FMSoft
Lecture 4 — Recursive equations and fixpoints

(lecture version)

Tadeusz Litak

Nov 6, 2018

Informatik 8, FAU Erlangen-Nürnberg

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- A partial order is a preorder which is antisymmetric
- A poset: partially ordered set. Notation $(W, \sqsubseteq)$

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- ...partial functions (identified with their graphs) with inclusion order ...?

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- Return now to partial functions example ...

This is where October 30 has ended ...

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- Easy blackboard (counter)examples ...

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- Partial functions ...?
- ...they do NOT form a lattice!

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- $\bigsqcup X = \bigcup X$ and $\bigsqcap X = \bigcap X$

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- A major example: partial functions

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- What if both $\top$ and $\bot$ exist?

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    1. at least monotonicity of f or perhaps even better properties
    2. existence of certain suprema/infima in $(W, \sqsubseteq)$ ...

       ...more than just those of $\varnothing$

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- ...the least and the greatest one!

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Theorem (Knaster-Tarski)

- *The least fixpoint of f exists and is obtained as $\bigsqcap Pre_f$*
- *The greatest fixpoint of f exists and is obtained as $\bigsqcup Post_f$*

Sketch.
To show the first item, it is enough to prove $\bigsqcap Pre_f \in Pre_f$. Full proof on the blackboard. Second shown analogously. □

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- Instead, note $\bigsqcap\{w \in W \mid f(w) \sqsubseteq w\}$ is not the most useful formula ...

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- This is the Kleene fixpoint theorem ...we may need it later

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- ...and $f_1(A) = [\![\phi]\!]^{\mathcal{M}} \cap (\mathsf{AX})A$ ...

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- ...and $f_1(A) = [\![\phi]\!]^{\mathcal{M}} \cap (\mathsf{AX})A$ ...
- ...now contrast it with $f_2(A) = [\![\phi]\!]^{\mathcal{M}} \cup (\mathsf{EX})A$

Equivalences for fixpoint computation

- $\mathsf{AG}\phi \equiv \phi \wedge \mathsf{AX}\,\mathsf{AG}\phi$
- $\mathsf{EG}\phi \equiv \phi \wedge \mathsf{EX}\,\mathsf{EG}\phi$
- $\mathsf{AF}\phi \equiv \phi \vee \mathsf{AX}\,\mathsf{AF}\phi$
- $\mathsf{EF}\phi \equiv \phi \vee \mathsf{EX}\,\mathsf{EF}\phi$
- $\mathsf{A}[\phi \mathsf{U} \psi] \equiv \psi \vee (\phi \wedge \mathsf{AX}\,\mathsf{A}[\phi \mathsf{U} \psi])$
- $\mathsf{E}[\phi \mathsf{U} \psi] \equiv \psi \vee (\phi \wedge \mathsf{EX}\,\mathsf{E}[\phi \mathsf{U} \psi])$

Denotations as fixpoins

- $[\![\mathsf{AG}\phi]\!]^{\mathcal{M}} = [\![\phi]\!]^{\mathcal{M}} \cap (\mathsf{AX})[\![\mathsf{AG}\phi]\!]^{\mathcal{M}}$
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- $[\![\mathsf{A}[\phi \mathsf{U} \psi]]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}} \cup ([\![\phi]\!]^{\mathcal{M}} \cap (\mathsf{AX})[\![\mathsf{A}[\phi \mathsf{U} \psi]]\!]^{\mathcal{M}})$
- $[\![\mathsf{E}[\phi \mathsf{U} \psi]]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}} \cup ([\![\phi]\!]^{\mathcal{M}} \cap (\mathsf{EX})[\![\mathsf{E}[\phi \mathsf{U} \psi]]\!]^{\mathcal{M}})$

- Where do we need greatest fixpoints?
- Where do we need least ones?