Towards Coherence for Guarded Traces

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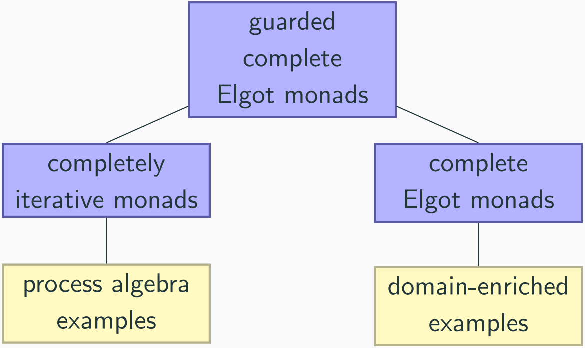
Introduction

- Recursion / iteration
  - order-theoretic / unguarded
  - process-theoretic / guarded
- Generic categorical models:
  - Total:
    - Axiomatic/synthetic domain theory (Hyland, Fiore, Taylor et al.)
    - let-ccc's with fixpoint objects (Crole/Pitts, Simpson)
    - Traced monoidal categories (Joyal/Street/Verity, Hasegawa)
    - Elgot monads/theories (Bloom/Esik, Adámek, Milius et al.)
  - Partial:
    - Completely iterative monads/theories (Bloom/Esik, Adámek, Milius et al.)
    - later-modality (Nakano, Appel, Melliès, Benton, Birkedal et al.)
    - Partial traced categories (Heghverdi, Scott, Malherbe, Selinger)
    - Functorial dagger (Milius, Litak)
- Here: Unifying framework for guarded and unguarded feedback in monoidal categories

Guarded Fixpoints: Overview



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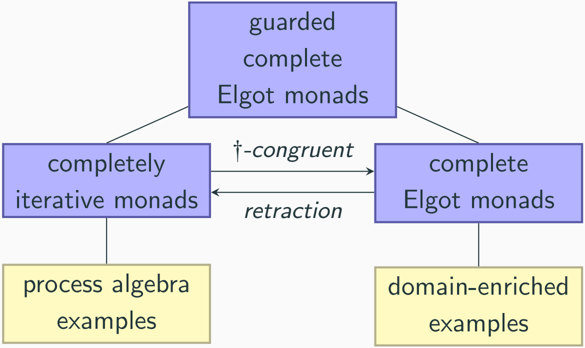
FoSSaCS17: G., Schröder, Rauch, Piróg, Unifying Guarded and Unguarded Iteration



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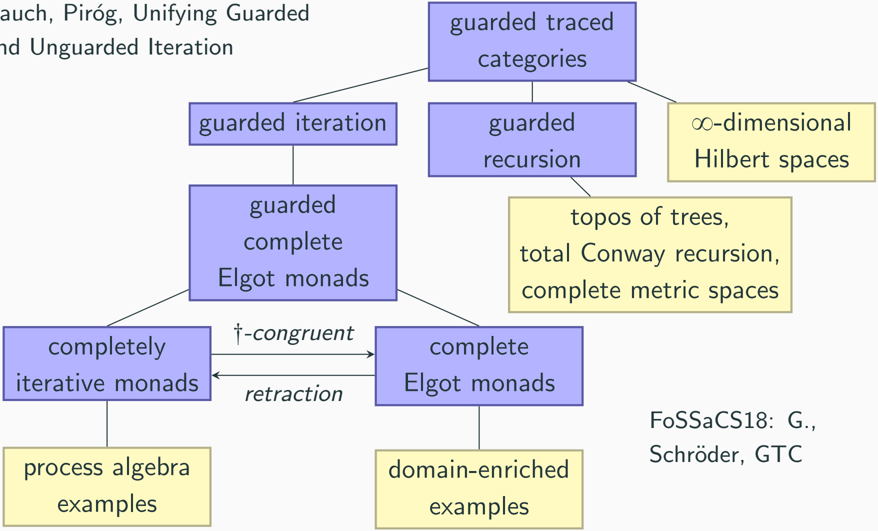


FoSSaCS18: G., Schröder, GTC

Motivating Example: Process Algebra

In process algebra, we solve tail-recursive process definitions, like

$$x = a.\,x + \tau.\,x + y$$

More abstractly, we involve a monad $T_\Sigma X = \nu\gamma.\,T(X + \Sigma\gamma)$ of infinite process trees and axiomatize guardedness of $f : X \to T_\Sigma Y$ in a coproduct summand $\sigma : Y' \hookrightarrow Y$ as follows (in Klesili):

$$\textbf{(vac}_+\textbf{)}\ \frac{f : X \to Z}{\mathsf{inl}\, f : X \to_{\mathsf{inr}} Z + Y} \qquad \textbf{(cmp}_+\textbf{)}\ \frac{\begin{array}{ll} f : & X \to_{\mathsf{inr}} Y + Z \\ g : & Y \to_\sigma V \qquad h : Z \to V \end{array}}{[g, h] \circ f : X \to_\sigma V}$$

$$\textbf{(par}_+\textbf{)}\ \frac{f : X \to_\sigma Z \qquad f : Y \to_\sigma Z}{[f, g] : X + Y \to_\sigma Z}$$

Guarded Iteration v.s. Guarded Recursion

Guarded iteration is a (partial) operation

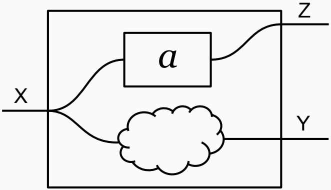
$$\frac{f : X \to Y + X}{f^{\dagger} : X \to Y}$$

with f guarded in X

Dualization should yield guarded recursion:

$$\frac{f : X \times Y \to X}{f_{\dagger} : Y \to X}$$

Can we make sense of this intuition? :



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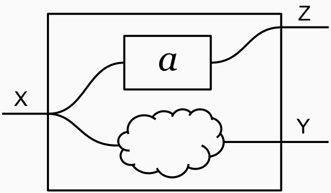
Pivotal Idea: Keep the notion of guardedness independent of fixpoint calculations



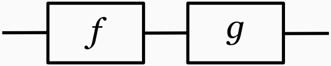
Going Monoidal

Going Monoidal

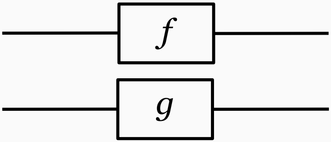
(We only consider symmetric monoidal categories, think of $\otimes = +, \times$)

Identity id:

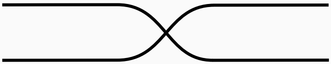
Composition $g \circ f$:



Tensor $g \otimes f$:

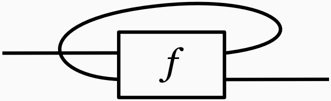


Symmetry:



Going Monoidal: Additional Structure

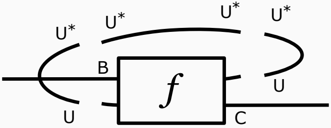
Trace $tr(f : U \otimes A \rightarrow B \otimes U)$[1]:
$A \rightarrow B$



Compact closure:
unit $\eta : I \rightarrow A \otimes A^{\star}$ and
counit $\epsilon : A^{\star} \otimes A \rightarrow I$

where $(-)^{\star}$ is a contravariant involutive endofunctor

In compact closed categories, trace is definable and unique, for:

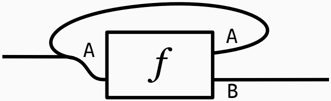


[1]The twist of input wires is nonstandard, but bear with me

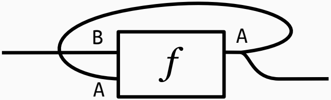
Iteration and Recursion

Iteration and recursion are typically viewed as corner cases:

- With $\otimes = +$, we obtain $(f : A \to B + A)^\dagger = tr(f \circ \nabla)$:



- With $\otimes = \times$, we obtain $(f : A \times B \to A)_\dagger = tr(\Delta \circ f)$:

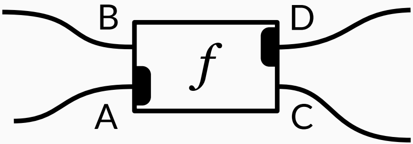


Corresponding converse definitions can also be produced. So, traces and (Conway) fixpoints are equivalent in the requisite cases!

Guarded Categories

Partially Guarded Morphisms

A monoidal category is guarded if it is equipped with distinguished families $\mathrm{Hom}^\bullet(A \otimes B, C \otimes D) \subseteq \mathrm{Hom}(A \otimes B, C \otimes D)$, drawn as follows



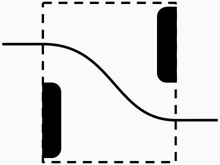
where

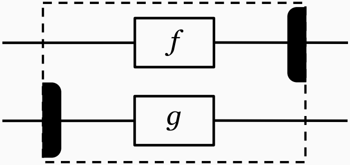
- A is unguarded input
- B is guarded input
- C is unguarded output
- D is guarded output

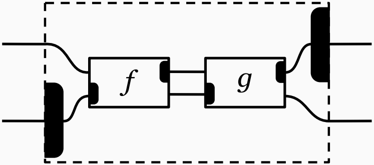
The idea is to allow feedback only on (A, D), which we call a guardedness profile of f.

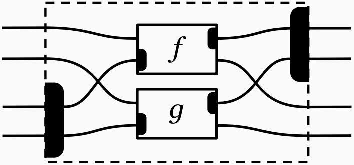
Hence we introduce axioms:

The Axioms



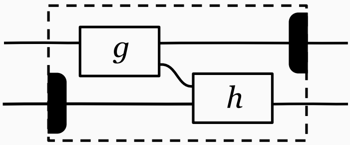






Some (Easy) Observations

- There is a greatest notion of guardedness, $\mathrm{Hom}^{\bullet}(A \otimes B, C \otimes D) = \mathrm{Hom}(A \otimes B, C \otimes D)$
- There is a least (vacuous) notion of guardedness,



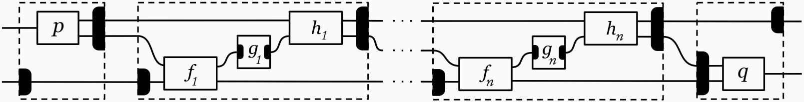
- Axioms are stable under 180°-rotations, hence $\mathbf{C}$ is guarded iff $\mathbf{C}^{op}$ is guarded, i.e. we maintain duality of recursion and iteration

Ideal Guardedness

A particularly common case is ideal guardedness

A guarded ideal is a family $\mathrm{Hom}^{\blacktriangleright}(X, Y) \subseteq \mathrm{Hom}(X, Y)$ closed under finite tensors and composition with any morphism on both sides

The general form of a partially guarded morphism over a guarded ideal is



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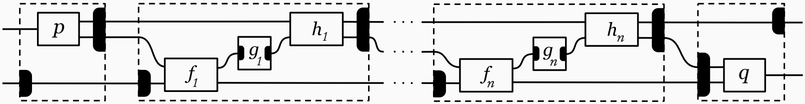


In the (co-)Cartesian case this simplifies greatly, generating standard notions, e.g. $f : X \to_2 Y + Z$ iff

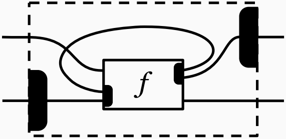
$$X \xrightarrow{h} Y + W \xrightarrow{[\mathsf{inl}, g]} Y + Z$$

with some $g \in \mathrm{Hom}^{\blacktriangleright}(W, Y + Z)$ and $h : X \to Y + W$

Guarded Traces

Guarded Traced Categories

A guarded category is guarded traced if it is equipped with a trace:



satisfying a collection of axioms adapted from the standard case

Guarded Conway iteration/recursion operators are obtained analogously to the standard case

Non-Ideal Case: Contractive Maps

Consider the category **CMS** of inhabited complete metric spaces and non-expansive maps

Let $f : X \times Y \to Z$ be guarded in Y if for all $x \in X$, $f(x, -)$ is contractive

This makes **CMS** into a guarded traced monoidal category (fixpoints calculated via Banach's fixpoint theorem) but not ideally guarded, because a contraction factor depends on x and may not be chosen uniformly

Question: Is there a parametrically contractive map in **CMS** that is not uniformly contractive?

Unguarded Recursion as Guarded Recursion

- A standard way to do recursion with monads is in the category $\mathbf{C}^{\mathbb{T}}_{\star}$ with $\mathbb{T}$-algebras as objects and $\mathbf{C}$-morphisms of carriers as morphisms

  Example: $\mathbf{C}$ = point-free dcpo's and continuous functions; $\mathbb{T}$ = lifting monad $X \mapsto X_\perp$

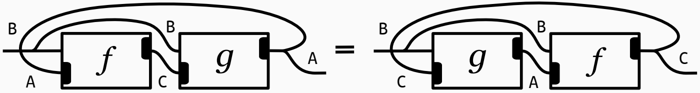
- Alternatively, following [Milius and Litak, 2013], we consider guarded recursion operators on $\mathbf{C}$ where $\mathbf{C}$ is ideally guarded over $\mathrm{Hom}^{\blacktriangleright}(X, Y) = \{f \circ \eta \mid f : TX \to Y\}$

  Example: with $\mathbf{C}$ and $\mathbb{T}$ as above, we allow only recursion on X of $f : X_\perp \times Y \to X$

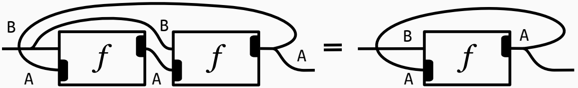
Unguarded Recursion as Guarded Recursion

We consider the following axioms:

- Dinaturality:



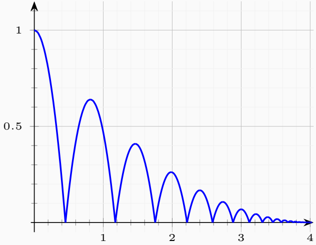
- Squaring (is not a property of Conway recursion but a property of Conway uniform recursion):



Theorem: There is a bijective correspondence between guarded squarable dinatural operators on $\mathbf{C}$ and unguarded squarable dinatural on $\mathbf{C}_{\star}^{\mathbb{T}}$

More Examples..

- The topos of trees (guardedness by later-operator)
- Non-pointed order-enriched monads (e.g. non-empty powerset, probability distributions)
- Hybrid iteration semantics[†] ("guardedness" = "progressiveness")



[†] Goncharov, Jakob, and Neves 2018, A Semantics for Hybrid Iteration

Guarded Traces in Hilbert Spaces

Finite-Dimensional Hilbert Spaces

Recall the multiplicative compact closed category of relations $(\mathbf{Rel}, \times, 1)$

Relations can be thought of as Boolean matrices, with transposition $(-)^\star$ and (unparamerized) trace being the trace of the square matrices

$$tr \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \sum_i b_{ii}$$

Analogously, linear operators on finite-dimensional Hilbert spaces can be represented as matrices over a field – we stick to the field of reals

Thus, Hilbert spaces are compact closed with tensors $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$, R as tensor unit, $X^\star = X$ on objects, $f^\star$ as the unique adjoint operator $\langle f(x), y \rangle = \langle x, f^\star(y) \rangle$ and unit/counit induced by inner products

Infinite-Dimensional Hilbert Spaces

More generally, Hilbert spaces are vector spaces with inner products, complete as a normed spaces under the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$

Category **Hilb**:

- Objects are Hilbert spaces
- Morphisms are bounded linear operators, i.e. $\|f(x)\| \leqslant c \cdot \|x\|$ for a fixed c and every x
- Monoidal structure as before
- Adjointness for operators still works and $(f \otimes g)^\star = f^\star \otimes g^\star$, $f^{\star\star} = f$, $\mathrm{id}^\star = \mathrm{id}$, $(f \circ g)^\star = g^\star \circ f^\star$

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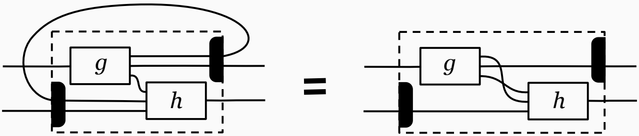
But there is no (total) trace, because the trace formula $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ may diverge ($\{e_i\}_i$ is any orthonormal basis)! E.g. it diverges with $f = \mathrm{id} : X \to X$ with infinite-dimensional X

Nontrivial Trace from Vacuous Guardedness

[Abramsky, Blute, and Panangaden, 1999] already give a construction of unparameterized partial traces in **Hilb** via nuclear ideals. We generalize and reconcile it with our approach by equipping **Hilb** with the vacuous guardedness structure:



What is nontrivial though is that this is independent of the decomposition into g and h!

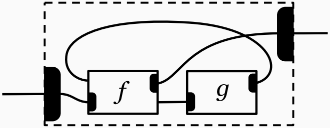
Morphisms $f : X \to X$ from the induced guarded ideal are precisely those for which $tr(f) = \sum_i \langle f(e_i), e_i \rangle$ absolutely converges for any choice of an orthonormal basis $(e_i)_i$; the sum is then independent of the basis

Towards Coherence

Structural Guardedness v.s. Geometric Guardedness

For guarded categories we have coherence of structural and geometric notions: a term is in $\mathrm{Hom}^\bullet(A \otimes B, C \otimes D)$ iff in the corresponding diagram every path from A to D runs through some atomic box via an unguarded input and a guarded output

This is no longer true for guarded traces:



Geometrically, this is OK but there is no structured way to derive it!

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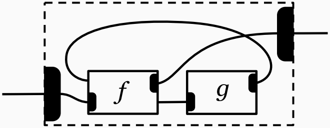


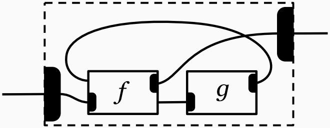
Geometrically, this is OK but there is no structured way to derive it!

But: This discrepancy does not arise in the ideal case

Conjecture: The same is true for the (co-)Cartesian case

Failure of Coherence

Differently put, the circuit



is not in the free guarded traced category

Possible ways to resolve it

1. Strengthen the geometric guardedness criterion
2. Weaken the definition of the guarded traced category
3. Do both 1. and 2.

Both approached have issues we fail to resolve, as of today

The Diversity of Further Avenues

- Resolve the pressing coherence issue $\Rightarrow$ possibly update the notion of guarded traced category
- Rebase on colored props (=symmetric concategories) + non-monoidal examples
- Deepen the theory of guarded traced categories: expressiveness, completeness (w.r.t. Hilbert spaces?), Int-construction
- Elaborate the relationship between guarded ideals and nuclear ideals
- Cover further examples from the Abramsky-Blute-Panangaden paper
- Metalanguages for guarded iteration and recursion

Questions?

References

S. Abramsky, R. Blute, and P. Panangaden. Nuclear and trace ideals in tensored *-categories. *J. Pure Appl. Algebra*, 143:3–47, 1999.

Sergey Goncharov, Julian Jakob, and Renato Neves. A semantics for hybrid iteration. In Sven Schewe and Lijun Zhang, editors, *29th International Conference on Concurrency Theory (CONCUR 2018)*, LNCS. Springer, 2018.

Stefan Milius and Tadeusz Litak. Guard your daggers and traces: On the equational properties of guarded (co-)recursion. In *Fixed Points in Computer Science, FICS 2013*, volume 126 of *EPTCS*, pages 72–86, 2013.