Completeness for coalgebraic μ-calculus: part 2

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(Joint work with Sebastian Enqvist and Yde Venema)

Overview

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- Completeness for μ-calculi: a coalgebraic approach

Coalgebraic μ-calculus

Definition

Given a set functor T and $n \in \omega$, an *n-place predicate lifting* λ for T is an assignment, to each set S, of a map

$$\lambda_S : (\mathcal{P}S)^n \to \mathcal{P}\mathsf{T}S,$$

such that for any map $f : S' \to S$ and any n-tuple $\overline{Z} = (Z_1, ..., Z_n) \in (\mathcal{P}S)^n$ we have, for all $\sigma \in \mathsf{T}S$:

$$\sigma \in \lambda_{S'}(f^{-1}[\overline{Z}]) \text{ iff } \mathsf{T}f(\sigma) \in \lambda_S(\overline{Z})$$

where $f^{-1}[\overline{Z}]$ abbreviates $(f^{-1}[Z_1], \ldots, f^{-1}[Z_n])$.

Definition

A predicate lifting $\lambda : \mathcal{P}^n \Rightarrow \mathcal{P}\mathsf{T}$ is *monotone* if for every set S, the map $\lambda_S : (\mathcal{P}S)^n \to \mathcal{P}\mathsf{T}S$ is order-preserving in each coordinate (with respect to the subset order).

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The induced predicate lifting $\lambda^\partial : \mathcal{P}^n \Rightarrow \mathcal{P}\mathsf{T}$, given by

$$\lambda^\partial{}_S(X_1, \ldots, X_n) := \mathsf{T}S \setminus \lambda_S(S \setminus X_1, \ldots, S \setminus X_1),$$

is called the *(Boolean) dual* of λ.

Coalgebraic μ-calculus

To each predicate lifting λ we associate a modality $\heartsuit_\lambda$ with the same arity as λ.
The semantics of $\heartsuit_\lambda$ in a T-model $\mathbb{S} = (S, \sigma, V)$ is:

$$\mathbb{S}, s \Vdash \heartsuit_\lambda(\overline{\varphi}) \text{ if } \sigma(s) \in \lambda_S([\![\varphi_1]\!]^{\mathbb{S}}, \ldots, [\![\varphi_n]\!]^{\mathbb{S}}).$$

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Example: $\lambda^\Diamond, \lambda^\Box : \mathcal{P}S \to \mathcal{P}\mathcal{P}S$:

$$\begin{aligned} \lambda^\Diamond : \quad & U \mapsto \{T \in \mathcal{P}S \mid T \cap U \neq \emptyset\} \\ \lambda^\Box : \quad & U \mapsto \{T \in \mathcal{P}S \mid T \subseteq U\}. \end{aligned}$$

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We can then formulate the semantics of $\Diamond$ via the map $\lambda^\Diamond$:

$$\mathbb{S}, s \Vdash \Diamond\varphi \text{ iff } \sigma_R(s) \in \lambda^\Diamond([\![\varphi]\!]^{\mathbb{S}})$$

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$$\begin{aligned} \epsilon_S : &\quad U \mapsto \{\alpha \in \mathcal{M}S \mid U \in \alpha\} \\ {\epsilon^\partial}_S : &\quad U \mapsto \{\alpha \in \mathcal{M}S \mid S \setminus U \notin \alpha\}. \end{aligned}$$

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The modal operators $\heartsuit_{\epsilon}$ and $\heartsuit_{\epsilon^{\partial}}$ coincide with the standard monotone modalities $\Box$ and $\Diamond$:

$$\begin{aligned} \mathbb{S}, s \Vdash \Box\varphi \quad & \text{iff} \quad U \subseteq [\![\varphi]\!]^{\mathbb{S}}, \text{ for some } U \in \sigma(s) \\ \mathbb{S}, s \Vdash \Diamond\varphi \quad & \text{iff} \quad U \cap [\![\varphi]\!]^{\mathbb{S}} \neq \emptyset, \text{ for all } U \in \sigma(s). \end{aligned}$$

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On arrows: given a map $f : S \to S'$ and $\sigma \in \mathcal{B}S$, we define the weight function $(\mathcal{B}f)\sigma : S' \to \omega$ by: $((\mathcal{B}f)\sigma)(s') := \sum\{\sigma(s) \mid f(s) = s'\}$.

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For a natural number k, we define the predicate liftings $\underline{k}$ and $\overline{k}$ by putting

$$\begin{aligned} \underline{k}_S : \quad & U \mapsto \{\sigma \in \mathcal{B}S \mid \textstyle\sum_{u \in U} \sigma(u) \geq k\} \\ \overline{k}_S : \quad & U \mapsto \{\sigma \in \mathcal{B}S \mid \textstyle\sum_{u \notin U} \sigma(u) < k\}, \end{aligned}$$

Coalgebraic μ-calculus

Definition

Given a set Λ of predicate liftings, the modal fixpoint language $\mu\mathtt{ML}_\Lambda$ is defined as follows:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_0 \vee \varphi_1 \mid \heartsuit_\lambda(\varphi_1, \ldots, \varphi_n) \mid \mu x.\varphi'$$

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Definition

Given a set Λ of predicate liftings and a set $\mathtt{X}$ of proposition letters, we let $\mu\mathtt{ML}_\Lambda(\mathtt{X})$ denote the set of $\mu\mathtt{ML}_\Lambda$-formulas φ of which all free variables belong to $\mathtt{X}$.

One-step logic

Definition

Given a set of predicate liftings Λ, and two disjoint sets $A, \mathtt{X}$ of variables, we define the set $\mathtt{Bool}(A)$ of *boolean formulas* over A and the set $\mathtt{ML}^1_\Lambda(A, \mathtt{X})$ of *one-step* Λ*-formulas* over A and parameters $\mathtt{X}$, by the following grammars:

$$\mathtt{Bool}(A) \ni \pi ::= a \mid \bot \mid \top \mid \pi \vee \pi \mid \pi \wedge \pi \mid \neg\pi$$

$$\mathtt{ML}^1_\Lambda(A, \mathtt{X}) \ni \alpha ::= p \mid \bot \mid \top \mid \heartsuit_\lambda \overline{\pi} \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha$$

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The *A-positive fragment* of $\mathtt{ML}^1_\Lambda(A, \mathtt{X})$ is denoted by $\mathtt{ML}^1_\Lambda(A, \mathtt{X})^+$.

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The negation-free fragment of $\mathtt{Bool}(A)$ is denoted by $\mathtt{Latt}(A)$ and refer to its elements as *lattice formulas* over A.

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Definition

A *one-step* T*-frame* over A is a triple $(X, \mathtt{Y}, \xi)$ where $\xi \in \mathsf{T}X$ and $\mathtt{Y} \subseteq \mathtt{X}$, or equivalently, a pair (X, ξ) with $\xi \in \mathsf{T}_{\mathtt{X}}X$.
A *one-step model* (X, ξ, m) is a one-step frame (X, ξ) together with a marking $m : X \to \mathcal{P}A$.

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The 1*-step interpretation* $[\![\alpha]\!]^1_m$ of $\alpha \in \mathtt{ML}^1_\Lambda(A, \mathtt{X})$ is defined as a subset of $\mathsf{T}_{\mathtt{X}}X$, with

$$[\![p]\!]^1_m := \{(\mathtt{Y}, \xi) \mid p \in \mathtt{Y}\},$$

$$[\![\heartsuit_\lambda(\pi_1, \ldots, \pi_n)]\!]^1_m := \{(\mathtt{Y}, \xi) \mid \xi \in \lambda_X([\![\pi_1]\!]^0_m, \ldots, [\![\pi_n]\!]^0_m)\},$$

and standard clauses for $\bot, \wedge, \vee$ and $\neg$.

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We write $X, \xi, m \Vdash^1 \alpha$ for $\xi \in [\![\alpha]\!]^1_m$.

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Definition

A monotone modal signature for T is *expressively complete* if, for every monotone n-place predicate lifting $\lambda \notin \Lambda$ and variables $a_1, \ldots, a_n$ there is a formula $\alpha \in \mathtt{ML}^1_\Lambda(\{a_1, \ldots, a_n\})$ which is equivalent to $\heartsuit_\lambda \overline{a}$.

Coalgebraic automa

Definition

A Λ-*automaton* over $\mathtt{X}$ is a quadruple (A, Θ, Ω, a_I) where A is a finite set of *states*, $\Omega : A \to \omega$ is the *priority map*, while the *transition map*

$$\Theta : A \to \mathtt{ML}^1_\Lambda(A, \mathtt{X})^+$$

maps states to (positive) one-step formulas over $\mathtt{X}$ and A.

Definition

Let $\mathbb{A} = (A, \Theta, \Omega, a_I)$ be a Λ-automaton, and let $\mathbb{S} = (S, \sigma, V)$ be a T-model, both over the set $\mathtt{X}$ of proposition letters. The *acceptance game* $\mathcal{A}(\mathbb{A}, \mathbb{S})$ for $\mathbb{A}$ with respect to $\mathbb{S}$ is defined as in the following table:

Position	Pl'r	Admissible moves
(a, s)	$\exists$	$\{m : S \to \mathcal{P}A \mid (S, \sigma(s), m) \Vdash^1 \Theta(a)\}$
m	$\forall$	$\{(b, t) \mid b \in m(t)\}$

Disjunctive bases

Definition

A formula $\alpha \in \mathtt{ML}^1_\Lambda(A)^+$ is said to be *disjunctive* if, for every one-step frame (X, ξ) and every marking $m : X \to \mathcal{P}A$, if $X, \xi, m \Vdash^1 \alpha$ then there is a one-step frame (X', ξ'), a map $f : X' \to X$ and a marking $m_0 : X' \to \mathcal{P}(A)$ such that:

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Example: The *cover modality* ∇ of standard modal logic:

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Example: Moss' modality ∇_T provides disjunctive formulas for every weak-pullback preserving functor T.

Disjunctive bases

For each finite set A let $\chi_A : \mathcal{P}(A) \to \mathtt{Latt}(A)$ be the substitution defined by $B \mapsto \bigwedge B$. Similarly, let $\theta_{A,B} : A \times B \to \mathtt{Latt}(A \cup B)$ be defined by $(a, b) \mapsto a \wedge b$.

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Let $\mathtt{D}$ be an assignment of a set of positive one-step formulas $\mathtt{D}(A) \subseteq \mathtt{ML}^1_\Lambda(A)^+$ for all finite sets A. Then $\mathtt{D}$ is called a *disjunctive basis* for Λ if each formula in $\mathtt{D}(A)$ is disjunctive, and the following conditions hold:

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(1) $\mathtt{D}(A)$ is closed under finite disjunctions.
(2) $\mathtt{D}$ is *distributive over* Λ: for every one-step formula $\heartsuit_\lambda \overline{\pi}$ there is a formula $\delta \in \mathtt{D}(\mathcal{P}(A))$ such that $\heartsuit_\lambda \overline{\pi} \equiv^1 \delta[\chi_A]$.

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(3) $\mathtt{D}$ *admits a distributive law*: for any two formulas $\alpha \in \mathtt{D}(A)$ and $\beta \in \mathtt{D}(B)$, there is a formula $\gamma \in \mathtt{D}(A \times B)$ such that $\alpha \wedge \beta \equiv^1 \gamma[\theta_{A,B}]$.

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Proposition. Let Λ be an expressively complete signature for a weak pullback-preserving functor T and suppose that T preserves finite sets. Then Λ admits a disjunctive basis.

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Question: What about preservation of finite sets and expressively completeness? Not necessary!

Proposition. $\mathcal{B}$ admits a disjunctive basis.

One-step soundness and completeness

Definition

A one-step formula α is *one-step valid*, notation $\Vdash^1 \alpha$, if $[\![\alpha]\!]^1_m = \mathsf{T}_{\mathtt{X}} X$ for all sets X and markings $m : X \to \mathcal{P}A$,

One-step soundness and completeness

Definition

A one-step formula α is *one-step valid*, notation $\Vdash^1 \alpha$, if $[\![\alpha]\!]^1_m = \mathsf{T}_{\mathtt{X}} X$ for all sets X and markings $m : X \to \mathcal{P}A$, and we say that β is a *one-step consequence* of α (written $\alpha \Vdash^1 \beta$) if $[\![\alpha]\!]^1_m \subseteq [\![\beta]\!]^1_m$ for all X, m.

A general completeness theorem

Definition

The *one-step derivation system* $\mathbf{H}^1$:

- (**H**) All formulas in $\mathbf{H}$ are axioms of $\mathbf{H}^1$.
- (MP) From $\alpha \to \beta$ and α, derive β, where $\alpha, \beta \in \mathtt{ML}^1_\Lambda(\mathtt{X})$.
- (CT) The formula $\alpha \in \mathtt{ML}^1_\Lambda(\mathtt{X})$ is an axiom if it is a substitution instance of some propositional tautology.
- (Cg) For all $\pi_1, \pi_2 \in \mathtt{Bool}(\mathtt{X})$, if $\pi_1 \leftrightarrow \pi_2$ is a substitution instance of a propositional tautology then $\heartsuit_\lambda \pi_1 \leftrightarrow \heartsuit_\lambda \pi_2$ is an axiom.
- (US) Given any map $\tau : \mathtt{X} \to \mathtt{Bool}(\mathtt{X})$ and $\alpha \in \mathtt{ML}^1_\Lambda(\mathtt{X})$, derive $\alpha[\tau]$ from α.
- (Du) the formula $\heartsuit_{\lambda^\partial} a \leftrightarrow \neg\heartsuit_\lambda \neg a$ is an axiom, for all $\lambda \in \Lambda$ and $a \in \mathtt{X}$;
- (Mon) For all $\lambda \in \Lambda$ and $a, b \in \mathtt{X}$, the formula $\heartsuit_\lambda a \to \heartsuit_\lambda (a \vee b)$ is an axiom.

A general completeness theorem

We add the *pre-fixpoint schema* and the *Kozen-Park induction rule* to obtain a define a Hilbert system $\mu\mathbf{H}$ for μ-calculus.

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We write $\vdash_{\mathbf{H}} \varphi$ to say that φ is provable in the system $\mu\mathbf{H}$, $\varphi \vdash_{\mathbf{H}} \psi$ for $\vdash_{\mathbf{H}} \varphi \to \psi$ and $\varphi \equiv_{\mathbf{H}} \psi$ for $\vdash_{\mathbf{H}} \varphi \leftrightarrow \psi$.

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A one-step axiomatization $\mathbf{H}$ is said to be *one-step sound* if $\Vdash^1 \alpha$ whenever $\vdash^1_{\mathbf{H}} \alpha$, for $\alpha \in \mathtt{ML}^1_\Lambda(A)$.

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A general completeness theorem

Theorem

Let T be a set functor, let Λ be a monotone modal signature for T, and let $\mathbf{H}$ be a one-step axiomatization for Λ and T. If $\mathbf{H}$ is one-step sound and complete and Λ admits a disjunctive basis, then $\mu\mathbf{H}$ is a sound and complete axiom system for the $\mu\mathtt{ML}_{\Lambda}$-formulas that are valid in the class of all T-coalgebras.

Applications

Theorem

The proof system $\mu\mathbf{H}$ is sound and complete for validity over T-models, where:

(1) $\mathsf{T} = \mathsf{Id}$ *and* $\mathbf{H} = \mathbf{I}$,

(2) $\mathsf{T} = \mathsf{Id}^k$ *and* $\mathbf{H} = \mathbf{I}^k$,

(3) $\mathsf{T} = \mathcal{P}$ *and* $\mathbf{H} = \mathbf{K}$,

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Uniform Interpolation

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We define the *supported companion functor* $\mathcal{M}_s$ of $\mathcal{M}$ as the subfunctor of $\mathcal{P} \times \mathcal{M}$, given, on objects, by
$\mathcal{M}_s S := \{(S_0, \gamma) \in \mathcal{P}S \times \mathcal{M}S \mid S_0 \text{ supports } \gamma\}$.

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A subset $S_0 \subseteq S$ *supports* an object $\gamma \in \mathcal{M}S$ whenever $U \in \gamma$ iff $U \cap S_0 \in \gamma$, for all $U \in \mathcal{P}S$.