LGruDat: Logical Foundations of Databases
Exercise 7–9, Deadline Thu, 30 Jan 2014

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This time aim for **16 points**. As usual, pick and mix as you please.

Sorry I didn't manage earlier in the weekend. But if the deadline is too tight, let me know and don't worry, we'll adjust.

The first five sections are corrected and redacted versions of what you have already seen in the Thursday draft. There are also new ones at the end.

1 Balanced trees

This is a leftover bonus exercise from the section on locality properties of first-order logic. A bit technical, but nevertheless an excellent overview.

Exercise **1**. **(5 pts)** We say that a tree is a *balanced binary tree* if all the non-leaves have exactly two children and all the branches are of the same length. Show that the property of being a balanced binary tree is not Hanf-local and henceforth not FOL-definable.

2 Some exercises on modal logic and Kripke validity

Warning: corrected typos from the draft version

Write

- $\Vdash \phi$ if for any $\mathfrak{A}, \mathsf{a}$, we have that $\mathfrak{A}, \mathsf{a} \Vdash \phi$.
- $\phi \models_{loc} \psi$ if $\mathsf{for\_all}\ \mathfrak{A}, \mathsf{a}.(\mathfrak{A}, \mathsf{a} \Vdash \phi\ \mathsf{implies}\ \mathfrak{A}, \mathsf{a} \Vdash \psi)$
- $\phi \models_{glo} \psi$ if $\mathsf{for\_all}\ \mathfrak{A}.(\ \mathsf{for\_all}\ \mathsf{a}.\mathfrak{A}, \mathsf{a} \Vdash \phi)\ \mathsf{implies}\ (\ \mathsf{for\_all}\ \mathsf{a}.\mathfrak{A}, \mathsf{a} \Vdash \psi)$

Exercise **2.a** **(2 pts)** Is it true that $\phi \models_{loc} \psi$ iff $\Vdash \phi \to \psi$? Prove or give a counterexample.

Exercise **2.b** **(2 pts)** Is it true that $\phi \models_{glo} \psi$ iff $\Vdash \phi \to \psi$? Prove or give a counterexample.

Exercise **2.c** **(2 pts)** Is it true that $\phi \models_{loc} \Box\phi$? Prove or give a counterexample.

Exercise **2.d** **(2 pts)** Is it true that $\phi \models_{glo} \Box\phi$? Prove or give a counterexample.

Warning: Removed the exercise 2.e from the draft version

Exercise **2.f** **(3 pts)** (bonus) Find counterexamples to $\Vdash \Box\phi \to \Box\Box\phi$, $\Vdash \phi \to \Box\Diamond\phi$, $\Vdash \Box(\Box\phi \to \phi) \to \Box\phi$ and $\Vdash \Box\phi \to \phi$.

Exercise **2.g** **(3 pts)** (bonus) How about the previous point when you consider only validity over relations which are, e.g., finite, transitive, reflexive, irreflexive, symmetric, linear orders or finite linear orders etc.? Can you think of necessary and sufficient conditions on binary relations interpreting modal connectives corresponding to validity of each of these formulas?

3 Beginning bisimulation equivalence

In all the exercises involving modal logic and bisimulations, we assume for simplicity, just like in the lecture, that we are dealing with modal similarity types only.

Do you remember the definition of $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{\infty} \mathfrak{B}, \mathsf{b}$ and $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$? (Note: unlike in the lecture, I'm using a superscript $\Diamond$, which you can freely drop if you want to) As I suggested, the bisimulation games play a similar role for ML (modal logic) as EF games for FOL. Just like in the first-order case, this can be made precise by showing that several superficially different notions are in fact equivalent.

Let us write $\mathfrak{A}, \mathsf{a} \leftrightsquigarrow^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$ to denote that $\mathfrak{A}, \mathsf{a}$ and $\mathfrak{B}, \mathsf{b}$ force exactly the same modal formulas of modal rank at most m.

Exercise **3.a** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \equiv_m \mathfrak{B}, \mathsf{b}$ implies $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$

Exercise **3.b** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$ implies $\mathfrak{A}, \mathsf{a} \leftrightsquigarrow^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$

4 Alternative characterization of bisimulation

Just like in case of FO, the game theoretic characterization can be modified to a one with a slightly more algebraic flavour. Let us begin with a counterpart of the unrestricted notion $\sim^{\Diamond}_{\infty}$. We write $\mathbf{Z} : \mathfrak{A}, \mathsf{a} \leftrightarrows^{\Diamond}_{\infty} \mathfrak{B}, \mathsf{b}$ to denote that for each $\mathsf{a} \in \mathsf{A}$ and $\mathsf{b} \in \mathsf{B}$, $\mathsf{a}Z\mathsf{b}$ implies that

- a and b agree on propositional symbols
- $\forall R_i \in \Sigma_{\Diamond}, (\mathsf{a}, \mathsf{a}') \in R_i^{\mathsf{A}}.\exists(\mathsf{b}, \mathsf{b}') \in R_i^{\mathfrak{B}}.\, \mathsf{a}'Z\mathsf{b}'$ (forth)
- $\forall R_i \in \Sigma_{\Diamond}, (\mathsf{b}, \mathsf{b}') \in R_i^{\mathsf{B}}.\exists(\mathsf{a}, \mathsf{a}') \in R_i^{\mathfrak{A}}.\, \mathsf{a}'Z\mathsf{b}'$ (back)

Now for the stratified version of this notion corresponding to $\sim_n^{\Diamond}$.

Let $\mathbf{Z} := \bigcup_{m \leq n} Z_m$. We write $\mathbf{Z} : \mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_n^{\Diamond} \mathfrak{B}, \mathsf{b}$ to denote that for each $m \leq n$, $\mathsf{a} \in \mathsf{A}$ and $\mathsf{b} \in \mathsf{B}$:

- $\mathsf{a} Z_0 \mathsf{b}$ implies a and b agree on propositional symbols
-

$$\mathsf{a} Z_{m+1} \mathsf{b} \text{ implies } \begin{cases} \forall R_i \in \Sigma_{\Diamond}, (\mathsf{a}, \mathsf{a}') \in R_i^{\mathsf{A}}. \exists (\mathsf{b}, \mathsf{b}') \in R_i^{\mathfrak{B}}.\, \mathsf{a}' Z_m \mathsf{b}' \text{ (forth)} \\ \text{and} \\ \forall R_i \in \Sigma_{\Diamond}, (\mathsf{b}, \mathsf{b}') \in R_i^{\mathsf{B}}. \exists (\mathsf{a}, \mathsf{a}') \in R_i^{\mathfrak{A}}.\, \mathsf{a}' Z_m \mathsf{b}' \text{ (back)} \end{cases}$$

In both cases, we can write $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_{\infty}^{\Diamond} \mathfrak{B}, \mathsf{b}$ ($\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_n^{\Diamond} \mathfrak{B}, \mathsf{b}$) to denote that there exists a suitable Z ($\mathbf{Z}$) satisfying these conditions.

Exercise **4.a** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_{\infty}^{\Diamond} \mathfrak{B}, \mathsf{b}$ iff $\mathfrak{A}, \mathsf{a} \sim_{\infty}^{\Diamond} \mathfrak{B}, \mathsf{b}$

Exercise **4.b** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_n^{\Diamond} \mathfrak{B}, \mathsf{b}$ iff $\mathfrak{A}, \mathsf{a} \sim_n^{\Diamond} \mathfrak{B}, \mathsf{b}$

5 What happens when games go on forever?

Warning/update: As Thorsten found a suitable example during the exercise session, all that is left is to show it works. Thorsten gets 3 bonus points for finding it.

Exercise **5.** **(3 pts)** Prove that there exist $\mathfrak{A}, \mathsf{a}$ and $\mathfrak{B}, \mathsf{B}$ s.t.

- $\mathfrak{A}, \mathsf{a} \equiv \mathfrak{B}, \mathsf{b}$ (and hence also for any n, $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_n^{\Diamond} \mathfrak{B}, \mathsf{b}$, $\mathfrak{A}, \mathsf{a} \sim_m^{\Diamond} \mathfrak{B}, \mathsf{b}$ and $\mathfrak{A}, \mathsf{a} \leftrightsquigarrow_m^{\Diamond} \mathfrak{B}, \mathsf{b}$) but
- it is **not** true that $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_{\infty}^{\Diamond} \mathfrak{B}, \mathsf{b}$.

6 Finishing the proof of van Benthem-Rosen

Exercise **6.a** **(3 pts)** Finish the remaining clauses of the normal forms lemma. Show that:

- the disjunction of all normal forms of the same degree is always valid
- two different modal normal forms of the same degree are mutually inconsistent
- if $nf_{\Sigma}^{l}(\mathsf{a}) = nf_{\Sigma}^{l}(\mathsf{b})$, then $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}_l^{\Diamond} \mathfrak{B}, \mathsf{b}$

Exercise **6.b** **(3 pts)** Finish the proofs of Disjointness Lemma, Locality of $\sim_l^{\Diamond}$ Lemma and Unravelling Lemma. That is, show that:

- $\mathfrak{A}, \mathsf{a} \sim_l^{\Diamond} \mathfrak{B}, \mathsf{b}$ iff for some/any arbitrarily chosen $\mathfrak{C}$, $\mathfrak{A} + \mathfrak{C}, \mathsf{a} \sim_l^{\Diamond} \mathfrak{B}, \mathsf{b}$. Similarly for $\sim_l^{\Diamond}$.

- two points a, b are l-bisimilar iff they are in this relation in the U^l-restrictions of their corresponding pointed models
- the graph of the canonical projection is a bisimulation relation
- every finite pointed model is $\sim^{\Diamond}_{\infty}$ bisimilar to another finite one which is l-locally a tree (i.e., prove that the construction suggested during the lecture works)

You don't have to be too verbose and you can omit details. It is enough to convince me you understand what is going on.

Exercise **6.c** **(5 pts)** Finish the proof of the Key Lemma: that for first-order formulas, bisimulation invariance implies ML-l-locality. More specifically, show that the two pointed structures (obtained as forests via the disjoint union operation) we drew on the blackboard are equivalent for formulas of quantifier rank at most q.

7 Hennessy-Milner theorem

Exercise **7**. **(4 pts)** Prove that it was essential to the counterexample we produced for Section 5 last Thursday that the structure was infinitely branching. That is, show that **if every point has only finitely many successors**,

$$(\forall n \in \omega.\, \mathfrak{A}, \mathsf{a} \leftrightsquigarrow^{\Diamond}_{n} \mathfrak{B}, \mathsf{b}) \text{ implies } \mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^{\Diamond}_{\infty} \mathfrak{B}, \mathsf{b}.$$

8 Finite validity

These are ambitious bonus exercises to put notions learned in the last lecture to heavy use.

Exercise **8.a** **(5 pts)** Prove that whenever a modal formula has a (counter-)model, there also exists a finite one.

If you're feeling even more ambitious:

Exercise **8.b** **(7 pts)** A Hilbert-style axiomatization for ML is obtained by addition of the axiom $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ and the one stating that $\Diamond$ is just $\neg\Box\neg$ (alternatively, we can simply take it as a syntactic abbreviation) to tautologies of classical propositional logic and closing the resulting set under both Modus Ponens and $\Box$-generalization (if you have ϕ, infer also $\Box\phi$). Sketch the argument that this axiomatization is complete. **Hint/request**: use normal forms. Don't use the Axiom of Choice and its consequences.

Bonus question: which of the consequence relations defined in Section 2 is sound wrt $\Box$-generalization rule?