The Locally Finite Fixpoint and its properties

A study of the semantics of finitely generated state and equation systems

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Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

\[ c \colon C \rightarrow HC \]
Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

$C \xrightarrow{c} HC$

$\nu H \xrightarrow{\tau} H\nu H$

Final Coalgebra

[AMV06]
Coalgebras

Coalgebras consist of a carrier, a functor, and a successor type along with a signature.

- **Carrier**: States, variables
- **Functor**: H
- **Successor type**: E.g., $\text{HX} = 2 \times (\Sigma - \varepsilon)$

The Coalgebra framework provides a unique solution for the fixpoint equation $Hc^\dagger$. This is illustrated in the diagram:

- $C \xrightarrow{c} HC$
- $\nu H \xrightarrow{\tau} H\nu H$
- $Hc^\dagger$

Additionally, the final coalgebra $\nu H$ is related to $H\nu H$ through a unique solution, mapping $c^\dagger$.

References:

[AMV06]
Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

E.g. $HX = 2 \times (\tau)^{\Sigma}$

Deterministic Automata

All Formal Languages

[AMV06]
Coalgebras

States, variables → Successor type, Signature

Carrier \rightarrow \text{Functor}

C\ f.p. \rightarrow \nu H \rightarrow H\nu H

E.g. \( HX = 2 \times (\neg)^\Sigma \)

Finite Deterministic Automata

All Formal Languages

[AMV06]
Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

C f.p. \( C \xrightarrow{c} H C \)

Finite Deterministic Automata

Rational Fixpoint

\( \rho H \xrightarrow{r} H \rho H \)

All Formal Languages

\( HX = 2 \times (\neg)_{\Sigma} \)

Final Coalgebra

\( \nu H \xrightarrow{\tau} H \nu H \)

[AMV06]
Coalgebras

States, variables

Successor type, Signature

Coalgebras $C$ f.p. $C \xrightarrow{c} HC$ Finite Deterministic Automata

$\varrho H \xrightarrow{r} H\varrho H$ Regular Languages

$\nu H \xrightarrow{\tau} H\nu H$ All Formal Languages

E.g. $HX = 2 \times (-)^{\Sigma}$

$\rho H \xleftarrow{c^+} HC \xleftarrow{Hc^+} H\rho H \xleftarrow{Hr^+} H\nu H$

[AMV06]
In some scenarios:

- $r^\dagger : \varrho H \to \nu H$ is not monic.
- Not all equal behaviours identified by $\varrho H$. 
In some scenarios:

- $r^† : \varrho H \rightarrow \nu H$ is not monic.
- Not all equal behaviours identified by $\varrho H$.

Wanted:

Collection of finite behaviours $\cong$ Subcoalgebra of $\nu H$ for finite behaviours
In some scenarios:

- $r^\dagger : \varrho H \longrightarrow \nu H$ is not monic.
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Wanted:

Collection of finite behaviours = Subcoalgebra of $\nu H$ for finite behaviours $\neq \varrho H$
Problems

In some scenarios:

- \( r^\dagger : \rho H \rightarrow \nu H \) is not monic.
- Not all equal behaviours identified by \( \rho H \).

Wanted:

\[
\text{Collection of finite behaviours} = \text{Subcoalgebra of } \nu H \text{ for finite behaviours} \neq \rho H
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Problems

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Problems

In some scenarios:

- $r^\dagger : \varrho H \rightarrow \nu H$ is not monic.
- Not all equal behaviours identified by $\varrho H$.

Wanted:

- Universal property?
- Fixpoint?

$?? = \text{Collection of finite behaviours} = \text{Subcoalgebra of } \nu H \text{ for finite behaviours } \neq \varrho H$
In some scenarios:

- $r^\dagger : \varrho H \rightarrow \nu H$ is not monic.
- Not all equal behaviours identified by $\varrho H$.

Wanted:

$$?? = \text{Collection of finite behaviours} = \text{Subcoalgebra of } \nu H \text{ for finite behaviours} \neq \varrho H$$
LFP Categories

Finitely presentable objects

\( X \text{ f.p. } \iff \text{Hom}(X, -) \) preserves filtered colimits

\[ \Downarrow \]

(Strong) Quotients

Finitely generated objects

\( X \text{ f.g. } \iff \text{Hom}(X, -) \) preserves directed colimits of monos

\[ \Downarrow \]

Filtered colimits

All other objects are . . .

assembled of finitely presentable objects

\( \iff \) assembled of finitely generated objects
Assumptions

- Base category \( B \) \( \text{lfp} \)
- Endofunctor \( H : B \to B \) finitary
- \( H \) preserves monos
LFG Coalgebras

\( \text{Coalg}_{\text{fg}} H \)

\( C \overset{c}{\to} HC \) with \( C \) f.g.

LFG Coalgebras \( \text{Coalg}_{\text{lfg}} H \)

\[ \forall \text{f.g. } S \overset{f_0}{\longrightarrow} P \overset{p}{\cong} HP P \text{ f.g.} \]

= filtered colimits of f.g.-carried \( H \)-Coalgebras
LFG Coalgebras

**Coalg\(_{\text{fg}}\) \(H\)**

\[ C \xrightarrow{c} HC \text{ with } C \text{ f.g.} \]

**LFG Coalgebras Coalg\(_{\text{lfg}}\) \(H\)**

\[ \forall \text{f.g. } S \xrightarrow{f_0} P \xrightarrow{p} HP \text{ P f.g.} \]

= filtered colimits of f.g.-carried \(H\)-Coalgebras

\[ \mathcal{L} \xrightarrow{l} H \mathcal{L} \text{ lfg final for } \text{Coalg}_{\text{fg}} \(H\) \]

\[ \mathcal{L} \xrightarrow{l} H \mathcal{L} \text{ lfg final for } \text{Coalg}_{\text{lfg}} \(H\) \]
**Construction**

Final lfg coalgebra \( = \text{colim} \text{Coalg}_{\text{fg}} H \)

**Proposition**

\( L \xrightarrow{\ell} H L \text{ lfg} \Rightarrow H L \xrightarrow{\ell} HH L \text{ lfg} \)

\( \Rightarrow \) Final lfg coalgebra is an isomorphism

**Definition**

\((\nu H, \ell) := \text{final lfg coalgebra}\)

The **locally finite fixpoint** (LFF) of \( H \)

Lfg coalgebras closed under quotients \( \Rightarrow \nu H \) subcoalgebra of \( \nu H \)
What about the inverse

**Definition**

**equation morphism** in an $A$:

$$X \rightarrow HX + A, \text{ with } X \text{ f.g.}$$

Algebra $HA \xrightarrow{a} \text{ is fg-iterative if:}$

$$\begin{align*}
X \xrightarrow{\exists!e^\dagger} & A \\
\forall e \downarrow & \quad [a,A] \\
HX + A \xrightarrow{He^\dagger + A} & HA + A
\end{align*}$$

Category of fg-iterative Algebras + solution preserving morphisms

= Category of fg-iterative Algebras + algebra homomorphisms
The rational image

\[ \text{LFF} = \text{Image of } \varrho H \text{ in } \nu H \]

\[ \begin{align*}
\varrho H & \xrightarrow{r} H \varrho H \\
q & \downarrow \cong \\
\vartheta H & \xrightarrow{\ell} H \vartheta H \\
\ell \dagger & \downarrow \cong \\
\nu H & \xrightarrow{\tau} H \nu H \\
\end{align*} \]

(under some stronger assumptions...)

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Summary

The locally finite fixpoint

- final lfg coalgebra
- final for f.g.-carried coalgebras
- union of images of f.g.-carried coalgebras
- initial fg-iterative Algebra
- rational image
Applications

Examples

All applications where the rational fixpoint does the job, already.
Applications

**Boring Examples**
All applications where the rational fixpoint does the job, already.

**Interesting Examples**
All applications in Categories with

\[ \text{f.p. objects} \subsetneq \text{f.g. objects} \]
Generalized powerset construction

$T$-Automaton

\[ X \xrightarrow{x} HTX \]

Computational side effect

\[ \text{Lifting} \]

\[ H : \text{Set} T \rightarrow \text{Set} T \]

sometimes f.p. $\subseteq$ f.g.
Generalized powerset construction

$T$-Automaton

\[ X \xrightarrow{x} HTX \]

Requirement

Lifting $H^T : \text{Set}^T \rightarrow \text{Set}^T$ of $H$ to $\text{Set}^T$.

Determinization

\[ X \xrightarrow{\eta_X} TX \xrightarrow{x^{#\dagger}} \nu H \]

\[ HTX \xrightarrow{Hx^{#\dagger}} H\nu H \]
Application of the LFF

Proposition

\[ \vartheta H^T = \bigcup_{x: X \to HTX \text{ finite}} \text{Im}(x^{\#^\dagger}) \subseteq \nu H \]

Examples (work in progress...)

- \[ T = \mathcal{P}_f((-)^*) \]
  \[ \Rightarrow X \to HTX = \text{grammars in Greibach normal form} \]
  \[ \Rightarrow \vartheta H^T = \text{context-free languages.} \]

- \[ T = \text{Stack monad} \]
  \[ \Rightarrow X \to HTX = \text{deterministic stack machines} \]
  \[ \Rightarrow \vartheta H^T \approx \text{deterministic context-free languages.} \]
Algebraic trees

Recursive program scheme

\[ e : H_\phi \rightarrow F^{H_\Sigma+H_\phi} \]

Example

For the signature \( \Sigma = \{f/2, g/1\} \):

\[ \varphi(x) = f(x, \varphi(g(x))) \]

Coalgebraic structure

On finitary Monads + \( H \)-pointing: \( H/Mnd_f(\text{Set}) \):

\[ e : H_\phi \rightarrow F^{H_\Sigma+H_\phi} \]

\[ F^{H_\Sigma+H_\phi} \rightarrow HF^{H_\Sigma+H_\phi} + \text{Id} \]

\[ \mathcal{H}_f B = HB + \text{Id} \]

[AMV11]
Application of the LFF

Coalgebraic structure

On finitary Monads \( + \) \( H \)-pointing: \( H/\text{Mnd}_f(\text{Set}) \):

\[
e : H_\phi \rightarrow F^{H_\Sigma + H_\phi}
\]

\[
F^{H_\Sigma + H_\phi} \rightarrow HF^{H_\Sigma + H_\phi} + \text{Id}
\]

\[
\mathcal{H}_f B = HB + \text{Id}
\]

Proposition

Algebraic trees = \( \wp \mathcal{H}_f \)
Thank you for your attention!

