Guarded Traced Categories for Recursion and Iteration

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Recursion / iteration
- order-theoretic / unguarded
- process-theoretic / guarded

Generic categorical models:
- Total:
  - Axiomatic/synthetic domain theory (Hyland, Fiore, Taylor et al.)
  - let-ccc’s with fixpoint objects (Crole/Pitts, Simpson)
  - Traced monoidal categories (Joyal/Street/Verity, Hasegawa)
  - Elgot monads/theories (Bloom/Esik, Adámek, Milius et al.)
- Partial:
  - Completely iterative monads/theories (Bloom/Esik, Adámek, Milius et al.)
  - later-modality (Nakano, Appel, Melliès, Benton, Birkedal et al.)
  - Partial traced categories (Heghverdi, Scott, Malherbe, Selinger)
  - Functorial dagger (Milius, Litak)
Guarded Fixpoints: Overview

Guarded traced categories

Guarded iteration

Elgot monads

Completely iterative monads

Process algebra examples

(complete) Elgot monads

Domain-enriched examples

8-dimensional Hilbert spaces

Topos of trees, total Conway recursion, complete metric spaces
Guarded Fixpoints: Overview

FoSSaCS17: G., Schröder, Rauch, Piróg, Unifying Guarded and Unguarded Iteration

Guarded Elgot monads

- 
  - Completely iterative monads
    - Process algebra examples
  - (Complete) Elgot monads
    - Domain-enriched examples

\[ \uparrow \text{-congruent} \]

Retraction
Guarded Fixpoints: Overview

FoSSaCS17: G., Schröder, Rauch, Piróg, Unifying Guarded and Unguarded Iteration

FoSSaCS18: G., Schröder, Guarded Traced Categories

- guarded traced categories
- guarded iteration
- guarded Elgot monads
- guarded recursion
- \(\infty\)-dimensional Hilbert spaces
- topos of trees, total Conway recursion, complete metric spaces

- (complete) Elgot monads
- domain-enriched examples
- process algebra examples

\(\dagger\)-congruent retraction
Unguarded Iteration on Monads
One Typical Scenario

\[
\text{fac } n = \text{if } n > 0 \text{ then } n \times \text{fac}(n - 1) \text{ else } 1
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\]

**Factorial (Recursively)**

\[
\text{fac} : \mathbb{N} \rightarrow \mathbb{N} = \left( (r, g) \mapsto \lambda n. \text{if } n > 0 \text{ then } n \times g(n - 1) \text{ else } r \right)^\dagger (1)
\]

where \((-)^\dagger\) is the least fixpoint of \(g \mapsto f \circ \langle \text{id}, g \rangle\). Alternatively,

**Factorial (Iteratively)**

\[
\text{fac } n = \left( (k, i) \mapsto \text{if } k > 0 \text{ then } \text{inr}(k - 1, k \times i) \text{ else } \text{inl } i \right)^\dagger (n, 1)
\]

where \((-)^\dagger\) is the least fixpoint of \(g \mapsto [\text{id}, g] \circ f\)
Iteration v.s. Recursion

Iteration is dual to (call by name) recursion:

\[
\begin{align*}
\frac{f : Y \times X \to X}{f^\dagger : Y \to X} & \quad \text{(rec)} \\
\frac{f : X \to Y + X}{f^\dagger : X \to Y} & \quad \text{(iter)}
\end{align*}
\]

E.g. \( X \to Y \) is the space of partial functions \( X \to Y_\perp \) on \( \text{Set} \)

More generally, \( X \to Y \) is \( X \to TY \) where \( T \) is a \text{monad}
Another Typical Scenario

Given an alphabet of actions $A = \{a, b\}$, equations

$$x_1 = a \cdot (x_2 + x_3) \quad x_2 = a \cdot x_1 + b \cdot x_3 \quad x_3 = a \cdot x_1 + \sqrt{\varepsilon}$$

specify processes $x_1, x_2, x_3$ of basic process algebra (BPA).

We can think of them as a map

$$X \to \mathcal{P}_\omega(\{\sqrt{\varepsilon}\} + \Sigma X)$$

where $X = \{x_1, x_2, x_3\}$, $\Sigma = A \times (-)$, and solve them by finding the unique $X \to T_\Sigma\{\sqrt{\varepsilon}\}$ in the domain of possibly non-wellfounded trees

$$T_\Sigma\{\sqrt{\varepsilon}\} = \nu \gamma. \mathcal{P}_\omega(\{\sqrt{\varepsilon}\} + A \times \gamma)$$

(final coalgebra). The original system must be guarded. An unguarded specification, e.g. $x = x$ may have arbitrary solutions

\(^\dagger\)Rutten and Turi 1994, Initial algebra and final coalgebra semantics for concurrency
In order to solve an unguarded system like

\[ x_1 = x_2 + a \cdot (x_2 + x_1) \quad x_2 = x_1 + a \cdot x_1 + b \cdot x_2 \]

we first need to guard it to obtain

\[ x_1 = a \cdot (x_2 + x_1) \quad x_2 = a \cdot x_1 + b \cdot x_2 \]

and solve the result. Equations like \( x = x \) must be replaced by \( x = \emptyset \) where \( \emptyset \) is unproductive divergence.

This induces a notion of iteration which is neither least nor unique.
Why Trees Can Not Be (Obviously) Ordered

Convexity Issue: a partial order would identify processes

\[ a.a.\emptyset + \emptyset \quad \text{and} \quad a.a.\emptyset + a.\emptyset + \emptyset. \]

For \( \sqsubseteq \): \( a.a.\emptyset + \emptyset = a.a.\emptyset + \emptyset + \emptyset \sqsubseteq a.a.\emptyset + a.\emptyset + \emptyset \)

(monotonicity of + and idempotence of +)

For \( \sqsupseteq \):

\[
\frac{\emptyset \sqsubseteq a.\emptyset}{a.a.\emptyset \sqsubseteq a.a.\emptyset} \quad \frac{a.\emptyset \sqsubseteq a.a.\emptyset}{a.a.\emptyset + a.\emptyset + \emptyset \sqsubseteq a.a.\emptyset + a.a.\emptyset + \emptyset} \quad \frac{\emptyset \sqsubseteq \emptyset}{\emptyset \sqsubseteq \emptyset} \quad \text{(monot. \ a.)} \quad \text{(monot. \ +)} \quad \text{(idemp. \ +)}
\]
## Definition (Monad)

A **monad** over a category \( \mathbf{C} \) is given by a **Kleisli triple** \( \mathbb{T} = (T, \eta, -^*) \) where

- \( T \) is an endomap on \( |\mathbf{C}| \)
- \( \eta \) is a family of morphisms \( \eta_X : X \to TX \), called **monad unit**
- \((-)^*\) assigns to each \( f : X \to TY \) a morphism \( f^* : TX \to TY \)

and the following laws hold:

\[
\begin{align*}
\eta^* &= \text{id} \\
f^* \circ \eta &= f \\
(f^* \circ g)^* &= f^* \circ g^*
\end{align*}
\]

This means that the hom-sets \( \text{Hom}(X, TY) \) form a category (**Kleisli category**) under Kleisli composition \( f \circ g = f^* \circ g \) and \( \eta_X \in \text{Hom}(X, TX) \)
**ω-Continuous Monads**

**Definition (ω-Continuous Monad)**

A monad \( T \) is \( ω \)-continuous if its Kleisli category is enriched over \( ω \)-complete partial orders with bottom \( \bot \) and (nonstrict) continuous maps, and

\[
f \diamond \bot = \bot \quad \bot \circ h = \bot
\]

For \( ω \)-continuous monads we can define iteration

\[
f : X \to T(Y + X) \\
f^\dagger : X \to TY
\]

and the lfp of \( g \mapsto [η, g] \diamond f \)

**Examples**

- \( TX = X + 1 \) (partiality), \( TX = PX \) (nondeterminism),
- \( TX = \{ ξ : X \to [0, 1] \mid \sum ξ \leq 1 \} \) (sub-probability), etc.
Let $\mathbb{T}$ be a monad with an iteration operator $-^\dagger$ satisfying fixpoint identity $f^\dagger = [\eta, f^\dagger] \diamond f$. It is called a Conway operator if it additionally satisfies

**Dinaturality:**

**Codiagonal:**
Naturality is a form of coherence:

Uniformity is the only non-equation axiom:
Axioms for Iteration: Elgot Monads

Definition
A monad $\mathbb{T}$ is a Elgot monad if it is equipped with a Conway iteration operator, which is natural and uniform.

Theorem (Ésik and Goncharov 2016)
Dinaturality is derivable

Theorem (Goncharov, Rauch, and Schröder 2015)
$\omega$-continuous monads are Elgot monads

Theorem (Goncharov, Rauch, and Schröder 2015)
Let $\mathbb{T}$ be a Elgot monad and $\Sigma$ and endofunctor. Then final coalgebras

$$T^\Sigma X = \nu_\gamma. T(X + \Sigma \gamma)$$

defines a Elgot monad $\mathbb{T}^\Sigma$ uniquely coherently extending $\mathbb{T}$.
Guared Iteration on Monads
Guarded v.s. Unguarded

Using the fact that $T$ is Elgot we can solve both guarded and unguarded definitions over $T_\Sigma$. Generally, we have:

<table>
<thead>
<tr>
<th></th>
<th>Canonical fixpoints</th>
<th>Unique fixpoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial fixpoint operators</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Total fixpoint operators</td>
<td>✓</td>
<td>—</td>
</tr>
</tbody>
</table>

If $T$ is not Elgot (e.g. nonempty powerset) we can no longer compute solutions of unguarded definitions (think of $x = x$), but we still can compute solutions of guarded ones.

More generally, guardedness does not guarantee uniqueness, e.g. under infinite trace semantics $x = a.x + 1$ has both $a^*$ and $a^* + a^\omega$ as solutions.
So, what means “guardedness” anyhow?

\[
f : X \rightarrow Y + X \quad \quad f^\dagger : X \rightarrow Y
\]

\[
f : X \times Y \rightarrow X \quad \quad f^\dagger : Y \rightarrow X
\]

Can we make sense of this intuition? :
Abstract Guardedness

So, what means “guardedness” anyhow?

\[
f : X \to Y + X \quad f : X \times Y \to X
\]

\[
f^\dagger : X \to Y \quad f^\dagger : Y \to X
\]

Can we make sense of this intuition? :

**Pivotal Idea:** Keep the notion of guardedness independent of fixpoint calculations
Abstract guardedness is a relation connecting $f : X \rightarrow TY$ with coproduct summands $\sigma : Y' \leftrightarrow Y$ in judgements $f : X \rightarrow_\sigma TY$, satisfying

$$
\text{(vac}_+\text{)} \quad \frac{f : X \rightarrow Z}{\text{inl } f : X \rightarrow_{\text{inr}} Z + Y}
$$

$$
\text{(par}_+\text{)} \quad \frac{f : X \rightarrow_\sigma Z \quad f : Y \rightarrow_\sigma Z}{[f, g] : X + Y \rightarrow_\sigma Z}
$$

$$
\text{(cmp}_+\text{)} \quad \frac{f : X \rightarrow_{\text{inr}} Y + Z \quad g : Y \rightarrow_\sigma V \quad h : Z \rightarrow V}{[g, h] \circ f : X \rightarrow_\sigma V}
$$

For example, $X \rightarrow_2 T\Sigma(Y + Z) = \nu \gamma. T((Y + Z) + \Sigma \gamma)$ iff

$$
\text{out } f = T(\text{inl } + \text{id})g : X \rightarrow T((Y + Z) + \Sigma T\Sigma(Y + Z))
$$

for suitable $g : X \rightarrow T(Y + \Sigma T\Sigma(Y + Z))$
Guarded Iteration Laws

Iteration:

\[ f \]

\[ X \]

\[ X \]

\[ Y \]

\[ f \]

\[ X \]

\[ X \]

\[ Y \]

\[ f \]

\[ X \]

\[ X \]

\[ Y \]

Naturality:

\[ f \]

\[ g \]

\[ X \]

\[ Y \]

\[ z \]

\[ f \]

\[ g \]

\[ X \]

\[ Y \]

\[ z \]

\[ f \]

\[ g \]

\[ X \]

\[ Y \]

\[ z \]

\[ f \]

\[ g \]

\[ X \]

\[ Y \]

\[ z \]
Guarded Iteration Laws (Continued)

Codiagonal:

\[ g \]

\[ X \]

\[ Y \]

\[ X \]

\[ X \]

\[ g \]

\[ X \]

\[ Y \]

\[ X \]

\[ X \]

Uniformity:

\[ h \]

\[ f \]

\[ Z \]

\[ Z \]

\[ Z \]

\[ Z \]
Some Results

- Guarded and unguarded iteration are instances of abstract guardedness.
- Every monad is “vacuously guarded.”
- (Unique) guarded iteration propagates along $T \mapsto \nu \gamma. T(- + \Sigma \gamma)$.
- Dinaturality and other laws are derivable.
- Guarded iteration is the exact dual of guarded recursion.
Example: Guarded Recursion

Consider the category CMS of inhabited complete metric spaces and non-expansive maps.

Let $f : X \times Y \to Z$ be guarded in $Y$ if for all $x \in X$, $f(x, -)$ is contractive.

This makes CMS into a guarded traced monoidal category (fixpoints calculated via Banach’s fixpoint theorem).
Going Monoidal
(We only consider symmetric monoidal categories, think of $\otimes = +, \times$)

Identity id:

Composition $g \circ f$:

Tensor $g \otimes f$:

Symmetry:
Trace \((f : U \otimes A \rightarrow B \otimes U) \leftrightarrow (tr_{A,B}^U f : A \rightarrow B)\)

is the “generalized fixpoint operator”
Iteration and recursion are typically viewed as corner cases:

- With $\otimes = +$, we obtain $(f : A \rightarrow B + A)^{\dagger} = tr(f \circ \nabla)$:

- With $\otimes = \times$, we obtain $(f : A \times B \rightarrow A)^{\dagger} = tr(\Delta \circ f)$:
Guarded Traced Categories
Partially Guarded Morphisms

A monoidal category is **guarded** if it is equipped with distinguished families \( \text{Hom}^*(A \otimes B, C \otimes D) \subseteq \text{Hom}(A \otimes B, C \otimes D) \), drawn as follows

```
  A   C
  |   |
  |   |
B---|--|--|---D
  |   |
  A   C
```

where

- \( A \) is unguarded input
- \( B \) is guarded input
- \( C \) is unguarded output
- \( D \) is guarded output

The idea is to allow feedback only on \((A, D)\), which we call a **guardedness profile** of \( f \).

Hence we introduce axioms:
The Axioms
A guarded category is **guarded traced** if it is equipped with a **trace**:

![Diagram of guarded traced category]

satisfying a collection of axioms adapted from the standard case

**Guarded iteration/recursion** operators are obtained analogously to the standard unguarded case.
Towards Coherence
For guarded categories we have coherence of structural and geometric notions: a term is in $\text{Hom}^\bullet(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from $A$ to $D$ runs through some atomic box via an unguarded input and a guarded output.

After adding traces, this is no longer true:

Geometrically, this is OK but there is no structured way to derive it!
For guarded categories we have **coherence** of structural and geometric notions: a term is in $\text{Hom}^*(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from $A$ to $D$ runs through some atomic box via an unguarded input and a guarded output.

After adding traces, this is no longer true:

Geometrically, this is OK but there is no structured way to derive it!

**But:** The are no natural examples when this actually materializes.
Differently put, the circuit

is not in the \textcolor{red}{free guarded traced category}

Possible ways to resolve it

1. Strengthen the geometric guardedness criterion
2. Weaken the definition of the guarded traced category
3. Do both 1. and 2.

Both approached have issues we fail to resolve, as of today
The Diversity of Further Avenues

• Resolve the pressing coherence issue ⇒ possibly update the notion of guarded traced category
• Rebase on colored props, cover non-monoidal examples
• Deepen the theory of guarded traced categories: expressiveness, completeness (w.r.t. Hilbert spaces?), Int-construction
• Metalanguages for guarded iteration and recursion
• Comonadic guarded recursion (with Tarmo)
Questions?
References


• There is a greatest notion of guardedness,
  \[ \text{Hom}^*(A \otimes B, C \otimes D) = \text{Hom}(A \otimes B, C \otimes D) \]
• There is a least (vacuous) notion of guardedness,

• Axioms are stable under 180°-rotations, hence $C$ is guarded iff $C^{op}$ is guarded, i.e. we maintain duality of recursion and iteration
A particularly common case is ideal guardedness

A guarded ideal is a family $\text{Hom} \uparrow(X, Y) \subseteq \text{Hom}(X, Y)$ closed under finite tensors and composition with any morphism on both sides.

The general form of a partially guarded morphism over a guarded ideal is

$$\begin{align*}
\begin{array}{c}
p \\
\hline
\end{array} & \begin{array}{c}
f_1 \\
\hline
\end{array} & \begin{array}{c}
g_i \\
\hline
h_i \\
\hline
\end{array} & \ldots & \begin{array}{c}
f_n \\
\hline
\end{array} & \begin{array}{c}
g_n \\
\hline
h_n \\
\hline
\end{array} & \begin{array}{c}
q \\
\hline
\end{array}
\end{align*}$$
A particularly common case is ideal guardedness

A guarded ideal is a family $\Hom^\triangleright(X, Y) \subseteq \Hom(X, Y)$ closed under finite tensors and composition with any morphism on both sides.

The general form of a partially guarded morphism over a guarded ideal is

In the (co-)Cartesian case this simplifies greatly, generating standard notions, e.g. $f : X \to Y + Z$ iff

$$X \xrightarrow{h} Y + W \xrightarrow{[\text{inl}, g]} Y + Z$$

with some $g \in \Hom^\triangleright(W, Y + Z)$ and $h : X \to Y + W$. 
We consider the following axioms:

- **Dinaturality:**

![Dinaturality Diagram]

- **Squaring** (is not a property of Conway recursion but a property of Conway *uniform* recursion):

![Squaring Diagram]

**Theorem:** There is a bijective correspondence between guarded squarable dinatural operators on $\mathbf{C}$ and unguarded squarable dinatural on $\mathbf{C}^*$.
Unguarded Recursion as Guarded Recursion

• A standard way to do recursion with monads is in the category $\mathcal{C}^\mathbb{T}$ with $\mathbb{T}$-algebras as objects and $\mathcal{C}$-morphisms of carriers as morphisms.

  Example: $\mathcal{C}$ = point-free dcpo’s and continuous functions; $\mathbb{T}$ = lifting monad $X \mapsto X_{\perp}$

• Alternatively, following [Milius and Litak, 2013], we consider guarded recursion operators on $\mathcal{C}$ where $\mathcal{C}$ is ideally guarded over $\text{Hom}^{\uparrow} (X, Y) = \{ f \circ \eta \mid f : TX \rightarrow Y \}$

  Example: with $\mathcal{C}$ and $\mathbb{T}$ as above, we allow only recursion on $X$ of $f : X_{\perp} \times Y \rightarrow X$
More Examples..

- The topos of trees (guardedness by later-operator)
- Non-pointed order-enriched monads (e.g. non-empty powerset, probability distributions)
- Hybrid iteration semantics\(^\dagger\) ("guardedness" = "progressiveness")

\(^\dagger\)Goncharov, Jakob, and Neves 2018, A Semantics for Hybrid Iteration
Guarded Traces in Hilbert Spaces
Finite-Dimensional Hilbert Spaces

Recall the multiplicative compact closed category of relations \((\text{Rel}, \times, 1)\).

Relations can be thought of as **Boolean matrices**, with transposition \((-)^*\) and (unparamerized) trace being the trace of the square matrices

\[
\begin{pmatrix}
  b_{11} & \cdots & b_{1n} \\
  \cdots & \cdots & \cdots \\
  b_{n1} & \cdots & b_{nn}
\end{pmatrix}
\]

\[
\text{tr} \begin{pmatrix}
  b_{11} & \cdots & b_{1n} \\
  \cdots & \cdots & \cdots \\
  b_{n1} & \cdots & b_{nn}
\end{pmatrix} = \sum_i b_{ii}
\]

Analogously, linear operators on finite-dimensional Hilbert spaces can be represented as matrices over a field – we stick to the field of **reals**.

Thus, Hilbert spaces are compact closed with tensors

\[(f \otimes g)(x \otimes y) = f(x) \otimes g(y),\]

\(R\) as tensor unit, \(X^* = X\) on objects,

\(f^*\) as the unique **adjoint operator** \(\langle f(x), y \rangle = \langle x, f^*(y) \rangle\) and unit/counit induced by **inner products**.
More generally, Hilbert spaces are vector spaces with inner products, complete as a normed spaces under the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Category $\text{Hilb}$:

- Objects are Hilbert spaces
- Morphisms are bounded linear operators, i.e. $\|f(x)\| \leq c \cdot \|x\|$ for a fixed $c$ and every $x$
- Monoidal structure as before
- Adjointness for operators still works and $(f \otimes g)^* = f^* \otimes g^*$, $f^{**} = f$, $\text{id}^* = \text{id}$, $(f \circ g)^* = g^* \circ f^*$

But there is no (total) trace, because the trace formula $\text{tr}(f) = \sum_i x f_p e_i q$ may diverge ($e_i$ is any orthonormal basis)! E.g. it diverges with $f = \text{id} : X \to X$ with infinite-dimensional $X$. 
More generally, Hilbert spaces are vector spaces with inner products, complete as a **normed spaces** under the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$

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E.g. it diverges with $f = \text{id} : X \to X$ with infinite-dimensional $X$
[Abramsky, Blute, and Panangaden, 1999] already give a construction of unparameterized partial traces in \textit{Hilb} via nuclear ideals. We generalize and reconcile it with our approach by equipping \textit{Hilb} with the vacuous guardedness structure:

What is nontrivial though is that this is independent of the decomposition into $g$ and $h$!

Morphisms $f : X \to X$ from the induced guarded ideal are precisely those for which $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ absolutely converges for any choice of an orthonormal basis $(e_i)_i$; the sum is then independent of the basis.