# Guarded Traced Categories

**Sergey Goncharov** and Lutz Schröder
Friedrich-Alexander-Universität Erlangen-Nürnberg

FoSSaCS 2018, 16-19 April 2018, Thessaloniki, Greece

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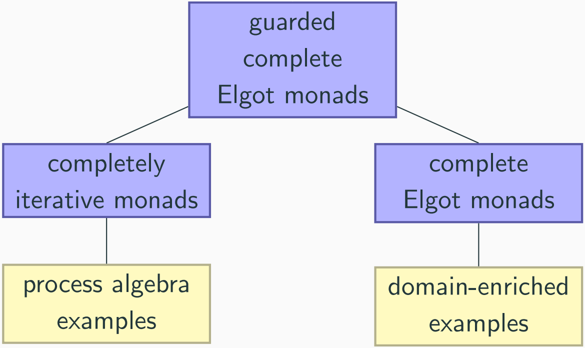
# Introduction

- Recursion / iteration
  - order-theoretic / unguarded
  - process-theoretic / guarded
- Generic categorical models:
  - Total:
    - Axiomatic/synthetic domain theory (Hyland, Fiore, Taylor et al.)
    - let-ccc's with fixpoint objects (Crole/Pitts, Simpson)
    - Traced monoidal categories (Joyal/Street/Verity, Hasegawa)
    - Elgot monads/theories (Bloom/Esik, Adámek, Milius et al.)
  - Partial:
    - Completely iterative monads/theories (Bloom/Esik, Adámek, Milius et al.)
    - *later*-modality (Nakano, Appel, Melliès, Benton, Birkedal et al.)
    - Partial traced categories (Heghverdi, Scott, Malherbe, Selinger)
    - Functorial dagger (Milius, Litak)
- Here: Unifying framework for guarded and unguarded feedback in monoidal categories

# Guarded Fixpoints: Overview



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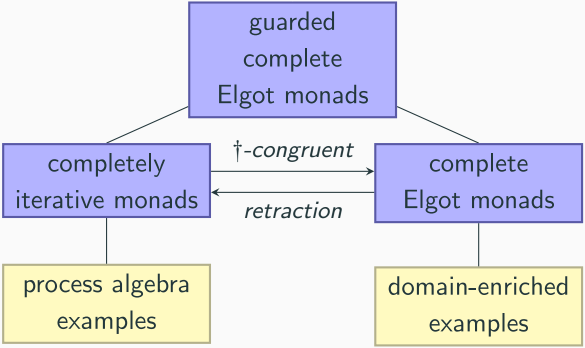
FoSSaCS17: G., Schröder, Rauch, Piróg, Unifying Guarded and Unguarded Iteration



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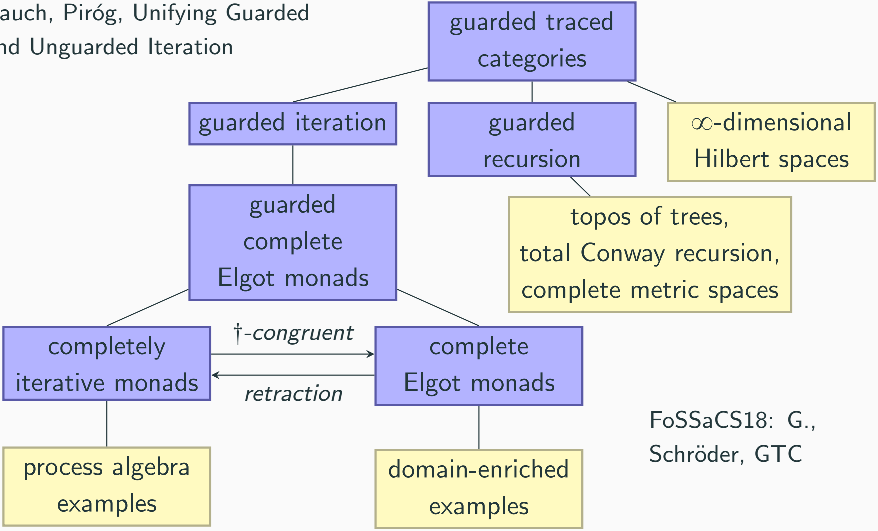



FoSSaCS18: G., Schröder, GTC

# Motivating Example: Process Algebra

In process algebra, we solve tail-recursive process definitions, like

$$x = a.\,x + \tau.\,x + y$$

More abstractly, we involve a monad $T_\Sigma X = \nu\gamma.\, T(X + \Sigma\gamma)$ of infinite process trees and axiomatize guardedness of $f : X \to T_\Sigma Y$ in a coproduct summand $\sigma : Y' \hookrightarrow Y$ as follows (in Klesili):

$$\textbf{(vac}_+\textbf{)}\ \frac{f : X \to Z}{\mathsf{inl}\, f : X \to_{\mathsf{inr}} Z + Y} \qquad \textbf{(cmp}_+\textbf{)}\ \frac{\begin{array}{ll} f : & X \to_{\mathsf{inr}} Y + Z \\ g : & Y \to_\sigma V \qquad h : Z \to V \end{array}}{[g, h] \circ f : X \to_\sigma V}$$

$$\textbf{(par}_+\textbf{)}\ \frac{f : X \to_\sigma Z \qquad f : Y \to_\sigma Z}{[f, g] : X + Y \to_\sigma Z}$$

# Guarded Iteration v.s. Guarded Recursion

Guarded iteration is a (partial) operation

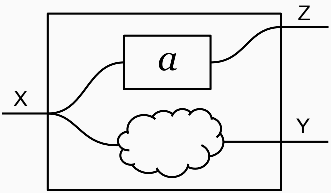
$$\frac{f : X \to Y + X}{f^{\dagger} : X \to Y}$$

with $f$ guarded in $X$

Dualization should yield guarded recursion:

$$\frac{f : X \times Y \to X}{f_{\dagger} : Y \to X}$$

Can we make sense of this intuition? :



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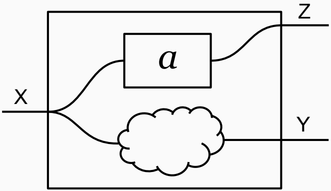



**Pivotal Idea:** Keep the notion of guardedness independent of fixpoint calculations

# Going Monoidal

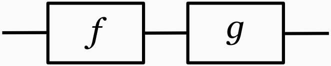
# Going Monoidal

(We only consider symmetric monoidal categories, think of $\otimes = +, \times$)

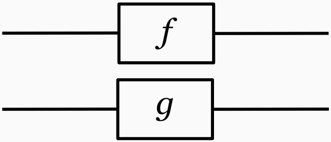
Identity id:



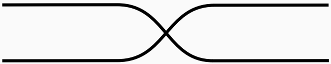
Composition $g \circ f$:




Tensor $g \otimes f$:

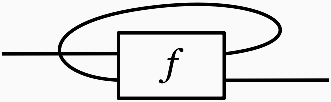



Symmetry:



# Going Monoidal: Additional Structure

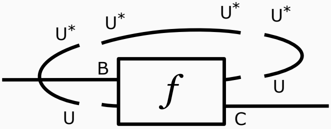
Trace $tr(f : U \otimes A \to B \otimes U)$[1]:
$A \to B$




Compact closure:
unit $\eta : I \to A \otimes A^\star$ and
counit $\epsilon : A^\star \otimes A \to I$

where $(-)^\star$ is a contravariant involutive endofunctor

In compact closed categories, trace is definable and unique, for:

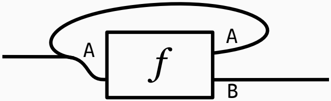



[1]The twist of input wires is nonstandard, but bear with me

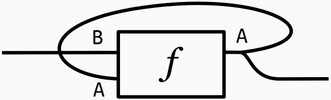
# Iteration and Recursion

Iteration and recursion are typically viewed as corner cases:

- With $\otimes = +$, we obtain $(f : A \to B + A)^\dagger = tr(f \circ \nabla)$:




- With $\otimes = \times$, we obtain $(f : A \times B \to A)_\dagger = tr(\Delta \circ f)$:

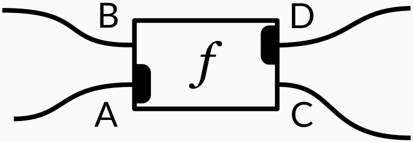



Corresponding converse definitions can also be produced. So, traces and (Conway) fixpoints are equivalent in the requisite cases!

# Guarded Categories

# Partially Guarded Morphisms

A monoidal category is guarded if it is equipped with distinguished families $\mathrm{Hom}^\bullet(A \otimes B, C \otimes D) \subseteq \mathrm{Hom}(A \otimes B, C \otimes D)$, drawn as follows

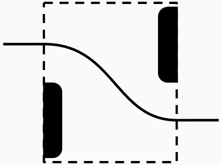


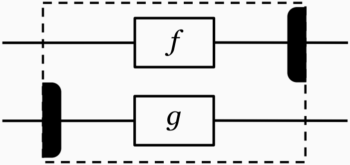

where

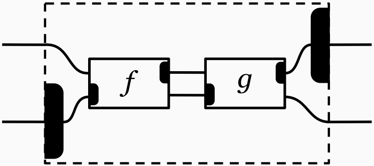
- $A$ is unguarded input
- $B$ is guarded input
- $C$ is unguarded output
- $D$ is guarded output

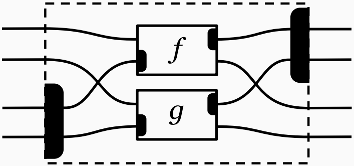
The idea is to prevent feedback on $(A, D)$. Hence we introduce axioms:

# The Axioms



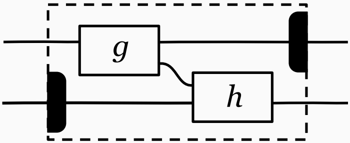


# Some (Easy) Observations

- There is a greatest notion of guardedness,
  $\mathsf{Hom}^\bullet(A \otimes B, C \otimes D) = \mathsf{Hom}(A \otimes B, C \otimes D)$
- There is a least (vacuous) notion of guardedness,



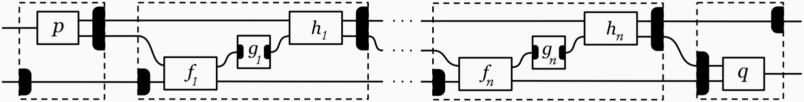

- Axioms are stable under 180°-rotations, hence **C** is guarded iff $\mathbf{C}^{op}$ is guarded, i.e. we maintain duality of recursion and iteration

# Ideal Guardedness

A particularly common case is ideal guardedness

A guarded ideal is a family $\mathrm{Hom}^{\blacktriangleright}(X, Y) \subseteq \mathrm{Hom}(X, Y)$ closed under finite tensors and composition with any morphism on both sides

The general form of a partially guarded morphism over a guarded ideal is



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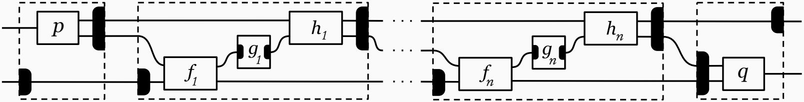



In the (co-)Cartesian case this simplifies greatly, generating standard notions, e.g. $f : X \to_2 Y + Z$ iff

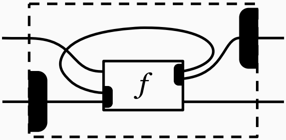
$$X \xrightarrow{h} Y + W \xrightarrow{[\mathsf{inl}, g]} Y + Z$$

with some $g \in \mathrm{Hom}^{\blacktriangleright}(W, Y + Z)$ and $h : X \to Y + W$

# Guarded Traces

# Guarded Traced Categories

A guarded category is guarded traced if it is equipped with a trace:



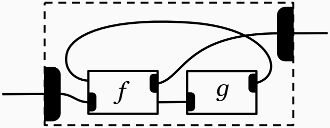

satisfying a collection of axioms adapted from the standard case

Guarded Conway iteration/recursion operators are obtained analogously to the standard case

# Structural Guardedness v.s. Geometric Guardedness

For guarded categories we have coherence of structural and geometric notions: a term is in $\mathrm{Hom}^\bullet(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from $A$ to $D$ runs through some atomic box via an unguarded input and a guarded output

This is no longer true for guarded traces:

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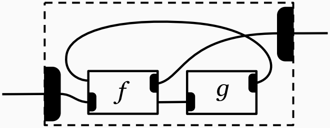



Geometrically, this is OK but there is no structured way to derive it!

**But:** This discrepancy does not arise in the ideal case

**Conjecture:** The same is true for the (co-)Cartesian case

# Non-Ideal Case: Contractive Maps

Consider the category **CMS** of inhabited complete metric spaces and non-expansive maps

Let $f : X \times Y \to Z$ be guarded in $Y$ if for all $x \in X$, $f(x, -)$ is contractive

This makes **CMS** into a guarded traced monoidal category (fixpoints calculated via Banach's fixpoint theorem) but not ideally guarded, because a contraction factor depends on $x$ and may not be chosen uniformly

# Unguarded Recursion as Guarded Recursion

- A standard way to do recursion with monads is in the category $\mathbf{C}^{\mathbb{T}}_{\star}$ with $\mathbb{T}$-algebras as objects and $\mathbf{C}$-morphisms of carriers as morphisms

  **Example:** $\mathbf{C}$ = point-free dcpo's and continuous functions;
  $\mathbb{T}$ = lifting monad $X \mapsto X_{\perp}$

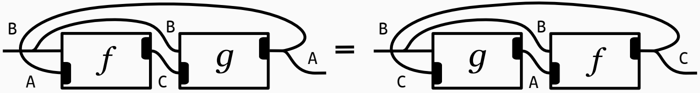
- Alternatively, following [Milius and Litak, 2013], we consider guarded recursion operators on $\mathbf{C}$ where $\mathbf{C}$ is ideally guarded over
  $\mathrm{Hom}^{\blacktriangleright}(X, Y) = \{f \circ \eta \mid f : TX \to Y\}$

  **Example:** with $\mathbf{C}$ and $\mathbb{T}$ as above, we allow only recursion on $X$ of $f : X_{\perp} \times Y \to X$

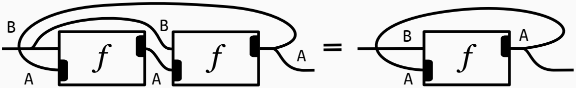
# Unguarded Recursion as Guarded Recursion

We consider the following axioms:

- Dinaturality:




- Squaring (is not a property of Conway recursion but a property of Conway uniform recursion):




**Theorem:** There is a bijective correspondence between guarded squarable dinatural operators on $\mathbf{C}$ and unguarded squarable dinatural on $\mathbf{C}^{\mathbb{T}}_{\star}$

# Guarded Traces in Hilbert Spaces

# Finite-Dimensional Hilbert Spaces

Recall the multiplicative compact closed category of relations $(\mathbf{Rel}, \times, 1)$

Relations can be thought of as Boolean matrices, with transposition $(-)^\star$ and (unparamerized) trace being the trace of the square matrices

$$tr \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \sum_i b_{ii}$$

Analogously, linear operators on finite-dimensional Hilbert spaces can be represented as matrices over a field – we stick to the field of reals

Thus, Hilbert spaces are compact closed with tensors $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$, $R$ as tensor unit, $X^\star = X$ on objects, $f^\star$ as the unique adjoint operator $\langle f(x), y \rangle = \langle x, f^\star(y) \rangle$ and unit/counit induced by inner products

# Infinite-Dimensional Hilbert Spaces

More generally, Hilbert spaces are vector spaces with inner products, complete as a normed spaces under the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$

Category **Hilb**:

- Objects are Hilbert spaces
- Morphisms are bounded linear operators, i.e. $\|f(x)\| \leqslant c \cdot \|x\|$ for a fixed $c$ and every $x$
- Monoidal structure as before
- Adjointness for operators still works and $(f \otimes g)^\star = f^\star \otimes g^\star$, $f^{\star\star} = f$, $\mathrm{id}^\star = \mathrm{id}$, $(f \circ g)^\star = g^\star \circ f^\star$

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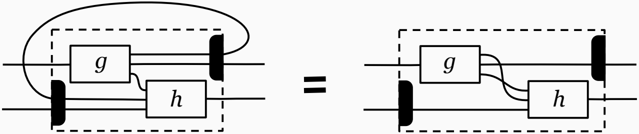
But there is no (total) trace, because the trace formula $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ may diverge ($\{e_i\}_i$ is any orthonormal basis)! E.g. it diverges with $f = \mathrm{id} : X \to X$ with infinite-dimensional $X$

# Nontrivial Trace from Vacuous Guardedness

[Abramsky, Blute, and Panangaden, 1999] already give a construction of unparameterized partial traces in **Hilb** via nuclear ideals. We generalize and reconcile it with our approach by equipping **Hilb** with the vacuous guardedness structure:




What is nontrivial though is that this is independent of the decomposition into $g$ and $h$!

Morphisms $f : X \to X$ from the induced guarded ideal are precisely those for which $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ absolutely converges for any choice of an orthonormal basis $(e_i)_i$; the sum is then independent of the basis

# The Diversity of Further Avenues

- Deepen the theory of guarded traced categories: coherence, expressiveness, completeness (w.r.t. Hilbert spaces?), Int-construction
- Elaborate the relationship between guarded ideals and nuclear ideals $\Rightarrow$ further examples
- Yet more examples: hybrid iteration, neural networks, ...
- Metalanguages for guarded iteration and recursion

Questions?

## References

S. Abramsky, R. Blute, and P. Panangaden. Nuclear and trace ideals in tensored *-categories. *J. Pure Appl. Algebra*, 143:3–47, 1999.

Stefan Milius and Tadeusz Litak. Guard your daggers and traces: On the equational properties of guarded (co-)recursion. In *Fixed Points in Computer Science, FICS 2013*, volume 126 of *EPTCS*, pages 72–86, 2013.