A [Monad-Based] Semantics for Hybrid Iteration

Sergey Goncharov\textsuperscript{a}  Julian Jakob\textsuperscript{a}  Renato Neves\textsuperscript{b}

CONCUR 2018, September 4-7, Beijing

\textsuperscript{a}Friedrich-Alexander-Universität Erlangen-Nürnberg
\textsuperscript{b}INESC TEC (HASLab) & University of Minho
Plan of the Talk

- Precursors:
  - Hybrid automata
  - Platzer’s dynamic logic
  - Höfner and Möller’s algebra of hybrid systems
  - Hybrid process calculi: $\text{ACP}_{\text{hs}}^\text{srt}$, hybrid $\chi$, HYPE, HyPA, hybrid CSP, . . .

- More recently: hybrid monad$^\dagger$ $\mathbb{H}_0$ (deterministic, abstract, composable)

- Here: two new monads $\mathbb{H}_+$ and $\mathbb{H}$ for hybrid iteration

$^\dagger$ Neves, Barbosa, Hofmann, and Martins 2016, Continuity as a computational effect
Bouncing ball is a simple Newtonian system specified by differential equation \( \ddot{h} = -g \) \((g \approx 9.8)\) whose solution is

\[
h(t) = h_0 + v_0 t - \frac{gt^2}{2}
\]

with initial values:

- \(v_0 = 0, h_0 \neq 0\) (peak height)
- \(h_0 = 0, v_0 \neq 0\) (zero height)

The velocity changes discretely at the bottom \(v \mapsto -cv\), but it changes continuously in the meanwhile. This is called hybrid behaviour.

The state of rest is only reachable in the limit. This is called Zeno behaviour.
Progressive Iteration vs. Singular Iteration

- Bouncing ball movement is **progressive** in the sense that each iteration is non-instantaneous.
- On the other hand

  \[
  \text{while true \{x := 1 - x\}}
  \]

  is not progressive and must yield the divergence \( \bot \).

  This disrupts the time flow structure, for we expect

  \[
  \text{while true \{x := 1 - x\} } \simeq \text{while true \{x := 1 - x\}; wait(1)}
  \]

**Idea:** progressiveness is a form of abstract guardedness†

---

†Goncharov and Schröder 2018, Guarded Traced Categories
Monads formalize generalized functions\(^\dagger\)

\[ f : X \to TY, \text{ like nondeterministic} \]

(with \(T = \mathcal{P}X\)) or partial (with \(TX = X + 1\))

\(T\) is a type constructor, together with operations

\[ \eta : X \to TX \text{ (unit) and } (f : X \to TY) \mapsto (f^* : TX \to TY) \text{ (lifting)}, \]

inducing the Klesili category of \(T\) under

\[ \text{id} = \eta : X \to TX \]

\[ f \circ g = (f : Y \to TZ)^* (g : X \to TY) \]

In Haskell’s point-full notation: \(\text{do } x \leftarrow p; f(x) = f^*(p)\)

\(^\dagger\)Moggi 1991, Notions of Computation and Monads
Abstract guardedness for a monad $T$ is a relation between Kleisi morphisms $f : X \to TY$ and summands $\sigma : Y' \leftrightarrow Y$ satisfying

\[(\text{trv})\quad \frac{f : X \to TY}{(T \text{ in}_1) f : X \to T(Y + Z)}\]

\[(\text{sum})\quad \frac{f : X \to T\sigma TZ \quad g : Y \to T\sigma TZ}{[f, g] : X + Y \to T\sigma TZ}\]

\[(\text{cmp})\quad \frac{f : X \to T(Y + Z) \quad g : Y \to T\sigma TV \quad h : Z \to TV}{[g, h]* f : X \to T\sigma TV}\]

where $f : X \to T\sigma TY$ means that $f$ and $\sigma$ are in the relation
Abstract Guardedness on Monads

A monad is **guarded Elgot** if it supports **partial iteration** sending each $f : X \to T(Y + X)$ to $f^\dagger : X \to TY$ satisfying the **fixpoint law**

$$f^\dagger = [\eta, f^\dagger]^* f$$

and other laws of iteration

This yields 2-dimensional classification:

<table>
<thead>
<tr>
<th></th>
<th>total</th>
<th>partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>iterative (unique $-^\dagger$)</td>
<td>degenerate, e.g. $T = 1$</td>
<td>completely iterative, e.g. $T = \nu \gamma. \mathcal{P}_\omega(- + A \times \gamma)$</td>
</tr>
<tr>
<td>Elgot (non-unique $-^\dagger$)</td>
<td>e.g. $T = \mathcal{P}$</td>
<td>e.g. $T = \mathcal{P}(A^* \times - + A^\omega)$</td>
</tr>
</tbody>
</table>
We identify **totally guarded** iteration with **unguarded** iteration

- $T = \mathcal{P}$: this supports unguarded iteration via least fixpoints
- $T = \mathcal{P}^+$, i.e. non-empty powerset: the iteration operator is inherited from $\mathcal{P}$ along injection $\mathcal{P}^+ \hookrightarrow \mathcal{P}$

The resulting operator is partial, for $\eta \text{inr} : 1 \rightarrow \mathcal{P}^+(1 + 1)$ does not have a fixpoint

This operator is guarded with $f : X \rightarrow_2 \mathcal{P}^+(Y + Z)$ iff

$\forall x. \exists y. \text{inl } y \in f(x)$
Basic Monad for Hybrid Computations
The Monad $\mathbb{H}_0$

1. A closed trajectory $t$ is a pair $(t_d \in \mathbb{R}_+, t_e : [0, t_d] \to X)$
2. An infinite trajectory $t$ is a pair $(\infty, t_e : [0, \infty) \to X)$
3. More uniformly, $t \in \overline{\mathbb{R}}_+ \times X^{\mathbb{R}_+} +$ flattening condition:

$$x \geq t_d \quad \text{implies} \quad t_e(x) = t_e(t_d)$$

Let $H_0 X$ be the set of trajectories identified by (3).
The Monad $H_0$

1. A closed trajectory $t$ is a pair $(t_d \in \mathbb{R}_+, t_e : [0, t_d] \to X)$
2. An infinite trajectory $t$ is a pair $(\infty, t_e : [0, \infty) \to X)$
3. More uniformly, $t \in \overline{\mathbb{R}}_+ \times X^{\mathbb{R}_+} +$ flattening condition:

$$x \geq t_d \quad \text{implies} \quad t_e(x) = t_e(t_d)$$

Let $H_0 X$ be the set of trajectories identified by (3). $H_0$ is a monad!\(^\dagger\)

- **Unit:** $\eta(x) = (0, x)$, where $x$ is the constant function $\lambda \to x$;
- **Kleisli lifting:** given $f : X \to H_0 Y$ and $(d, e) \in H_0 X$,

$$f^*(d, e))_d = d + f_d(e^d) \triangle d \in \mathbb{R}_+ \triangleright \infty$$
$$f^*(d, e))_e = f^0_e(e^t) \triangle t < d \triangleright f^t_{e-d}(e^d)$$

where $x^t$ reads as $x(t)$ and $x \triangle y \triangleright z$ as “if $y$ then $x$ else $z$”

\(^\dagger\)Neves, Barbosa, Hofmann, and Martins 2016, Continuity as a computational effect
Why $\mathbb{H}_0$ Does Not Suffice?

Closure under iteration $\implies$ closure under Zeno behaviour
Why $H_0$ Does Not Suffice?

Closure under iteration $\Rightarrow$ closure under Zeno behaviour

However,

1. Zeno argued that it is impossible to reach the limit;)
   So, it is impossible to close the trajectory even
   if there is a unique closed trajectory candidate
Why $H_0$ Does Not Suffice?

Closure under iteration $\Rightarrow$ closure under Zeno behaviour

However,

1. Zeno argued that it is impossible to reach the limit;
   So, it is impossible to close the trajectory even
   if there is a unique closed trajectory candidate

2. It is certainly possible to have hybrid behaviour without a closed trajectory candidate whatsoever

Solution: open trajectories
Iterating Hybrid Computations
Let \( \mathbb{M} \) be the **maybe monad**: \( MX = X + 1 \)

**Proposition:** Every monad \( \mathbb{T} = (T, \eta, (-)^*) \) induces a monad \( \mathbb{T}\mathbb{M} \) whose underlying functor is \( X \mapsto T(X + 1) \)

**Theorem:** The hom-sets \( \text{Hom}(X, TM) \) can be suitably ordered, so that

1. Every \( H_0MX \) is an \( \omega \)-complete partial order under \( \sqsubseteq \), and \( (0, \bot) \) is the bottom element
2. Kleisli composition is monotone and continuous on both sides
3. Kleisli composition is right-strict: \( f^*(0, \bot) = (0, \bot) \)

However, Kleisli composition is not left-strict:

\[
(0, \bot)^*(1, \text{id}) = (1, \bot) \neq (0, \bot)!
\]

**Theorem:** By (1)–(3), \( \mathbb{H}_0\mathbb{M} \) is a (total) Elgot monad
$\mathbb{H}_0M$ contains lots of junk trajectories, while we only need open and closed ones: $(d \in \mathbb{R}_+, e : [0, d) \to X)$ and $(d \in \mathbb{R}_+, e : [0, d] \to X)$

Let $H_+$ be the following subfunctor of $H_0M$:

$$(d, e) \in H_+X \quad \text{iff} \quad e \neq \bot \quad \text{and} \quad e^t \downarrow \quad \text{for all} \quad t \in [0, d)$$

This induces a monad $\mathbb{H}_+$ with

$$(f^*(d, e))_d = d, \quad (f^*(d, e))_e^t = f_e^0(e^t) \triangleq t < d \triangleright \bot \quad \text{if} \quad e^d \uparrow$$

$$(f^*(d, e))_d = d + f_d(e^d), \quad (f^*(d, e))_e^t = f_e^0(e^t) \triangleq t < d \triangleright f_e^{t-d}(e^d) \quad \text{if} \quad e^d \downarrow$$

Total iteration of $\mathbb{H}_0M$ restricts to guarded iteration of $\mathbb{H}_+$ with guardedness being progressiveness: $(d, e) : X \to H_+(Y + Z)$ if $e^0 : X \to Y + Z$ factors through $\text{inl}$
In $\mathbb{H}_+$ we excluded the “black hole” trajectory $(0, \bot)$, for it brings up “space-time artefacts” like $(1, \bot)$

Let $H$ be the following subfunctor of $H_0M$:

$$(d, e) \in HX \text{ iff } e \neq \bot \text{ and } e^t \downarrow \text{ for all } t \in [0, d)$$

Now, the induced injection $\nu : H \hookrightarrow H_0M$ is not a monad morphism: $(0, \bot)^*(1, \text{id})$ must be $(0, \bot)$ in $\mathbb{H}$, but not in $H_0M$

However, $\nu$ has a retraction $\rho : H_0M \rightarrow H$, which is a monad morphism. Moreover $(\nu, \rho)$ is an iteration-congruent retraction$^\dagger$

**Theorem:** $\mathbb{H}$ is an iteration-congruent retract of $\mathbb{H}_0M$, hence an Elgot monad

$^\dagger$Goncharov, Schröder, Rauch, and Piróg 2017, Unifying Guarded and Unguarded Iteration
• $\mathbb{H}_+$ is a submonad of $\mathbb{H}$ via $\rho \iota$

• progressive (partial) iteration on $\mathbb{H}_+$ is a restriction of total iteration on $\mathbb{H}$ (analogously to $\mathcal{P}^+$ vs. $\mathcal{P}$)
Further Work

- What is a topological semantic domain for hybrid computations?
- Designing a hybrid programming language, relating to the metalanguage for guarded iteration
- Program logics for hybrid iteration, apposing Platzer’s differential dynamic logic
- Process semantics via generalized coalgebraic resumption monad transform
- Guarded Lawvere theories, guarded PROPs and applications to $\mathbb{H}_0$, $\mathbb{H}_+$, $\mathbb{H}$

---

†Goncharov, Rauch, and Schröder 2018, A Metalanguage for Guarded Iteration
‡Goncharov, Rauch, and Schröder 2015, Unguarded Recursion on Coinductive Resumptions
References


Hybrid iteration occurring in the literature is a Kleene-style iteration: it is supposed to send $f : X \to H_+ X$ to $f^\# : X \to H_+ X$ (e.g. this is the case with bouncing ball, because there is no “final result”)

We recover such basic iteration $(d, e)^\# : X \to H_+ X$ as a progressive one:

$$
\left( (d, \lambda x. \lambda t. \mathrm{inl} \ e^0(x) \triangleright t = 0 \triangleright \mathrm{inr} \ e^t(x)) : X \to_2 H_+(X + X) \right)^\dagger
$$

Unlike $\mathbb{H}_0$, we cannot move $\mathbb{H}_+$ (even more so $\mathbb{H}$) from Set to Top because of instability: given continuous $(d, e) : \mathbb{R}_+ \to H_+ \mathbb{R}_+$ with $d(x) = x$, $e^t(x) = x$, $(d, e)^\#_d(0) = 0$, $(d, e)^\#_d(\epsilon) = \infty$ for any $\epsilon > 0$