f exists and is obtained as $\sqcup Post_f$*

Sketch.

To show the first item, it is enough to prove $\sqcap Pre_f \in Pre_f$. **Full proof on the blackboard.** Second shown analogously. \square

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- The original Knaster-Tarski in fact more general
- Fixpoints form a **complete lattice**
- We do not need it here
- Instead, note $\sqcap\{w \in W \mid f(w) \sqsubseteq w\}$ is not the most useful formula ...

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- At least in the finite case, it is possible to compute these more efficiently
- **(discussion on blackboard)**
- Note: using this finitary method, we approach least fixpoint using **postfixpoints!**
- Using classical set theory, this can be generalized to the infinite case, but requires theory of **ordinals** and **transfinite induction**
- We won't do it here
- But if we assume a little more than monotonicity, we only need natural numbers
- ...and we don't need completeness then, or even lattice structure: only very few suprema are required to exist!
- This is the **Kleene fixpoint theorem** ...we may need it later

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- Why both smallest and greatest fixpoints are going to be important for us?
- Recall our goal: computing

$$[[\phi]]^{\mathcal{M}} := \{s \in \mathcal{M} \mid s \models \phi\}$$

- There are some obvious functions $2^S \rightarrow 2^S$
- Consider $(\text{EX})A := \{s \in S \mid \exists t. s \longrightarrow t \ \& \ t \in A\}$...
- ...and $(\text{AX})A := \{s \in S \mid \forall t. s \longrightarrow t \Rightarrow t \in A\}$
- ...and $f_1(A) = [[\phi]]^{\mathcal{M}} \cap (\text{AX})A$...
- ...now contrast it with $f_2(A) = [[\phi]]^{\mathcal{M}} \cup (\text{EX})A$

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Equivalences for fixpoint computation

- $\text{AG} \phi \equiv \phi \wedge \text{AXAG} \phi$
- $\text{EG} \phi \equiv \phi \wedge \text{EXEG} \phi$
- $\text{AF} \phi \equiv \phi \vee \text{AXAF} \phi$
- $\text{EF} \phi \equiv \phi \vee \text{EXEF} \phi$
- $\text{A}[\phi \text{U} \psi] \equiv \psi \vee (\phi \wedge \text{AXA}[\phi \text{U} \psi])$
- $\text{E}[\phi \text{U} \psi] \equiv \psi \vee (\phi \wedge \text{EXE}[\phi \text{U} \psi])$

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- $[\text{AG}\phi]^{\mathcal{M}} = [\phi]^{\mathcal{M}} \cap (\text{AX})[\text{AG}\phi]^{\mathcal{M}}$
- $[\text{EG}\phi]^{\mathcal{M}} = [\phi]^{\mathcal{M}} \cap (\text{EX})[\text{EG}\phi]^{\mathcal{M}}$
- $[\text{AF}\phi]^{\mathcal{M}} = [\phi]^{\mathcal{M}} \cup (\text{AX})[\text{AF}\phi]^{\mathcal{M}}$
- $[\text{EF}\phi]^{\mathcal{M}} = [\phi]^{\mathcal{M}} \cup (\text{EX})[\text{EF}\phi]^{\mathcal{M}}$
- $[\text{A}[\phi\text{U}\psi]]^{\mathcal{M}} = [\psi]^{\mathcal{M}} \cup ([\phi]^{\mathcal{M}} \cap (\text{AX})[\text{A}[\phi\text{U}\psi]]^{\mathcal{M}})$
- $[\text{E}[\phi\text{U}\psi]]^{\mathcal{M}} = [\psi]^{\mathcal{M}} \cup ([\phi]^{\mathcal{M}} \cap (\text{EX})[\text{E}[\phi\text{U}\psi]]^{\mathcal{M}})$

- Where do we need greatest fixpoints?
- Where do we need least ones?