A [Monad-Based] Semantics for Hybrid Iteration

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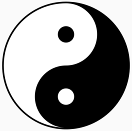

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Plan of the Talk

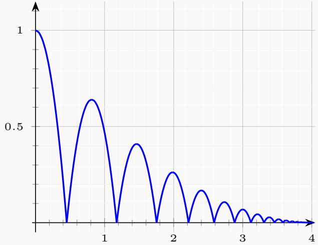
- Precursors:
  - Hybrid automata
  - Platzer's dynamic logic
  - Höfner and Möller's algebra of hybrid systems
  - Hybrid process calculi: $\mathsf{ACP}^{\mathsf{srt}}_{\mathsf{hs}}$, hybrid χ, HYPE, HyPA, hybrid CSP, ...
- More recently: hybrid monad[†] $\mathbb{H}_0$ (deterministic, abstract, composable)
- Here: two new monads $\mathbb{H}_+$ and $\mathbb{H}$ for hybrid iteration

[†] Neves, Barbosa, Hofmann, and Martins 2016, Continuity as a computational effect

Quick Intro: Bouncing Ball and Zeno Behaviour

Bouncing ball is a simple Newtonian system specified by differential equation $\ddot{h} = -g$ ($g \approx 9.8$) whose solution is

$$h(t) = h_0 + v_0 t - \frac{gt^2}{2}$$



with initial values:

- $v_0 = 0$, $h_0 \neq 0$ (peak height)
- $h_0 = 0$, $v_0 \neq 0$ (zero height)

The velocity changes discretely at the bottom $v \mapsto -cv$, but it changes continuously in the meanwhile. This is called hybrid behaviour

The state of rest is only reachable in the limit. This is called Zeno behaviour

Progressive Iteration vs. Singular Iteration

- Bouncing ball movement is progressive in the sense that each iteration is non-instantaneous
- On the other hand

$$\mathtt{while\ true}\ \{\mathtt{x} := 1 - \mathtt{x}\}$$

is not progressive and must yield the divergence $\perp$

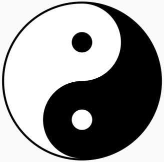
This disrupts the time flow structure, for we expect

$$\mathtt{while\ true}\ \{\mathtt{x} := 1 - \mathtt{x}\} \simeq \mathtt{while\ true}\ \{\mathtt{x} := 1 - \mathtt{x}\}; \mathtt{wait}(1)$$

Idea: progressiveness is a form of abstract guardedness[†]

[†] Goncharov and Schröder 2018, Guarded Traced Categories

Monads in Semantics



- Monads formalize generalized functions[†] $f : X \to TY$, like nondeterministic (with $T = \mathcal{P}X$) or partial (with $TX = X + 1$)
- T is a type constructor, together with operations $\eta : X \to TX$ (unit) and $(f : X \to TY) \mapsto (f^\star : TX \to TY)$ (lifting), inducing the Klesili category of T under

$$\mathrm{id} = \eta : X \to TX \qquad f \diamond g = (f : Y \to TZ)^\star\,(g : X \to TY)$$

In Haskell's point-full notation: do $x \leftarrow p; f(x) = f^\star(p)$

[†]Moggi 1991, Notions of Computation and Monads

Abstract Guardedness on Monads

Abstract guardedeness for a monad T is a relation between Kleisi morphisms $f : X \to TY$ and summands $\sigma : Y' \hookrightarrow Y$ satisfying

$$\textbf{(trv)} \qquad \frac{f : X \to TY}{(T\,\mathsf{in}_1)\, f : X \to_{\mathsf{in}_2} T(Y+Z)}$$

$$\textbf{(sum)} \qquad \frac{f : X \to_\sigma TZ \qquad g : Y \to_\sigma TZ}{[f,g] : X + Y \to_\sigma TZ}$$

$$\textbf{(cmp)} \qquad \frac{f : X \to_{\mathsf{in}_2} T(Y+Z) \qquad g : Y \to_\sigma TV \qquad h : Z \to TV}{[g,h]^\star f : X \to_\sigma TV}$$

where $f : X \to_\sigma TY$ means that f and σ are in the relation

Abstract Guardedness on Monads

A monad is guarded Elgot if it supports partial iteration sending each $f : X \to_2 T(Y + X)$ to $f^\dagger : X \to TY$ satisfying the fixpoint law

$$f^\dagger = [\eta, f^\dagger]^\star f$$

and other laws of iteration

This yields 2-dimensional classification:

	total	partial
iterative (unique $-^\dagger$)	degenerate, e.g. $T = 1$	completely iterative, e.g. $T = \nu\gamma.\, \mathcal{P}_\omega(-+A \times \gamma)$
Elgot (non-unique $-^\dagger$)	e.g. $T = \mathcal{P}$	e.g. $T = \mathcal{P}(A^\star \times - + A^\omega)$

Comparing Guarded and Unguarded Iteration

We identify totally guarded iteration with unguarded iteration

- $T = \mathcal{P}$: this supports unguarded iteration via least fixpoints
- $T = \mathcal{P}^+$, i.e. non-empty powerset: the iteration operator is inherited from $\mathcal{P}$ along injection $\mathcal{P}^+ \hookrightarrow \mathcal{P}$

  The resulting operator is partial, for $\eta\,\mathsf{inr} : 1 \to \mathcal{P}^+(1+1)$ does not have a fixpoint

  This operator is guarded with $f : X \to_2 \mathcal{P}^+(Y+Z)$ iff $\forall x.\, \exists y.\ \mathsf{inl}\, y \in f(x)$

Basic Monad for Hybrid Computations

The Monad $\mathbb{H}_0$

1. A closed trajectory t is a pair $(t_\mathsf{d} \in \mathbf{R}_+, t_\mathsf{e} : [0, t_\mathsf{d}] \to X)$
2. An infinite trajectory t is a pair $(\infty, t_\mathsf{e} : [0, \infty) \to X)$
3. More uniformly, $t \in \overline{\mathbf{R}}_+ \times X^{\mathbf{R}_+}$ + flattening condition:

$$x \geqslant t_\mathsf{d} \qquad \text{implies} \qquad t_\mathsf{e}(x) = t_\mathsf{e}(t_\mathsf{d})$$

Let $H_0 X$ be the set of trajectories identified by (3).

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Let H_0X be the set of trajectories identified by (3). H_0 is a monad![†]

- Unit: $\eta(x) = (0, \underline{x})$, where $\underline{x}$ is the constant function $\lambda - . x$;
- Kleisli lifting: given $f : X \to H_0Y$ and $(d, e) \in H_0X$,

$$(f^\star(d,e))_\mathsf{d} = d + f_\mathsf{d}(e^d) \lhd d \in \mathbf{R}_+ \rhd \infty$$
$$(f^\star(d,e))_\mathsf{e}^t = f_\mathsf{e}^0(e^t) \lhd t < d \rhd f_\mathsf{e}^{t-d}(e^d)$$

where x^t reads as $x(t)$ and $x \lhd y \rhd z$ as "if y then x else z"

[†] Neves, Barbosa, Hofmann, and Martins 2016, Continuity as a computational effect

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Closure under iteration $\Rightarrow$ closure under Zeno behaviour

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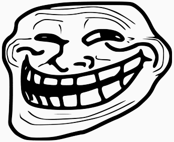

However,

1. Zeno argued that it is impossible to reach the limit;)
   So, it is impossible to close the trajectory even
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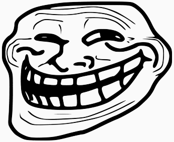
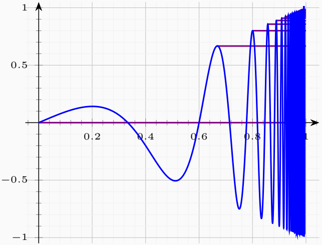


Closure under iteration $\Rightarrow$ closure under Zeno behaviour

However,

1. Zeno argued that it is impossible to reach the limit;)
   So, it is impossible to close the trajectory even if there is a unique closed trajectory candidate
2. It is certainly possible to have hybrid behaviour without a closed trajectory candidate whatsoever



Solution: open trajectories

Iterating Hybrid Computations

Partial Trajectories

Let $\mathbb{M}$ be the maybe monad: $MX = X + 1$

Proposition: Every monad $\mathbb{T} = (T, \eta, (\text{–})^\star)$ induces a monad $\mathbb{TM}$ whose underlying functor is $X \mapsto T(X + 1)$

Theorem: The hom-sets $\mathrm{Hom}(X, TM)$ can be suitably ordered, so that

1. Every $H_0 MX$ is an ω-complete partial order under $\sqsubseteq$, and $(0, \bot)$ is the bottom element
2. Kleisli composition is monotone and continuous on both sides
3. Kleisli composition is right-strict: $f^\star(0, \bot) = (0, \bot)$

However, Kleisli composition is not left-strict:

$$(0, \bot)^\star(1, \mathrm{id}) = (1, \bot) \neq (0, \bot)!$$

Theorem: By (1)–(3), $\mathbb{H}_0\mathbb{M}$ is a (total) Elgot monad

Open Trajectories

$\mathbb{H}_0\mathbb{M}$ contains lots of junk trajectories, while we only need open and closed ones: $(d \in \overline{\mathbf{R}}_+, e : [0, d) \to X)$ and $(d \in \mathbf{R}_+, e : [0, d] \to X)$

Let H_+ be the following subfunctor of $H_0 M$:

$$(d, e) \in H_+ X \qquad \text{iff} \qquad \boxed{e \neq \bot} \qquad \text{and} \qquad e^t{\downarrow} \quad \text{for all} \quad t \in [0, d)$$

This induces a monad $\mathbb{H}_+$ with

$$\begin{aligned}
&(f^\star(d, e))_\mathsf{d} = d, && (f^\star(d, e))^t_\mathsf{e} = f^0_\mathsf{e}(e^t) \lhd t < d \rhd \bot && \text{if } e^d{\uparrow} \\
&(f^\star(d, e))_\mathsf{d} = d + f_\mathsf{d}(e^d), && (f^\star(d, e))^t_\mathsf{e} = f^0_\mathsf{e}(e^t) \lhd t < d \rhd f^{t-d}_\mathsf{e}(e^d) && \text{if } e^d{\downarrow}
\end{aligned}$$

Total iteration of $\mathbb{H}_0\mathbb{M}$ restricts to guarded iteration of $\mathbb{H}_+$ with guardedness being progressiveness: $(d, e) : X \to_2 H_+(Y + Z)$ if $e^0 : X \to Y + Z$ factors through inl

Full Hybrid Monad $\mathbb{H}$

In $\mathbb{H}_+$ we excluded the "black hole" trajectory $(0, \bot)$, for it brings up "space-time artefacts" like $(1, \bot)$

Let H be the following subfunctor of $H_0 M$:

$$(d, e) \in HX \qquad \text{iff} \qquad \cancel{e \neq \bot \quad \text{and}} \quad e^t{\downarrow} \quad \text{for all} \quad t \in [0, d)$$

Now, the induced injection $\upsilon : H \hookrightarrow H_0 M$ is **not** a monad morphism: $(0, \bot)^\star(1, \mathrm{id})$ must be $(0, \bot)$ in $\mathbb{H}$, but not in $H_0 M$

However, υ has a retraction $\rho : H_0 M \to H$, which **is** a monad morphism. Moreover (υ, ρ) is an iteration-congruent retraction[†]

Theorem: $\mathbb{H}$ is an iteration-congruent retract of $\mathbb{H}_0\mathbb{M}$, hence an Elgot monad

[†] Goncharov, Schröder, Rauch, and Piróg 2017, Unifying Guarded and Unguarded Iteration

Big Picture

$$
\begin{tikzcd}
& \mathbf{H}_0\mathbf{M} \arrow[dr, shift right, "\rho"'] & \\
\mathbf{H}_+ \arrow[ur, hook, "\iota"] \arrow[rr, hook, dashed, "\rho\iota"] & & \mathbf{H} \arrow[ul, hook, shift right, "\upsilon"']
\end{tikzcd}
$$

- $\mathbb{H}_+$ is a submonad of $\mathbb{H}$ via $\rho\iota$
- progressive (partial) iteration on $\mathbb{H}_+$ is a restriction of total iteration on $\mathbb{H}$ (analogously to $\mathcal{P}^+$ vs. $\mathcal{P}$)

Further Work

- What is a topological semantic domain for hybrid computations?
- Designing a hybrid programming language, relating to the metalanguage for guarded iteration[†]
- Program logics for hybrid iteration, apposing Platzer's differential dynamic logic
- Process semantics via generalized coalgebraic resumption monad transform[‡] $\nu\gamma.\, H(\text{–} + A \times \gamma)$
- Guarded Lawvere theories, guarded PROPs and applications to $\mathbb{H}_0$, $\mathbb{H}_+$, $\mathbb{H}$

[†] Goncharov, Rauch, and Schröder 2018, A Metalanguage for Guarded Iteration
[‡] Goncharov, Rauch, and Schröder 2015, Unguarded Recursion on Coinductive Resumptions

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Simple Iteration and Instability

Hybrid iteration occurring in the literature is a Kleene-style iteration: it is supposed to send $f : X \to H_+ X$ to $f^\# : X \to H_+ X$ (e.g. this is the case with bouncing ball, because there is no "final result")

We recover such basic iteration $(d, e)^\# : X \to H_+ X$ as a progressive one:

$$\Big((d, \lambda x.\, \lambda t.\, \mathsf{inl}\, e^0(x) \lhd t = 0 \rhd \mathsf{inr}\, e^t(x)) : X \to_2 H_+(X + X)\Big)^\dagger$$

Unlike $\mathbb{H}_0$, we cannot move $\mathbb{H}_+$ (even more so $\mathbb{H}$) from **Set** to **Top** because of instability: given continuous $(d, e) : \mathbf{R}_+ \to H_+ \mathbf{R}_+$ with $d(x) = x$, $e^t(x) = x$, $(d, e)^\#_{\mathsf{d}}(0) = 0$, $(d, e)^\#_{\mathsf{d}}(\epsilon) = \infty$ for any $\epsilon > 0$