A Metalanguage for Guarded Iteration

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Computations from different perspectives

Domain theory

- computations are identified with final result (if any)
- programs either terminate with a value, or they diverge
- *extensional* paradigm

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Process algebra

- computations are processes unfolding in time
- main attribute: behaviour (of an infinitely running process)
- *intensional* paradigm

Guardedness

General idea

ensure progress or productivity in (co)recursive definitions

In process algebra

- recursive process specification $X = t$ guarded if every occurrence of X in t is under an action
- e.g. in *CCS* under bisimulation semantics: guarded recursive specifications have unique solutions [Milner '83]
- for example,

$$P = a.P$$

keeps performing the action a

Guardedness

General idea

ensure progress or productivity in (co)recursive definitions

More recently: in (co)programming

- guardedness analysis in Coq for corecursive definitions, proofs by corecursion:

  are corecursive calls (directly!) under constructors?

- guarded recursion by typing/functorial guardedness
  [Birkedal, Møgelberg '13],[Vezzosi '15],[Milius, Litak '13], [Clouston et al. '15] and others

(Abstractly) Guarded co-Cartesian Categories

Inference rules

$$\frac{f : X \to Y}{\mathsf{in}_1\, f : X \to_{\mathsf{in}_2} Y + Z} \qquad \frac{f : X \to_\sigma Z \quad g : Y \to_\sigma Z}{[f, g] : X + Y \to_\sigma Z}$$

$$\frac{f : X \to_{\mathsf{in}_2} Y + Z \qquad g : Y \to_\sigma V \qquad h : Z \to V}{[g, h] \circ f : X \to_\sigma V}$$

Definition (Guarded co-Cartesian category)

A co-Cartesian category $\mathbf{C}$ equipped with distinguished subsets $\mathrm{Hom}_\sigma(X, Y) \subseteq \mathrm{Hom}(X, Y)$ of *partially guarded morphisms* for $A, B \in |\mathbf{C}|$, and any summand $\sigma : Y_1 \to Y_1 + Y_2 \simeq Y$ satisfying the rules above is called *guarded* ($f : X \to_\sigma Y$ means $f \in \mathrm{Hom}_\sigma(X, Y)$).

Guardedness for Monads

Abstract guardedness for a strong monad T

- guarded: Kleisli category of T is a guarded category
- $$\frac{f : X \to_\sigma TY}{\tau \circ \langle \mathsf{id}, f \rangle : X \to_{\mathsf{id} \times \sigma} T(X \times Y)}$$
- underlying category distributive
- guarded pre-iterative: every $\sigma +$ id-guarded morphism has a solution w.r.t.

  $$f^\dagger = [\eta, f^\dagger]^\star \circ f$$

  (and $f^\dagger$ is σ-guarded)

Guardedness for Monads

Examples

- Finitely branching processes: $TX = \nu\gamma.\, \mathcal{P}_\omega(X + \mathrm{Act} \times \gamma)$
- $TX = \mathcal{P}X$ admits total iteration
- $TX = (\nu\gamma.\, X + S \times \gamma)^S$, stateful traces

Guardedness for Monads

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Metalanguages

- Moggi: metalanguage for strong monads to model effectful programs
- Here: metalanguage for strong guarded (pre-)iterative monads to model effectful programs with guarded iteration

Setup

Labelled iteration

- imperative languages: break/continue for loops
- [Geron, Levy '16]: enable break/continue in Fine-Grain Call-By-Value
- modeled using exceptions
- guardedness: also indicated via coproducts; similar idea can be used
- organise guarded control flow using labels/exceptions

Fine-Grain Call-By-Value [Levy et al. '02]

- separate judgements for values and computations
- more suitable for operational semantics than Moggi's *monadic metalanguage*

Language overview

Design

- parametric in a guarded pre-iterative monad T
- rules for *guardedness* mirrored in the calculus
- exception model suggests breaking up composition into
  - (unguarded) composition
  - (guarded) "exception handling"
- guarded iteration construct

Language overview

Implementation

- signature containing the operations of the monad: $[\![f : A \to B[C]]\!] : A \to_2 T(B + C)$
- *two* contexts: Δ for guardedness information, Γ for program variables
- Δ stores typed "exception" names tagged with guardedness info ("g" or "u" added to the type)
- value judgements: $[\![\Gamma \vdash_v v : A]\!] : \Gamma \to A$
- computation judgements: $[\![\Delta, \Gamma \vdash_c p : B]\!] : \Gamma \to_{!+\sigma_\Delta} T(B + \Delta)$

Program example

Guessing numbers

handleit $e = *$ in
  handle u in
    $(print\ (\texttt{"think of a number"})\ \&\ raise\_u\ *\ )$
  with
    (do $y \leftarrow random();$
        $z \leftarrow read();$
        if $(y = z)$ then ret $*$ else $raise\_e\ *)$

- handle: guarded "composition"
- handleit: iteration construct
- $random$, $read$: unguarded signature operations
- $print$: guarded operation (accompanied by $raise\_u$)

Language Features

Contexts

- we want to work with multiple variables
- definable *strong iteration* operator

$$\frac{f : W \times X \to_2 T(Y + X)}{f^\ddagger = (T((\mathsf{pr}_2 + \mathsf{id}) \circ \mathsf{dist}) \circ \tau \circ \langle \mathsf{pr}_1, f \rangle)^\dagger : W \times X \to TY}$$

- . . . which is iteration in the co-Kleisli category $\mathbf{C}[W]$ of the context comonad $W \times -$

Theorem

Guarded pre-iterative monads on $\mathbf{C}$ *extend to guarded pre-iterative monads on* $\mathbf{C}[W]$ *under the same definition of guardedness and iteration defined as above.*

Language Features

Algebraic operations via Generic effects

- symbols $f : A \to B[0]$ correspond to Kleisli morphisms $A \to TB$
- for A, B natural numbers n, m, also called *generic effects*
- *algebraic operations*: certain natural transformations $T^m \to T^n$
- e.g. nondeterministic choice $\oplus : T^2 \to T^1$
- correspondence via (eliding isomorphisms):

$$TX^m \xrightarrow{\alpha} TX^n \quad \Rightarrow \quad f = \alpha(m \xrightarrow{\eta} Tm) : n \to Tm$$

$$n \xrightarrow{f} Tm \quad \Rightarrow \quad \alpha(g : m \to TX) = g^\star \circ f : v \to TX$$

Language Features

Explicit guards

- given $f : A \to B[C]$:

  $\mathsf{gcase}\, f(v)\ \mathsf{of}\ \mathsf{inl}\, x \mapsto p;\ \mathsf{inr}\, y \mapsto \mathsf{raise}_e\, q$
- raise a *guarded* exception, i.e.
  - call a guard f from the signature
  - run p if we end up in the unguarded part
  - otherwise raise e
- e^g is added to Δ

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Unguarded exceptions

- $\mathsf{raise}_e\, v$
- adds ordinary exceptions e^u to Δ

Language Features

Exception handling

- handle e in p with q
- currently handles only *guarded* exceptions e occurring in p
- q acts as an exception handler

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Iteration

- intended as iterated exception handling
- handleit $x = v$ in p
- iteratively runs p (depending on x, which is initialised to v)

Language Features

λ-abstraction and application

- abstraction turns a computation into a value
- values do not have guardedness annotation
- we store Δ in the type:

$$\frac{\Delta \mid \Gamma, x : A \vdash_c p : B}{\Gamma \vdash_v \lambda x.\, p : A \to_\Delta B} \qquad \frac{\Gamma \vdash_v w : A \qquad \Gamma \vdash_v v : A \to_\Delta B}{\Delta \mid \Gamma \vdash_c v\, w : B}$$

Denotational Semantics

syntactic feature	semantic equivalent
composition, handling	Kleisli composition + guardedness axioms
iterated handling	strong iteration operator
λ-abstraction, application	???

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λ-abstraction, application	???

- the latter are problematic: what is $[\![A \to_\Delta B]\!]$?
- need to assume additional properties of our category ...

Greatest σ-algebras

Representable guarded homs

Assume that the contravariant functors $X \mapsto \mathsf{Hom}_\sigma(X, TA)$ are representable, i.e. there exist $A_\sigma \in |\mathbf{C}|$ and natural isomorphisms

$$\xi : \mathsf{Hom}(X, A_\sigma) \simeq \mathsf{Hom}_\sigma(X, TA)$$

Greatest σ-algebras

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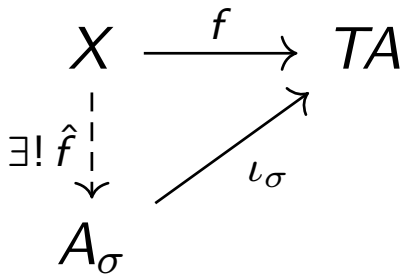
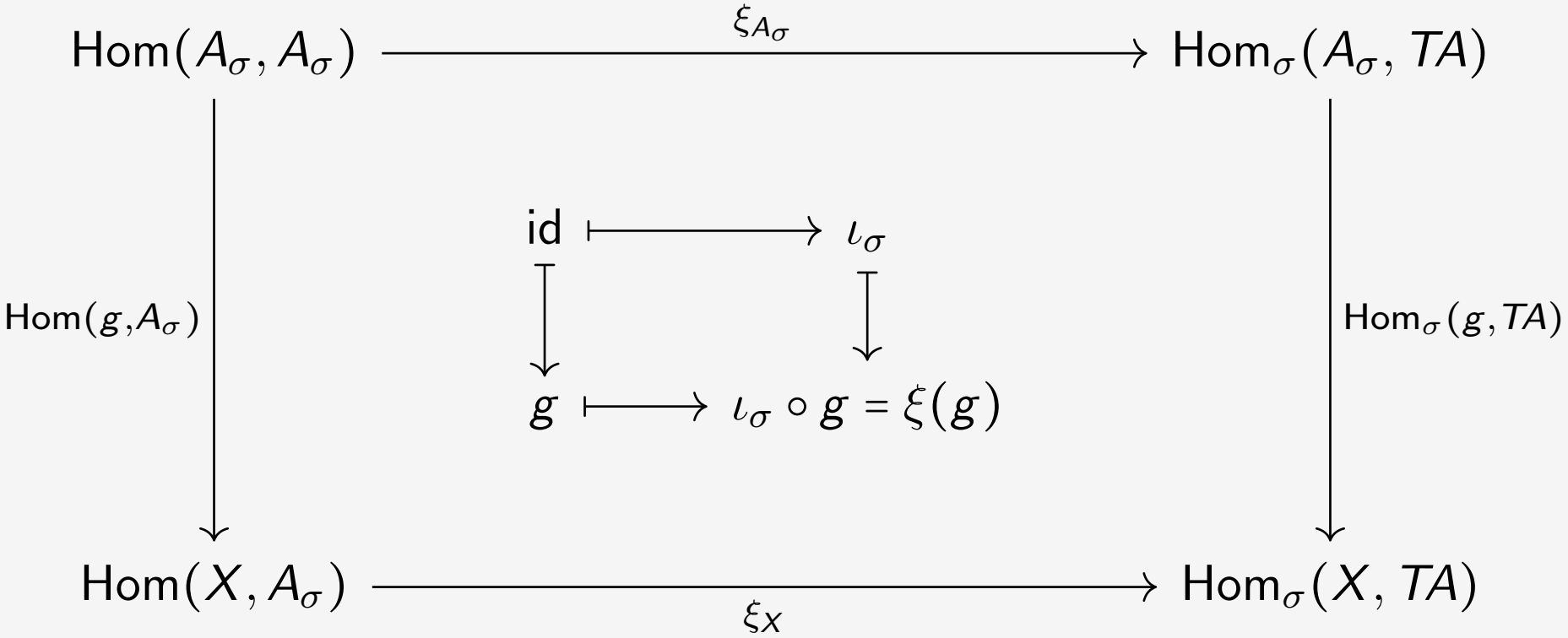
Equivalently:



(A_σ, ι_σ) *greatest* σ*-algebra*

where $f \in \mathrm{Hom}_\sigma(X, TA)$, $\iota_\sigma \in \mathrm{Hom}_\sigma(A_\sigma, TA)$

Proof


$\mathsf{Hom}(A_\sigma, A_\sigma)$
ξ_{A_σ}
$\mathsf{Hom}_\sigma(A_\sigma, TA)$
$\mathsf{Hom}(g, A_\sigma)$
$\mathsf{Hom}_\sigma(g, TA)$
id
ι_σ
g
$\iota_\sigma \circ g = \xi(g)$
$\mathsf{Hom}(X, A_\sigma)$
ξ_X
$\mathsf{Hom}_\sigma(X, TA)$

Greatest σ-algebras

Proposition

Suppose that (A_σ, ι_σ) *is a greatest* σ*-algebra. Then there is a unique* $\alpha_\sigma : TA_\sigma \to A_\sigma$ *such that* $\iota_\sigma \alpha_\sigma = \iota_\sigma^\star$. *The pair* $(A_\sigma, \alpha_\sigma)$ *is a* T*-subalgebra of* (TA, μ).

Greatest σ-algebras

Proposition

Suppose that (A_σ, ι_σ) is a greatest σ-algebra. Then there is a unique $\alpha_\sigma : TA_\sigma \to A_\sigma$ such that $\iota_\sigma \alpha_\sigma = \iota_\sigma^\star$. The pair $(A_\sigma, \alpha_\sigma)$ is a T-subalgebra of (TA, μ).

Example: $T_\Sigma X = \nu\gamma.\, T(X + \Sigma\gamma)$

For $\sigma : A' \hookrightarrow A$, $\overline{\sigma} : A'' \hookrightarrow A$, we have

$$A_\sigma = T(A'' + \Sigma T_\Sigma A) \qquad \iota_\sigma = \mathrm{out}^{-1} \circ T(\overline{\sigma} + \mathrm{id})$$

Recall that a morphism $f : X \to T_\Sigma A$ is σ-guarded iff it factors as $f = \mathrm{out}^{-1} \circ T(\overline{\sigma} + \mathrm{id}) \circ u$ for some u.

Semantics of λ-abstraction

Abstraction

Given $f := [\![\Delta \mid \Gamma, x : A \vdash_c p : B]\!]$, we let

$$[\![\Gamma \vdash_v \lambda x.\, t : A \to_\Delta B]\!] = \mathrm{curry}(\hat{f}).$$

Application

Given $g := [\![\Gamma \vdash_v v : A \to_\sigma B]\!]$ as well as some $w := [\![\Gamma \vdash_v w : A]\!]$, we let

$$[\![\Delta \mid \Gamma \vdash_c v\, w : B]\!] := \iota_\sigma \circ \mathrm{uncurry}(g) \circ \langle \mathrm{id}, w \rangle.$$

Operational Semantics

Simple trace example

- $TX = X \times \mathbb{N}^\star + \mathbb{N}^\omega$
- operation $put : \mathbb{N} \to 0[1]$ adds a number to the *trace*
- gcase construct simplifies to

$$\mathsf{gcase}\, put(n)\ \mathsf{of}\ \mathsf{inl}\, x \mapsto \mathsf{init}\, x;\ \mathsf{inr}\, \_ \mapsto \mathsf{raise}_e\, v =: put(n)\ \&\ \mathsf{raise}_e\, v$$

- guardedness: at least one output operation performed
- operationally, a (guarded!) program can either
  - terminate with a value and a finite trace
  - produce an infinite trace

Operational Semantics

Exemplary rules

$$\frac{p \Downarrow \pi}{\text{handle } x \text{ in } p \text{ with } q \Downarrow \pi}$$

$$\frac{p \Downarrow \text{raise}_x\, v, \tau \quad q[v/x] \Downarrow \pi}{\text{handle } x \text{ in } p \text{ with } q \Downarrow \tau +\!\!+ \pi}$$

$$\frac{p \Downarrow \text{ret}\, v, \tau}{\text{handle } x \text{ in } p \text{ with } q \Downarrow \text{ret}\, v, \tau}$$

$$\frac{p \Downarrow \text{raise}_y\, v, \tau}{\text{handle } x \text{ in } p \text{ with } q \Downarrow \text{raise}_y\, v, \tau}$$

where τ denotes a finite trace, π an infinite trace

Operational Semantics

Theorem (Adequacy)

Let $\Delta \mid - \vdash_c p : B$. *Then,*

1. *If* $p \Downarrow \mathsf{ret}\, v, \tau$ *then* $[\![\Delta \mid - \vdash_c p : B]\!] = (\mathsf{in}_1\, v, \tau) \in (B + \Delta) \times \mathbb{N}^\star$.
2. *If* $p \Downarrow \mathsf{raise}_x\, v, \tau$ *and* $x : E^{\mathsf{g}}$ *is in* Δ *then* $[\![\Delta \mid - \vdash_c p : B]\!] = (\mathsf{in}_2\, \mathsf{in}_x\, v, \tau) \in (B + \Delta) \times \mathbb{N}^+$.
3. *If* $p \Downarrow \mathsf{raise}_x\, v, \tau$ *and* $x : E^{\mathsf{u}}$ *is in* Δ *then* $[\![\Delta \mid - \vdash_c p : B]\!] = (\mathsf{in}_2\, \mathsf{in}_x\, v, \tau) \in (B + \Delta) \times \mathbb{N}^\star$.
4. *If* $p \Downarrow \pi$, *then* $[\![\Delta \mid - \vdash_c p : B]\!] = \pi \in \mathbb{N}^\omega$.

Related and Further Work

- Adequacy result generalises [Geron, Levy '16]
- [Nakata, Uustalu '09] give a similar semantics for a WHILE language
- $M_S X = (\nu\gamma.\, X + S \times \gamma)^S$ provides a denotational background for op.cit.
- M_S is guarded pre-iterative
- how does our calculus, instantiated with M_S, relate to [Nakata, Uustalu '09]?

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