The Many Faces of Modal Logic
Day 2: Unifying Semantics

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Modal Logics are Coalgebraic

Why Coalgebras?

- because they provide a uniform semantics for a large class of modal logics

So which logics are amenable to colgebraic semantics?

- Of course, K. But also: neighbourhood logic, coalition logic, probabilistic logic, graded logic, conditional logic . . .

And what can you do at this level of generality?

- The usual stuff: completeness, complexity, Hennessy-Milner, interpolation – but generically and compositionally

Yesterday's Cook's Tour Through Modal Logics

Kripke Modal Logic

- $\Diamond\phi$
- ϕ can be true

Conditional Logic

- $\phi \Rightarrow \psi$
- ψ if ϕ

Coalition Logic

- $[C]\phi$
- C can force ϕ

Graded Modal Logic

- $\Diamond_k\phi$
- ϕ in $> k$ successors

Probabilistic Modal Logic

- $L_p\phi$
- ϕ holds with probability $\geq p$

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Similarities

- same *questions:* Completeness, decidability, complexity, ...
- *combinations:* probabilities and non-determinism, uncertainty in games

Graded Modal Logic

- $\Diamond_k\phi$
- ϕ in $> k$ successors

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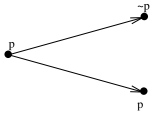
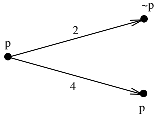
A Cook's Tour Through Modal Semantics

Kripke Frames.



$$C \to \mathcal{P}(C)$$

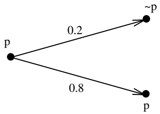
Multigraph Frames.



$$C \to \mathcal{B}(C)$$

$$\mathcal{B}(X) = \{f : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$$

Probabilistic Frames.



$$C \to \mathcal{D}(C)$$

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

More Examples

Neighbourhood Frames.

$$C \to \mathcal{P}\mathcal{P}(C) = \mathcal{N}(C)$$

mapping each world $c \in C$ to a set of neighbourhoods

Game Frames over a set N of agents

$$C \to \{((S_n)_{n \in N}, f) \mid f : \prod_n S_n \to C\} = \mathcal{G}(C)$$

associating to each state $c \in C$ a *strategic game* with strategy sets S_n and outcome function f

Conditional Frames.

$$C \to \{f : \mathcal{P}(C) \to \mathcal{P}(C) \mid f \text{ a function}\} = \mathcal{C}(C)$$

where every state yields a *selection function* that assigns successor sets to conditions

Coalgebras and Modalites: A Non-Definition

Coalgebras are about successors. T-coalgebras are pairs (C, γ) where

$$\gamma : C \to TC$$

maps states to successors. Coalg(T) is the class of T-coalgebras.

$$\frac{\text{states: elts } c \in C}{\text{succ's: elts } \gamma(c) \in TC} \qquad \frac{\text{prop's of states: subsets } P \subseteq C}{\text{prop's of successors: subsets } [\![\heartsuit]\!](P) \subseteq TC}$$

Modalities are about properties of successors: predicate liftings

$$[\![\heartsuit]\!]_C : \mathcal{P}(C) \to \mathcal{P}(TC)$$

Intended use:

$$c \models \heartsuit\phi \quad \text{iff} \quad \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi]\!]_C)$$

Example: Kripke Frames

Intuition. In $\gamma : C \to \mathcal{P}(C)$ think of $\gamma(c)$ as "the" successor. Then:

$$\begin{aligned} c \models \Box\phi &\iff \text{all elements of "the" successor } \gamma(c) \text{ of } c \text{ satisfy } \phi \\ &\iff \text{"the" successor } \gamma(c) \text{ of } c \text{ is a subset of } [\![\phi]\!] \\ &\iff \gamma(c) \in \{B \subseteq C \mid B \subseteq [\![\phi]\!]\} \end{aligned}$$

$\to$ **Predicate lifting**

$$\begin{aligned} [\![\Box]\!]_C : \mathcal{P}(C) &\to \mathcal{P}\mathcal{P}(C) \\ A &\mapsto \{B \subseteq C \mid B \subseteq A\}. \end{aligned}$$

$\to$ **Satisfaction**

$$c \models \Box\phi \iff \gamma(c) \in [\![\Box]\!]_C([\![\phi]\!])$$

Another Example: Neighbourhood Frames

Intuition. In $\gamma : C \to \mathcal{P}\mathcal{P}(C)$, $\gamma(c)$ contains the neighbourhoods of c

$$\begin{aligned} c \models \Box\phi &\iff [\![\phi]\!] \in \gamma(c) \\ &\iff \gamma(c) \in \{N \in \mathcal{N}(C) \mid [\![\phi]\!] \in N\}. \end{aligned}$$

$\to$ **Predicate lifting**

$$[\![\Box]\!]_C : \mathcal{P}(C) \to \mathcal{P}(\mathcal{N}(C)), \quad A \mapsto \{N \in \mathcal{N}(C) \mid A \in N\}$$

$\to$ **Satisfaction**

$$c \models \Box\phi \iff \gamma(c) \in [\![\Box]\!]_C([\![\phi]\!]_C)$$

(Recall the definition for Kripke Frames?)

Probabilistic Frames

Intuition. In $\gamma : C \to \mathcal{D}(C)$, $\gamma(c)$ is a random successor of c.

$$\begin{aligned} c \models L_p\phi &\iff \gamma(c)([\![\phi]\!]) \geq p \\ &\iff \gamma(c) \in \{\mu \in \mathcal{D}(C) \mid \mu([\![\phi]\!]) \geq p\}. \end{aligned}$$

$\to$ **Predicate lifting**

$$[\![L_p]\!]_C : \mathcal{P}(C) \to \mathcal{P}(\mathcal{D}C), \quad A \mapsto \{\mu \in \mathcal{D}(C) \mid \mu(A) \geq p\}$$

$\to$ **Satisfaction**

$$c \models L_p\phi \iff \gamma(c) \in [\![L_p]\!]_C([\![\phi]\!]_C)$$

(Recall Kripke frames and neighbourhood frames?)

Conditional Frames

Intuition. In $\gamma : C \to (\mathcal{P}(C) \to \mathcal{P}(C))$, $\gamma(c)(A)$ are the most typical alternatives to c under (non-monotonic) condition A

$$
\begin{aligned}
c \models \phi \Rightarrow \psi &\iff \gamma(c)([\![\phi]\!]) \subseteq [\![\psi]\!] \\
&\iff \gamma(c) \in \{f \in \mathcal{C}W \mid f([\![\phi]\!]) \subseteq [\![\psi]\!]\}.
\end{aligned}
$$

→ **Binary predicate lifting**

$$[\![\Rightarrow]\!]_W : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(\mathcal{C}W), \quad (A, B) \mapsto \{f \in \mathcal{C}W \mid f(A) \subseteq B\}$$

→ **Satisfaction**

$$c \models \phi \Rightarrow \psi \iff \gamma(c) \in [\![\Rightarrow]\!]_C([\![\phi]\!]_C, [\![\psi]\!]_C)$$

More Examples

Graded Modal Logic over multigraph frames

$$\gamma : C \to \mathcal{B}C = \{f : C \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$$

Predicate lifting for “more than k successors validate …”

$$[\![\Diamond_k]\!]_C(A) = \{f : C \to \mathbb{N} \mid \sum_{a \in A} f(a) \geq k\}$$

Coalition Logic over game frames

$$\gamma : C \to \mathcal{G}C = \{(f, (S_n)_{n \in N} \mid f : \prod_n S_n \to C\}$$

Predicate lifting for “coalition $K \subseteq N$ can force …”

$$[\![[K]]\!]_C(A) = \{(f, (S_n)) \in \mathcal{G}C \mid \exists \sigma_K \in \prod_K S_i . \forall \sigma_{N-K} \in \prod_{N-K} S_k . (f(\sigma_K, \sigma_{N-K}) \in A)\}$$

Satisfaction in either case:

$$c \models \heartsuit\phi \iff \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi]\!]_C)$$

Kripke Frames: Bisimilarity

$S \subseteq C \times D$ simulation if for all cSd

- $c \in \pi(p) \Longrightarrow d \in \pi(p)$
- $\forall c' \in \gamma(c). \exists d' \in \gamma(d). c' S d'$

S bisimulation $\iff$ S, S^{-} simulations

c, d bisimilar $\iff$ cSd for some bisimulation S.

Lemma: Bisimilar states satisfy the same modal formulae

Kripke Frames: *p*-Morphisms and Bisimilarity

p-morphisms = functions $f : C \to D$ s.t.

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \gamma \downarrow & & \downarrow \delta \\ \mathcal{P}(C) & \xrightarrow[\mathcal{P}(f)]{} & \mathcal{P}(D) \end{array}$$

where $\mathcal{P}(f)(A) = f[A] = \{f(a) \mid a \in A\}$.

Lemma: Two states are bisimilar if and only if they can be identified by *p*-morphisms:

$$\begin{array}{ccc} c \in C & & D \ni d \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle g} & \\ & f(c) = g(d) \in E & \end{array}$$

Behavioural Equivalence, Coalgebraically

Defn. $f : (C, \gamma) \to (D, \delta)$ (coalgebra homo-)morphism if

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ TC & \xrightarrow[\quad Tf \quad]{} & TD \end{array}$$

States c, d are behaviourally equivalent ($c \simeq d$) if they can be identified by a morphism.

Ooops! How is $T(f)$ defined *in general*?

Answer: Require T to be a functor, i.e. $Tf : TA \to TB$ if $f : A \to B$ and

$$T(id_A) = id_{TA} \quad T(g \circ f) = Tg \circ Tf$$

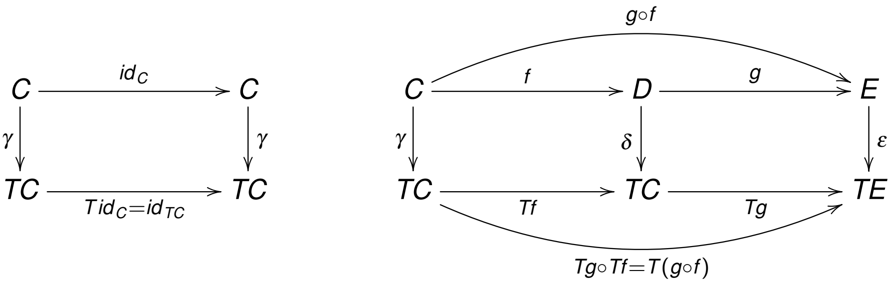
for $A \in \mathbf{Set}$ and composable functions f, g.

Good for us. The action of T on functions is usually canonical.

A Wee Bit of Structure Theory

Abstract Coalgebra: Prove statements about T-coalgebras without knowing T.

Simple Stuff: The identity is a morphism and morphisms compose.



Harder Stuff Behavioural equivalence is transitive and preserved by morphisms. (Try to prove it!)

Modalities and Behavioural Equivalence

"Good" Modalities are **compatible**. This one is from hell:

$$[\![\Box]\!]_C(A) = \begin{cases} \emptyset & C = \mathbb{N} \\ TC & \text{o/w} \end{cases}$$

Defn. An *n-ary predicate lifting* for $T : \mathbf{Set} \to \mathbf{Set}$ is a family $(\lambda_X)_{X \in \mathbf{Set}}$ of functions $\lambda_X : \mathcal{P}(X)^n \to \mathcal{P}(TX)$ such that

$$\begin{array}{ccc} (\mathcal{P}X)^n & \xrightarrow{\lambda_X} & \mathcal{P}(TX) \\ \uparrow{\scriptstyle (f^{-1})^n} & & \uparrow{\scriptstyle (Tf)^{-1}} \\ (\mathcal{P}Y)^n & \xrightarrow[\lambda_Y]{} & \mathcal{P}(TY) \end{array}$$

commutes for all $f : X \to Y$.

(For category buffs: $\lambda : \mathcal{Q}^n \to \mathcal{Q} \circ T^{op}$ is natural
for $\mathcal{Q} : \mathbf{Set}^{op} \to \mathbf{Set}$ contravariant powerset)

Naturality: Examples

- **Kripke frames** / K:

$$\mathcal{P}(f)(A) \in [\![\Box]\!](B) \iff f[A] \subseteq B$$
$$\iff A \subseteq f^{-1}[B] \iff A \in [\![\Box]\!](f^{-1}[B]).$$

- **Kripke frames** / **Graded Modal Logic**: $f : \{0,1\} \to \{0\}$:

$$\mathcal{P}(f)(\{0,1\}) \notin [\![\Diamond_1]\!](\{0\}) \quad \text{but} \quad \{0,1\} \in [\![\Diamond_1]\!](f^{-1}[\{0\}])$$

but

$$\mathcal{B}(f)(\mu) \in [\![\Diamond_k]\!](B) \iff \mathcal{B}(f)(\mu)(B) = \mu(f^{-1}[B]) > k$$
$$\iff \mu \in [\![\Diamond_k]\!](f^{-1}[B])$$

Finally: Proper Definitions

Defn. (Modal) signature Λ = set of finitary modal operators (for readability: unary). Λ-formulas:

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \heartsuit(\phi_1, \dots, \phi_n) \qquad (p \in V, \heartsuit \in \Lambda \text{ } n\text{-ary})$$

Λ-structure:

- Functor $T : \mathbf{Set} \to \mathbf{Set}$
- n-ary predicate lifting $[\![\heartsuit]\!]$ for $\heartsuit \in \Lambda$ n-ary

Modal Semantics over $\gamma : C \to TC$, $\sigma : \mathcal{V} \to \mathcal{P}(C)$:

$$C, c, \sigma \models p \text{ iff } c \in \sigma(p)$$
$$C, c, \sigma \models \heartsuit(\phi_1, \dots, \phi_n) \text{ iff } \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi_1]\!]_{C,\sigma}, \dots, [\![\phi_n]\!]_{C,\sigma})$$
$$C, \sigma \models \phi \text{ iff } C, c\sigma \models \phi \text{ for all } c \in C$$
$$C \models \phi \text{ iff } C, \sigma \models \phi \text{ for all } \sigma : \mathcal{V} \to \mathcal{P}(C)$$

where $[\![\phi]\!]_{C,\sigma} = \{c \in C \mid C, c, \sigma \models \phi\}$

Logical Equivalence vs Behavioural Equivalence

Defn. States c, d logically equivalent if

$$c \models \phi \iff d \models \phi \qquad \text{for all } \phi \in \mathcal{F}(\Lambda)$$

Lemma. Morphisms f preserve semantics: $c, f(c)$ logically equivalent
Proof:

$$\begin{aligned} f(c) \models \heartsuit\phi &\iff \delta(f(c)) = Tf(\gamma(c)) \in [\![\heartsuit]\!]_D [\![\phi]\!]_D \\ &\iff \gamma(c) \in [\![\heartsuit]\!]_D f^{-1}[[\![\phi]\!]_D] \overset{\text{IH}}{=} [\![\heartsuit]\!]_D [\![\phi]\!]_C \iff c \models \heartsuit\phi \end{aligned}$$

Cor. Modal logic is invariant under behavioural equivalence:

$$c \simeq d \implies c, d \text{ logically equivalent}$$

(Converse, i.e. the Hennessy-Milner property, holds over finitely branching systems if there are enough modalities)

E.g. graded modalities are invariant over $\mathcal{B}$ (but not over $\mathcal{P}$!)

Coalgebras and Their Logics

What are the laws for reasoning about
probabilities / strategic games / non-monotonic conditionals . . . ?

Defn. A logic over a signature Λ is a set $L \subseteq \mathcal{F}(\Lambda)$ of formulae that

- contains all propositional tautologies
- is closed under modus ponens, uniform substitution, and congruence

$$\frac{\phi \leftrightarrow \psi \in L}{\heartsuit\phi \leftrightarrow \heartsuit\psi \in L}$$

$$\mathrm{Log}(T) = \{\phi \in \mathcal{F}(\Lambda) \mid C \models \phi \text{ for all } T\text{-coalgebras } C\}$$

is a logic. Write $T \models \phi$ for $\phi \in \mathrm{Log}(T)$.

Logics, Syntactically Defined

Defn. A Λ-rule is a pair ϕ/ψ of Λ-formulae, and ϕ/ψ is admissible in a logic L if $\psi\sigma \in L$ whenever $\phi\sigma \in L$ and σ is a substitution.

$$\mathrm{Log}(\mathcal{R}) = \bigcap\{L \subseteq \mathcal{F}(\Lambda) \mid L \text{ a logic}, \phi/\psi \text{ admissible in } L \text{ for all } \phi/\psi \in \mathcal{R}\}$$

is the logic generated by the set $\mathcal{R}$ of rules. Write $\mathcal{R} \vdash \phi$ if $\phi \in \mathrm{Log}(\mathcal{R})$.

Goal. Given a Λ-structure T, find rules $\mathcal{R}$ such that $\mathrm{Log}(T) = \mathrm{Log}(\mathcal{R})$.

Roadmap.

- solve this generically, i.e. without looking into T
- instead, postulate coherence conditions linking $\mathcal{R}$ and T.

Soundness

$\mathcal{R}$ sound $:\Longleftrightarrow$ Everything that is provable is true: $\mathrm{Log}(\mathcal{R}) \subseteq \mathrm{Log}(T)$

Lemma. $\mathcal{R}$ is sound if

$$C, \sigma \models \psi \text{ whenever } C, \sigma \models \phi \text{ for all } \phi/\psi \in \mathcal{R},\ \sigma : \mathcal{V} \to \mathcal{P}(C)$$

> **Slogan.** The system is sound if each rule is sound

Aside on $\mathrm{Log}(\mathcal{R})$. $\mathrm{Log}(\mathcal{R})$ = provable formulae, inductively:

- all instances of propositional tautologies are provable.
- if $\mathcal{R} \vdash \phi\sigma$, then $\mathcal{R} \vdash \psi\sigma$ for all $\phi/\psi \in \mathcal{R}$
- if $\mathcal{R} \vdash \phi \to \psi$, $\mathcal{R} \vdash \phi$, then $\mathcal{R} \vdash \psi$
- if $\mathcal{R} \vdash \phi \leftrightarrow \psi$ then $\mathcal{R} \vdash \heartsuit\phi \leftrightarrow \heartsuit\psi$

Examples

Modal Logic K

$$\frac{p}{\Box p} \qquad \frac{p \wedge q \to r}{\Box p \wedge \Box q \to \Box r}$$

Neighbourhood Frames

$$\frac{p \leftrightarrow q}{\Box p \to \Box q}$$

Probabilistic Modal Logic

$$\frac{}{L_0 p} \qquad \frac{p}{L_u p} \qquad \frac{\neg p \vee \neg q}{\neg L_u p \vee \neg L_v q}\,(u+v>1) \qquad \frac{p \vee q}{L_u p \vee L_v q}\,(u+v=1)$$

$$\frac{\sum_{i=1}^{r} 1_{p_i} = \sum_{j=1}^{s} 1_{\bar{q}_j}}{\bigwedge_{i=1}^{r} L_{u_i} p_i \wedge \bigwedge_{j=2}^{s} L_{(1-v_j)} q_j \to L_{v_1} q_1}\left(\sum_{j=1}^{s} v_j = \sum_{i=1}^{r} u_i\right)$$

where $\bar{d}_1 = d_1$ and $\bar{d}_j = \neg d_j$ for $j \geq 2$.

Observation. Propositional premiss, modal conclusion.

More Examples

Coalition Logic for pairwise disjoint sets C_i of coalitions:

$$\frac{\bigvee_{i=1,\dots,n} \neg p_i}{\bigvee_{i=1,\dots,n} \neg [C_i] p_i} \qquad \frac{p}{[C]p} \qquad \frac{p \vee q}{[\emptyset]p \vee [N]q} \qquad \frac{\bigwedge_{i=1,\dots,n} p_i \to q}{\bigwedge_{i=1,\dots,n} [C_i] p_i \to [\bigcup C_i] q}$$

Graded Modal Logic

$$\frac{p \to q}{\Diamond_{n+1} p \to \Diamond_n q} \qquad \frac{r \to p \vee q}{\Diamond_{n+k} r \to \Diamond_n p \vee \Diamond_k q} \qquad \frac{p \leftrightarrow q}{\Diamond_k p \to \Diamond_k q}$$

$$\frac{(p \vee q \to r) \wedge (p \wedge q \to s)}{\Diamond_n p \wedge \Diamond_k q \to \Diamond_{n+k} r \vee \Diamond_0 s} \qquad \frac{\neg p}{\neg \Diamond_0 p}$$

Conditional Logic

$$\frac{q}{p \Rightarrow q} \qquad \frac{q_1 \wedge q_2 \to q_0}{(p \Rightarrow q_1) \wedge (p \Rightarrow q_2) \to (p \Rightarrow q_0)} \qquad \frac{p_1 \leftrightarrow p_2}{(p_1 \Rightarrow q) \to (p_2 \Rightarrow q)}$$

Observation. Again, propositional premiss, modal conclusion

Examples: Soundness

Propn. All above rule sets are sound, i.e. $\mathrm{Log}(\mathcal{R}) \subseteq \mathrm{Log}(T)$.

Observation. All rules have the form ϕ/ψ where $\phi \in \mathrm{Prop}(\mathcal{V})$ and $\psi \in \mathrm{Prop}(\Lambda(\mathcal{V}))$ where

- $\mathrm{Prop}(F)$ are propositional combinations of elements of $F \subseteq \mathcal{F}(\Lambda)$
- $\Lambda(F) = \{\heartsuit\phi \mid \heartsuit \in \Lambda, \phi \in F\}$

Defn. Rules of the above form are *one-step rules*.

Coherence Conditions 1: Soundness

Goal. Using the form of the rules, can we find a simpler condition that ensures soundness?

One-Step Logics = sets $L \subseteq \text{Prop}(\Lambda(\text{Prop}(\mathcal{V})))$ that

- contain all instances of propositional tautologies
- are closed under modus ponens
- contain $\heartsuit\phi \leftrightarrow \heartsuit\psi$ whenever $\phi \leftrightarrow \psi$ is a propositional tautology

One-Step Semantics. Given $\sigma : \mathcal{V} \to \mathcal{P}(X)$ we have

- $[\![\phi]\!]\sigma \subseteq X$ for $\phi \in \text{Prop}(\mathcal{V})$
- $[\![\psi]\!]\sigma \subseteq TX$ for $\psi \in \text{Prop}(\Lambda(\text{Prop}(\mathcal{V})))$ where

$$[\![\heartsuit\phi]\!]\sigma = [\![\heartsuit]\!]_X([\![\phi]\!]\sigma)$$

(NB: one-step semantics doesn't refer to models)

One-Step Soundness

$$\mathrm{Log}_1(T) = \{\psi \in \mathrm{Prop}(\Lambda(\mathrm{Prop}(\mathcal{V}))) \mid [\![\psi]\!]\sigma = TX \text{ for all } \sigma : \mathcal{V} \to \mathcal{P}(X)\}$$

is a one-step logic.

Coherence for Soundness. One-step rule ϕ/ψ admissible in one-step logic L if

$$\phi\sigma \text{ tautology } \implies \psi\sigma \in L.$$

For $\mathcal{R}$ set of one-step rules,

$$\mathrm{Log}_1(\mathcal{R}) := \bigcap\{L \text{ one-step logic} \mid \phi/\psi \text{ admissible in } L \text{ for all } \phi/\psi \in \mathcal{R}\}$$

$\mathcal{R}$ one-step sound if $\mathrm{Log}_1(\mathcal{R}) \subseteq \mathrm{Log}_1(T)$.

Soundness Theorem. One-step soundness implies soundness, that is $\mathrm{Log}(\mathcal{R}) \subseteq \mathrm{Log}(T)$ whenever $\mathrm{Log}_1(\mathcal{R}) \subseteq \mathrm{Log}_1(T)$.

One-Step Soundness: Examples

Equivalent Characterisation. $\mathcal{R}$ one-step sound iff

$$[\![\phi]\!]\sigma = X \implies [\![\psi]\!]\sigma = TX$$

for all $\phi/\psi \in \mathcal{R}$, $\sigma : \mathcal{V} \to \mathcal{P}(X)$.

Example. For $T = \mathcal{P}$, the rule $p/\Box p$ is one-step sound: pick $\sigma : \mathcal{V} \to \mathcal{P}(X)$ such that $\sigma(p) = \top = X$. Then

$$[\![\Box p]\!]\sigma = \{Y \subseteq X \mid Y \subseteq [\![p]\!]\sigma\} = \top = \mathcal{P}(X)$$

Observations.

- all rules that we have presented are one-step sound
- one-step soundness (a little) easier to check than soundness
- N.B.: Sound $\not\Rightarrow$ one-step sound: E.g. $\bot/\bot$ or probabilistic modal logic without propositional atoms

Tomorrow

Key Question. Do we have enough rules to axiomatise all valid formulae?

Tomorrow: Completeness

- algebraic semantics: completeness is easy
- duality: from algebraic to coalgebraic models
- coherence conditions for completeness