LGruDat: Logical Foundations of Databases Lecture 3 — FORC: A reminder on first-order logic continued

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In this set of notes, will denote corrections as compared with handouts distributed during the lecture, not homework. See separate document on the webpage for homework

1 Diagrams, homomorphisms, isomorphism

A comment to exercise solutions

One of you was very careful to take care of models with empty universes ...

What happens if we allow these?

- Do we have that whenever $v\#\alpha$, $\vDash \alpha \vee \exists v.\beta \equiv \exists v.(\alpha \vee \beta)$? $\vDash \alpha \wedge \forall v.\beta \equiv \forall v.(\alpha \wedge \beta)$? Careful with prenex normal form!
- Standard axioms like $\vDash \forall v.\alpha \to \alpha$?
- Also remember: no valuation into an empty domain

Useful syntactic convention

First, recall once again our lemma: whenever

- $free(\alpha) = \{v_1, \ldots, v_n\}$
- $\kappa(v_1) = \kappa'(v_1)$, ..., $\kappa(v_n) = \kappa'(v_n)$

then $\mathfrak{A}, \kappa \vDash \alpha$ iff $\mathfrak{A}, \kappa' \vDash \alpha$.
and its corollary:

$$\begin{aligned}\phi(\mathfrak{A}) &:= \{(\mathsf{a}_1, \ldots \mathsf{a}_n) \in \mathsf{A}^n \mid \text{ exists } \kappa.\kappa(\overline{\mathbf{v}}) = \overline{\mathsf{a}} \text{ and } \mathfrak{A}, \kappa \vDash \alpha\} \\ &= \{(\mathsf{a}_1, \ldots \mathsf{a}_n) \in \mathsf{A}^n \mid \text{ for\_all } \kappa.\kappa(\overline{\mathbf{v}}) = \overline{\mathsf{a}} \text{ implies } \mathfrak{A}, \kappa \vDash \alpha\} \\ &= \{(\mathsf{a}_1, \ldots \mathsf{a}_n) \in \mathsf{A}^n \mid \text{ for\_all } \kappa.\mathfrak{A}, \kappa[v_1 := \mathsf{a}_1] \ldots [v_n := \mathsf{a}_n] \vDash \alpha\} \\ &= \{(\mathsf{a}_1, \ldots \mathsf{a}_n) \in \mathsf{A}^n \mid \text{ exists } \kappa.\mathfrak{A}, \kappa[v_1 := \mathsf{a}_1] \ldots [v_n := \mathsf{a}_n] \vDash \alpha\}\end{aligned}$$

Our new syntactic conventions: $\mathfrak{A} \models \alpha[\overline{a}]$ and $(\mathfrak{A}, \overline{a}) \models \alpha(\overline{a})$.

Not introduced explicitly in the lecture, but the meaning is hopefully clear: $\mathfrak{A} \models \alpha[\overline{a}]$ means that some/any valuation sending the free variables of α to $\overline{a}$ makes α hold. The other notation removes the need for valuations by expanding the signature with (constants for) satisfying elements, so we need to . . .

- . . . Define reducts and expansions
- Define notions proposed by A. Robinson:
  - Positive diagram: $\mathsf{diagp}^{+}(\mathfrak{A})$
  - Diagram: $\mathsf{diag}(\mathfrak{A})$
  - Elementary diagram: $\mathsf{eldiag}(\mathfrak{A})$
- Our additions: $\exists\mathsf{diagp}^{+}(\mathfrak{A})$, $\exists\mathsf{diag}(\mathfrak{A})$ and $\exists\mathsf{eldiag}(\mathfrak{A})$.
- (Equivalent to) single sentences for finite $\mathfrak{A}$!
- During the lecture I forgot at first to introduce $\exists\mathsf{eldiag}(\mathfrak{A})$. Yet, as you recall, for a finite $\mathfrak{A}$ it's precisely (the single sentence encoding) $\exists\mathsf{eldiag}(\mathfrak{A})$ which encodes the isomorphism type of $\mathfrak{A}$

Last time we finished with:

- Notion of homomorphism, surjective homomorphism, embedding and isomorphism
- Primitive-positive (CQ) and existential-positive (UCQ—but careful here!) $\longrightarrow \mathsf{diagp}^{+} A$

  Easy direction of h.p.t.
- Positive

  Easy direction of Lyndon: surjective homomorphism
- Existential (dual Łoś-Tarski, embeddings) $\longrightarrow \mathsf{diag} A$

WE WILL RETURN TO THIS

- For the time being: restate the easy directions in terms of diagrams
- Dualize (reflecting rather than preserving)

The role of isomorphism

- Contrast with elementary equivalence
- How about the finite case?
- Return to elementary diagrams for a while
- Classes of models always closed under isomorphism from now on
- Invariance of queries under isomorphism
- Enumerating isomorphism types for finite signatures
- Notions of Th and Mod
- Elementarity and Δ-elementarity Now called EC and EC_Δ
- Trivialization of the latter in the finite (for finite signature)

Examples of FO Theories and queries

- We haven't done this during the lecture, but by now they should be too easy even to be made a part of the homework
- Finite cardinalities.
- Graph-like examples: isolated nodes, having at least two succ, having exactly two neighbours etc.
- Strict linear order, with smallest and greatest element

Back to universal validity—the last remaining old exercise. Recall it was:

Exercise 4 (8 pts) How about the converse of 3.b? Assume that Σ contains a single binary symbol. Can you think of any α s.t. not ($\overset{\text{unr}}{\models} \alpha$), but $\overset{\text{fin}}{\models} \alpha$? If so, give an explicit example of such a formula and prove both statements.

- Most of you solved the exercise and solved it well
- (perhaps one more example)
- What happens if all symbols in Σ are at most unary?
- What if we have at most two variables?

2 Validity and inference

Entailment: finite and unrestricted, global and local

These definitions were not discussed during the lecture. Do you understand them? See homework.

$$\Gamma \models^{\text{unr}}_{loc} \alpha \ \textsf{iff} \ \textsf{for\_all}\ \textbf{unrestricted}\ \mathfrak{A}, \kappa.(\mathfrak{A}, \kappa \models \Gamma\ \textsf{implies}\ \mathfrak{A}, \kappa \models \alpha)$$

$$\Gamma \models^{\text{fin}}_{loc} \alpha \ \textsf{iff} \ \textsf{for\_all}\ \textbf{finite}\ \mathfrak{A}, \kappa.(\mathfrak{A}, \kappa \models \Gamma\ \textsf{implies}\ \mathfrak{A}, \kappa \models \alpha)$$

$$\Gamma \models^{\text{unr}}_{glo} \alpha \ \textsf{iff} \ \textsf{for\_all}\ \textbf{unrestricted}\ \mathfrak{A}.$$

$$(\ \textsf{for\_all}\ \kappa.\mathfrak{A}, \kappa \models \Gamma)\ \textsf{implies}\ (\ \textsf{for\_all}\ \kappa.\mathfrak{A}, \kappa \models \alpha)$$

$$\Gamma \models^{\text{fin}}_{glo} \alpha \ \textsf{iff} \ \textsf{for\_all}\ \textbf{finite}\ \mathfrak{A}.(\forall \kappa.\mathfrak{A}, \kappa \models \Gamma)\ \textsf{implies}\ (\ \textsf{for\_all}\ \kappa.\mathfrak{A}, \kappa \models \alpha)$$

where $\mathfrak{A}, \kappa \models \Gamma$ iff $\textsf{for\_all}\ \gamma \in \Gamma$, $\mathfrak{A}, \kappa \models \gamma$

- ¿ $\models_{loc} \quad \subseteq \quad \models_{glo}$?
- ¿ $\models_{glo} \quad \subseteq \quad \models_{loc}$?
- Define universal closure $\overline{\gamma}^{\forall}$ and $\overline{\Gamma}^{\forall}$
- What if $\Gamma = \overline{\Gamma}^{\forall}$? (empty, sentences only ...) See homework for this and the following two points
- Can we characterize the relationship between $\models_{loc}$ and $\models_{glo}$ in the terms of universal closure?
- $\vdash^{\text{unr}}_{loc}$ and $\vdash^{\text{unr}}_{glo}$: inference systems for $\models^{\text{unr}}_{loc}$ and $\models^{\text{unr}}_{glo}$
- The notion of deductive theory and *Post-complete theory (MCS)*
- Is $Th(\mathfrak{A})$ a deductive theory? A Post-complete one?
- Brief sketch of completeness:

$$\vdash^{\text{unr}}_{loc} = \models^{\text{unr}}_{loc} \quad \textsf{and} \quad \vdash^{\text{unr}}_{glo} = \models^{\text{unr}}_{glo}$$

- What is the relationship between Post-complete theories and those of the form $Th(\mathfrak{A})$? Not discussed, see homework
- No f.m.p.: this we already know ...
- How about countable model property? Not discussed
- Other consequences: compactness ...
- This is where we finished Lecture 3, removed the remaining part