The Many Faces of Modal Logic

Course notes for NASSLLI 2014

Dirk Pattinson   Lutz Schröder

Contents

1 Modal Logics

We understand modal logics as extensions of propositional logic with additional operators, and kick off with examples that illustrate the many faces of modal logic: the logic of necessity and possibility, deontic logic, temporal logic, conditional logic, coalition logic, the modal logic of probability, and various description logics. Our discussion is centered around the tension between syntactic reasoning principles and their semantic justification.

1.1 Propositional Logic

1.2 Relational Modal Logic

The most well-known systems of modal logic are those with two modalities $\Box$ and $\Diamond$, with a variety of readings including

- 'necessarily' ($\Box$) / 'possibly' ($\Diamond$)
- 'it is known that' ($\Box$)

- 'it is believed that' ($\Box$)
- 'it is obligatory that' ($\Box$).

In the simplest version extending classical propositional logic, the formulae of modal logic are thus given by the grammar

$$\phi, \psi ::= \bot \mid p \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid \Diamond\psi$$

where p ranges over a given set P of proposition symbols. We define additional propositional connectives $\top$, $\vee$, $\rightarrow$, $\leftrightarrow$ by abbreviation in the usual fashion.

In the established *relational semantics* of modal logic, these formulae are interpreted as follows. The underlying type of *semantic structures* are *Kripke models* $M = (X, R, V)$ consisting of

- a set X of *states* (or *worlds*);
- a binary *accessibility relation* $R \subseteq X \times X$; and
- a *valuation* $V : P \rightarrow \mathcal{P}(X)$.

The pair (X, R) is also called a *Kripke frame*. When $(x, y) \in R$ then we say that y is a *successor* of x.

Over this class of models, one defines a notion of *satisfaction* of a formula ϕ in a state $x \in X$, written $x \models_M \phi$, by

$$\begin{aligned}
x &\not\models_M \bot \\
x \models_M p &\iff x \in V(p) \\
x \models_M \phi \wedge \psi &\iff x \models_M \phi \text{ and } x \models_M \psi \\
x \models_M \neg\phi &\iff x \not\models_M \phi \\
x \models_M \Box\phi &\iff y \models_M \phi \text{ for all } y \in X \text{ such that } (x, y) \in R \\
x \models_M \Diamond\phi &\iff y \models_M \phi \text{ for some } y \in X \text{ such that } (x, y) \in R.
\end{aligned}$$

Thus, $\Box$ and $\Diamond$ are *duals*, i.e. we have that $x \models_M \Diamond\phi$ iff $x \models_M \neg \Box \neg\phi$. We write $[\![\phi]\!]_M = \{x \in X \mid x \models_M \phi\}$. A formula ϕ is *valid* ($\models \phi$) if $x \models_M \phi$ for all M, x.

A relational modal logic is defined by fixing a class of frames; here, we shall focus on the modal logic of *all* Kripke frames, standardly referred to as K. Standard reasoning systems for modal logics are given in *Hilbert* style, i.e. comprise only few deduction rules and otherwise focus on axioms. The rules are typically, and in the case of K,

- *Modus ponens:*

$$\frac{\phi \qquad \phi \to \psi}{\psi}$$

- *Necessitation:*

$$\frac{\phi}{\Box\phi}.$$

Another rule that is often included is *uniform substitution*, which allows for replacing proposition symbols in formulas with arbitrary formulas. We keep this rule implicit here, by formulating all axioms as *axiom schemes.* The axioms of K are, then,

- all substitution instances of propositional tautologies
- all instances of the axioms scheme K (*normality*)

$$\Box(\phi \to \psi) \to \Box\phi \to \Box\psi.$$

We write $\vdash \phi$ if ϕ can be derived in the system. It is fairly easy to see that the system is *sound*:

Theorem 1.1. *If* $\vdash \phi$ *then* $\vDash \phi$*.*

The proof is by a simple induction over the derivation of ϕ; the main point to note is that all instances of K are valid.

It is rather harder to see that the deduction system is also *complete*, i.e. whenever $\vDash \phi$ then $\vdash \phi$. One way to prove this fact is to apply generic completeness theorems that we establish later. We note for now that completeness proofs are typically by *model construction*: call ϕ *satisfiable* if $\neg\phi$ is not valid, i.e. there exist M and x such that $x \vDash_M \phi$, and *consistent* if $\nvdash \neg\phi$. Then completeness can equivalently be phrased as *every consistent formula is satisfiable.*

1.3 Graded Modal Logic

Generalizing the operators $\Box$ and $\Diamond$ that just check for existence or non-existence of successors with certain properties, one may also decide to *count* such successors in Kripke models, thus arriving at *graded* modalities [4]:

- $\Box_k\phi$ states that ϕ holds in all but at most k successors, and
- $\Diamond_k\phi$ states that ϕ holds in more than (!) k successors.

Formally, given a state x in a Kripke model (X, R, V), we extend the previous definition of satisfaction by clauses

$$x \vDash_M \Box_k \phi \iff |\{y \in X \mid (x, y) \in R \text{ and } y \vDash_M \neg\phi\}| \leq k$$
$$x \vDash_M \Diamond_k \phi \iff |\{y \in X \mid (x, y) \in R \text{ and } y \vDash_M \phi\}| > k.$$

We obtain the operators $\Box$ and $\Diamond$ as the special cases $\Box = \Box_0$, $\Diamond = \Diamond_0$.

A natural generalization of the semantics is to let n equivalent successor states of x be represented by a single successor state that is a successor of x with *multiplicity* n [3]. One thus arrives at a new underlying system type for the semantics: a *multigraph* consists of a set X of states and a *multiplicity function* $\mu : X \times X \to \mathbb{N} \cup \{\infty\}$; *models* consist of a multigraph (replacing Kripke frames) and a valuation. Note that a Kripke frame may be seen as a multigraph where $\mu(x, y) \leq 1$ for all x, y. We understand $\mu(x, \cdot)$ as an integer-valued measure, and correspondingly write $\mu(x, A) = \sum_{y \in A} \mu(x, y)$ for $A \subseteq X$. The semantic clauses for $\Box_k$ and $\Diamond_k$ are then generalized to

$$x \vDash \Box_k \phi \iff \mu(x, [\![\neg\phi]\!]) \leq k$$
$$x \vDash \Diamond_k \phi \iff \mu(x, [\![\phi]\!]) > k.$$

Of course, it is not clear a priori that the semantics are equivalent; this is taken care of by

Lemma 1.2. *Every graded modal formula ϕ that is satisfiable over multigraphs is also satisfiable over Kripke frames.*

Proof. One can turn a multigraph into a semantically equivalent Kripke frame by making copies of elements according to their multiplicity; explicitly: Let X be a multigraph. Construct a Kripke frame $\bar{X}$ with transition relation R by taking as states all pairs $(y, j) \in X \times \mathbb{N}$ for which there exists x such that y is a successor of x with multiplicity $n > j$ in X, and in this case put $(x, i)R(y, j)$ for all i such that (x, i) is a state in $\bar{X}$. By induction over graded modal formulas ϕ, one shows easily that $x \vDash \phi$ in X iff $(x, i) \vDash \phi$ in $\bar{X}$. ∎

Note that the Kripke frame $(\bar{X}, R)$ constructed in the proof is finite if the original multigraph is finite and has only finite multiplicities.

One axiomatization of graded modal logic is due to Fine [4]; besides propositional tautologies (which we will henceforth omit to mention explicitly), it contains the axioms

$$\Box_0(\phi \to \psi) \to \Box_0\phi \to \Box_0\psi$$
$$\Diamond_k\phi \to \Diamond_l\phi \qquad (l < k)$$
$$\Diamond_k\phi \leftrightarrow \bigvee_{i=-1}^{k} (\Diamond_i(\phi \wedge \psi) \wedge \Diamond_{k-1-i}(\phi \wedge \neg\psi))$$
$$\Box_0(\phi \to \psi) \to \Diamond_k\phi \to \Diamond_k\psi$$

where by convention $\Diamond_{-1}\phi \equiv \top$, and moreover the rules modus ponens (no longer mentioned explicitly from now on) and necessitation, $\phi / \Box_0\,\phi$. That is, $\Box_0$ is normal, and all $\Diamond_k$ (hence, all $\Box_k$) are *monotone*, i.e. $\Diamond_k\phi \to \Diamond_k\psi$ is valid whenever $\phi \to \psi$ is valid (in fact, the last axiom above is a combination of this fact and the fact that $\Box_0$ and the $\Diamond_k$ are interpreted over the same multigraph). The key axiom capturing gradedness is the third axiom, which exploits that there are only $k+2$ ways to write $k+1$ as a binary sum.

1.4 Probabilistic Modal Logic

Having seen the idea of writing numbers on the edges in a model, it is a natural next idea to think about using real numbers, e.g. probabilities. The structures that one arrives at are (history-free, single-action) *Markov chains*: they consist of essentially the same data as a multigraph, except now we require that for each x, $\mu(x, \cdot)$ is a real-valued probability measure on X. A Markov chain can be read as a reactive system that evolves probabilistically, but one can also interpret it as a model of uncertain knowledge: in each state, the agent has subjectively uncertain beliefs about possible epistemic alternatives, modelled as a probability distribution over states. The operators of *probabilistic modal logic* are written in various styles, e.g. $\Diamond_p$ [7] or L_p [5], for $p \in \mathbb{Q} \cap [0, 1]$, and read 'with probability at least p, the next state satisfies' or, epistemically, 'the agent believes with confidence at least p that'. Their semantics is analogous to that of the graded modalities:

$$x \models L_p\phi \iff \mu(x, [\![\phi]\!]) \geq p.$$

The axiomatization of probabilistic modal logic, however, poses greater problems than that of graded modal logic, essentially because there are infinitely many ways of writing a real number as a binary sum. That is, for $p + q \leq 1$ one does have valid formulae

$$L_p(\phi \wedge \psi) \wedge L_q(\phi \wedge \neg\psi) \to L_{p+q}\phi$$

but no candidate for a converse implication as for graded modal logic. Heifetz and Mongin [5] prove completeness of a system including a modality M_p ('at most p') with semantics given by

$$x \models M_p\phi \iff \mu(x, [\![\phi]\!]) \leq p,$$

with axioms

$$\begin{gathered} L_o\phi \\ L_p\top \\ L_p\phi \to \neg L_q\neg\phi \quad (p+q>1) \\ \neg L_p\phi \to M_p\phi \end{gathered}$$

and rules

- *Replacement of equivalents:*

$$\frac{\phi \leftrightarrow \psi}{L_p\phi \leftrightarrow L_p\psi}$$

- *Rule (B):*

$$\frac{(\phi_1, \ldots, \phi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)}{\bigwedge_{i=1}^m L_{p_1}\phi_i \wedge \bigwedge_{j=2}^n M_{q_j}\psi_j \to L_{p_1+\ldots+p_m-q_2-\ldots-q_n}\psi_1}$$

  where $(\phi_1, \ldots, \phi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)$ abbreviates

$$\bigwedge_{k=1}^{\max(m,n)} \phi^{(k)} \leftrightarrow \psi^{(k)},$$

  with $\phi^{(k)}$, in turn, being shorthand for

$$\bigvee_{1\leq l_1<\cdots<l_k\leq m} (\phi_{l_1} \vee \cdots \vee \phi_{l_k}).$$

The premise of Rule (B) states that every state validates as many ϕ_i as ψ_j; soundness of the rule is based on the observation that in this case, the sum of the probabilities of the ϕ_i must coincide with the sum of the probabilities of the ψ_j.

1.5 Neighbourhood Logics

The Scott-Montague *neighbourhood semantics* of modal logic strips modal operators to their essence, seeing a modal operator essentially just as a compositional operator on formulae, where by *compositional* we meant that the semantics of the operator does not look into the structure of its argument formula but instead depends only on the *semantics* of its argument. That is, the interpretation of a modal operator $\Box$ in a model with carrier set X should be a map

$$\mathcal{P}(X) \to \mathcal{P}(X),$$

or, rewriting $\mathcal{P}(X)$ to the set 2^X of 2-valued predicates on X,

$$2^X \to 2^X.$$

Rearranging arguments, this is the same as giving a map

$$X \to 2^{(2^X)},$$

and indeed this is exactly the definition of a neighbourhood frame: A *neighbourhood frame* is a pair $(X, \mathfrak{N})$ consisting of a set X and a map $\mathfrak{N} : X \to 2^{(2^X)}$ assigning to each $x \in X$ a system $\mathfrak{N}(x) \subseteq 2^X$ of *neighbourhoods* of x. Then, we put

$$x \models \Box\phi \iff [\![\phi]\!] \in \mathfrak{N}(x),$$

in words: x satisfies $\Box\phi$ if the extension of ϕ is a neighbourhood of x.

This logic supports no reasoning principles beyond propositional reasoning and replacement of equivalents,

$$\frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}.$$

One can easily impose additional reasoning principles by restricting the admissible neighbourhood systems. E.g. *monotone neighbourhood frames* are those neighbourhood frames $(X, \mathfrak{N})$ where neighbourhoods are upwards closed under set inclusion, i.e. $A \in \mathfrak{N}(x)$ and $A \subseteq B$ implies $B \in \mathfrak{N}(x)$. Over monotone neighbourhood frames, one obtains a complete axiomatization by the *monotony* rule

$$\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

(from which, of course, we can derive replacement of equivalents).

1.6 Conditional Logics

Conditional logic offers alternatives to classical *material implication* that, e.g., try to avoid some properties that are felt to be 'paradoxes of implication' (e.g. $(b \wedge a) \to b$, a failure of *relevance*) or capture real-life properties such as defeasibility. Focusing for the moment on the latter, we denote a *defeasible conditional* by $\cdot \Rightarrow \cdot$, read 'if – then normally'. E.g. attempting to model our daily work life, we might postulate that on Mondays, we normally go to work:

$$\mathsf{Monday} \Rightarrow \mathsf{Work}. \tag{1}$$

From this, we would not wish to conclude (as we would have to if $\Rightarrow$ was material implication) that we normally go to work on Mondays if we're sick, i.e.

$$(\mathsf{Monday} \wedge \mathsf{Sick}) \Rightarrow \mathsf{Work}. \tag{2}$$

Given this purely negative observation, there arises rather a variety of reasoning principles that we might or might not want to retain, e.g. the disjunction property

$$(\phi \Rightarrow \psi) \wedge (\chi \Rightarrow \psi) \to ((\phi \vee \chi) \Rightarrow \psi)$$

(where $\to$ continues to denote material impliation); modus ponens

$$(\phi \Rightarrow \psi) \to \phi \to \psi$$

(not a property we expect of a reading of $\Rightarrow$ as a default conditional but a reasonable axiom if $\Rightarrow$ is meant to resemble relevant implication); *conditional excluded middle*

$$(\phi \Rightarrow \psi) \vee (\phi \Rightarrow \neg\psi);$$

identity

$$\phi \Rightarrow \phi;$$

or *cautious monotony*

$$(\phi \Rightarrow \chi) \to (\phi \Rightarrow \psi) \to ((\phi \wedge \chi) \Rightarrow \psi).$$

As an application of the latter reasoning principle, we can extend our running example: if we assume that due to our recreational habits we are normally sick on Mondays, cautious monotony will allow us after all to conclude (2) from (1).

Different semantics of conditional logics have been developed with a view to supporting various combinations of these reasoning principles. One of these is *selection function semantics*, in which models M consist of a set X of worlds, a valuation on X, and for each subset A of X a Kripke frame R_A

on X. One way to understand R_A is as relating each world x to those worlds that are most typical for condition A. Then, $\phi \Rightarrow \cdot$ is interpreted as a modal box over $R_{[\![\phi]\!]}$, i.e.

$$M, x \models \phi \Rightarrow \psi \quad \text{iff} \quad \forall y. \, (x R_\phi y \to M, y \models \psi).$$

This semantics supports only replacement of equivalents in the first argument, and normality in the second argument. The above-mentioned reasoning principles can be enabled by imposing additional conditions on the semantics, e.g. identity amounts to requiring that each R_A is reflexive; and conditional excluded middle is sound when each R_A is *functional*, i.e. each world has at most one R_A-successor. For more advanced reasoning principles such as cautious monotony, however, the arising conditions on the semantics can become rather complex and hard to harness. An alternative is to use *preference semantics*, which is based on models consisting of a set X of worlds and a ternary relation R such that for all x, y, z, w,

$$Rxyz \to Rxyy$$
$$(Rxyz \wedge Rxzw) \to Rxyw$$

– that is, $Rx\cdot\cdot$ is transitive and reflexive on $\{y \mid \exists z. Rxyz\}$. Intuitively, $Rxyz$ holds if y is preferred to / closer than / more typical than z as an alternative to x. Then, we interpret the conditional $\Rightarrow$ by

$$x \models \phi \Rightarrow \psi \quad \text{iff} \quad \forall y. \, (Rxyy \to \exists z. \, (Rxzy \wedge \forall t. \, (Rxtz \to M, t \models \psi))).$$

This semantics validates exactly the so-called *KLM (Kraus/Lehmann/Magidor) postulates* [6], i.e. (besides replacement of equivalents on the left and normality on the right) identity, the disjunction property, and cautious monotony [2].

1.7 Coalition Logic / Alternating-Time Logic

As our final example, we consider a logic for reasoning about the coalitional power of agents; it appears in the literature in two very similar versions, *alternating-time temporal logic (ATL)* [1] and *coalition logic* [8]. Both logics are parametrized over a finite set $N = \{1, \dots, n\}$ of *agents*; subsets of N are called *coalitions*. In coalition logic, one has a modal operator $[C]$ for each coalition C (denoted $\langle\!\langle C \rangle\!\rangle \bigcirc$ in ATL), with $[C]\phi$ informally read 'C can force ϕ'. These modalities are interpreted over structures called *game frames* (or *concurrent game structures* in ATL), consisting of

- a set X of states (and a valuation V), and

- at each state $x \in X$, a concurrent game, which in turn consists of
  - for each agent $i \in N$, a set S_i^x of available *moves*
  - an *outcome function* $f_x : (\prod_{i \in N} S_i^x) \to X$ determining the successor state of x, given a choice of a move by each agent.

(The semantics of ATL differs from this in assuming the sets S_i^x to be finite, and then w.l.o.g. initial segments of the natural numbers.) For a coalition C and joint choices $\sigma_C \in \prod_{i \in C} S_i^x$ and $\sigma_{N-C} \in \prod_{i \in N-C} S_i^x$ of moves by the agents in C and the agents outside C, respectively, we let $\langle \sigma_C, \sigma_{N-C} \rangle$ represent the obvious induced element of $\prod_{i \in N} S_i^x$, i.e. the arising overall choice of moves. The semantics of the operators $[C]$ is then determined, for a state x with data S_i^x, f_x as above, by

$$x \models [C]\phi \quad \text{iff} \quad \exists \sigma_C \in \prod_{i \in C} S_i^x.\ \forall \sigma_{N-C} \in \prod_{i \in N-C} S_i^x.\ f_x \langle \sigma_C, \sigma_{N-C} \rangle \models \phi.$$

(The interpretation of the ATL next operator $\langle\langle C \rangle\rangle \bigcirc$ is identical. ATL additionally has temporal operators defined as fixpoints over these basic operators, discussed later.) Coalition logic is completely axiomatized [8] by

$$\begin{array}{ll} \neg[C]\bot & \\ [C]\top & \\ \neg[\varnothing]\neg\phi \to [N]\phi & \\ [C](\phi \wedge \psi) \to [C]\phi & \\ [C]\phi \wedge [D]\psi \to [C \cup D](\phi \wedge \psi) & \qquad \text{if } C \cap D \neq \varnothing. \end{array}$$

The most interesting axiom is the last one, which states that two coalitions jointly force the conjunction of two formulas that the respective coalitions can force separately, but only if the coalitions are disjoint – otherwise, agents in the intersection of the two coalitions might need conflicting moves to function correctly in the two coalitions.

References

[1] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49:672–713, 2002.

[2] J. Burgess. Quick completeness proofs for some logics of conditionals. *Notre Dame J. Formal Logic*, 22:76–84, 1981.

[3] G. D'Agostino and A. Visser. Finality regained: A coalgebraic study of Scott-sets and multisets. *Arch. Math. Logic*, 41:267–298, 2002.

[4] K. Fine. In so many possible worlds. *Notre Dame J. Formal Logic*, 13:516–520, 1972.

[5] A. Heifetz and P. Mongin. Probabilistic logic for type spaces. *Games and Economic Behavior*, 35:31–53, 2001.

[6] S. Kraus, D. J. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intell.*, 44:167–207, 1990.

[7] K. Larsen and A. Skou. Bisimulation through probabilistic testing. *Inf. Comput.*, 94:1–28, 1991.

[8] M. Pauly. A modal logic for coalitional power in games. *J. Log. Comput.*, 12:149–166, 2002.