Guarded Traced Categories for Recursion and Iteration

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(joint ongoing work with Lutz Schröder[a] and Paul Blain Levy[b])

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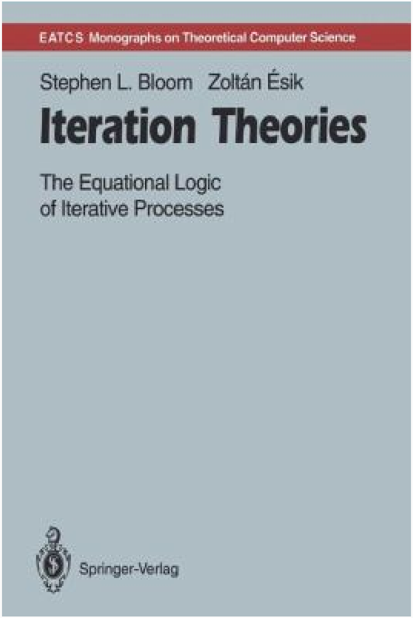
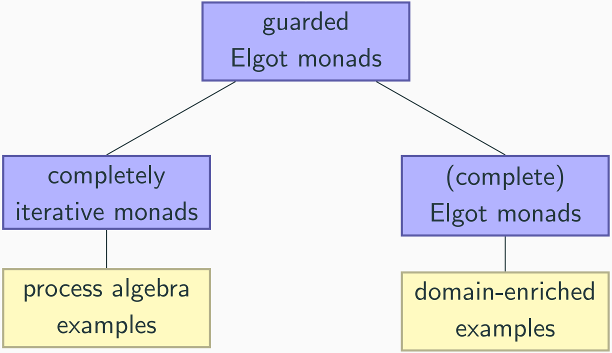
Context

- Recursion / iteration
  - order-theoretic / unguarded
  - process-theoretic / guarded
- Generic categorical models:
  - Total:
    - Axiomatic/synthetic domain theory (Hyland, Fiore, Taylor et al.)
    - let-ccc's with fixpoint objects (Crole/Pitts, Simpson)
    - Traced monoidal categories (Joyal/Street/Verity, Hasegawa)
    - Elgot monads/theories (Bloom/Esik, Adámek, Milius et al.)
  - Partial:
    - Completely iterative monads/theories (Bloom/Esik, Adámek, Milius et al.)
    - later-modality (Nakano, Appel, Melliès, Benton, Birkedal et al.)
    - Partial traced categories (Heghverdi, Scott, Malherbe, Selinger)
    - Functorial dagger (Milius, Litak)

Context

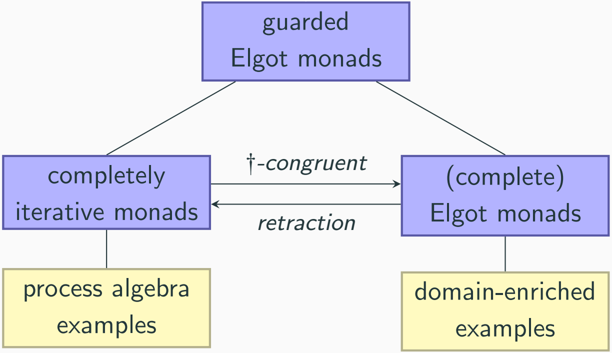

EATCS Monographs on Theoretical Computer Science
Stephen L. Bloom Zoltán Ésik
Iteration Theories
The Equational Logic
of Iterative Processes
Springer-Verlag

Guarded Fixpoints: Overview



Guarded Fixpoints: Overview

FoSSaCS17: G., Schröder, Rauch, Piróg, Unifying Guarded and Unguarded Iteration



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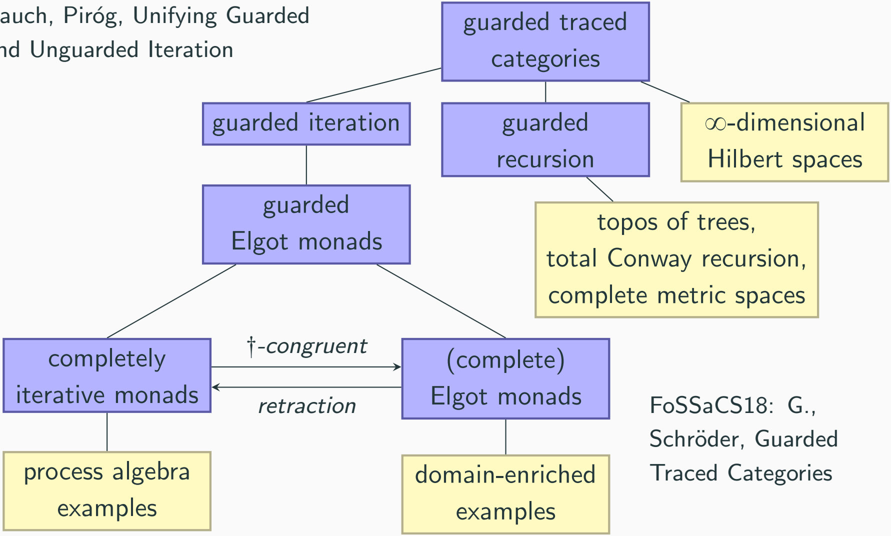

FoSSaCS18: G., Schröder, Guarded Traced Categories

Unguarded Iteration on Monads

One Typical Scenario

$$\textbf{fac}\, n = \textbf{if}\ n > 0\ \textbf{then}\ n * \textbf{fac}(n-1)\ \textbf{else}\ 1$$

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Factorial (Recursively)

$$\mathbf{fac} : \mathbb{N} \rightharpoonup \mathbb{N} = \Big((r, g) \mapsto \lambda n.\, \mathbf{if}\ n > 0\ \mathbf{then}\ n * g(n-1)\ \mathbf{else}\ r\Big)_{\dagger}(1)$$

where $(-)_{\dagger}$ is the least fixpoint of $g \mapsto f \circ \langle \mathrm{id}, g \rangle$. Alternatively,

Factorial (Iteratively)

$$\mathbf{fac}\, n = \Big((k, i) \mapsto \mathbf{if}\ k > 0\ \mathbf{then}\ \mathrm{inr}(k-1, k * i)\ \mathbf{else}\ (\mathrm{inl}\ i)\Big)^{\dagger}(n, 1)$$

where $(-)^{\dagger}$ is the least fixpoint of $g \mapsto [\mathrm{id}, g] \circ f$

Iteration v.s. Recursion

Iteration is dual to (call by name) recursion:

$$\frac{f : Y \times X \rightharpoonup X}{f_{\dagger} : Y \rightharpoonup X} \quad \textbf{(rec)} \qquad\qquad \frac{f : X \rightharpoonup Y + X}{f^{\dagger} : X \rightharpoonup Y} \quad \textbf{(iter)}$$

E.g. $X \rightharpoonup Y$ is the space of partial functions $X \to Y_{\perp}$ on **Set**

More generally, $X \rightharpoonup Y$ is $X \to TY$ where T is a monad

Another Typical Scenario

Given an alphabet of actions $A = \{a, b\}$, equations

$$x_1 = a.\,(x_2 + x_3) \qquad x_2 = a.\,x_1 + b.\,x_3 \qquad x_3 = a.\,x_1 + \checkmark$$

specify processes x_1, x_2, x_3 of basic process algebra (BPA)

We can think of them as a map

$$X \to \mathcal{P}_\omega(\{\checkmark\} + \Sigma X)$$

where $X = \{x_1, x_2, x_3\}$, $\Sigma = A \times (-)$, and solve them by finding the unique $X \to T_\Sigma\{\checkmark\}$ in the domain of possibly non-wellfounded trees

$$T_\Sigma\{\checkmark\} = \nu\gamma.\,\mathcal{P}_\omega(\{\checkmark\} + A \times \gamma)$$

(final coalgebra). The original system must be guarded. An unguarded specification, e.g. $x = x$ may have arbitrary solutions

[†] Rutten and Turi 1994, Initial algebra and final coalgebra semantics for concurrency

Unguarded Iteration

In order to solve an unguarded system like

$$x_1 = x_2 + a.\,(x_2 + x_1) \qquad\qquad x_2 = x_1 + a.\,x_1 + b.\,x_2$$

we first need to guard it to obtain

$$x_1 = a.\,(x_2 + x_1) \qquad\qquad x_2 = a.\,x_1 + b.\,x_2$$

and solve the result. Equations like $x = x$ must be replaced by $x = \varnothing$ where $\varnothing$ is unproductive divergence

This induces a notion of iteration which is neither least nor unique

Why Trees Can Not Be (Obviously) Ordered

Convexity Issue: a partial order would identify processes

$$a.a.\emptyset + \emptyset \quad \text{and} \quad a.a.\emptyset + a.\emptyset + \emptyset.$$

For $\sqsubseteq$: $a.a.\emptyset + \emptyset = a.a.\emptyset + \emptyset + \emptyset \sqsubseteq a.a.\emptyset + a.\emptyset + \emptyset$

(monotonicity of $+$ and idempotence of $+$)

For $\sqsupseteq$:

$$\dfrac{\dfrac{\dfrac{}{a.a.\emptyset \sqsubseteq a.a.\emptyset} \quad \dfrac{\dfrac{}{\emptyset \sqsubseteq a.\emptyset}}{a.\emptyset \sqsubseteq a.a.\emptyset}\ (monot.\ a.-) \quad \dfrac{}{\emptyset \sqsubseteq \emptyset}}{a.a.\emptyset + a.\emptyset + \emptyset \sqsubseteq a.a.\emptyset + a.a.\emptyset + \emptyset}\ (monot.\ +)}{a.a.\emptyset + a.\emptyset + \emptyset \sqsubseteq a.a.\emptyset + \emptyset}\ (idemp.\ +)$$

Monads

Definition (Monad)

A monad over a category $\mathbf{C}$ is given by a Kleisli triple $\mathbb{T} = (T, \eta, -^\star)$ where

- T is an endomap on $|\mathbf{C}|$
- η is a family of morphisms $\eta_X : X \to TX$, called monad unit
- $(-)^\star$ assigns to each $f : X \to TY$ a morphism $f^\star : TX \to TY$

and the following laws hold:

$$\eta^\star = \mathrm{id} \qquad f^\star \circ \eta = f \qquad (f^\star \circ g)^\star = f^\star \circ g^\star$$

This means that the hom-sets $\mathrm{Hom}(X, TY)$ form a category (Kleisli category) under Kleisli composition $f \diamond g = f^\star \circ g$ and $\eta_X \in \mathrm{Hom}(X, TX)$

ω-Continuous Monads

Definition (ω-Continuous Monad)

A monad $\mathbb{T}$ is ω-continuous if its Kleisli category is enriched over ω-complete partial orders with bottom $\bot$ and (nonstrict) continuous maps, and

$$f \diamond \bot = \bot \qquad\qquad \bot \circ h = \bot$$

For ω-continuous monads we can define iteration

$$\frac{f : X \to T(Y + X)}{f^{\dagger} : X \to TY}$$

and the lfp of $g \mapsto [\eta, g] \diamond f$

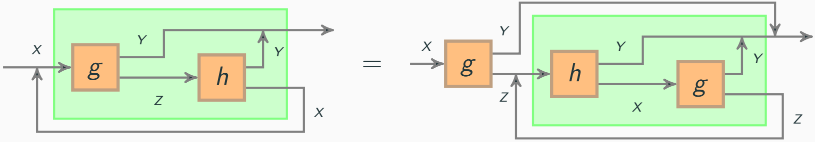
Examples

$TX = X + 1$ (partiality), $TX = \mathcal{P}X$ (nondeterminism),
$TX = \{\xi : X \to [0, 1] \mid \sum \xi \leqslant 1\}$ (sub-probability), etc.

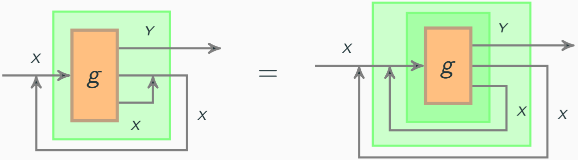
Axioms for Iteration: Conway Operators

Let $\mathbb{T}$ be a monad with an iteration operator $-^{\dagger}$ satisfying fixpoint identity $f^{\dagger} = [\eta, f^{\dagger}] \diamond f$. It is called a Conway operator if it additionally satisfies

Dinaturality:



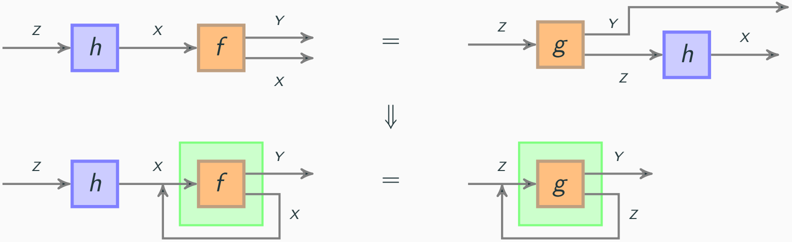
Codiagonal:



Axioms for Iteration: Naturality and Uniformty

Naturality is a form of coherence:

Uniformity is the only non-equation axiom:



Axioms for Iteration: Elgot Monads

Definition

A monad $\mathbb{T}$ is a Elgot monad if it is equipped with a Conway iteration operator, which is natural and uniform

Theorem (Ésik and Goncharov 2016)

Dinaturality is derivable

Theorem (Goncharov, Rauch, and Schröder 2015)

ω-continuous monads are Elgot monads

Theorem (Goncharov, Rauch, and Schröder 2015)

Let $\mathbb{T}$ be a Elgot monad and Σ and endofunctor. Then final coalgebras

$$T_\Sigma X = \nu\gamma.\, T(X + \Sigma\gamma)$$

defines a Elgot monad $\mathbb{T}_\Sigma$ uniquely coherently extending $\mathbb{T}$

Guared Iteration on Monads

Guarded v.s. Unguarded

Using the fact that $\mathbb{T}$ is Elgot we can solve both guarded and unguarded definitions over $\mathbb{T}_\Sigma$. Generally, we have:

	Canonical fixpoints	Unique fixpoints
Partial fixpoint operators	✓	✓
Total fixpoint operators	✓	—

If $\mathbb{T}$ is not Elgot (e.g. nonempty powerset) we can no longer compute solutions of unguarded definitions (think of $x = x$), but we still can compute solutions of guarded ones

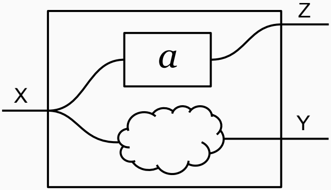
More generally, guardedness does not guarantee uniqueness, e.g. under infinite trace semantics $x = a.x + 1$ has both $a^\star$ and $a^\star + a^\omega$ as solutions

Abstract Guardedness

So, what means "guardedness" anyhow?

$$\frac{f : X \to Y + X}{f^{\dagger} : X \to Y} \qquad\qquad \frac{f : X \times Y \to X}{f_{\dagger} : Y \to X}$$

Can we make sense of this intuition? :



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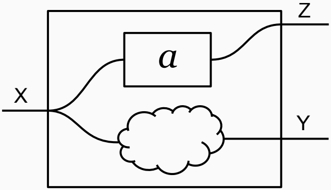


Pivotal Idea: Keep the notion of guardedness independent of fixpoint calculations

Abstract Guardedness for Monads

Abstract guardedness is a relation connecting $f : X \to TY$ with coproduct summands $\sigma : Y' \hookrightarrow Y$ in judgements $f : X \to_\sigma TY$, satisfying

$$\textbf{(vac}_+\textbf{)}\ \frac{f : X \to Z}{\mathsf{inl}\, f : X \to_{\mathsf{inr}} Z + Y} \qquad \textbf{(par}_+\textbf{)}\ \frac{f : X \to_\sigma Z \qquad f : Y \to_\sigma Z}{[f, g] : X + Y \to_\sigma Z}$$

$$\textbf{(cmp}_+\textbf{)}\ \frac{f : X \to_{\mathsf{inr}} Y + Z \qquad g : Y \to_\sigma V \qquad h : Z \to V}{[g, h] \circ f : X \to_\sigma V}$$

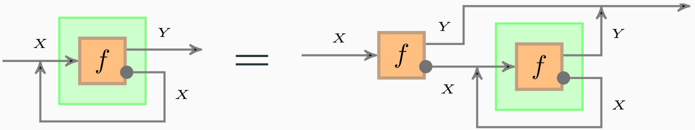
For example, $X \to_2 T_\Sigma(Y + Z) = \nu\gamma.\, T((Y + Z) + \Sigma\gamma)$ iff

$$\mathsf{out}\, f = T(\mathsf{inl} + \mathsf{id})g : X \to T((Y + Z) + \Sigma T_\Sigma(Y + Z))$$

for suitable $g : X \to T(Y + \Sigma T_\Sigma(Y + Z))$

Guarded Iteration Laws

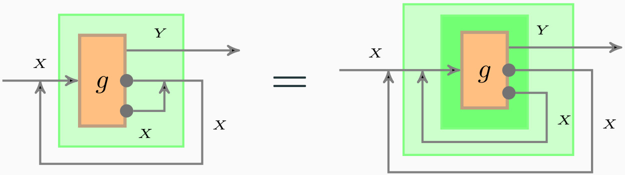
Iteration:



Naturality:

Guarded Iteration Laws (Continued)

Codiagonal:



Uniformity:

Some Results

- Guarded and unguarded iteration are instances of abstract guardedness
- Every monad is "vacuously guarded"
- (Unique) guarded iteration propagates along $T \mapsto \nu\gamma.\, T(\text{–} + \Sigma\gamma)$
- Dinaturality and other laws are derivable
- Guarded iteration is the exact dual of guarded recursion

Example: Guarded Recursion

Consider the category **CMS** of inhabited complete metric spaces and non-expansive maps

Let $f : X \times Y \to Z$ be guarded in Y if for all $x \in X$, $f(x, -)$ is contractive

This makes **CMS** into a guarded traced monoidal category (fixpoints calculated via Banach's fixpoint theorem)

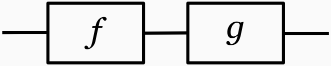
Going Monoidal

Going Monoidal

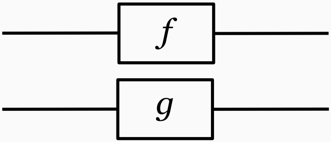
(We only consider symmetric monoidal categories, think of $\otimes = +, \times$)

Identity id:

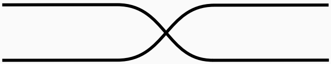
Composition $g \circ f$:



Tensor $g \otimes f$:

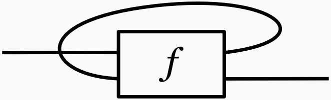


Symmetry:



Trace

Trace $(f : U \otimes A \to B \otimes U) \mapsto (tr^U_{A,B} f : A \to B)$

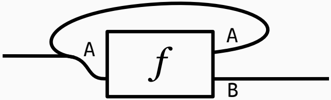


is the "generalized fixpoint operator"

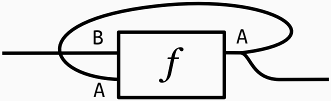
Iteration and Recursion

Iteration and recursion are typically viewed as corner cases:

- With $\otimes = +$, we obtain $(f : A \to B + A)^{\dagger} = tr(f \circ \nabla)$:



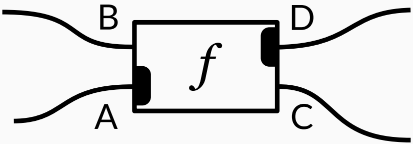
- With $\otimes = \times$, we obtain $(f : A \times B \to A)_{\dagger} = tr(\Delta \circ f)$:



Guarded Traced Categories

Partially Guarded Morphisms

A monoidal category is guarded if it is equipped with distinguished families $\mathrm{Hom}^\bullet(A \otimes B, C \otimes D) \subseteq \mathrm{Hom}(A \otimes B, C \otimes D)$, drawn as follows



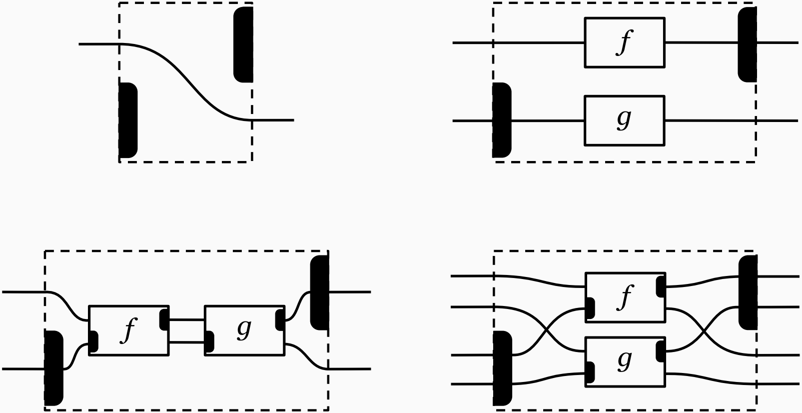
where

- A is unguarded input
- B is guarded input
- C is unguarded output
- D is guarded output

The idea is to allow feedback only on (A, D), which we call a guardedness profile of f.

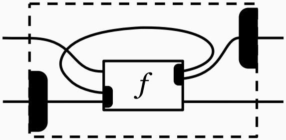
Hence we introduce axioms:

The Axioms



Guarded Traced Categories

A guarded category is guarded traced if it is equipped with a trace:



satisfying a collection of axioms adapted from the standard case

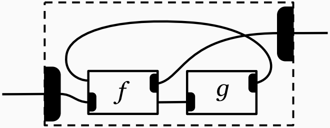
Guarded iteration/recursion operators are obtained analogously to the standard unguarded case

Towards Coherence

Structural Guardedness v.s. Geometric Guardedness

For guarded categories we have coherence of structural and geometric notions: a term is in $\mathrm{Hom}^{\bullet}(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from A to D runs through some atomic box via an unguarded input and a guarded output

After adding traces, this is no longer true:



Geometrically, this is OK but there is no structured way to derive it!

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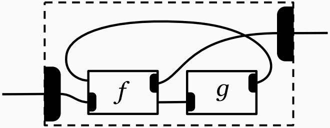
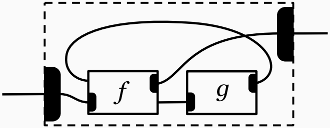


Geometrically, this is OK but there is no structured way to derive it!

But: The are no natural examples when this actually materializes

Failure of Coherence

Differently put, the circuit



is not in the free guarded traced category

Possible ways to resolve it

1. Strengthen the geometric guardedness criterion
2. Weaken the definition of the guarded traced category
3. Do both 1. and 2.

Both approached have issues we fail to resolve, as of today

The Diversity of Further Avenues

- Resolve the pressing coherence issue $\Rightarrow$ possibly update the notion of guarded traced category
- Rebase on , cover non-monoidal examples
- Deepen the theory of guarded traced categories: expressiveness, completeness (w.r.t. Hilbert spaces?), Int-construction
- Metalanguages for guarded iteration and recursion
- Comonadic guarded recursion (with Tarmo)

Questions?

References

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Zoltán Ésik and Sergey Goncharov. Some remarks on Conway and iteration theories. *CoRR*, abs/1603.00838, 2016. URL http://arxiv.org/abs/1603.00838.

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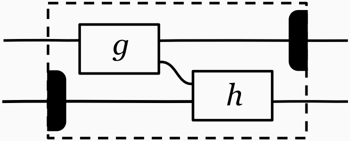
Jan Rutten and Daniele Turi. Initial algebra and final coalgebra semantics for concurrency. pages 530–582. Springer-Verlag, 1994.

Some (Easy) Observations

- There is a greatest notion of guardedness, $\mathrm{Hom}^\bullet(A \otimes B, C \otimes D) = \mathrm{Hom}(A \otimes B, C \otimes D)$
- There is a least (vacuous) notion of guardedness,



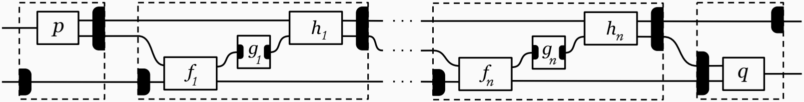
- Axioms are stable under 180°-rotations, hence $\mathbf{C}$ is guarded iff $\mathbf{C}^{op}$ is guarded, i.e. we maintain duality of recursion and iteration

Ideal Guardedness

A particularly common case is ideal guardedness

A guarded ideal is a family $\mathrm{Hom}^{\blacktriangleright}(X, Y) \subseteq \mathrm{Hom}(X, Y)$ closed under finite tensors and composition with any morphism on both sides

The general form of a partially guarded morphism over a guarded ideal is



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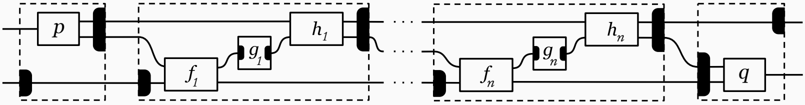


In the (co-)Cartesian case this simplifies greatly, generating standard notions, e.g. $f : X \to_2 Y + Z$ iff

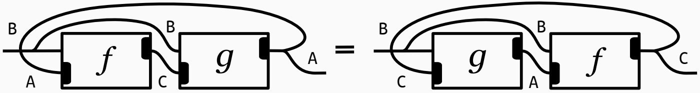
$$X \xrightarrow{h} Y + W \xrightarrow{[\mathsf{inl}, g]} Y + Z$$

with some $g \in \mathrm{Hom}^{\blacktriangleright}(W, Y + Z)$ and $h : X \to Y + W$

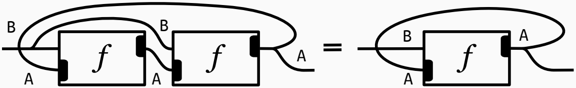
Unguarded Recursion as Guarded Recursion

We consider the following axioms:

- Dinaturality:



- Squaring (is not a property of Conway recursion but a property of Conway uniform recursion):



Theorem: There is a bijective correspondence between guarded squarable dinatural operators on $\mathbf{C}$ and unguarded squarable dinatural on $\mathbf{C}_{\star}^{\mathbb{T}}$

Unguarded Recursion as Guarded Recursion

- A standard way to do recursion with monads is in the category $\mathbf{C}_\star^{\mathbb{T}}$ with $\mathbb{T}$-algebras as objects and $\mathbf{C}$-morphisms of carriers as morphisms

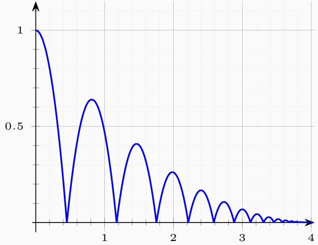
  Example: $\mathbf{C}$ = point-free dcpo's and continuous functions; $\mathbb{T}$ = lifting monad $X \mapsto X_\perp$

- Alternatively, following [Milius and Litak, 2013], we consider guarded recursion operators on $\mathbf{C}$ where $\mathbf{C}$ is ideally guarded over $\mathrm{Hom}^{\blacktriangleright}(X, Y) = \{f \circ \eta \mid f : TX \to Y\}$

  Example: with $\mathbf{C}$ and $\mathbb{T}$ as above, we allow only recursion on X of $f : X_\perp \times Y \to X$

More Examples..

- The topos of trees (guardedness by later-operator)
- Non-pointed order-enriched monads (e.g. non-empty powerset, probability distributions)
- Hybrid iteration semantics[†] (“guardedness” = “progressiveness”)



[†] Goncharov, Jakob, and Neves 2018, A Semantics for Hybrid Iteration

Guarded Traces in Hilbert Spaces

Finite-Dimensional Hilbert Spaces

Recall the multiplicative compact closed category of relations $(\mathbf{Rel}, \times, 1)$

Relations can be thought of as Boolean matrices, with transposition $(-)^\star$ and (unparamerized) trace being the trace of the square matrices

$$tr \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \sum_i b_{ii}$$

Analogously, linear operators on finite-dimensional Hilbert spaces can be represented as matrices over a field – we stick to the field of reals

Thus, Hilbert spaces are compact closed with tensors $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$, R as tensor unit, $X^\star = X$ on objects, $f^\star$ as the unique adjoint operator $\langle f(x), y\rangle = \langle x, f^\star(y)\rangle$ and unit/counit induced by inner products

Infinite-Dimensional Hilbert Spaces

More generally, Hilbert spaces are vector spaces with inner products, complete as a normed spaces under the induced norm $\|x\| = \sqrt{\langle x, x\rangle}$

Category **Hilb**:

- Objects are Hilbert spaces
- Morphisms are bounded linear operators, i.e. $\|f(x)\| \leqslant c \cdot \|x\|$ for a fixed c and every x
- Monoidal structure as before
- Adjointness for operators still works and $(f \otimes g)^\star = f^\star \otimes g^\star$, $f^{\star\star} = f$, $\mathrm{id}^\star = \mathrm{id}$, $(f \circ g)^\star = g^\star \circ f^\star$

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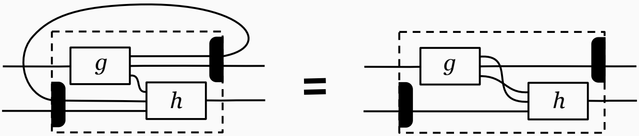
But there is no (total) trace, because the trace formula $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ may diverge ($\{e_i\}_i$ is any orthonormal basis)!
E.g. it diverges with $f = \mathrm{id} : X \to X$ with infinite-dimensional X

Nontrivial Trace from Vacuous Guardedness

[Abramsky, Blute, and Panangaden, 1999] already give a construction of unparameterized partial traces in **Hilb** via nuclear ideals. We generalize and reconcile it with our approach by equipping **Hilb** with the vacuous guardedness structure:



What is nontrivial though is that this is independent of the decomposition into g and h!

Morphisms $f : X \to X$ from the induced guarded ideal are precisely those for which $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ absolutely converges for any choice of an orthonormal basis $(e_i)_i$; the sum is then independent of the basis