the subkripkean, the subcanonical and the completeness hypercuboid
take Jónsson&Tarski'51 as a starting point

overview mutual relations between notions of strong completeness, canonicity and closure under various form of completions, and propose a way of classifying them

explain how we were led to revisit these problems in a recent collaboration with Holliday

point out potential connections: the theory of coalgebraic logics, algebraic proof theory and possibility semantics (recently, an interesting perspective involving non-well-founded proofs by Shamkanov, WoLLiC’17)

as an aside: proofs and counterexamples as a rule more constructive than these for canonical extensions, though more general
and counterexamples often among “natural” logics, sometimes with axioms rank ≤2
any set of formulas under all CPC-tautologies...

... all instances of \([i](A \rightarrow B) \rightarrow [i]A \rightarrow [i]B\) ...
i \in \Sigma, where \Sigma is the supply of modalities

... Modus Ponens ...

... substitution ...

... and Necessitation a.k.a. Gödel rule: from A, infer \([i]A\)
Equivalent algebraic semantics: Boolean Algebras with Operators (BAOs):

Modal Algebras (MAs) if just one unary operator

BAOs with additional operations distributing over finite joins, i.e.,
\(<i>(A \lor B) = <i>A \lor <i>B\)
\(<i>\bot = \bot\)

(equivalently defined in terms of boxes \([i]\) – dual operators)
alternative semantics: Kripke frames
(I presume everybody is familiar with it)

With any Kripke frame $F = (W, (R_i)_{i \in \Sigma})$, we can associate a BAO validating same formulas:

take powerset of $W$ as the boolean reduct

interpret $[i]$ as the satisfaction relation tells you, i.e., $[i]A := \{ \omega \mid \forall u. (\omega R_i u \Rightarrow u \in A) \}$

( $<i>A = \{ \omega \mid \exists u. (\omega R_i u \& u \in A) \}$ )
As it was realized already by Jónsson & Tarski (prior to formal invention of Kripke frames!), such an algebra will have three important—in fact, defining—properties:

- **C**: lattice-completeness (think of arbitrary unions)
- **A**: Atomicity (think of singletons)
- **V**: complete additivity: $\langle \vee a_i \rangle = \vee \langle a_i \rangle$

When $[]$ is primitive: $[] \land a_i = \land [!]a_i$

Crucially, the notion makes sense in the absence of $C$. Diamonds have to distribute over all existing infinite joins.
Thus, being Kripke-complete obtains a precise semantic sense: being HSP-generated by CAV-BAOs.

Jónsson & Tarski came up with a technique of embedding every BAO $\Lambda$ into a CAV-BAO defined on the $\text{Pow}(Uf(\Lambda))$, i.e.,

a way to define operator(s) on the Stone representation of $\Lambda$ so that Stone’s BA-embedding becomes BAO-embedding.

They were talking about perfect extensions. These days, we rather talk about canonical extensions (influence of the Lemmon-Scott notes, I guess) or perhaps canonical completions.
In hindsight, this provides a brutal way to prove that the minimal normal logic $K$ is Kripke complete (every nontheorem refuted in a Kripke model = every non-derivable equation refutable on a $CAV$-$BAO$ = being $HSP$-generated from $CAV$-$BAOs = weak CAV-completeness)

Furthermore, so is any logic whose defining axioms/equations are preserved under canonical extensions!

Jónsson & Tarski’s original paper identified the first class of such equations: the first study of canonicity. Bjarni revisited the field in 1994 (Studia Logica)
Jónsson & Tarski technology proves more than the fact that $V(K)$ (and other canonical logics/varieties) are \textbf{HSP-generated} from its \textit{CAV-BAOs}:

$$K = \text{HSP}(K \cap \text{CAV})$$

or even that they are \textbf{$S$-generated} by such algebras:

$$K = S(K \cap \text{CAV})$$

This last property, intermediate between weak completeness and canonicity, is a.k.a. \textit{(CAV-)complexity} (dual algebras of Kripke frames sometimes called “complex algebras”)

But in the 1970’s, people realized that many logics fail to be even weakly \textit{CAV-complete}
My 2005 PhD investigated (in-)completeness wrt classes of BAOs broader than CAV unifying, expanding, and building on earlier results by Thomason, Fine, Gerson, van Benthem, Blok, Chagrova, Chagrov, Wolter, Zakharyaschev, Venema and other researchers.

As it turned out, almost (?) every possible combination of C, A, V and related properties leads to a different non-trivial weak completeness notion (see L. AiML 2004, AU 2008 or Holliday and L., submitted).

So here’s our first guest of the day ...
THE SUBKRIKPKEAN HIERARCHY

At first approximation, just the boolean cube...
Weak (in-)completeness landscape AD 2005
(simplified version)

CAV-completeness
(Kripke frames)

CA-completeness
(discrete frames)

CA-completeness
(neighbourhood frames)

V-completeness
(possibility frames)

C-completeness

A-completeness

BAO-completeness

converse known to fail AD 2005

converse was an open question then
Weak (in-)completeness landscape AD 2017
(simplified version)

- **CAV-completeness** (Kripke frames)
  - **CV-completeness** (possibility frames)
  - **AV-completeness** (discrete frames)
  - **CA-completeness** (neighbourhood frames)

- **V-completeness**
  - **C-completeness**
  - **A-completeness**

- **BAO-completeness**

**Converse known to fail AD 2017**

**Converse is an open question**
\textbf{V-incompleteness: highlights}

- The solved problem also stated in Venema's "Algebras and coalgebras" chapter of the "Handbook of Modal Logic".

- Solution involved an elementary reformulation of $V$
  - Write " $a \in (b]"$ for $\bot < a \leq b$ and " $a \uparrow b"$ for $a \wedge b \neq \bot$.
  - $V$ defined by $\forall a. \forall b \uparrow \lhd a. \exists c \in (a]. \forall d \in (c]. b \uparrow \lhd a.$

- Parallel approach to complete additivity by Andréka, Gyenis, and Némethi.

- Van Benthem provided a nice proof of this reformulation in the spirit of "generalized correspondence theory".

- We were led to it by Wesley's work on possibility semantics, currently becoming a hot trend (also at this TACL).

- Our most spectacular counterexample involved the provability logic GLB (previously only known to be Kripke-incomplete).

- We lifted also, e.g., the so-called "Blok Dichotomy" to degrees of $V$-incompleteness.
Some additional properties

- $T$: admissibility of residuals/conjugates
  ($T$-completeness = conservativity of minimal tense extensions;
  similarly, $\mathcal{AV}$-completeness = hybrid conservativity)

- $E$: being (the dual of) a frame closed under weak SO with FO comprehension
  (van Benthem'79)
Weak (in-)completeness landscape AD 2017 (simplified version)

**CAV-completeness** (Kripke frames)

**CV-completeness** (possibility frames)

**AV-completeness** (discrete frames)

**CA-completeness** (neighbourhood frames)

**V-completeness**

**C-completeness**

**A-completeness**

**BAO-completeness**

Converse known to fail AD 2017

Converse seems an open question
converse known to fail AD 2017

converse seems an open question
More problems re weak (in-)completeness
(see also Ch. 9 of my PhD or conclusions of Holliday&L.)

- Better results for smaller lattices of modal logics—e.g., any $\mathcal{AV}$-incomplete logics over $K4$?
- “The Blok Dichotomy” for $A$-completeness? In smaller lattices of logics?
- Analogous analysis in the superintuitionistic or substructural setting?
- But ...
Using canonicity just to show weak completeness is often a drastic overkill.

If you’re not interested in strong completeness and duality, this uncountable detour is pedagogically and philosophically problematic.

For more, see e.g. Moss’ JPL’07 paper on and references therein (also Fine 1976, Ghilardi 1995, Bezhanishvili and Kurz 2007, Coumans and van Gool 2013, Bezhanishvili and Ghilardi 2014... or my joint work on modal aspects of XPath with ten Cate, Fontaine and Marx).

But as said before, there are stronger completeness notions, and these cannot be investigated using the normal forms technology.
Another notion inspired by the Kripke semantics: strong completeness, i.e., completeness for the associated consequence relation. Two possible relations (cf. Fine’s distinction between “case-to-case” and “truth-to-truth”)

- **Local consequence** over a logic $\Lambda$:
  closing $\Gamma$ only under members of $\Lambda$ and MP
  (of course, $\Lambda$ itself is closed under Necessitation and Substitution)

  Equivalent to $y_1 \land \ldots \land y_n \rightarrow \alpha \in \Lambda$ for some $y_1 \ldots y_n \in \Gamma$

  Over CPC reducible to consistency of $\Gamma \cup \neg \alpha$ over $\Lambda$

- **Global** for the global consequence over a logic $\Lambda$:
  closing $\Gamma$ under members of $\Lambda$ and MP and Necessitation
strong local $\kappa$-completeness:
“every $\Lambda$-consistent set $\kappa$-satisfiable”:
for every $\kappa$ and every proper filter $F$ in the $\kappa$-generated free algebra $A_\kappa$ in $V(\Lambda)$, there exists $\kappa$-generated $B \in V(\Lambda) \cap \mathcal{X}$ (i.e., with some $f: A_\kappa \rightarrow B$) s.t. $f[F]$ is contained in a principal filter

strong global $\kappa$-completeness: if $\alpha$ is not a global consequence of $\Gamma$ over $\Lambda$, then there is $B \in V(\Lambda) \cap \mathcal{X}$ and a valuation $v$ in $B$ s.t. $v(\Gamma) = \{T\}$ and $v(\alpha) \neq T$
As for canonicity, an adequate generalization (perhaps even too general a generalization) has been proposed by Chellas’80 “Modal Logic” book (so, long before the dawn of coalgebraic logic).

A logic $\Lambda$ is (Chellas) $\mathcal{X}$-canonical if for any algebra $A$ in the associated variety $V(\Lambda)$ there is a way of defining an operator on $\text{Pow}(Uf(\Lambda))$ s.t.

- the Stone embedding becomes a BAO-embedding
- the resulting algebra belongs to $V(\Lambda) \cap \mathcal{X}$

But, of course, such an algebra is always CA.

So in fact this provides just a neighbourhood generalization of canonicity.
Existence of neighbourhood incomplete logics

(Gerson'75; in hindsight, original Thomason'72 could’ve been used)

shows, a fortiori, that not every variety of BAOs can be neighbourhood canonical, \( \mathcal{EA} \)-complex etc.

But how different are those notions? For “natural” \( \mathfrak{A} \), Is there a real hierarchy of notions properly refining weak \( \mathfrak{A} \)-completeness?

Kripke-wise, crucial insights provided by Wolter

I analyzed them algebraically (cf. e.g. my TANCL’05 talk)
(some insights can be found also in AAL literature, esp. Czelakowski’s book—not sufficiently familiar with it)
THE SUBCANONICAL HIERARCHY
$X$-complex

- strongly globally $X$-complete
  - $X = \mathcal{P}X$ (trivial)
  - $X \subseteq \mathcal{V} \& X = HcX$
  - $X = \mathcal{P}_0X$

- strongly locally $X$-complete

- weakly $X$-complete

Kripke, possibility and discrete frames collapse these strong completeness notions

neighbourhood breakdown
(Shamkanov: move to non-well-founded deductions...)
THE COMPLETENESS
HYPERCUBOID
MacNeille-closed

\[ \downarrow \]

canonical

\[ \downarrow \]

weakly $X$-complete

\[ \downarrow \]

strongly locally $X$-complete

\[ \downarrow \]

strongly globally $X$-complete

\[ \overset{\text{canonical}}{\Rightarrow} \]

\[ X \subseteq V \land X = H_0 X \]
\[ \mathcal{X} \text{-complex} = \text{strongly globally } \mathcal{X} \text{-complete} \]

\[ \text{strongly locally } \mathcal{X} \text{-complete} \]

\[ \text{weakly } \mathcal{X} \text{-complete} \]

\[ \text{MacNeille-closed} \]

\[ \mathcal{J}(\mathcal{C}A\mathcal{V}) \]

\[ \text{canonical} \]

\[ \mathcal{C}A\mathcal{V} = \text{CT} \]

\[ \text{HSP-generated} \]

\[ \text{HSP-generated} \]

\[ \mathcal{C}A\mathcal{V} = \text{CT} \]
c-oriented projection

(actually, not quite a projection: Chellas canonicity did not fit in our “hypercuboid with opremium”)
MacNeille canonical admitting completions strongly globally Kripke complete strongly locally Kripke complete

(CChellas) $C^4$-canonical $(CAV)$ complete

$C^4$-complex strongly globally neighbourhood complete

$C^4$-complex strongly locally neighbourhood complete

$c$-complex = admitting completions

strongly globally neighbourhood complete

strongly locally neighbourhood complete

(weakly) $C^4$-complete

(weakly) $C^4$-complete

(fmp)

(weakly) $C^4$-complete

$C^4$-complete

(weakly) neighbourhood $(C^4)$ complete

(weakly) $c$-complete

Converse is known to fail

Converse seems an open question
+ follows from Fine, generalized by Gehrke&Harding&Venema

- MacNeille canonical

- Surendonk (2001): McKinsey logic is a counterexample

- L., generalizing Wolter’s reals example

- Wang’92 via Surendonk

- Wolter: tense logic of reals

- Surendonk (weakly) Kripke complete

- Wolter (weakly) complete

- Wolter strongly locally Kripke complete

- C4-complex strongly globally Kripke complete

- C4-complex = admitting completions

- C4-complex (weakly) neighbourhood complete

- (weakly) neighbourhood (C4) complete

- (weakly) C-complete

- converse is known to fail

- converse seems an open question
Conclusions & Questions

- Not-so-well-known or understood subtle hierarchy of completeness notions
  (many can be distinguished already with equations of rank ≤2!)

- We don’t have a complete classification of proper inclusions in the “hypercuboid”

- Surendonk’s “Chellas-canonicity” result never analyzed coalgebraically or otherwise

- Possibility semantics: new insights and questions
  Is the McKinsey logic $CV$-complex? Almost certainly no, but no proof

- Shamkanov: positive global neighbourhood completeness results possible when switching to non-well-founded consequence any insights on this, coalgebraists?

- Superintuitionistic or substructural analogues...?!
  Strengthening of Kuznetsov’s 1975 question: is every variety of Heyting algebras complex wrt spatial locales??
An intuitionistic analogue?
$C$-complex = admitting completions

converse is known to fail

converse seems an open question

$\epsilon$-complex = $e$-complete

$\text{spatially-complete} \leftrightarrow \text{strongly topologically complete} \leftrightarrow \text{aHA-complex} \leftrightarrow \text{strongly Kripke complete} \leftrightarrow \text{fmp}$

$\text{(weakly) Kripke (aHA) complete} \leftrightarrow \text{(weakly) topologically complete} \leftrightarrow \text{complete} \leftrightarrow \text{MacNeille canonical}$