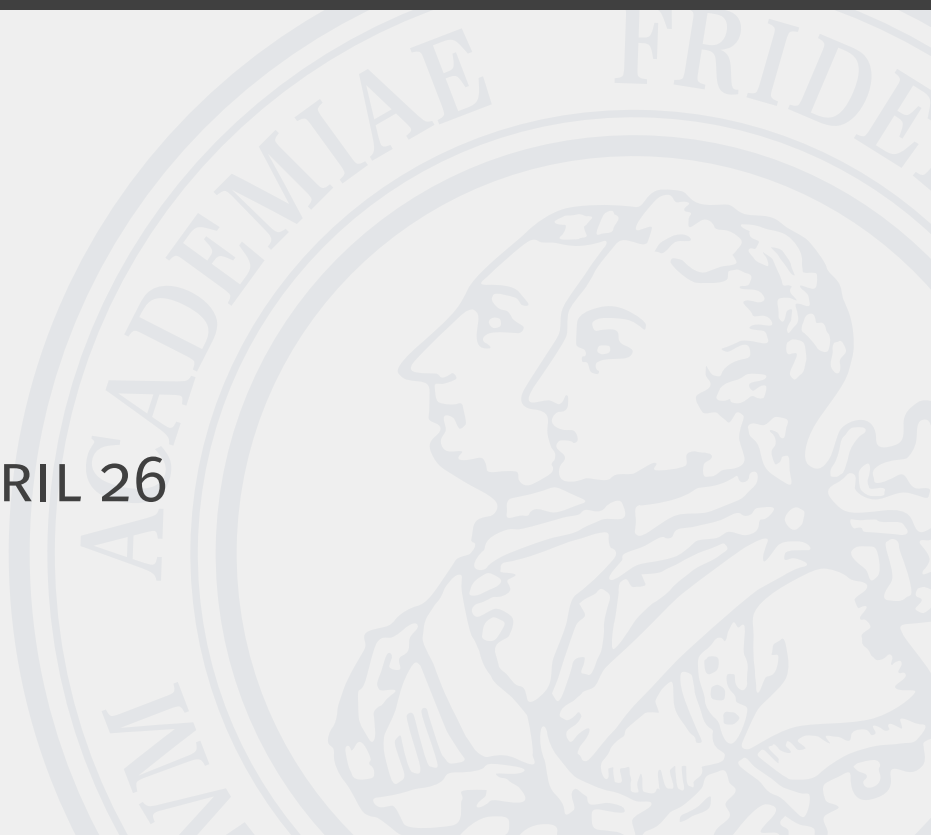
Unifying Notions of Feedback

Sergey Goncharov



FAU Tag der Informatik 2019, April 26

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Outline

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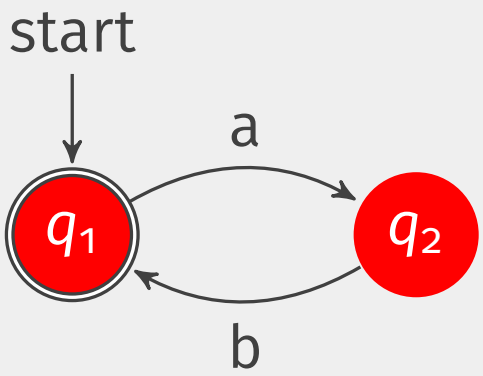
- Total (but not partial!) feedback operators are categorically unified with traced (monoidal) categories
- Grand unification:

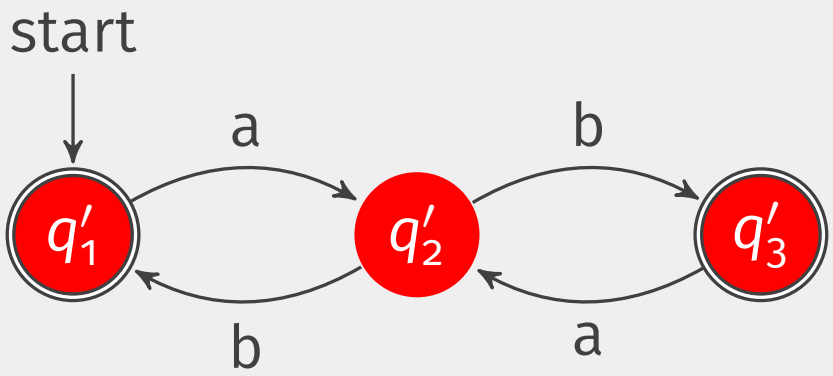
Guarded Traced Categories

Why Semantics?

A Well-known Scenario

How do we know that automata





are equivalent?

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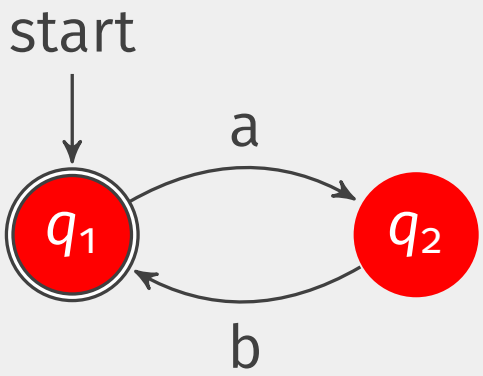

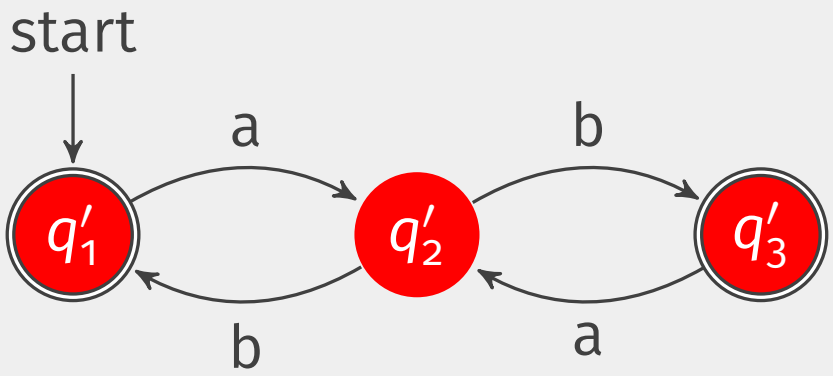

are equivalent?

dinaturality identity

Because $[\![q_1]\!] = (ab)^\star$, $[\![q'_1]\!] = a(ba)^\star b + 1$ and: $(ab)^\star = a(ba)^\star b + 1$

semantic brackets

Semantics in Computer Science

Semantics

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Effectiveness
(of presentation, decidability)

Semantics

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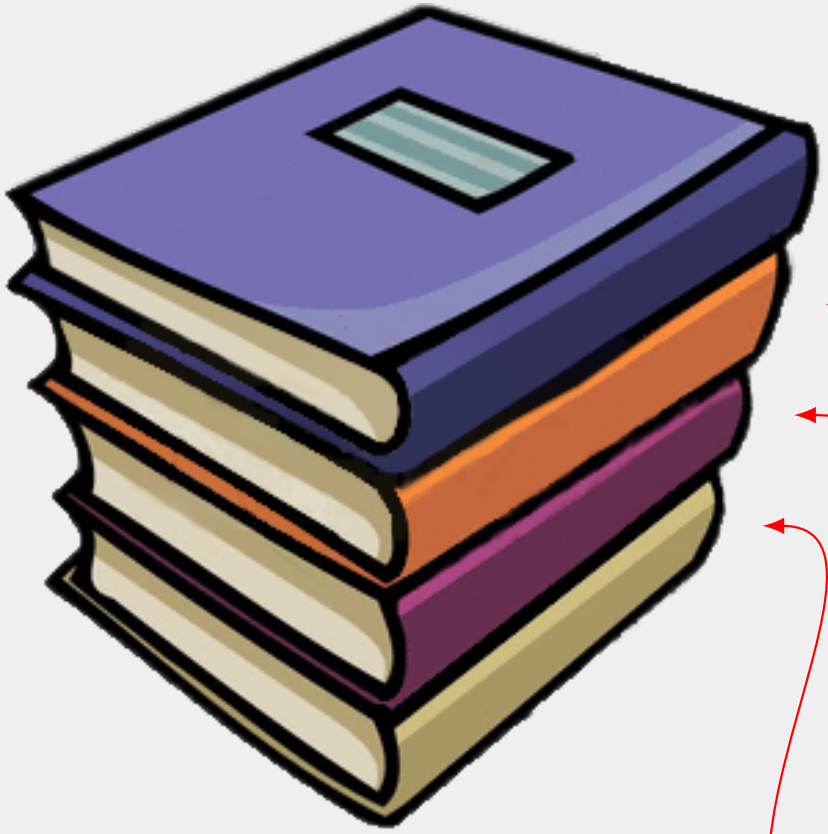


Algorithms, implementations, concrete programming languages, compilers, interpreters, simulators, proof assistants, ...

Efficiency (complexity of computations, efficient data structures)

Effectiveness (of presentation, decidability)

Semantics

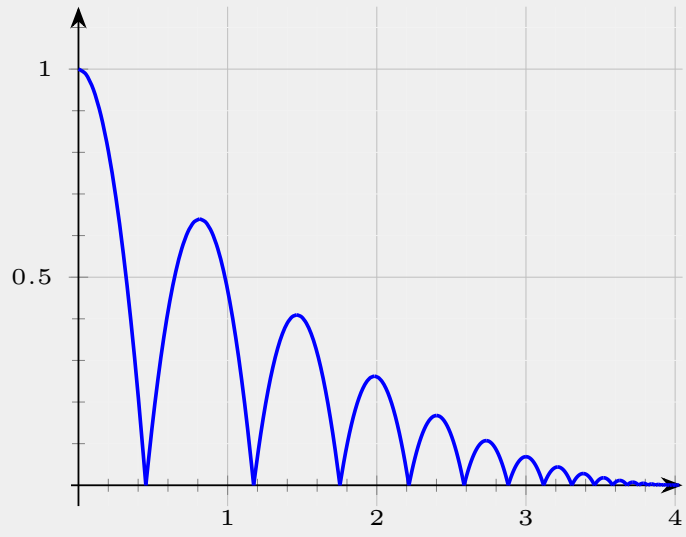
A Less Known Scenario

Bouncing ball is a simple Newtonian system specified by differential equation $\ddot{h} = -g$ ($g \approx 9.8$) whose solution is

$$h(t) = h_0 + v_0 t - \frac{gt^2}{2}$$

with initial values:

- $v_0 = 0, h_0 \neq 0$ (peak height)
- $h_0 = 0, v_0 \neq 0$ (zero height)



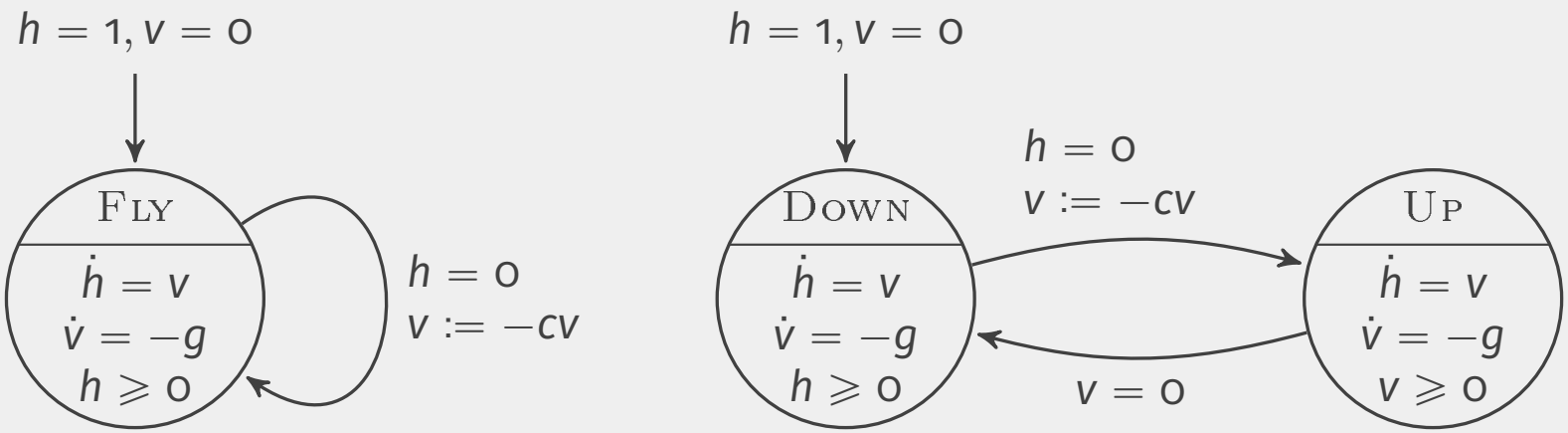
Features:

- deterministic
- hybrid: the velocity changes discretely at the bottom $v \mapsto -cv$, but it changes continuously in the meanwhile
- Zeno behaviour: the state of rest is only reachable in the limit

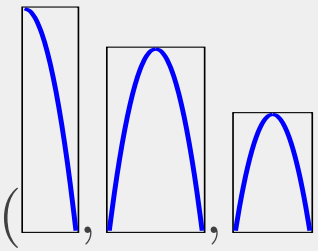
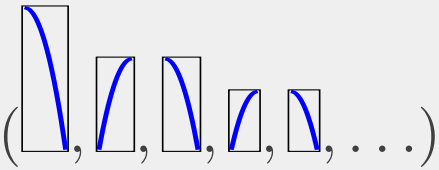
damping factor

Hybrid Automata

The following hybrid automata "A" and "B" capture the bouncing ball behaviour:



These automata are not equivalent under standard semantics, because

$[\![A]\!] = ($  $, \ldots)$, but $[\![B]\!] = ($  $, \ldots)$

Impact of Semantics

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- We can optimize programs, e.g.

```
while b do ...                    while b do ...
  if b then  ...          →         ...
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  and verify them (since, we know what they mean!)
- Principled semantic foundations improve design of languages, software and hardware systems (types, compositionality, Curry-Howard correspondence, etc)

Why Category Theory?

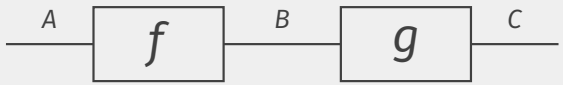
[...] Kategorientheorie – ein sehr komplexes Gebiet mit tiefen mathematischen Wurzeln, und mit relativ wenigen Experten auf diesem Gebiet

— Anonymous referee

Categories: Quick Intro

- A category $\mathbf{C}$ consists of wires (=objects) $|\mathbf{C}|$ and boxes (=morphisms) $\mathbf{C}(A, B)$ with $A, B \in |\mathbf{C}|$, which can be combined:

identity wire



sequential composition

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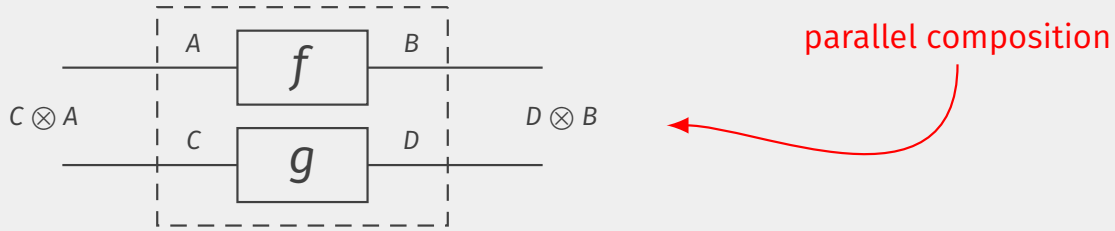
- A category is monoidal if morphisms can be tensored:



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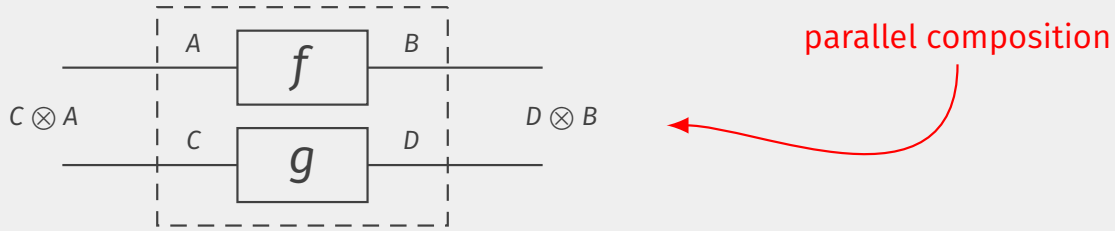

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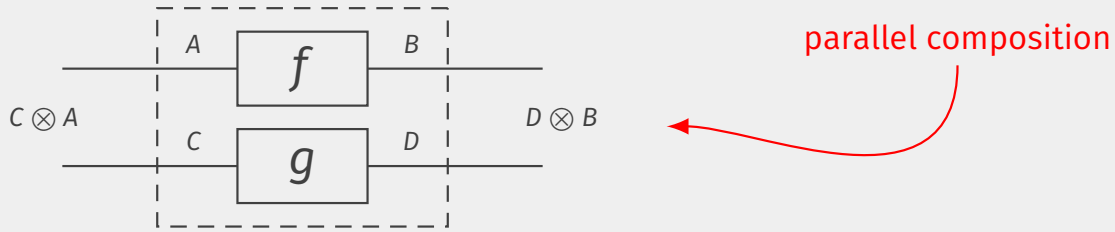
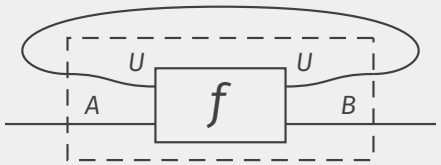


- A monoidal category is symmetric if wires can be crossed:

Boxes are intuitively: programs, processes, components, automata; wires are types, communication channels

Traced Categories

Traced categories additionally allow feedback loops, called traces:



Traced categories provide a unifying framework for

- Iteration (roughly: while-loops)
- Recursion (roughly: fixpoint combinators of λ-calculus)
- Knot theory
- Operator theory (e.g. traces model quantum measurements)

Why Guarded Traces?

Automata, Revisited

Consider again the regular expressions $(ab)^\star$ and $a(ba)^\star b + 1$
Here, Kleene star $e^\star$ is the unique fixpoint of

$$x \mapsto ex + 1$$

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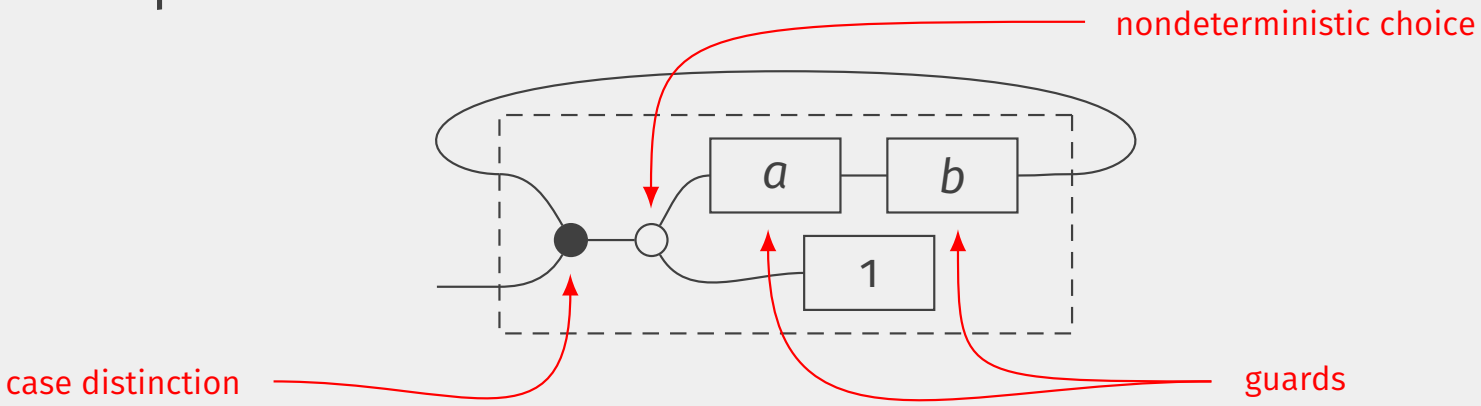
This only works because the map $x \mapsto abx + 1$ is guarded

Contrastingly, the map $x \mapsto (a + 1)x + 1$ is unguarded and has infinitely many fixpoints

Guarded Traces

Automata can be organized into a traced symmetric category, e.g. $(ab)^\star$ is expressed as

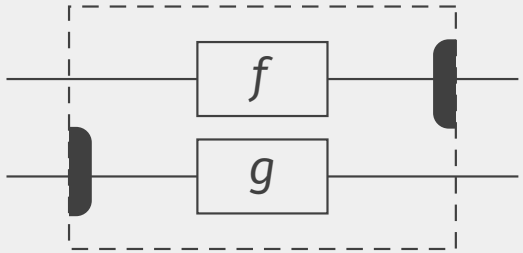


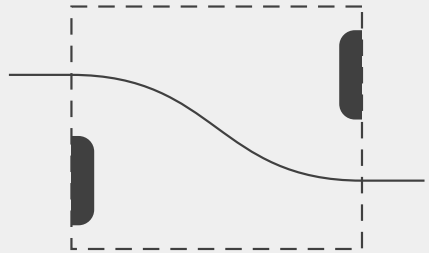
But we do not have an explicit distinction between guarded and unguarded loops

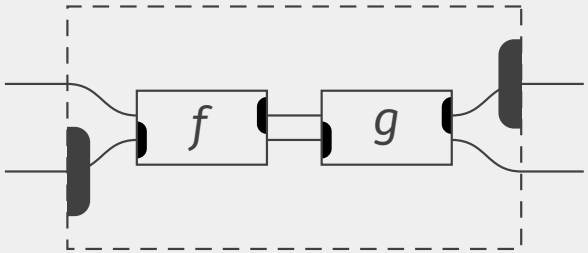
- Guarded traces are simpler and better behaved
- They often have special useful properties, e.g. uniqueness
- Sometimes, there is no candidate for a total trace operator

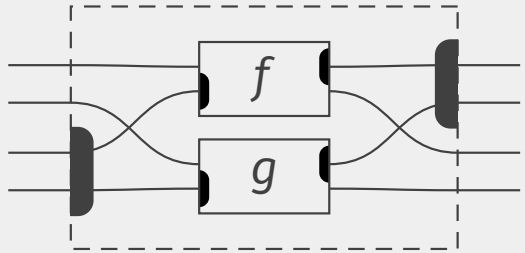
Guarded Categories

Guarded (monoidal) categories are defined in terms of decorated diagrams, subject to the axioms:





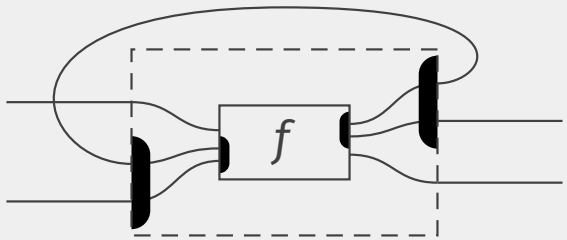




Only the paths from left bars to right bars are considered guarded

Guarded Traces

A guarded category is guarded traced if it supports guarded traces:



Intuitively, we are allowed to close the loop because it only runs through the interface of f indicated by the black bars

Total Guardedness and Vacuous Guardedness

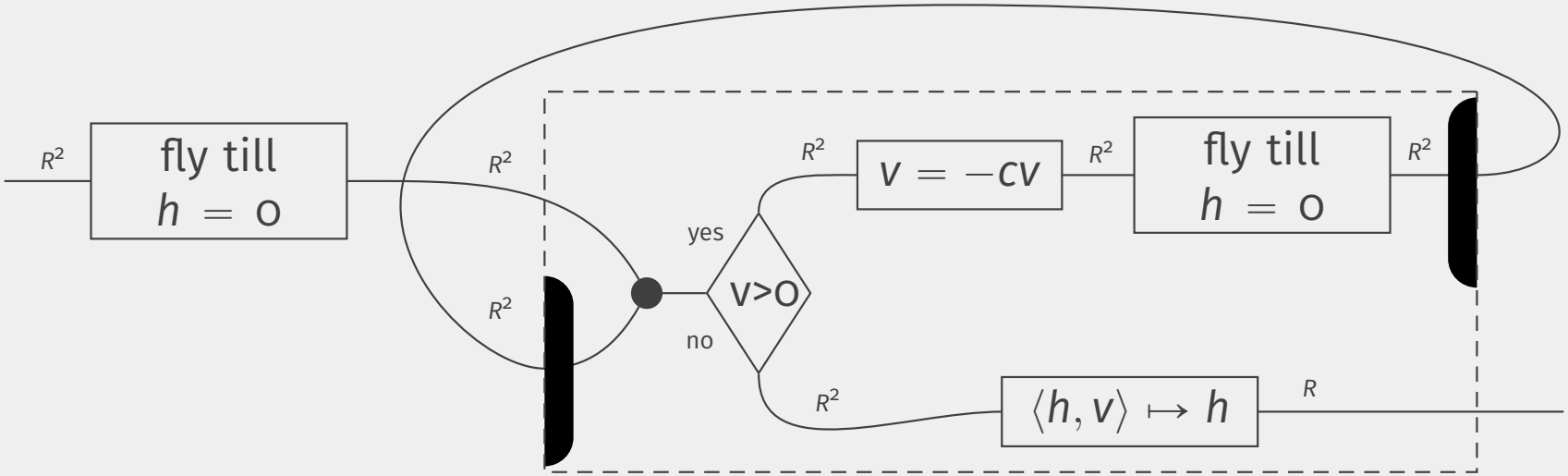
With guarded categories we mediate between symmetric monoidal categories and traced symmetric ones with

- vacuous guardedness: there are no guarded paths at all (symmetric monoidal categories)
- total guardedness: every path is guarded (traced symmetric categories)
- motivating examples (such as automata) occur properly between these two extremes

Bouncing Ball Revisited

Guardedness in hybrid semantics means progressiveness, i.e. consuming non-zero time

The behaviour of bouncing ball is described in terms of velocity and height $\langle v, h \rangle \in R^2$:



In contrast to general traces, guarded ones are computed as least fixpoints

Further Examples

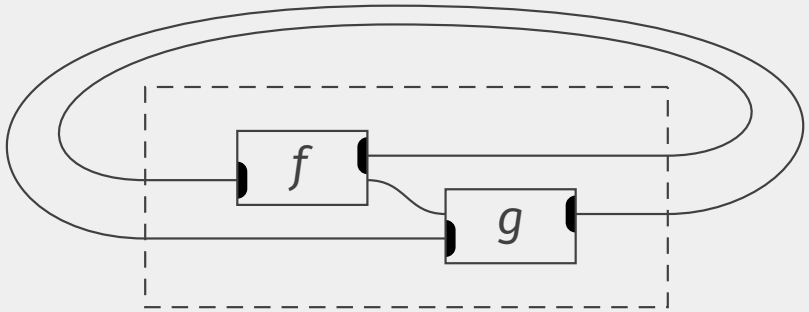
- Complete metric spaces: guardedness = contractiveness, fixpoints are computed via Banach's fixpoint theorem
- Infinite-dimentional Hilbert spaces under vacuous guardedness, traces = traces of bounded linear operators
- Infinite trace semantics: guardedness by actions, traces are greatest fixpoints, e.g. $a^\star + a^\omega$ is the canonical fixpoint of

$$x \mapsto ax + 1,$$

and not just $a^\star$

Completeness Problem

Q: Can we make sense of the following diagram?



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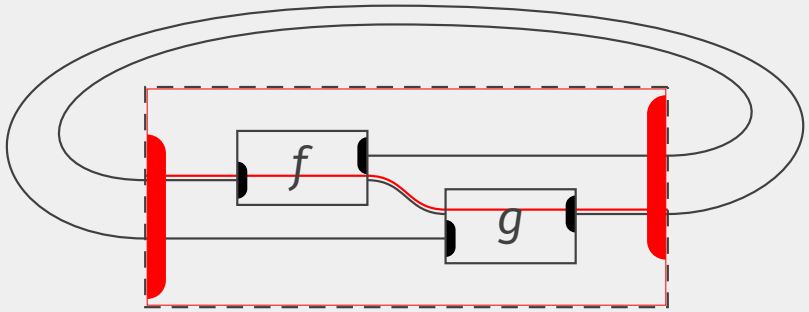

Completeness Problem

Q: Can we make sense of the following diagram?
A: Yes, because we can redraw it as

Completeness Problem

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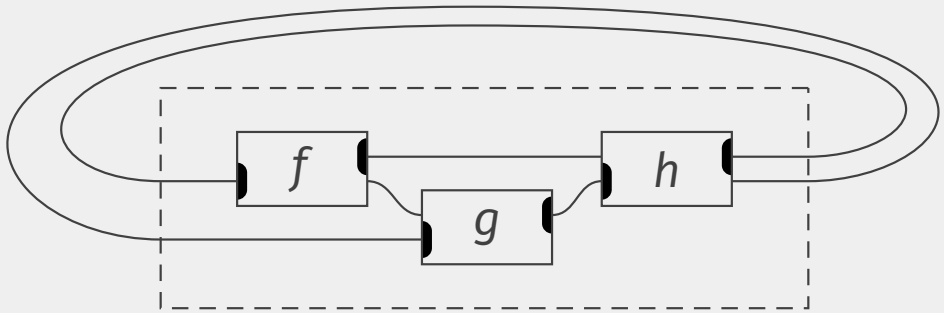

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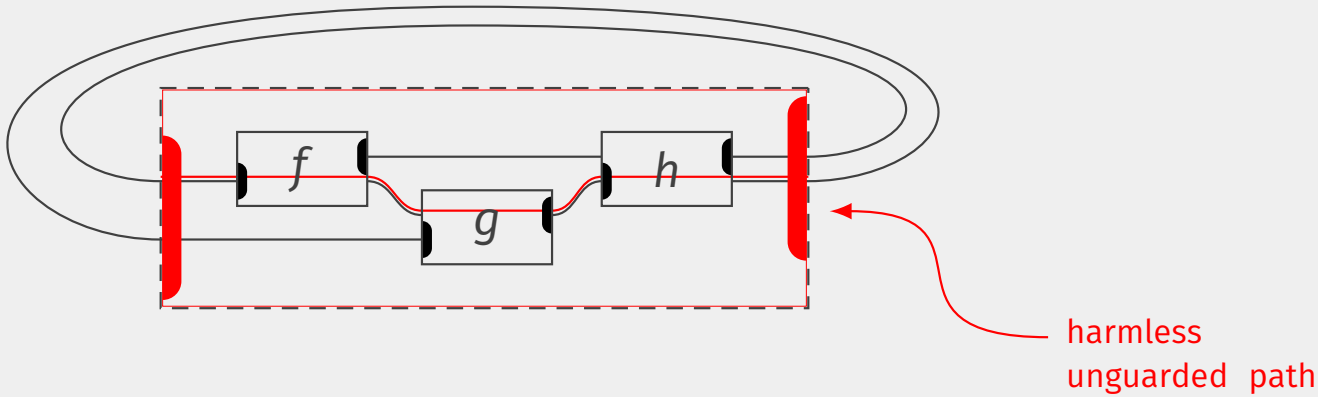

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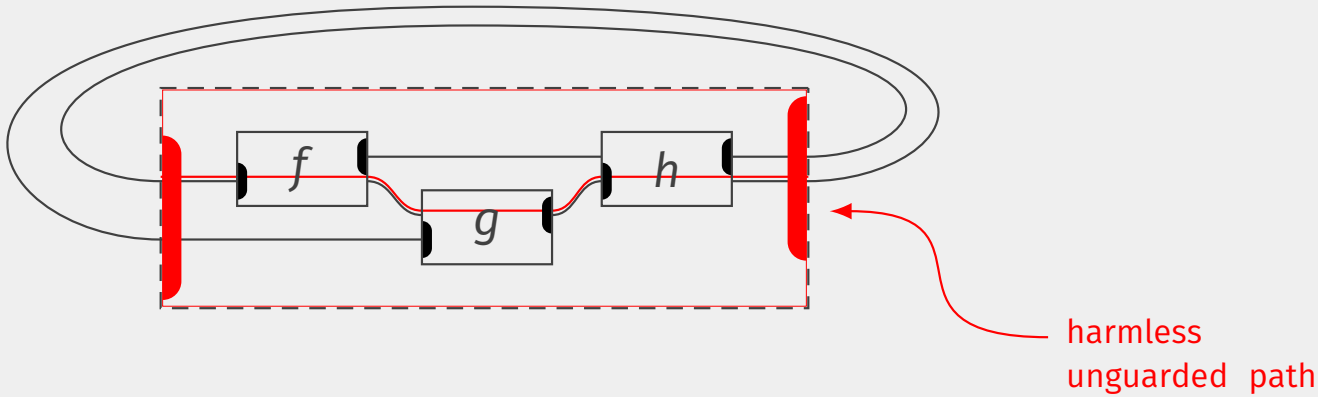

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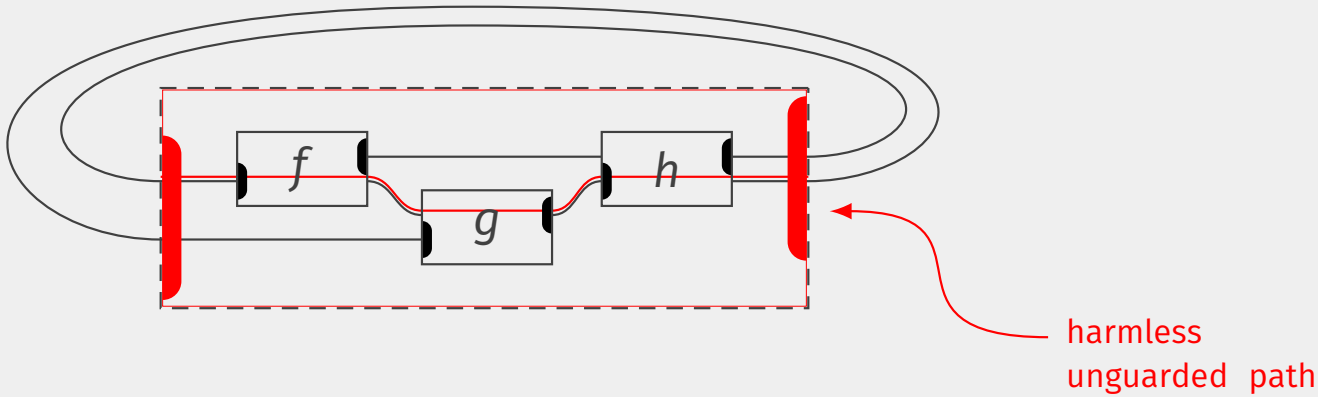

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Open Problem: resolve the discrepancy between geometric guardedness and structural guardedness

Conclusions

- Guarded traced categories are abound in semantics and beyond
- Abstract notion of guardedness unifies various principles behind partiality of feedback: delay, progress, contractivity, etc
- Guarded categories come together with a semantically justified unifying diagrammatic metalanguage, suitable for visual programming and modelling (e.g. hybrid systems)
- Abstract guardedness helps to identify well-behaved notions of feedback (unique, least, greatest), which come together with the corresponding reasoning principles (co-induction, Scott induction)