LGruDat: Logical Foundations of Databases
Exercise 4 **and** 5, Deadline 6 Dec 2013

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- The present exercise sheet comprises the previous one and two exercises added after the introductory lecture on E-F games in Sections 6 below. Exercise 5 has been corrected after Thorsten spotted a nasty mistake in the previous version (he will be awarded additional points for that).

  The new lecture abstract for week 5 (beginning EF games) has been published online.

  Please submit your **pick of exercises that would sum up to 19 points**, this obviously including whatever you managed to do from the Blatt 4 alone so far.

  If you can, please have a look at these two newly added exercises, especially 6. Things may be a bit quicker next week if you warm up on them.

1 Connectedness

Exercise **1**. **(5 pts)** Take a signature $\Sigma = \{R\}$, where R is a binary relation symbol. Take the standard notion of *connectedness* from graph theory (email me if this is unclear ... or just google it out ...). Use compactness to show that the unrestricted class of connected graphs is not EC_{Δ}.

Hint. In the proof, you'll be well-advised to consider a language extending Σ with two individual constants.

2 Finite validity

Exercise **2**. **(5 pts)** During the lecture, we mentioned without proof Trakthenbrot's Theorem, according to which the set of sentences valid over finite structures is not recursively enumerable. Discuss whether this set is or can possibly be *co*-recursively enumerable, i.e., whether there is a Turing machine outputting all sentences (in a fixed finite signature Σ) with a finite *countermodel*.

3 Trivial cases of decidability in the finite

Exercise **3**. **(4 pts)** In our reminder lectures on unrestricted model theory, we often used examples of theories containing a sentence

"there are no more than n elements"

(for some $n \in \mathbb{N}$) as particularly well-behaved ones; for example, we noted that **given our standing assumption of finiteness of** Σ they only have finitely many maximal consistent extensions. Actually, the **decidability** of theories containing such a sentence can be proved in a fairly straightforward way. Sketch the argument.

4 Lemma on constants

This is a little and rather obvious technicality, but often used in model theory. Most of the time when we write "$\Gamma \mid\!\frac{\text{unr}}{loc}\, \alpha$", "$\Gamma \mid\!\frac{\text{unr}}{glo}\, \alpha$", "$\Gamma \models\!\frac{\text{unr}}{loc}\, \alpha$", "$\Gamma \models\!\frac{\text{unr}}{glo}\, \alpha$", we implicitly assume α and Γ are in the same signature Σ. Indeed, otherwise, e.g., definitions such as $\mathsf{Ded}(\Gamma) := \{\alpha \mid \Gamma \mid\!\frac{\text{unr}}{loc}\, \alpha\}$ and $\mathsf{Ded}^0(\Gamma) := \{\alpha \text{ a sentence } \mid \Gamma \mid\!\frac{\text{unr}}{loc}\, \alpha\}$ would be ambiguous or simply wrong. However, we might sometimes be interested in a situation when the signature of α properly extends that of Γ, especially with finitely many new constants; indeed, this is precisely what happens when we consider (positive) diagrams of finite structures, for example.

Either ...

Exercise **4.a** **(2 pts)** Show first a lemma which is useful in its own right:

Take a model $\mathfrak{A}$ adequate for Σ, a sequence of elements $\mathsf{a}_1, \ldots, \mathsf{a}_n \in \mathsf{A}$ and a formula $\alpha(x_1, \ldots, x_n) \in \mathsf{FORC}(\Sigma)$ where the notation $\alpha(x_1, \ldots, x_n)$ denotes as usual that $FV(\alpha) = \{x_1, \ldots, x_n\}$. Define $(\mathfrak{A}, \mathsf{a}_1, \ldots, \mathsf{a}_n)$ to be a model adequate for the signature $\Sigma \cup \{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$ where the new constants in the language are (names of) elements $\mathsf{a}_1, \ldots, \mathsf{a}_n$; in fact, it is customary to identify the element and the corresponding constant in this context. Show that

$$\mathfrak{A}, (\kappa[x_1 := \mathsf{a}_1] \ldots)[x_n := \mathsf{a}_n] \models \alpha$$

iff

$$(\mathfrak{A}, \mathsf{a}_1, \ldots, \mathsf{a}_n), \kappa \models \alpha(\mathsf{a}_1/x_1, \ldots \mathsf{a}_n/x_n)$$

and derive that whenever $\Gamma \models\!\frac{\text{unr}}{loc}\, \alpha(c_1/x_1, \ldots, c_n/x_n)$ for some constants $c_1, \ldots, c_n$ **not occurring** in Γ, then $\Gamma \models\!\frac{\text{unr}}{glo}\, \forall x_1 \ldots x_n.\alpha(x_1, \ldots, x_n)$.

... or ...

Exercise **4.b** **(2 pts)** Show a syntactical counterpart of this result: that whenever $\Gamma \vdash^{\text{unr}}_{loc} \alpha(c_1/x_1, \ldots, c_n/x_n)$ for some constants $c_1, \ldots, c_n$ **not occurring** in Γ, then $\Gamma \vdash^{\text{unr}}_{glo} \forall x_1 \ldots x_n . \alpha(x_1, \ldots, x_n)$

Of course, if you want you can prove both cases separately. Were we informed of any results that allow to derive one from the other?

5 Unrestricted preservation theorems

As announced before, the originally posted version of this exercise had a nasty mistake

In case you didn't notice: we have enough apparatus now to attack difficult directions of unrestricted preservation theorems!

Given a class of models $\mathcal{K}$ (recall again that our classes of models are always closed under isomorphism), denote by $\mathsf{S}(\mathcal{K})$ the closure of $\mathcal{K}$ under (isomorphic copies of) submodels. Also, denote by $\mathsf{Ded}_\forall(\Gamma)$ the subset of $\mathsf{Ded}^0(\Gamma)$ consisting of universal sentences only.

Exercise **5.a** **(5 pts)** Show that for any set of sentences Γ

$$\mathsf{Mod}(\mathsf{Ded}_\forall(\Gamma)) = \mathsf{S}(\mathsf{Mod}(\Gamma)).$$

The previous version of this one involving *submodel-insensitivity* was nonsensical, with an unnecessary equivalence which blatantly failed to hold. Thanks to Thorsten for spotting that!

Hint. The exercise consists of two inclusions, the easy and the difficult one. The proof of the easy one is in fact the same as that of the easy direction of Łoś-Tarski. In the difficult one, you may like to use: the notion of diagram, compactness and Exercise 4 above.

Exercise **5.b** **(3 pts)** Deduce as a corollary the following somewhat generalized form of the **Łoś-Tarski Theorem**: for any FORC theory T and any sentence γ, the following conditions are equivalent:

- there exists an universal sentence γ' s.t. $T \vdash \gamma \leftrightarrow \gamma'$
- $\mathsf{Mod}(T) \cap \mathsf{S}(\mathsf{Mod}(\{\gamma\})) = \mathsf{Mod}(T \cup \{\gamma\})$

Exercise **5.c** **(2 pts)** Use contraposition and derive a dual version of (b) above, with universal sentences replaced by existential ones and (isomorphic copies of) submodels suitably replaced by embeddings.

Exercise **5.d** **(7 pts)** Do you see how to adapt to adapt notions, definitions and proofs used above to obtain analogous characterizations for sentences reflected or preserved by **homomorphisms**? If so, state and prove formally counterparts of clauses (a)—(c) above. Pay attention to details, both syntactic and semantic ones (although, as usual, you don't have to be too verbose if it's really clear what you mean).

6 The easy direction of the Ehrenfeucht Theorem

Exercise **6**. **(3 pts)** Prove the easy direction of Ehrenfeucht Theorem:

$$\mathfrak{A}, \bar{\mathsf{a}} \simeq_m \mathfrak{B}, \bar{\mathsf{b}} \text{ implies } \mathfrak{A}, \bar{\mathsf{a}} \equiv_m \mathfrak{B}, \bar{\mathsf{b}}$$

The back-and-forth clause of the Lemma we proved during the last lecture and which you can find in newly posted lecture abstract for week 5 may prove very useful here.

7 EF on linear orders

At the end of the lecture we discussed how many elements a linear order must contain to guarantee duplicator's win. For simplicity, let $\Sigma = \{<\}$ and consider $\underline{n}$ to be the a model for this signature whose underlying domain is $\{0, \ldots, n-1\}$ and the order is the natural strict order.

Exercise **7**. **(4 pts)** Pick any two of the items below you like and try to answer:

- Is $\underline{30} \simeq_5 \underline{1042}$?
- Is $\underline{31} \simeq_5 \underline{142}$?
- Is $\underline{62} \simeq_6 \underline{1042}$?
- Is $\underline{63} \simeq_6 \underline{142}$?