LGruDat: Logical Foundations of Databases
Exercise 7–8 (draft), Deadline : so far none

Tadeusz Litak

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By way of reviewing for tomorrow lecture: have a look at these exercises and decide if you can try to do some of the 2pt/1pt ones tomorrow in the classroom on the blackboard (I'd give you normal points for that). Of course, if any of you feels ready to solve the harder ones on the spot, you're very welcome. Then I'll prepare a final version with the deadline for the next week.

1 Balanced trees

This is a leftover bonus exercise from the section on locality properties of first-order logic. A bit technical, but nevertheless an excellent overview. This is left for the next week, we will not do it this week in the classroom, as it does not have much to do with our subject this week: modal logic.

Exercise **1**. **(4 pts)** We say that a tree is a *balanced binary tree* if all the non-leaves have exactly two children and all the branches are of the same length. Show that the property of being a balanced binary tree is not Hanf-local and henceforth not FOL-definable.

2 Some exercises on modal logic and Kripke validity

Write

- $\psi \Vdash \phi$ if for any $\mathfrak{A}, \bar{\mathsf{a}}$, we have that $\mathfrak{A}, \bar{\mathsf{a}} \Vdash \phi$.
- $\phi \models_{loc} \psi$ if $\mathsf{for\_all}\ \mathfrak{A}, \bar{\mathsf{a}}.(\mathfrak{A}, \bar{\mathsf{a}} \Vdash \phi\ \mathsf{implies}\ \mathfrak{A}, \bar{\mathsf{a}} \Vdash \psi)$
- $\phi \models_{glo} \psi$ if $\mathsf{for\_all}\ \mathfrak{A}.(\ \mathsf{for\_all}\ \bar{\mathsf{a}}.\mathfrak{A}, \bar{\mathsf{a}} \Vdash \phi)\ \mathsf{implies}\ (\ \mathsf{for\_all}\ \bar{\mathsf{a}}.\mathfrak{A}, \bar{\mathsf{a}} \Vdash \psi)$

Exercise **2.a** **(2 pts)** Is it true that $\phi \models_{loc} \psi$ iff $\Vdash \phi \to \psi$? Prove or give a counterexample.

Exercise **2.b** **(2 pts)** Is it true that $\phi \models_{glo} \psi$ iff $\Vdash \phi \to \psi$? Prove or give a counterexample.

Exercise **2.c** **(2 pts)** Is it true that $\phi \models_{loc} \Box\phi$? Prove or give a counterexample.

Exercise **2.d** **(2 pts)** Is it true that $\phi \models_{glo} \Box\phi$? Prove or give a counterexample.

Exercise **2.e** **(2 pts)** Example from the previous lecture: is it true that

$$\Box(\Diamond\phi \to \Diamond\psi) \models_{loc} \Diamond\Diamond\phi \to \Diamond\Diamond\psi$$

? Prove or give a counterexample.

Exercise **2.f** **(3 pts)** (bonus) Find counterexamples to $\Vdash \Box\phi \to \Box\Box\phi$, $\Vdash \phi \to \Box\Diamond\phi$, $\Vdash \Box(\Box\phi \to \phi) \to \Box\phi$ and $\Vdash \Box\phi \to \phi$.

Exercise **2.g** **(3 pts)** (bonus) How about the previous point when you consider only validity over relations which are, e.g., finite, transitive, reflexive, irreflexive, symmetric, linear orders or finite linear orders etc.? Can you think of necessary and sufficient conditions on binary relations interpreting modal connectives corresponding to validity of each of these formulas?

3 Beginning bisimulation equivalence

In all the exercises involving modal logic and bisimulations, we assume for simplicity, just like in the lecture, that we are dealing with modal similarity types only.

Do you remember the definition of $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{\infty} \mathfrak{B}, \mathsf{b}$ and $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$? (Note: unlike in the lecture, I'm using a superscript $\Diamond$, which you can freely drop if you want to) As I suggested, the bisimulation games play a similar role for ML (modal logic) as EF games for FOL. Just like in the first-order case, this can be made precise by showing that several superficially different notions are in fact equivalent.

Let us write $\mathfrak{A}, \mathsf{a} \leftrightsquigarrow^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$ to denote that $\mathfrak{A}, \mathsf{a}$ and $\mathfrak{B}, \mathsf{b}$ force exactly the same modal formulas of modal rank at most m.

Exercise **3.a** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \equiv_m \mathfrak{B}, \mathsf{b}$ implies $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$

Exercise **3.b** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \sim^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$ implies $\mathfrak{A}, \mathsf{a} \leftrightsquigarrow^{\Diamond}_{m} \mathfrak{B}, \mathsf{b}$

4 Alternative characterization of bisimulation

Just like in case of FO, the game theoretic characterization can be modified to a one with a slightly more algebraic flavour. Let us begin with a counterpart of the unrestricted notion $\sim^{\Diamond}_{\infty}$. We write $\mathbf{Z} : \mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^{\Diamond} \mathfrak{B}, \mathsf{b}$ to denote that for each $\mathsf{a} \in \mathsf{A}$ and $\mathsf{b} \in \mathsf{B}$, $\mathsf{a}Z\mathsf{b}$ implies that

- a and b agree on propositional symbols
- $\forall R_i \in \Sigma_\Diamond, (\mathsf{a}, \mathsf{a}') \in R_i^{\mathsf{A}}.\exists(\mathsf{b}, \mathsf{b}') \in R_i^{\mathfrak{B}}.\, \mathsf{a}' Z \mathsf{b}'$ (forth)
- $\forall R_i \in \Sigma_\Diamond, (\mathsf{b}, \mathsf{b}') \in R_i^{\mathsf{B}}.\exists(\mathsf{a}, \mathsf{a}') \in R_i^{\mathfrak{A}}.\, \mathsf{a}' Z \mathsf{b}'$ (back)

Now for the stratified version of this notion corresponding to $\sim_n^\Diamond$.

Let $\mathbf{Z} := \bigcup_{m \leq n} Z_m$. We write $\mathbf{Z} : \mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond{}_n \mathfrak{B}, \mathsf{b}$ to denote that for each $m \leq n$, $\mathsf{a} \in \mathsf{A}$ and $\mathsf{b} \in \mathsf{B}$:

- $\mathsf{a} Z_0 \mathsf{b}$ implies a and b agree on propositional symbols
- $$\mathsf{a} Z_{m+1} \mathsf{b} \text{ implies } \begin{cases} \forall R_i \in \Sigma_\Diamond, (\mathsf{a}, \mathsf{a}') \in R_i^{\mathsf{A}}.\exists(\mathsf{b}, \mathsf{b}') \in R_i^{\mathfrak{B}}.\, \mathsf{a}' Z_m \mathsf{b}' \text{ (forth)} \\ \text{and} \\ \forall R_i \in \Sigma_\Diamond, (\mathsf{b}, \mathsf{b}') \in R_i^{\mathsf{B}}.\exists(\mathsf{a}, \mathsf{a}') \in R_i^{\mathfrak{A}}.\, \mathsf{a}' Z_m \mathsf{b}' \text{ (back)} \end{cases}$$

In both cases, we can write $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond \mathfrak{B}, \mathsf{b}$ ($\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond{}_n \mathfrak{B}, \mathsf{b}$) to denote that there exists a suitable Z ($\mathbf{Z}$) satisfying these conditions.

Exercise **4.a** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond \mathfrak{B}, \mathsf{b}$ iff $\mathfrak{A}, \mathsf{a} \sim_\infty^\Diamond \mathfrak{B}, \mathsf{b}$

Exercise **4.b** **(2 pts)** Show that $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond{}_n \mathfrak{B}, \mathsf{b}$ iff $\mathfrak{A}, \mathsf{a} \sim_m^\Diamond \mathfrak{B}, \mathsf{b}$

5 Bonus exercise: what happens when games go on forever?

Exercise **5**. **(5 pts)** Can you produce examples of $\mathfrak{A}, \mathsf{a}$ and $\mathfrak{B}, \mathsf{B}$ s.t.

- $\mathfrak{A}, \mathsf{a} \equiv \mathfrak{B}, \mathsf{b}$ (and hence also for any n, $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond{}_n \mathfrak{B}, \mathsf{b}$, $\mathfrak{A}, \mathsf{a} \sim_m^\Diamond \mathfrak{B}, \mathsf{b}$ and $\mathfrak{A}, \mathsf{a} \leftrightsquigarrow_m^\Diamond \mathfrak{B}, \mathsf{b}$) but
- it is **not** true that $\mathfrak{A}, \mathsf{a} \underline{\leftrightarrow}^\Diamond \mathfrak{B}, \mathsf{b}$?